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Best proximity points for alternative p-contractions

Mi Zhou[1,2,3,4], Nicolae Adrian Secelean[5], Naeem Saleem[6,7*] and Mujahid Abbas[8*]

*Correspondence:
naeem.saleem2@gmail.com;
abbas.mujahid@gcu.edu.pk
[6]Department of Mathematics, University of Management and Technology, Lahore 54700, Pakistan
[8]Department of Mathematics, Government College University, Katchery Road, Lahore 54000, Pakistan
Full list of author information is available at the end of the article

Abstract

Cyclic mappings describe fixed paths for which each point is sequentially transmitted from one set to another. Cyclic mappings satisfying certain cyclic contraction conditions have been used to obtain the best proximity points, which constitute a suitable framework for the mirror reflection model. Alternative contraction mappings introduced by Chen (Symmetry 11:750, 2019) built a new model containing several mirrors in which the light reflected from a mirror does not go to the next mirror sequentially, and its path may diverge to any other mirror. The aim of this paper is to present a new variant of alternative contraction called alternative p-contraction and study its properties. The best proximity point result for such contractions under the alternative UC property is proved. An example to support the result proved herein is provided.

Mathematics Subject Classification: Primary 47H10; secondary 47H09; 54H25

Keywords: Best proximity point; Alternative UC property; Alternative p-contraction

1 Introduction and preliminaries

The Banach contraction principle (BCP) [8] plays a significant role in metric fixed point theory and has been extended and generalized in several directions. For example, Kirk *et al.* [21] presented an interesting generalization of the Banach contraction principle by introducing cyclic mappings.

Definition 1.1 [21] Let $(\mathfrak{W}, \mathfrak{d})$ be a metric space, $\mathbb{G}_i$ be nonempty subsets of $\mathfrak{W}$, where $i = 1, 2, \ldots, m$. A mapping $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is a cyclic mapping if $\Gamma(\mathbb{G}_i) \subset \mathbb{G}_{i+1}$ for $i = 1, 2, \ldots, m$ and $\mathbb{G}_{m+1} = \mathbb{G}_1$.

Theorem 1.2 [21] *Let $(\mathfrak{W}, \mathfrak{d})$ be a complete metric space, $\mathbb{G}$, $\mathbb{J}$ be two closed subsets of $\mathfrak{W}$, and $\Gamma : \mathbb{G} \cup \mathbb{J} \to \mathbb{G} \cup \mathbb{J}$ be a cyclic mapping, that is, $\Gamma(\mathbb{G}) \subset \mathbb{J}$ and $\Gamma(\mathbb{J}) \subset \mathbb{G}$. If there exists $k \in (0, 1)$ such that $\mathfrak{d}(\Gamma\mathfrak{x}, \Gamma\mathfrak{y}) \le k\mathfrak{d}(\mathfrak{x}, \mathfrak{y})$ for all $\mathfrak{x} \in \mathbb{G}$ and $\mathfrak{y} \in \mathbb{J}$, then Γ has a fixed point in $\mathbb{G} \cap \mathbb{J}$.*

Cyclic mappings in the above theorem are a dynamic system which describes the trajectories between two state spaces along which each state moves from one state space to another iteratively. Fixed point theorems for such mappings give necessary conditions for existence of an equilibrium state that is common to both state spaces (see [15, 18, 19, 23, 24, 29, 31] and the references given there).

By considering non-self mappings, another interesting extension of *BCP* was obtained in [9].

Let $\emptyset \neq \mathbb{G}, \mathbb{J}$ be two sets in $(\mathfrak{W}, \mathfrak{d})$ and $\Gamma : \mathbb{G} \to \mathbb{J}$. If $\mathbb{G} \cap \mathbb{J} = \emptyset$, then Γ becomes a fixed point free mapping. Then it is desired to find a point $\mathfrak{x}^*$ in $\mathbb{G}$ which is closest to $\Gamma\mathfrak{x}^*$ in $\mathbb{J}$, that is,

$$\mathfrak{x}^* = \arg\min \mathfrak{d}(\mathfrak{x}, \Gamma\mathfrak{x}).$$

Since $\mathfrak{d}(\mathfrak{x}, \Gamma\mathfrak{x}) \geq \mathfrak{d}(\mathbb{G}, \mathbb{J}) := \inf\{\mathfrak{d}(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a} \in \mathbb{G}, \mathfrak{b} \in \mathbb{J}\}$ holds for all $\mathfrak{x} \in \mathbb{G}$, it follows that $\mathfrak{d}(\mathbb{G}, \mathbb{J})$ is the lower bound of the set $\{\mathfrak{d}(\mathfrak{x}, \Gamma\mathfrak{x}) : \mathfrak{x} \in A\}$. A point $\mathfrak{x}^*$ in A such that $\mathfrak{d}(\mathfrak{x}^*, \Gamma\mathfrak{x}^*) = \mathfrak{d}(\mathbb{G}, \mathbb{J})$ is called a best proximity point of Γ and is an optimal solution to the following optimization problem:

$$\min_{\mathfrak{x} \in \mathbb{G}} \mathfrak{d}(\mathfrak{x}, \Gamma\mathfrak{x}).$$

For more results in this direction, we refer to [2, 4, 5, 14, 20, 27, 28].

In case $\mathbb{G} = \mathbb{J} = \mathfrak{W}$, the best proximity point becomes a fixed point of Γ and hence the best proximity point results can be viewed as a potential generalization of fixed point results.

Combining the ideas of both cyclic mappings and best proximity point, Eldred and Veeramani [12] introduced the notion of cyclic contraction mapping for two subsets of $\mathfrak{W}$ and then obtained a best proximity theorem.

Definition 1.3 [12] Let A and B be nonempty subsets of a metric space $(\mathfrak{W}, \mathfrak{d})$. A mapping $\Gamma : \mathbb{G} \cup \mathbb{J} \to \mathbb{G} \cup \mathbb{J}$ is called a cyclic contraction if $\Gamma(\mathbb{G}) \subset \mathbb{J}$ and $\Gamma(\mathbb{J}) \subset \mathbb{G}$ and the following condition holds:

$$\mathfrak{d}(\Gamma\mathfrak{x}, \Gamma\mathfrak{y}) \leq \alpha\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) + (1 - \alpha)\mathfrak{d}(\mathbb{G}, \mathbb{J})$$

for all $\mathfrak{x} \in \mathbb{G}$ and $\mathfrak{y} \in \mathbb{J}$, where $\alpha \in (0, 1)$ and $\mathfrak{d}(\mathbb{G}, \mathbb{J}) = \inf\{\mathfrak{d}(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a} \in \mathbb{G}, \mathfrak{b} \in \mathbb{J}\}$.

Theorem 1.4 [12] *Let $\mathbb{G}$ and $\mathbb{J}$ be nonempty closed subsets of a metric space $(\mathfrak{W}, \mathfrak{d})$ and $\Gamma : \mathbb{G} \cup \mathbb{J} \to \mathbb{G} \cup \mathbb{J}$ be a cyclic contraction. If either $\mathbb{G}$ or $\mathbb{J}$ is boundedly compact, then there exists $\mathfrak{x} \in \mathbb{G} \cup \mathbb{J}$ with $\mathfrak{d}(\mathfrak{x}, \Gamma\mathfrak{x}) = \mathfrak{d}(\mathbb{G}, \mathbb{J})$.*

Various cyclic contractions were defined for n subsets using a sequential pattern, as outlined by Kirk *et al.* [21] and Eldred and Veeramani [12], employing the 2-sets methodology. Examples of such contractions can be found in [1, 16, 17, 22, 25, 30] and the relevant literature.

In contrast, Chen [11] introduced the concepts of alternative maps and alternative contractions, which extend beyond the scope of cyclic mapping and do not necessitate a sequential pattern. Furthermore, drawing inspiration from the *UC* property stated in [29] and Fan's best proximity point theorem mentioned in [13], the alternative *UC* condition was introduced and the nonsequential best proximity point theorem was derived.

Definition 1.5 [29] Let $\mathbb{G}$ and $\mathbb{J}$ be nonempty subsets of a metric space $(\mathfrak{W}, \mathfrak{d})$. Then $(\mathbb{G}, \mathbb{J})$ is said to satisfy the *UC* property if for every two sequences $\{\mathfrak{x}_n\}$ and $\{\mathfrak{x}'_n\}$ in $\mathbb{G}$ and each $\{\mathfrak{y}_n\}$ in $\mathbb{J}$ such that $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{x}_n, \mathfrak{y}_n) = \mathfrak{d}(\mathbb{G}, \mathbb{J})$ and $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{x}'_n, \mathfrak{y}_n) = \mathfrak{d}(\mathbb{G}, \mathbb{J})$, then $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{x}_n, \mathfrak{x}'_n) = 0$.

It is obvious that if $\mathfrak{d}(\mathbb{G}, \mathbb{J}) = 0$, then $(\mathbb{G}, \mathbb{J})$ satisfies the *UC* property.

In light of the *UC* condition proposed in reference [29], Chen [11] introduced a novel approach by extending this concept to a finite collection of nonempty subsets within a metric space, thus offering an alternative *UC* condition.

