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New midpoint-type inequalities in the context of the proportional Caputo-hybrid operator

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Abstract

Fractional calculus is a crucial foundation in mathematics and applied sciences, serving as an extremely valuable tool. Besides, the new hybrid fractional operator, which combines proportional and Caputo operators, offers better applications in numerous fields of mathematics and computer sciences. Due to its wide range of applications, we focus on the proportional Caputo-hybrid operator in this research article. Firstly, we begin by establishing a novel identity for this operator. Then, based on the newfound identity, we establish some integral inequalities that are relevant to the left-hand side of Hermite–Hadamard-type inequalities for the proportional Caputo-hybrid operator. Furthermore, we show how the results improve upon and refine many previous findings in the setting of integral inequalities. Later, we present specific examples together with their related graphs to offer a better understanding of the newly obtained inequalities. Our results not only extend previous studies but also provide valuable viewpoints and methods for tackling a wide range of mathematical and scientific problems.

Mathematics Subject Classification: 26D07; 26D10; 26D15; 26A33

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1 Introduction

In mathematics, convex analysis holds great importance as it can be applied to numerous areas such as control theory, optimization theory, energy systems, physics, engineering applications, economics, and finance. Also, there is a strong relationship between convex analysis and integral inequalities, and these two concepts complement each other closely in terms of the properties they provide. One of the most famous inequalities in convex theory is the Hermite–Hadamard inequality, which was independently investigated by Charles Hermite and Jacques Hadamard [16, 19]. This inequality can be expressed as follows:

$$\psi\left(\frac{\zeta+\vartheta}{2}\right) \leq \frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta}\psi(\varkappa)\,d\varkappa \leq \frac{\psi(\zeta)+\psi(\vartheta)}{2}, \qquad (1)$$

where $\psi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\zeta, \vartheta \in I$ with $\zeta < \vartheta$. If ψ is concave, then both inequalities in the statement hold in the reverse direction.

The Hermite–Hadamard inequality gives both upper and lower bounds for the average value of a convex function over a compact interval. This inequality finds many applications in various fields, including integral calculus, probability theory, statistics, optimization, and number theory. Additionally, it is a useful tool for solving physical and engineering problems that require determining function averages. The Hermite–Hadamard inequality is extensively researched and utilized in different areas of mathematics. As new problems arise, their applications continue to expand, making them a valuable tool for solving a broad range of mathematical problems. Also, the Hermite–Hadamard inequality is characterized by the trapezoidal and midpoint inequalities on its right and left parts. These categories of inequality have been the focus of researchers' work. Trapezoid-type inequalities for the case of convex functions were first established by Dragomir and Agarwal in [15], whereas midpoint-type inequalities for the case of convex functions were first proved by Kırmacı in [25]. Since the appearance of these inequalities, there has been a lot of activity in this area [2, 8, 21].

Fractional calculus has a significant historical foundation. The beginnings of fractional calculus can be traced back to the correspondence between Leibniz and L'Hopital. Fractional calculus allows us to describe the behavior of complex systems more accurately, especially those that exhibit noninteger order dynamics. It extends the concepts of traditional calculus to include fractional orders. It has gained more importance and has found applications in various fields of science and engineering. Chu et al. [12] explored a deterministic-stochastic malnutrition model involving nonlinear perturbation via piecewise fractional operators techniques. Also, Chu et al. [11] put forward a numerical approach for solving an assortment of fractional-order chaotic systems. In the recent times, fractional calculus is a developing branch of mathematics that plays a significant role in capturing the dynamics of intricate systems across diverse fields of science and engineering (see [28, 30, 31, 33]) because of the new fractional integral and derivative such as Caputo–Fabrizio [10], Atangana–Baleanu [5], tempered [34], etc.

The following is the definition of Riemann–Liouville integral operators, which are one of the fundamental fractional integral operators [24]:

Definition 1 For $\psi \in L_1[\zeta, \vartheta]$, the Riemann–Liouville integrals of order $\varrho > 0$ are given by

$$J_{\zeta+}^{\varrho}\psi(\varkappa) = \frac{1}{\Gamma(\varrho)} \int_{\zeta}^{\varkappa} (\varkappa - \mathfrak{s})^{\varrho-1}\psi(\mathfrak{s})\, d\mathfrak{s}, \quad \varkappa > \zeta,$$

and

$$J_{\vartheta-}^{\varrho}\psi(\varkappa) = \frac{1}{\Gamma(\varrho)} \int_{\varkappa}^{\vartheta} (\mathfrak{s} - \varkappa)^{\varrho-1}\psi(\mathfrak{s})\, d\mathfrak{s}, \quad \varkappa < \vartheta.$$

Here, $\Gamma(\varrho)$ is the gamma function and $J_{\zeta+}^{0}\psi(\varkappa) = J_{\vartheta-}^{0}\psi(\varkappa) = \psi(\varkappa)$. Obviously, Riemann–Liouville integrals will be equal to classical integrals for the condition $\varrho = 1$.

Sarıkaya and Yıldırım [38] presented the different representation of the Hermite–Hadamard inequality in terms of fractional integrals in the following manner.

Theorem 1 *Let $\psi : [\zeta,\vartheta] \to \mathbb{R}$ be a function with $0 \le \zeta < \vartheta$ and $\psi \in L_1[\zeta,\vartheta]$. If ψ is a convex function on $[\zeta,\vartheta]$, then the following inequalities for fractional integrals hold*:

$$\psi\left(\frac{\zeta+\vartheta}{2}\right) \le \frac{\Gamma(\varrho+1)}{2(\vartheta-\zeta)^{\varrho}}\left[J^{\varrho}_{(\frac{\zeta+\vartheta}{2})+}\psi(\vartheta) + J^{\varrho}_{(\frac{\zeta+\vartheta}{2})-}\psi(\zeta)\right] \le \frac{\psi(\zeta)+\psi(\vartheta)}{2}$$

with $\varrho > 0$.

Later, Sarıkaya et al. [37] and Iqbal et al. [22] introduced several inequalities of fractional midpoint-type inequalities and the trapezoid-type inequalities for the convex functions, respectively. For other papers about fractional integral inequalities, see [7, 9, 17, 26] and the references cited therein.

Another significant definition in fractional analysis is the following [