Definition 1.6 [11] Let $(\mathfrak{W}, \mathfrak{d})$ be a metric space, $\mathbb{G}_i, i = 1, 2, \dots, m$, be nonempty subsets of $\mathfrak{W}$ and, for every $n \in \mathbb{N}$, $\mathfrak{y}_n^1, \mathfrak{y}_n^\nu \in \mathbb{G}_r$ and $\mathfrak{y}_n^2, \mathfrak{y}_n^3, \dots, \mathfrak{y}_n^{\nu-1} \in \bigcup_{j\neq r} \mathbb{G}_j$ for some positive integer $\nu \geq 2$. We say that the family $\{\mathbb{G}_i\}_{i=1}^m$ satisfies the alternative *UC* condition if the following holds: if

$$\lim_{n\to\infty} \mathfrak{d}\big(\mathfrak{y}_n^1, \mathfrak{y}_n^2\big) = \lim_{n\to\infty} d\big(\mathfrak{y}_n^2, \mathfrak{y}_n^3\big) = \cdots = \lim_{n\to\infty} d\big(\mathfrak{y}_n^{\nu-1}, \mathfrak{y}_n^\nu\big) = \mathfrak{d}(\mathbb{G}, \mathbb{J})$$
$$\text{for some } \mathbb{G}, \mathbb{J} \in \{\mathbb{G}_i\}_{i=1}^m,$$

then $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{y}_n^1, \mathfrak{y}_n^\nu) = 0$.

To give an example, we need the following elementary result.

Lemma 1.7 *Let us consider $a, b \in \mathbb{R}$ and two sequences of real numbers $\{x_n\}_n$, $\{y_n\}_n$ such that $x_n \leq a$, $y_n \geq b$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathfrak{d}(y_n - x_n) = b - a$. Then*

$$\lim_{n\to\infty} x_n = a \quad \textit{and} \quad \lim_{n\to\infty} y_n = b.$$

Proof

$$0 \leq (a - x_n) + (y_n - b) = (y_n - x_n) - (b - a) \longrightarrow 0.$$

Since $a - x_n \geq 0$ and $y_n - b \geq 0$, we deduce that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$. □

Example 1.8 Let $\mathfrak{W} = \mathbb{R}$ be endowed with the Euclidean metric σ and $\mathbb{G}_1 = [0, 1]$, $\mathbb{G}_2 = [-2, -1]$, $\mathbb{G}_3 = \{0\}$, $\mathbb{G}_4 = [1, 2]$. Then $\{\mathbb{G}_i\}_{i=1}^4$ satisfies the alternative *UC* condition.

Proof Let us consider the following sequences: $\{\mathfrak{y}_n^1\}, \{\mathfrak{y}_n^5\} \subset \mathbb{G}_1$, $\{\mathfrak{y}_n^2\} \subset \mathbb{G}_2$, $\{\mathfrak{y}_n^3\} \subset \mathbb{G}_3$, $\{\mathfrak{y}_n^4\} \subset \mathbb{G}_4$, and suppose that

$$\lim_{n\to\infty} \big|\mathfrak{y}_n^2 - \mathfrak{y}_n^1\big| = \lim_{n\to\infty} \big|\mathfrak{y}_n^3 - \mathfrak{y}_n^2\big| = \lim_{n} \big|\mathfrak{y}_n^4 - \mathfrak{y}_n^3\big| = \lim_{n\to\infty} \big|\mathfrak{y}_n^5 - \mathfrak{y}_n^4\big| = \mathfrak{d}(\mathbb{G}_3, \mathbb{G}_4) = 1.$$

Thus, by Lemma 1.7, one obtains successively

$$\mathfrak{y}_n^1 \to 0, \qquad \mathfrak{y}_n^2 \to -1, \qquad \mathfrak{y}_n^3 = 0, \qquad \mathfrak{y}_n^4 \to 1, \qquad \mathfrak{y}_n^5 \to 0.$$

Therefore $\lim_{n\to\infty} |\mathfrak{y}_n^1 - \mathfrak{y}_n^5| = 0$. □

Definition 1.9 [11] Let $(\mathfrak{W}, \mathfrak{d})$ be a metric space, $\mathbb{G}_i, i = 1, 2, \ldots, m$, be nonempty subsets of $\mathfrak{W}$. A map $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is called an alternative map if $\Gamma(\mathbb{G}_i) \subseteq \bigcup_{j \neq i} \mathbb{G}_j$ for $i = 1, 2, \ldots, m$. A mapping $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is called an alternative contraction (AC) if Γ is an alternative map and there exists a constant $\alpha \in [0, 1)$ such that for any $\mathfrak{x} \in \mathbb{G}_j$, $\mathfrak{y} \in \mathbb{G}_k$ for some $j, k \in \{1, \ldots, m\}$, the following condition holds:

$$\mathfrak{d}(\Gamma\mathfrak{x}, \Gamma\mathfrak{y}) \leq \alpha \mathfrak{d}(\mathfrak{x}, \mathfrak{y}) + (1 - \alpha)\mathfrak{d}(\mathbb{G}_j, \mathbb{G}_k). \tag{1}$$

As cited in [11], the classical result of the best proximity point can be described using a model of two mirrors reflecting each other, where a cyclic contraction is employed to specify the path that the light takes from one mirror to the other. Suzuki *et al.* [29] demonstrated that with cyclic contraction and the UC condition, the best proximity points are the brightest points obtained through repeated reflection between the two mirrors and infinite bouncing. Conversely, the alternative map transforms the two-mirror model into one that involves multiple mirrors, where the light, after being reflected from a mirror, may not necessarily return and its path may diverge in any given direction. Chen [11] proved that under the conditions of the proposed alternative contraction and the UC condition, the best proximity points of the alternative map will manifest as shining points in the multi-mirror reflection model.

Theorem 1.10 [11, *Theorem* 7] *Let* $(\mathfrak{W}, \mathfrak{d})$ *be a complete metric space and* $\{\mathbb{G}_i\}_{i=1}^m$ *be a family of nonempty and closed subsets of* $\mathfrak{W}$. *Assume that* $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ *is an AC with UC condition. Then there exists a best proximity point of* Γ.

On the other side, Popescu [26] introduced a new type of contraction mapping that generalized BCP, as follows.

Definition 1.11 [26] Let $(\mathfrak{W}, \mathfrak{d})$ be a metric space. A mapping $\Gamma : \mathfrak{W} \to \mathfrak{W}$ is said to be a p-contraction if there exists a real number $k \in [0, 1)$ such that

$$\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) \leq k\big[\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) + \big|\mathfrak{d}(\mathfrak{x}, \Gamma\mathfrak{x}) - \mathfrak{d}(\mathfrak{y}, \Gamma\mathfrak{y})\big|\big] \tag{2}$$

for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{W}$.

Note that if we set $k = 1$ in (2) and the inequality is strict for all $\mathfrak{x} \neq \mathfrak{y} \in \mathfrak{W}$, then the mapping Γ is said to be a p-contractive mapping introduced in [3].

Theorem 1.12 [26, *Theorem* 2.2] *If* $(\mathfrak{W}, \mathfrak{d})$ *is a complete metric space and* Γ *is a p-contraction on* $\mathfrak{W}$, *then* Γ *has a unique fixed point* $\mathfrak{x}^*$ *and the sequence* $\{\Gamma^n\mathfrak{x}\}$ *converges to* $\mathfrak{x}^*$ *for every* $\mathfrak{x} \in \mathfrak{W}$.

Recently, Aslantas *et al.* [6] introduced a new concept called cyclic p-contraction pair and demonstrated its application in obtaining best proximity point results. They unified the concepts of p-contraction and cyclic contraction to achieve this. The authors also established conditions that ensure the existence and uniqueness of a common solution for a system of second order boundary value problems. These conditions are based on the fixed point consequence of their main results. For further information, please refer to [6].

The objective of this paper is to present the concept of alternative p-contraction by merging the ideas of p-contraction and alternative contraction. Additionally, this paper aims to provide a comprehensive presentation of best proximity point results based on the work of Chen [11]. The outcomes of this paper expand, unify, and generalize various fixed point and best proximity point theorems found in current academic literature.

2 Alternative *p*-contractions and its properties

Inspired by [11], in this section we introduce the alternative p-contraction and study its properties. Throughout this section $(\mathfrak{W},\mathfrak{d})$ will be a metric space and $\{\mathbb{G}_i\}_{i=1}^m$ is a family of nonempty subsets of it, where $m\geq 2$.

Definition 2.1 A map $\Gamma:\bigcup_{i=1}^m\mathbb{G}_i\to\bigcup_{i=1}^m\mathbb{G}_i$ is called an alternative p-contraction if Γ is an alternative map and there exists a right-continuous mapping $\alpha:[0,\infty)\to[0,1)$ with $\alpha(0)=0$ such that, for every $j,k\in\{1,\dots,m\}$ and $\mathfrak{x}\in\mathbb{G}_j$, $\mathfrak{y}\in\mathbb{G}_k$, the following condition holds:

$$\mathfrak{d}(\Gamma\mathfrak{x},\Gamma\mathfrak{y})\leq\alpha\big(\mathfrak{d}(\mathfrak{x},\mathfrak{y})\big)\big(\mathfrak{d}(\mathfrak{x},\mathfrak{y})+\big|\mathfrak{d}(\mathfrak{x},\Gamma\mathfrak{x})-\mathfrak{d}(\mathfrak{y},\Gamma\mathfrak{y})\big|\big)+\big[1-\alpha\big(\mathfrak{d}(\mathfrak{x},\mathfrak{y})\big)\big]\mathfrak{d}(\mathbb{G}_j,\mathbb{G}_k). \tag{3}$$

Lemma 2.2 *If $\Gamma:\bigcup_{i=1}^m\mathbb{G}_i\to\bigcup_{i=1}^m\mathbb{G}_i$ is an alternative p-contraction, then for each $\mathfrak{x}\in\bigcup_{i=1}^m\mathbb{G}_i$, the sequence $\{\mathfrak{d}(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})\}$ is decreasing.*

Proof As Γ is an alternative p-contraction, for every $\mathfrak{x}\in\bigcup_{i=1}^m\mathbb{G}_i$ and $n\in\mathbb{N}$, one can find $i(n)\in\{1,\dots,m\}$ such that $\Gamma^n\mathfrak{x}\in\mathbb{G}_{i(n)}$.

By hypothesis, there exists a mapping $\alpha:[0,\infty)\to[0,1)$ such that

$$\begin{aligned}\mathfrak{d}\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)&=\mathfrak{d}\big(\Gamma\Gamma^n\mathfrak{x},\Gamma\Gamma^{n-1}\mathfrak{x}\big)\\&\leq\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)+\big|\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n+1}\mathfrak{x}\big)-\mathfrak{d}\big(\Gamma^{n-1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big|\big)\\&\quad+\big[1-\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big]\mathfrak{d}(\mathbb{G}_{i(n-1)},\mathbb{G}_{i(n)}).\end{aligned}\tag{4}$$

We divide the proof into two cases as follows:

Case 1. If $\mathfrak{d}(\Gamma^n\mathfrak{x},\Gamma^{n+1}\mathfrak{x})\geq\mathfrak{d}(\Gamma^{n-1}\mathfrak{x},\Gamma^n\mathfrak{x})$, then (4) becomes

$$\begin{aligned}\mathfrak{d}\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)&=\mathfrak{d}\big(\Gamma\Gamma^n\mathfrak{x},\Gamma\Gamma^{n-1}\mathfrak{x}\big)\\&\leq\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)+\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n+1}\mathfrak{x}\big)-\mathfrak{d}\big(\Gamma^{n-1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big)\\&\quad+\big[1-\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big]\mathfrak{d}(\mathbb{G}_{i(n-1)},\mathbb{G}_{i(n)})\\&\leq\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n+1}\mathfrak{x}\big)+\big[1-\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big]\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big),\end{aligned}$$

which implies that $\mathfrak{d}(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})\leq\mathfrak{d}(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x})$.

Case 2. If $\mathfrak{d}(\Gamma^n\mathfrak{x},\Gamma^{n+1}\mathfrak{x})<\mathfrak{d}(\Gamma^{n-1}\mathfrak{x},\Gamma^n\mathfrak{x})$, then (4) becomes

$$\begin{aligned}\mathfrak{d}\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)&=\mathfrak{d}\big(\Gamma\Gamma^n\mathfrak{x},\Gamma\Gamma^{n-1}\mathfrak{x}\big)\\&\leq\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)-\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n+1}\mathfrak{x}\big)+\mathfrak{d}\big(\Gamma^{n-1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big)\\&\quad+\big[1-\alpha\big(\mathfrak{d}\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big]\mathfrak{d}(\mathbb{G}_{i(n-1)},\mathbb{G}_{i(n)})\end{aligned}$$

$$
\begin{aligned}
&\leq 2\alpha\big(\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)-\alpha\big(\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\\
&\quad+\big[1-\alpha\big(\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big]\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\\
&=\big(1+\alpha\big(\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\big)\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)-\alpha\big(\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x}\big)\big)\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big),
\end{aligned}
$$

which also implies that $\eth(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})\leq\eth(\Gamma^n\mathfrak{x},\Gamma^{n-1}\mathfrak{x})$.

Hence $\{\eth(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})\}$ is a decreasing sequence. □

Proposition 2.3 *Let $\Gamma:\bigcup_{i=1}^m\mathbb{G}_i\to\bigcup_{i=1}^m\mathbb{G}_i$ be an alternative p-contraction. Then, for each $\mathfrak{x}\in\bigcup_{i=1}^m\mathbb{G}_i$, the sequence $\{\eth(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})\}_n$ is convergent.*

Proof The sentence follows by Lemma 2.2 taking into account that $\{\eth(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})\}_n$ is a sequence of nonnegative numbers. □

Proposition 2.4 *Let $\Gamma:\bigcup_{i=1}^m\mathbb{G}_i\to\bigcup_{i=1}^m\mathbb{G}_i$ be an alternative p-contraction. Then, for every $\mathfrak{x}\in\bigcup_{i=1}^m\mathbb{G}_i$, we have*

$$\lim_{n\to\infty}\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\in\big\{\eth(\mathbb{G}_\alpha,\mathbb{G}_\beta)\big\}_{1\leq\alpha\neq\beta\leq m}.$$

Proof Assume that $\Gamma^n\mathfrak{x}\in\mathbb{G}_{i(n)}$, $\Gamma^{n+1}\mathfrak{x}\in\mathbb{G}_{i(n+1)}$ for some $i(n),i(n+1)\in\{1,2,\dots,m\}$. Since Γ is an alternative map, it follows that $\Gamma^{n+1}\mathfrak{x}\in\bigcup_{j\neq i(n)}\mathbb{G}_j$, and so we can suppose that $i(n)\neq i(n+1)$.

Consider the right-continuous map $\alpha:[0,\infty)\to(0,1)$ such that (3) is satisfied for $(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})$. One has

$$
\begin{aligned}
\eth\big(\Gamma^{n+2}\mathfrak{x},\Gamma^{n+1}\mathfrak{x}\big)&=\eth\big(\Gamma\Gamma^{n+1}\mathfrak{x},\Gamma\Gamma^n\mathfrak{x}\big)\\
&\leq\alpha\big(\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big)\big(\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)+\big|\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^{n+2}\mathfrak{x}\big)-\eth\big(\Gamma^n\mathfrak{x},\Gamma^{n+1}\mathfrak{x}\big)\big|\big)\\
&\quad+\big[1-\alpha\big(\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big)\big]\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)})\\
&=2\alpha\big(\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big)\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)-\alpha\big(\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big)\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^{n+2}\mathfrak{x}\big)\\
&\quad+\big[1-\alpha\big(\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)\big)\big]\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)}).
\end{aligned}
\tag{5}
$$

On taking an upper limit on the both sides of (5) and $c=\lim_{n\to\infty}\eth(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})$, we obtain

$$c\leq\alpha(c)\cdot c+\big(1-\alpha(c)\big)\limsup_{n\to\infty}\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)})\quad\Leftrightarrow\quad c\leq\limsup_{n\to\infty}\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)}).\tag{6}$$

For every $n\in\mathbb{N}$, since $\Gamma^n\mathfrak{x}\in\mathbb{G}_{i(n)}$, $\Gamma^{n+1}\mathfrak{x}\in\mathbb{G}_{i(n+1)}$, we have $\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)})\leq\eth(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x})$ and

$$\limsup_{n\to\infty}\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)})\leq\limsup_{n\to\infty}\eth\big(\Gamma^{n+1}\mathfrak{x},\Gamma^n\mathfrak{x}\big)=c.\tag{7}$$

By (5) and (7), one obtains

$$c=\lim_{n\to\infty}\sup\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)}).\tag{8}$$

Since the family $\{\mathbb{G}_i\}_{i=1}^m$ has at most $\frac{m(m-1)}{2}$ elements, it follows that $\limsup_{n\to\infty}\eth(\mathbb{G}_{i(n+1)},\mathbb{G}_{i(n)})=\eth(\mathbb{G}_\alpha,\mathbb{G}_\beta)$ for some $1\leq\alpha\neq\beta\leq m$. Consequently, $c=\eth(\mathbb{G}_\alpha,\mathbb{G}_\beta)$. □

Proposition 2.5 *Assume that $\mathbb{G}_i, i = 1, 2, \dots, m$, are subsets of $\mathfrak{W}$ and $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is an alternative p-contraction. If there exists $l \in \mathbb{N}$ such that $\lim_{k\to\infty} \Gamma^{n_k+r}\mathfrak{x} = p_r(\mathfrak{x})$, $r = 0, 1, 2, \dots, l$, for some $\mathfrak{x} \in \mathfrak{W}$ and some subsequence $\{n_k\} \subset \mathbb{N}$, then the following conditions are equivalent*:

(i) $\eth(p_0(\mathfrak{x}), p_1(\mathfrak{x})) \le \lim_{n\to\infty} \eth(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x})$;
(ii) $\eth(p_0(\mathfrak{x}), p_1(\mathfrak{x})) = \lim_{n\to\infty} \eth(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x})$;
(iii) $\eth(p_l(\mathfrak{x}), p_{l+1}(\mathfrak{x})) \le \cdots \le \eth(p_2(\mathfrak{x}), p_1(\mathfrak{x})) \le \eth(p_1(\mathfrak{x}), p_0(\mathfrak{x})) \le \lim_{n\to\infty} \eth(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x})$;
(iv) $\eth(p_l(\mathfrak{x}), p_{l+1}(\mathfrak{x})) = \cdots = \eth(p_2(\mathfrak{x}), p_1(\mathfrak{x})) = \eth(p_1(\mathfrak{x}), p_0(\mathfrak{x})) = \lim_{n\to\infty} \eth(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x})$.

Proof It is straightforward to show that (ii) ⇒ (i), (iv) ⇒ (iii), (iv) ⇒ (ii) and (iii) ⇒ (i).

We now prove that (i) ⇒ (iv). Since, by Lemma 2.2, $\{\eth(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x})\}_n$ is a decreasing sequence, it follows that

$$\begin{aligned}\eth\bigl(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}\bigr) &\le \eth\bigl(\Gamma^{n_k+l}\mathfrak{x}, \Gamma^{n_k+l-1}\mathfrak{x}\bigr) \le \cdots \le \eth\bigl(\Gamma^{n_k+2}\mathfrak{x}, \Gamma^{n_k+1}\mathfrak{x}\bigr) \\ &\le \eth\bigl(\Gamma^{n_k+1}\mathfrak{x}, \Gamma^{n_k}\mathfrak{x}\bigr), \quad \forall n \ge n_k.\end{aligned} \tag{9}$$

Since $p_r(\mathfrak{x}) = \lim_{k\to\infty} \Gamma^{n_k+r}\mathfrak{x}$, $r = 0, 1, 2, \dots, l$, passing the limit in (9), we get

$$\lim_{n\to\infty} \eth\bigl(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}\bigr) \le \eth\bigl(p_l(\mathfrak{x}), p_{l-1}(\mathfrak{x})\bigr) \le \cdots \le \eth\bigl(p_2(\mathfrak{x}), p_1(\mathfrak{x})\bigr) \le \eth\bigl(p_1(\mathfrak{x}), p_0(\mathfrak{x})\bigr). \tag{10}$$

Hence, combining (10) and condition (i), we obtain

$$\eth\bigl(p_l(\mathfrak{x}), p_{l-1}(\mathfrak{x})\bigr) = \cdots = \eth\bigl(p_2(\mathfrak{x}), p_1(\mathfrak{x})\bigr) = \eth\bigl(p_1(\mathfrak{x}), p_0(\mathfrak{x})\bigr) = \lim_{n\to\infty} \eth\bigl(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}\bigr). \qquad \square$$

Proposition 2.6 *Suppose that the sets $\mathbb{G}_i$, $i = 1, \dots, m$, are closed and satisfy the alternative UC condition, and $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is an alternative p-contraction. If there exist $\mathfrak{x} \in \mathfrak{W}$ and a subsequence $\{n_k\} \subset \mathbb{N}$ such that $p_r(\mathfrak{x}) = \lim_{k\to\infty} \Gamma^{n_k+r}\mathfrak{x}$, $r = 0, 1, 2, \dots, m$, with $\eth(p_0(\mathfrak{x}), p_1(\mathfrak{x})) \le \lim_{n\to\infty} \eth(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x})$, then one can find $s, t \in \{0, 1, \dots, m\}$, $t \ge s + 2$, such that $p_s(\mathfrak{x}) = p_t(\mathfrak{x})$.*

Proof Let $\mathfrak{x} \in \mathfrak{W}$ as in hypothesis. By Proposition 2.5, we have

$$\eth\bigl(p_m(\mathfrak{x}), p_{m-1}(\mathfrak{x})\bigr) = \cdots = \eth\bigl(p_2(\mathfrak{x}), p_1(\mathfrak{x})\bigr) = \eth\bigl(p_1(\mathfrak{x}), p_0(\mathfrak{x})\bigr) = \lim_{n\to\infty} \eth\bigl(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}\bigr).$$

Since $p_0(\mathfrak{x}), p_1(\mathfrak{x}), \dots, p_m(\mathfrak{x}) \in \bigcup_{i=1}^m \mathbb{G}_i$, the cardinal number of $\{p_i(\mathfrak{x})\}_{i=0}^m$ is $m + 1$ and the cardinal number of $\{\mathbb{G}_i\}_{i=1}^m$ is m, by the pigeonhole principle, there exist two distinct numbers $s, t \in \{0, 1, 2, \dots, m\}$ such that $p_s(x), p_t(\mathfrak{x}) \in \mathbb{G}_j$ for some j. Without loss of generality, we assume that $s < t$. Since Γ is an alternative map and $p_s(\mathfrak{x}) \in \mathbb{G}_j$, it follows that $p_{s+1}(\mathfrak{x}) \in \mathbb{G}_k$, $k \ne j$. Hence, to have $p_t(\mathfrak{x}) \in \mathbb{G}_j$, it is necessary that $t \ge s + 2$.

On the other hand,

$$\eth\bigl(p_t(\mathfrak{x}), p_{t-1}(\mathfrak{x})\bigr) = \cdots = \eth\bigl(p_{s+1}(\mathfrak{x}), p_s(\mathfrak{x})\bigr) = \lim_{n\to\infty} \eth\bigl(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}\bigr). \tag{11}$$

By Proposition 2.4, we have $\lim_{n\to\infty} \eth(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}) = \eth(\mathbb{G}_\alpha, \mathbb{G}_\beta)$ for some $\mathbb{G}_\alpha, \mathbb{G}_\beta \in \{\mathbb{G}_i\}_{i=1}^m$. So, by (11) and the alternative *UC* condition, we have $p_s(\mathfrak{x}) = p_t(\mathfrak{x})$. $\square$

Proposition 2.7 *Assume that $\mathbb{G}_i$, $i = 1, 2, \dots, m$, are closed and $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is an alternative p-contraction. If, for some $\mathfrak{x} \in \bigcup_{i=1}^m \mathbb{G}_i$ and some subsequence $\{n_k\}_k$ of $\mathbb{N}$, there exists $\lim_{k\to\infty} \Gamma^{n_k}\mathfrak{x}$, then*

$$\lim_{k\to\infty} \Gamma^{n_k+1}\mathfrak{x} = \Gamma\Bigl(\lim_{k\to\infty} \Gamma^{n_k}\mathfrak{x}\Bigr).$$

Proof Let $\mathfrak{x} \in \bigcup_{i=1}^m \mathbb{G}_i$ and $\{n_k\} \subset \mathbb{N}$ be such that $\omega := \lim_{k\to\infty} \Gamma^{n_k}\mathfrak{x}$ exists. We have $\{\Gamma^{n_k}\mathfrak{x}\}_{k=1}^\infty \subset \{\mathbb{G}_i\}_{i=1}^m$. Since $\bigcup_{i=1}^m \mathbb{G}_i$ is closed, it follows that $\lim_{k\to\infty} \Gamma^{n_k}\mathfrak{x} \in \mathbb{G}_r$ for some $r \in \{1, \dots, m\}$. By the pigeonhole principle, we can choose a subsequence $\{n_{k_l}\}$ of $\{n_k\}$ and $s \in \{1, \dots, m\}$ such that $\Gamma^{n_{k_l}}\mathfrak{x} \in \mathbb{G}_s$ for all $l \in \mathbb{N}$. Also, $\lim_{l\to\infty} \Gamma^{n_{k_l}}\mathfrak{x}$ exists and

$$\lim_{l\to\infty} \Gamma^{n_{k_l}}\mathfrak{x} = \lim_{k\to\infty} \Gamma^{n_k}\mathfrak{x} = \omega. \tag{12}$$

It follows that

$$\eth(\mathbb{G}_r, \mathbb{G}_s) \le \eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr), \quad \forall l \in \mathbb{N}. \tag{13}$$

Taking into account the properties of the function α in Definition 2.1 and by (3), (13), and (12), we obtain

$$\begin{aligned}
&\lim_{l\to\infty} \eth\bigl(\Gamma\omega, \Gamma^{n_{k_l}+1}\mathfrak{x}\bigr) \\
&\quad\le \lim_{l\to\infty} \alpha\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr)\bigr)\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr) + \bigl|\eth(\omega, \Gamma\omega) - \eth\bigl(\Gamma^{n_{k_l}}\mathfrak{x}, \Gamma^{n_{k_l}+1}\mathfrak{x}\bigr)\bigr|\bigr) \\
&\qquad + \lim_{l\to\infty} \bigl[1 - \alpha\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr)\bigr)\bigr]\eth(\mathbb{G}_r, \mathbb{G}_s) \\
&\quad\le \lim_{l\to\infty} \alpha\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr)\bigr)\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr) + \bigl|\eth(\omega, \Gamma\omega) - \eth\bigl(\Gamma^{n_{k_l}}\mathfrak{x}, \Gamma^{n_{k_l}+1}\mathfrak{x}\bigr)\bigr|\bigr) \\
&\qquad + \lim_{l\to\infty} \bigl[1 - \alpha\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr)\bigr)\bigr]\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr) \\
&\quad\le \lim_{l\to\infty} \alpha\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr)\bigr)\bigl|\eth(\omega, \Gamma\omega) - c\bigr| + \lim_{l\to\infty} \bigl[1 - \alpha\bigl(\eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr)\bigr)\bigr] \lim_{l\to\infty} \eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr) \\
&\quad\le \lim_{l\to\infty} \eth\bigl(\omega, \Gamma^{n_{k_l}}\mathfrak{x}\bigr) \\
&\quad= 0,
\end{aligned}$$

where we denoted $c = \lim_{l\to\infty} \eth(\Gamma^{n_{k_l}}\mathfrak{x}, \Gamma^{n_{k_l}+1}\mathfrak{x})$ according to Proposition 2.3.

The conclusion now follows from (12). The proof is complete. □

Proposition 2.8 *Assume that $\mathbb{G}_i$, $i = 1, 2, \dots, m$, are closed and $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is an alternative p-contraction. If, for every $\mathfrak{x} \in \bigcup_{i=1}^m \mathbb{G}_i$, one can find a subsequence $\{n_k\}$ of $\mathbb{N}$ such that there exists $p_0(\mathfrak{x}) = \lim_{k\to\infty} \Gamma^{n_k}\mathfrak{x}$, then there exist $p_r(\mathfrak{x}) = \lim_{k\to\infty} \Gamma^{n_k+r}\mathfrak{x}$ for $r = 1, 2, \dots, m$ and $p_r(\mathfrak{x}) = \Gamma^r p_0(\mathfrak{x})$.*

Proof From Proposition 2.7 it follows that, for every $r = 1, 2, \dots, m$,

$$\begin{aligned}
p_r(\mathfrak{x}) &= \lim_{k\to\infty} \Gamma^{n_k+r}\mathfrak{x} \\
&= \lim_{k\to\infty} \Gamma\bigl(\Gamma^{n_k+r-1}\mathfrak{x}\bigr)
\end{aligned}$$

$$= \Gamma\Big(\lim_{k\to\infty}\big(\Gamma^{n_k+r-1}\mathfrak{x}\big)\Big)$$
$$= \Gamma p_{r-1}(\mathfrak{x}).$$

By induction, we infer that $p_r(\mathfrak{x}) = \Gamma^r p_0(\mathfrak{x})$. □

3 Best proximity point for alternative *p*-contractions

Chen [11] defined the best proximity point of an alternative map as follows. We assume that we are in the context of Sect. 2.

Definition 3.1 [11] An element $\mathfrak{x}^* \in \mathfrak{W}$ is called a best proximity point of an alternative map Γ if there exist a positive integer $l \geq 2$, and $r_0, r_1, \ldots, r_{l-1} \in \{1, \ldots, m\}$ such that $\Gamma^j\mathfrak{x}^* \in \mathbb{G}_{r_j}$, $j = 0, 1, \ldots, l-1$, (where $\Gamma^0\mathfrak{x}^* = \mathfrak{x}^*$) satisfying the following three conditions:

1. $\mathfrak{x}^* = \Gamma^l\mathfrak{x}^*$;
2. $\mathfrak{d}(\Gamma^j\mathfrak{x}^*, \Gamma^{j+1}\mathfrak{x}^*) = \mathfrak{d}(\mathbb{G}_{r_j}, \mathbb{G}_{r_{j+1}})$, $j = 0, 1, 2, \ldots, l-1$;
3. $\mathfrak{d}(\mathbb{G}_{r_0}, \mathbb{G}_{r_1}) = \mathfrak{d}(\mathbb{G}_{r_1}, \mathbb{G}_{r_2}) = \cdots = \mathfrak{d}(\mathbb{G}_{r_{l-1}}, \mathbb{G}_{r_0})$.

We now present some best proximity point results for alternative p-contractions.

Theorem 3.2 *Let $(\mathfrak{W}, \mathfrak{d})$ be a metric space and $\{\mathbb{G}_i\}_{i=1}^m$ be a family of nonempty closed subsets of $\mathfrak{W}$ satisfying alternative UC condition. Suppose that $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is an alternative p-contraction. If there exist $\mathfrak{x} \in \bigcup_{i=1}^m \mathbb{G}_i$ and a subsequence $\{n_k\}$ of $\mathbb{N}$ such that* $\lim_{k\to\infty}\Gamma^{n_k}\mathfrak{x} = p_0(\mathfrak{x})$, *then there exist some best proximity points of Γ.*

Proof Note that the existence of $p_j(\mathfrak{x}) = \lim_{k\to\infty}\Gamma^{n_k+j}$, $j = 1, \ldots, m$, follows from Proposition 2.7.

From the given assumptions and Propositions 2.3, 2.7, 2.8, we get

$$\mathfrak{d}\big(p_0(\mathfrak{x}), \Gamma p_0(\mathfrak{x})\big) = \lim_{k\to\infty}\mathfrak{d}\big(\Gamma^{n_k}\mathfrak{x}, \Gamma^{n_k+1}\mathfrak{x}\big) = \lim_{n\to\infty}\mathfrak{d}\big(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}\big)$$

and

$$p_j(\mathfrak{x}) = \Gamma^j p_0(\mathfrak{x}), \quad j = 1, 2, \ldots, m.$$

Again, by Proposition 2.6, we have

$$p_s(\mathfrak{x}) = p_t(\mathfrak{x}) \quad \text{for some } s, t \in \{1, \ldots, m\}, s+2 \leq t. \tag{14}$$

Moreover, by Proposition 2.5, we obtain

$$\mathfrak{d}\big(p_s(\mathfrak{x}), p_{s+1}(\mathfrak{x})\big) = \mathfrak{d}\big(p_{s+1}(\mathfrak{x}), p_{s+2}(\mathfrak{x})\big) = \cdots = \mathfrak{d}\big(p_{t-1}(\mathfrak{x}), p_t(\mathfrak{x})\big) = \lim_{n\to\infty}\mathfrak{d}\big(\Gamma^n\mathfrak{x}, \Gamma^{n+1}\mathfrak{x}\big).$$

If we set $p(\mathfrak{x}) = p_s(\mathfrak{x})$, $l = t - s$ and $q_j(\mathfrak{x}) = p_{s+j}(\mathfrak{x})$, $j = 1, 2, \ldots, l$, then $l \geq 2$ and one has

$$q_j(\mathfrak{x}) = p_{s+j}(\mathfrak{x}) = \Gamma^j p_s(\mathfrak{x}) = \Gamma^j p(\mathfrak{x}). \tag{15}$$

By (14) and (15), we conclude that

$$p(\mathfrak{x}) = p_s(\mathfrak{x}) = p_t(\mathfrak{x}) = \Gamma^l p(\mathfrak{x}). \tag{16}$$

Also, from Proposition 2.4, we have $\lim_{n\to\infty} \mathfrak{d}(\Gamma^n \mathfrak{x}, \Gamma^{n+1}\mathfrak{x}) = \mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta)$ for some $1 \le \alpha \ne \beta \le m$, which together with Proposition 2.5 implies

$$\mathfrak{d}\big(p(\mathfrak{x}), q_1(\mathfrak{x})\big) = \mathfrak{d}\big(q_1(\mathfrak{x}), q_2(\mathfrak{x})\big) = \cdots = \mathfrak{d}\big(q_{l-1}(\mathfrak{x}), q_l(\mathfrak{x})\big) = \mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta). \tag{17}$$

Then, by (15) and (17), one obtains

$$\mathfrak{d}\big(p(\mathfrak{x}), \Gamma p(\mathfrak{x})\big) = \mathfrak{d}\big(\Gamma p(\mathfrak{x}), \Gamma^2 p(\mathfrak{x})\big) = \cdots = \mathfrak{d}\big(\Gamma^{l-1} p(\mathfrak{x}), \Gamma^l p(\mathfrak{x})\big) = \mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta). \tag{18}$$

Let $\Gamma^j p(\mathfrak{x}) \in \mathbb{G}_{r_j}$ for some $\mathbb{G}_{r_j} \in \{\mathbb{G}_i\}_{i=1}^m$, $j \in \{0, 1, \ldots, l\}$. Note that $\Gamma^l p(\mathfrak{x}) = p(\mathfrak{x}) \in \mathbb{G}_{r_0}$, $\Gamma^{l+1} p(\mathfrak{x}) \in \mathbb{G}_{r_1}$, so one can write $r_l := r_0$, $r_{l+1} := r_1$. We have

$$\begin{aligned}
&\mathfrak{d}\big(\Gamma^{j+2} p(\mathfrak{x}), \Gamma^{j+1} p(\mathfrak{x})\big) \\
&\quad\le \alpha\big(\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big)\big)\big[\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big) \\
&\qquad + \big|\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^{j+2} p(\mathfrak{x})\big) - \mathfrak{d}\big(\Gamma^j p(\mathfrak{x}), \Gamma^{j+1} p(\mathfrak{x})\big)\big|\big] \\
&\qquad + \big[1 - \alpha\big(\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big)\big)\big]\mathfrak{d}(\mathbb{G}_{r_{j+1}}, \mathbb{G}_{r_j}) \\
&\quad\Rightarrow \quad \mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta) \le \alpha\big(\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big)\big)\mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta) \\
&\qquad\qquad\qquad + \big[1 - \alpha\big(\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big)\big)\big]\mathfrak{d}(\mathbb{G}_{r_{j+1}}, \mathbb{G}_{r_j}) \\
&\quad\Rightarrow \quad \big[1 - \alpha\big(\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big)\big)\big]\mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta) \\
&\qquad\quad \le \big[1 - \alpha\big(\mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big)\big)\big]\mathfrak{d}(\mathbb{G}_{r_{j+1}}, \mathbb{G}_{r_j}) \\
&\quad\Rightarrow \quad \mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta) \le \mathfrak{d}(\mathbb{G}_{r_{j+1}}, \mathbb{G}_{r_j}).
\end{aligned}$$

Again, from (18), we have

$$\mathfrak{d}(\mathbb{G}_{r_{j+1}}, \mathbb{G}_{r_j}) \le \mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big) = \mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta).$$

Hence

$$\mathfrak{d}(\mathbb{G}_{r_{j+1}}, \mathbb{G}_{r_j}) = \mathfrak{d}\big(\Gamma^{j+1} p(\mathfrak{x}), \Gamma^j p(\mathfrak{x})\big) \tag{19}$$

for $j = 0, 1, \ldots, l$. By (16), (18), and (19), we conclude that $p(\mathfrak{x})$ is a best proximity point of Γ. □

Theorem 3.3 *Let $(\mathfrak{W}, \mathfrak{d})$ be a complete metric space. Suppose that the collection $\{\mathbb{G}_i\}_{i=1}^m$ of nonempty closed subsets of $\mathfrak{W}$ satisfies the alternative UC condition and $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is an alternative p-contraction. Then there exists a best proximity point of Γ.*

Proof Choose $\mathfrak{x} \in \mathfrak{W}$. For any subsequence $\{n_k\}$ of $\mathbb{N}$, we have $\Gamma^{n_k}\mathfrak{x} \in \bigcup_{i=1}^m \mathbb{G}_i$. One can find a subsequence $\{n_{k_p}\}_p \subset \{n_k\}$ such that $\Gamma^{n_{k_p}}\mathfrak{x} \in \mathbb{G}_r$ for all $p \in \mathbb{N}$ and some $r \in \{1, 2, \ldots, m\}$.

Moreover, since Γ is an alternative p-contraction, by Proposition 2.4, we have

$$\begin{aligned}
&\lim_{p<q\to\infty} \mathfrak{d}\bigl(\Gamma^{n_{k_p}}\mathfrak{x}, \Gamma^{n_{k_p}+1}\mathfrak{x}\bigr)\\
&\quad = \lim_{p<q\to\infty} \mathfrak{d}\bigl(\Gamma^{n_{k_p}+1}\mathfrak{x}, \Gamma^{n_{k_p}+2}\mathfrak{x}\bigr)\\
&\quad = \cdots\\
&\quad = \lim_{p<q\to\infty} \mathfrak{d}\bigl(\Gamma^{n_{k_q}-1}\mathfrak{x}, \Gamma^{n_{k_q}}\mathfrak{x}\bigr)\\
&\quad = \lim_{n\to\infty} \mathfrak{d}\bigl(\Gamma^{n+1}\mathfrak{x}, \Gamma^{n}\mathfrak{x}\bigr)\\
&\quad = \mathfrak{d}(\mathbb{G}_\alpha, \mathbb{G}_\beta)
\end{aligned}$$

for some $\mathbb{G}_\alpha, \mathbb{G}_\beta \in \{\mathbb{G}_i\}_{i=1}^m$.

By alternative *UC* condition, we deduce that $\lim_{p,q\to\infty} \mathfrak{d}(\Gamma^{n_{k_p}}\mathfrak{x}, \Gamma^{n_{k_q}}\mathfrak{x}) = 0$, which implies that $\{\Gamma^{n_{k_p}}\mathfrak{x}\}_p$ is a Cauchy sequence. Due to the completeness of $\mathfrak{W}$, there exists $z \in \mathfrak{W}$ such that $\lim_{p\to\infty} \Gamma^{n_{k_p}}\mathfrak{x} = z$. Therefore, the existence of the fixed point of Γ follows using similar arguments as in the proof of Theorem 3.2. □

Corollary 3.4 *Let $(\mathfrak{W}, \mathfrak{d})$ be a compact metric space. Suppose that the collection $\{\mathbb{G}_i\}_{i=1}^m$ of nonempty closed subsets of $\mathfrak{W}$ satisfies the alternative UC condition and $\Gamma : \bigcup_{i=1}^m \mathbb{G}_i \to \bigcup_{i=1}^m \mathbb{G}_i$ is an alternative p-contraction. Then there exists a best proximity point of Γ.*

Proof Since every compact metric space is complete, the conclusion follows from Theorem 3.3. □

Example 3.5 Let $\mathfrak{W} = \mathbb{R}$ endowed with the Euclidean metric $\mathfrak{d}(\mathfrak{x}, \mathfrak{y}) = |\mathfrak{x} - \mathfrak{y}|$ and a family of subsets $\{\mathbb{G}_i\}_{i=1}^4$ given by $\mathbb{G}_1 = \{2, 3, \dots, n, \dots\}$, $\mathbb{G}_2 = \{-2, -3, \dots, -n, \dots\}$, $\mathbb{G}_3 = \{0\}$, $\mathbb{G}_4 = \{1\}$.

Define $\Gamma : \bigcup_{i=1}^4 \mathbb{G}_i \to \bigcup_{i=1}^4 \mathbb{G}_i$ by

$$\Gamma\mathfrak{x} = \begin{cases} -\mathfrak{x}, & \mathfrak{x} \in \mathbb{G}_1,\\ [\frac{-\mathfrak{x}+1}{2}], & \mathfrak{x} \in \mathbb{G}_2,\\ 1, & \mathfrak{x} \in \mathbb{G}_3,\\ 0, & \mathfrak{x} \in \mathbb{G}_4 \end{cases}$$

and $\alpha : [0, \infty) \to [0, 1)$ by $\alpha(t) = \frac{t}{1+t}$, where [ ] means the integer part. Then:

a) Γ has no fixed points, hence it is not a p-contraction on the complete metric space $(\mathbb{Z} \setminus \{-1\}, \mathfrak{d})$;
b) Γ is an alternative p-contraction;
c) the family $\{\mathbb{G}_i\}_{i=1}^4$ satisfies the alternative UC condition;
d) Γ has a best proximity point.

Proof a) By definition it is clear that Γ has no fixed points, hence, according to Theorem 1.12, it is not p-contraction.

b) It is also obvious that $\Gamma(\mathbb{G}_i) \subset \bigcup_{j\neq i} \mathbb{G}_j$ for all $i = 1, \dots, 4$, so Γ is an alternative map. Let us consider $\mathfrak{x}, \mathfrak{y} \in \bigcup_{i=1}^4 \mathbb{G}_i$. If $\mathfrak{x} = \mathfrak{y}$, then (3) is trivially verified.

Assume that $\mathfrak{x} \neq \mathfrak{y}$.

We distinguish the following cases:

1. If $\mathfrak{x}, \mathfrak{y} \in \mathbb{G}_1$, then $\eth(\Gamma\mathfrak{x}, \Gamma\mathfrak{y}) = |\mathfrak{y} - \mathfrak{x}|$, $\eth(\mathfrak{x}, \mathfrak{y}) + |\eth(\mathfrak{x}, \Gamma\mathfrak{x}) - \eth(\mathfrak{y}, \Gamma\mathfrak{y})| = |\mathfrak{x} - \mathfrak{y}| + |2\mathfrak{x} - 2\mathfrak{y}| = 3|\mathfrak{x} - \mathfrak{y}|$.

So

$$\alpha\big(\eth(\mathfrak{x}, \mathfrak{y})\big)\big(\eth(\mathfrak{x}, \mathfrak{y}) + \big|\eth(\mathfrak{x}, \Gamma\mathfrak{x}) - \eth(\mathfrak{y}, \Gamma\mathfrak{y})\big|\big) = \frac{3|\mathfrak{x} - \mathfrak{y}|^2}{1 + |\mathfrak{x} - \mathfrak{y}|}.$$

To prove (3), it is enough to see that

$$|\mathfrak{x} - \mathfrak{y}| \leq \frac{3|\mathfrak{x} - \mathfrak{y}|^2}{1 + |\mathfrak{x} - \mathfrak{y}|} \Leftrightarrow |\mathfrak{x} - \mathfrak{y}| + 1 \leq 3|\mathfrak{x} - \mathfrak{y}| \Leftrightarrow |\mathfrak{x} - \mathfrak{y}| \geq \frac{1}{2}.$$

2. Let $\mathfrak{x}, \mathfrak{y} \in \mathbb{G}_2$ and, without loss of generality, we can assume that $\mathfrak{x} < \mathfrak{y}$. Then $\mathfrak{y} - \mathfrak{x} - 1 \geq 0$.

2.1. If $\mathfrak{x}$ is even, $\mathfrak{y}$ is odd, one has

$$\begin{aligned}
&|\Gamma\mathfrak{x} - \Gamma\mathfrak{y}| \leq \alpha\big(|\mathfrak{y} - \mathfrak{x}|\big)\left(|\mathfrak{y} - \mathfrak{x}| + \left|-\frac{3\mathfrak{x}}{2} + \frac{3\mathfrak{y} - 1}{2}\right|\right) \\
&\Leftrightarrow \quad \frac{-\mathfrak{x} + \mathfrak{y} - 1}{2} \leq \alpha(\mathfrak{y} - \mathfrak{x})\left((\mathfrak{y} - \mathfrak{x}) + \frac{1}{2}(-3\mathfrak{x} + 3\mathfrak{y} - 1)\right) \\
&\Leftrightarrow \quad \mathfrak{y} - \mathfrak{x} - 1 \leq 5\alpha(\mathfrak{y} - \mathfrak{x})(\mathfrak{y} - \mathfrak{x}) - \alpha(\mathfrak{y} - \mathfrak{x}) \\
&\qquad 0 \leq \big(5(\mathfrak{y} - \mathfrak{x}) - (\mathfrak{y} - \mathfrak{x}) - 1\big)(\mathfrak{y} - \mathfrak{x}) + 1 \\
&\qquad 0 \leq \big(4(\mathfrak{y} - \mathfrak{x} - 1) + 3\big)(\mathfrak{y} - \mathfrak{x}) + 1,
\end{aligned}$$

which is obvious.

2.2. If $\mathfrak{x}$ is odd and $\mathfrak{y}$ is even, then (3) is equivalent to

$$\begin{aligned}
&|\Gamma\mathfrak{x} - \Gamma\mathfrak{y}| \leq \alpha(\mathfrak{y} - \mathfrak{x})\left((\mathfrak{y} - \mathfrak{x}) + \left|\frac{-3\mathfrak{x} + 1}{2} + \frac{3\mathfrak{y}}{2}\right|\right) \\
&\Leftrightarrow \quad \frac{\mathfrak{y} - \mathfrak{x} + 1}{2} \leq \alpha(\mathfrak{y} - \mathfrak{x})\left((\mathfrak{y} - \mathfrak{x}) + \frac{1}{2}\big(3(\mathfrak{y} - \mathfrak{x}) + 1\big)\right) \\
&\Leftrightarrow \quad 1 \leq \big(5\alpha(\mathfrak{y} - \mathfrak{x}) - 1\big)(\mathfrak{y} - \mathfrak{x}) + \alpha(\mathfrak{y} - \mathfrak{x}) \\
&\Leftrightarrow \quad 0 \leq (\mathfrak{y} - \mathfrak{x})(4\mathfrak{y} - 4\mathfrak{x} - 1) - 1 \\
&\Leftrightarrow \quad 0 \leq (\mathfrak{y} - \mathfrak{x})\big(4(\mathfrak{y} - \mathfrak{x} - 1) + 3\big) - 1,
\end{aligned}$$

which is also obvious.

The other subcases, where both $\mathfrak{x}$ and $\mathfrak{y}$ are odd or even, are simple and treated analogously.

3. If $\mathfrak{x} \in \mathbb{G}_1$, $\mathfrak{y} \in \mathbb{G}_2$, then one has $\eth(\mathbb{G}_1, \mathbb{G}_2) = 4$ and the following subcases:

3.1. If $\mathfrak{y}$ is even, then $\mathfrak{x} - \mathfrak{y} \geq 4$ and

$$|\Gamma\mathfrak{x} - \Gamma\mathfrak{y}| = \mathfrak{x} - \frac{\mathfrak{y}}{2},$$

$$\begin{aligned}\mathfrak{d}(\mathfrak{x},\mathfrak{y})+\left|\mathfrak{d}(\mathfrak{x},\Gamma\mathfrak{x})-\mathfrak{d}(\mathfrak{y},\Gamma\mathfrak{y})\right| &= |\mathfrak{x}-\mathfrak{y}|+\left|2\mathfrak{x}+\frac{3}{2}\mathfrak{y}\right| \\ &= (\mathfrak{x}-\mathfrak{y})+\frac{1}{2}|4\mathfrak{x}+3\mathfrak{y}|.\end{aligned}$$

We need to show that

$$2\mathfrak{x}-\mathfrak{y}\le\alpha(\mathfrak{x}-\mathfrak{y})\bigl(2(\mathfrak{x}-\mathfrak{y})+|4\mathfrak{x}+3\mathfrak{y}|\bigr)+8\bigl(1-\alpha(\mathfrak{x}-\mathfrak{y})\bigr). \tag{20}$$

3.1.1. $4\mathfrak{x}+3\mathfrak{y}\ge 0\Leftrightarrow -3\mathfrak{y}\le 4\mathfrak{x}\Leftrightarrow 6\mathfrak{x}-3\mathfrak{y}\le 10\mathfrak{x}$. To prove (20), it is sufficient to show that

$$\begin{aligned}&10\mathfrak{x}\le 3\alpha(\mathfrak{x}-\mathfrak{y})\bigl(2(\mathfrak{x}-\mathfrak{y})+4\mathfrak{x}+3\mathfrak{y}\bigr)+24\bigl(1-\alpha(\mathfrak{x}-\mathfrak{y})\bigr)\\ &\quad\Leftrightarrow\quad 10\mathfrak{x}\le\frac{3(\mathfrak{x}-\mathfrak{y})(6\mathfrak{x}+\mathfrak{y})+24}{\mathfrak{x}-\mathfrak{y}+1}\\ &\quad\Leftrightarrow\quad 10\mathfrak{x}(\mathfrak{x}-\mathfrak{y})+10\mathfrak{x}\le(\mathfrak{x}-\mathfrak{y})(18\mathfrak{x}+3\mathfrak{y})+24\\ &\quad\Leftrightarrow\quad 10\mathfrak{x}\le(\mathfrak{x}-\mathfrak{y})(8\mathfrak{x}+3\mathfrak{y})+24=(\mathfrak{x}-\mathfrak{y})(4\mathfrak{x}+3\mathfrak{y})+4\mathfrak{x}(\mathfrak{x}-\mathfrak{y})+24\\ &\quad\Leftrightarrow\quad 0\le(\mathfrak{x}-\mathfrak{y})(4\mathfrak{x}+3\mathfrak{y})+2\mathfrak{x}\bigl(2(\mathfrak{x}-\mathfrak{y})-5\bigr)+24,\end{aligned}$$

which is obvious as a sum of positive numbers.

3.1.2. $4\mathfrak{x}+3\mathfrak{y}<0\Leftrightarrow 4\mathfrak{x}-2\mathfrak{y}\le -5\mathfrak{y}$. We will show as above that

$$\begin{aligned}&-5\mathfrak{y}\le 2\alpha(\mathfrak{x}-\mathfrak{y})\bigl(2(\mathfrak{x}-\mathfrak{y})-(4\mathfrak{x}+3\mathfrak{y})\bigr)+16\bigl(1-\alpha(\mathfrak{x}-\mathfrak{y})\bigr)\\ &\quad\Leftrightarrow\quad -5\mathfrak{y}\le\frac{(\mathfrak{x}-\mathfrak{y})(-4\mathfrak{x}-10\mathfrak{y})+16}{\mathfrak{x}-\mathfrak{y}+1}\\ &\quad\Leftrightarrow\quad -5\mathfrak{y}(\mathfrak{x}-\mathfrak{y})-5\mathfrak{y}\le(\mathfrak{x}-\mathfrak{y})(-4\mathfrak{x}-10\mathfrak{y})+16\\ &\quad\Leftrightarrow\quad -5\mathfrak{y}\le(\mathfrak{x}-\mathfrak{y})(-4\mathfrak{x}-3\mathfrak{y})-2\mathfrak{y}(\mathfrak{x}-\mathfrak{y})+16\\ &\quad\Leftrightarrow\quad 0\le(\mathfrak{x}-\mathfrak{y})(-4\mathfrak{x}-3\mathfrak{y})+\bigl(5-2(\mathfrak{x}-\mathfrak{y})\bigr)\mathfrak{y}+16.\end{aligned}$$

3.2. If $\mathfrak{y}$ is odd, then $\mathfrak{x}-\mathfrak{y}\ge 5$ and, after some simple computations, (3) is equivalent to

$$2\mathfrak{x}-\mathfrak{y}+1\le\alpha(\mathfrak{x}-\mathfrak{y})\bigl(2(\mathfrak{x}-\mathfrak{y})+|4\mathfrak{x}+3\mathfrak{y}-1|\bigr)+8\bigl(1-\alpha(\mathfrak{x}-\mathfrak{y})\bigr). \tag{21}$$

3.2.1. If $4\mathfrak{x}+3\mathfrak{y}-1\ge 0$, then $6\mathfrak{x}-3\mathfrak{y}+3\le 10\mathfrak{x}+2$ and, to prove (21), it suffices to show that

$$10\mathfrak{x}+2\le 3\alpha(\mathfrak{x}-\mathfrak{y})\bigl(2(\mathfrak{x}-\mathfrak{y})+4\mathfrak{x}+3\mathfrak{y}-1\bigr)+24\bigl(1-\alpha(\mathfrak{x}-\mathfrak{y})\bigr).$$

This is equivalent to

$$\begin{aligned}&10\mathfrak{x}\le\frac{3(\mathfrak{x}-\mathfrak{y})(6\mathfrak{x}+\mathfrak{y}-1)-2(\mathfrak{x}-\mathfrak{y})+24}{\mathfrak{x}-\mathfrak{y}+1}\\ &\quad\Leftrightarrow\quad 10\mathfrak{x}\le(\mathfrak{x}-\mathfrak{y})(8\mathfrak{x}+3\mathfrak{y}-5)+24\\ &\quad\Leftrightarrow\quad 0\le(\mathfrak{x}-\mathfrak{y})(4\mathfrak{x}+3\mathfrak{y}-1)+4(\mathfrak{x}-1)(\mathfrak{x}-\mathfrak{y})-10\mathfrak{x}+24\\ &\quad\Leftrightarrow\quad 0\le(\mathfrak{x}-\mathfrak{y})(4\mathfrak{x}+3\mathfrak{y}-1)+4(\mathfrak{x}-1)(\mathfrak{x}-\mathfrak{y}-5)+10(\mathfrak{x}-2)+24,\end{aligned}$$

which is obvious.

3.2.2. If $4\mathfrak{x} + 3\mathfrak{y} - 1 < 0$, then $4\mathfrak{x} - 2\mathfrak{y} + 2 < -5\mathfrak{y} + 3$. To establish (21), we will prove that

$$-5\mathfrak{y} + 3 \leq 2\alpha(\mathfrak{x} - \mathfrak{y})\big(2(\mathfrak{x} - \mathfrak{y}) - 4\mathfrak{x} - 3\mathfrak{y} + 1\big) + 16\big(1 - \alpha(\mathfrak{x} - \mathfrak{y})\big).$$

One has

$$\begin{aligned} -5\mathfrak{y} + 3 &\leq \frac{2(\mathfrak{x} - \mathfrak{y})(-2\mathfrak{x} - 5\mathfrak{y} + 1) + 16}{\mathfrak{x} - \mathfrak{y} + 1} \\ \Leftrightarrow \quad & -5\mathfrak{y} \leq (\mathfrak{x} - \mathfrak{y})(-4\mathfrak{x} - 5\mathfrak{y} - 1) + 13 \\ \Leftrightarrow \quad & 0 \geq (\mathfrak{x} - \mathfrak{y})(4\mathfrak{x} + 3\mathfrak{y} - 1) + 2(\mathfrak{x} - \mathfrak{y} - 5)(\mathfrak{y} + 1) + 5\mathfrak{y} - 3, \end{aligned}$$

obvious.

4. If $\mathfrak{x} \in \mathbb{G}_1$, $\mathfrak{y} \in \mathbb{G}_3$, one has $\eth(\mathbb{G}_1, \mathbb{G}_3) = 2$, $|\Gamma\mathfrak{x} - \Gamma\mathfrak{y}| = \mathfrak{x} + 1$ and

$$\begin{aligned} &\alpha\big(\eth(\mathfrak{x},\mathfrak{y})\big)\big(\eth(\mathfrak{x},\mathfrak{y}) + \big|\eth(\mathfrak{x},\Gamma\mathfrak{x}) - \eth(\mathfrak{y},\Gamma\mathfrak{y})\big|\big) + \big(1 - \alpha\big(\eth(\mathfrak{x},\mathfrak{y})\big)\big)\eth(\mathbb{G}_1,\mathbb{G}_3) \\ &\quad = \alpha\big(|\mathfrak{x}|\big)\big(|\mathfrak{x}| + |2\mathfrak{x} - 1|\big) + 2\big(1 - \alpha\big(|\mathfrak{x}|\big)\big) \\ &\quad = \frac{3\mathfrak{x}^2 - \mathfrak{x} + 2}{1 + \mathfrak{x}}. \end{aligned}$$

Thus (3) is equivalent to

$$\begin{aligned} 1 + \mathfrak{x} &\leq \frac{3\mathfrak{x}^2 - \mathfrak{x} + 2}{1 + \mathfrak{x}} \\ \Leftrightarrow \quad & 0 \leq 2\mathfrak{x}^2 - 3\mathfrak{x} + 1 \\ \Leftrightarrow \quad & 0 \leq 2(\mathfrak{x} - 1)\left(\mathfrak{x} - \frac{1}{2}\right). \end{aligned}$$

5. If $\mathfrak{x} \in \mathbb{G}_1$, $\mathfrak{y} \in \mathbb{G}_4$, we have $\eth(\mathbb{G}_1, \mathbb{G}_4) = 1$, $\eth(\Gamma\mathfrak{x}, \Gamma\mathfrak{y}) = \mathfrak{x}$, $\eth(\mathfrak{x}, \mathfrak{y}) = \mathfrak{x} - \mathfrak{y} = \mathfrak{x} - 1$.
We have

$$\begin{aligned} &\alpha\big(\eth(\mathfrak{x},\mathfrak{y})\big)\big(\eth(\mathfrak{x},\mathfrak{y}) + \big|\eth(\mathfrak{x},\Gamma\mathfrak{x}) - \eth(\mathfrak{y},\Gamma\mathfrak{y})\big|\big) + \big(1 - \alpha\big(\eth(\mathfrak{x},\mathfrak{y})\big)\big)\eth(\mathbb{G}_1,\mathbb{G}_4) \\ &\quad = \frac{|\mathfrak{x} - 1|(|\mathfrak{x} - 1| + |2\mathfrak{x} - 1|) + 1}{1 + |\mathfrak{x} - 1|} \\ &\quad = \frac{3(\mathfrak{x}^2 - \mathfrak{x} + 1)}{\mathfrak{x}}. \end{aligned}$$

Thus (3) is equivalent to

$$0 \leq 2\mathfrak{x}^2 - 3\mathfrak{x} + 3,$$

which holds for all $\mathfrak{x}$.

6. If $\mathfrak{x} \in \mathbb{G}_2$, $\mathfrak{y} \in \mathbb{G}_3$, we have $\mathfrak{y} = 0$ and $\eth(\mathbb{G}_2, \mathbb{G}_3) = 2$.

6.1. Assume that $\mathfrak{x}$ is even. Then

$$\begin{aligned} &\alpha\big(\eth(\mathfrak{x},\mathfrak{y})\big)\big(\eth(\mathfrak{x},\mathfrak{y}) + \big|\eth(\mathfrak{x},\Gamma\mathfrak{x}) - \eth(\mathfrak{y},\Gamma\mathfrak{y})\big|\big) + \big(1 - \alpha\big(\eth(\mathfrak{x},\mathfrak{y})\big)\big)\eth(\mathbb{G}_2,\mathbb{G}_3) \\ &\quad = \frac{|\mathfrak{x}|}{1 + |\mathfrak{x}|}\left(|\mathfrak{x}| + \left|-\frac{3\mathfrak{x}}{2} - 1\right|\right) + \frac{2}{1 + |\mathfrak{x}|} \end{aligned}$$

$$= \frac{-\mathfrak{x}}{1-\mathfrak{x}}\left(-\frac{5\mathfrak{x}}{2}-1\right) + \frac{2}{1-\mathfrak{x}}$$
$$= \frac{\frac{5\mathfrak{x}^2}{2}+\mathfrak{x}+2}{1-\mathfrak{x}}.$$

Thus (3) is equivalent to

$$(-2-\mathfrak{x})(1-\mathfrak{x}) \le 5\mathfrak{x}^2 + 2\mathfrak{x} + 4$$
$$\Leftrightarrow \quad 0 \le 4\mathfrak{x}^2 + \mathfrak{x} + 6,$$

which holds for all $\mathfrak{x}$.

6.2. Assume that $\mathfrak{x}$ is odd. Then, as before,

$$|\Gamma\mathfrak{x} - \Gamma\mathfrak{y}| \le \alpha(\mathfrak{y}-\mathfrak{x})[(\mathfrak{y}-\mathfrak{x}) + \left|\left|\frac{-\mathfrak{x}+1}{2}\right| - 1\right| + 2\big(1-\alpha(\mathfrak{y}-\mathfrak{x})\big)$$
$$\Leftrightarrow \quad -\frac{\mathfrak{x}+1}{2} \le -\alpha(-\mathfrak{x})\mathfrak{x} - \frac{\alpha(-\mathfrak{x})}{2}(3\mathfrak{x}-1) + 2\big(1-\alpha(-\mathfrak{x})\big)$$
$$\Leftrightarrow \quad 0 \le 4\mathfrak{x}^2 - \mathfrak{x} + 5.$$

7. If $\mathfrak{x} \in \mathbb{G}_2$, $\mathfrak{y} \in \mathbb{G}_4$, we have $\mathfrak{y} = 1$, $\mathfrak{d}(\mathbb{G}_2, \mathbb{G}_4) = 3$.

7.1. If $\mathfrak{x}$ is even, then $\Gamma\mathfrak{x} = \frac{-\mathfrak{x}}{2}$ and (3) is equivalent to

$$\frac{-\mathfrak{x}}{2} \le \alpha(1-\mathfrak{x})\left((1-\mathfrak{x}) + \left|\frac{-3\mathfrak{x}}{2} - 1\right|\right) + 3\big(1-\alpha(1-\mathfrak{x})\big)$$
$$\Leftrightarrow \quad -\mathfrak{x} \le \frac{-5(1-\mathfrak{x})\mathfrak{x}+6}{2-\mathfrak{x}}$$
$$\Leftrightarrow \quad 0 \le 4\mathfrak{x}^2 - 3\mathfrak{x} + 6,$$

which is true for all $\mathfrak{x}$.

7.2. If $\mathfrak{x}$ is odd, an easy computation similar to those above shows that (3) is verified.

8. The cases $\mathfrak{x}, \mathfrak{y} \in \mathbb{G}_3$ and $\mathfrak{x}, \mathfrak{y} \in \mathbb{G}_4$ are trivial.

Therefore, Γ is an alternative p-contraction.

c) Let us consider $\mathfrak{y}_n^1, \mathfrak{y}_n^5 \in \mathbb{G}_1$ and $\mathfrak{y}_n^2, \mathfrak{y}_n^3, \mathfrak{y}_n^4 \in \bigcup_{j\neq 1} \mathbb{G}_j$ such that

$$\lim_n \left|\mathfrak{y}_n^1 - \mathfrak{y}_n^2\right| = \lim_n \left|\mathfrak{y}_n^2 - \mathfrak{y}_n^3\right| = \lim_n \left|\mathfrak{y}_n^3 - \mathfrak{y}_n^4\right| = \lim_n \left|\mathfrak{y}_n^4 - \mathfrak{y}_n^5\right| = 2 = \mathfrak{d}(\mathbb{G}_2, \mathbb{G}_3).$$

It follows that $\mathfrak{y}_n^1 = 2$, $\mathfrak{y}_n^2 = 0 \in \mathbb{G}_4$, $\mathfrak{y}_n^3 = -2 \in \mathbb{G}_2$, $\mathfrak{y}_n^4 = 0 \in \mathbb{G}_4$, $\mathfrak{y}_n^5 = 2$. Therefore $\lim_n \mathfrak{y}_n^1 = \lim_n \mathfrak{y}_n^5 = 2$.

d) The assertion follows from Theorem 3.3. □

Remark 3.6 To establish the existence of best proximity points of the mapping Γ from the previous example Theorem 1.10 is inapplicable because, for example, taking some $\mathfrak{x}, \mathfrak{y} \in \mathbb{G}_1$, $\mathfrak{x} \neq \mathfrak{y}$, or some $\mathfrak{x} \in \mathbb{G}_1$, $\mathfrak{y} \in \mathbb{G}_3$, relation (1) is not satisfied for any $\alpha < 1$.

4 Conclusion

This paper presents and investigates the concept of alternative p-contraction, which serves as a unification of the notions of p-contraction and alternative contraction. By exploring the alternative UC property, we establish the existence of the best proximity point for mappings of this nature. Additionally, we assert that the significance of our main findings extends to both compact spaces and complete metric spaces. To further demonstrate the effectiveness of our proven result, we provide an illustrative example. Furthermore, it is worth considering whether similar conclusions can be drawn regarding the existence of (coupled) best proximity points for other types of contractions concerning alternative mappings [7, 10, 16]. In a related study, Zhelinski *et al.* [32] introduced a new type of UC property known as UC^* property, which is included into the UC property. It may be possible to extend our best proximity point results to various types of alternative contractions under the condition of the UC^* property.

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Author details
[1]School of Science and Technology, University of Sanya, Sanya 572022, China. [2]Center for Mathematical Research, University of Sanya, Sanya 572022, China. [3]Academician Guoliang Chen Team Innovation Center, University of Sanya, Sanya 572022, China. [4]Academician Chunming Rong Workstation, University of Sanya, Sanya 572022, China. [5]Department of Mathematics and Computer Science, Faculty of Sciences, Lucian Blaga University of Sibiu, Sibiu 550012, Romania. [6]Department of Mathematics, University of Management and Technology, Lahore 54700, Pakistan. [7]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa 0204, South Africa. [8]Department of Mathematics, Government College University, Katchery Road, Lahore 54000, Pakistan.

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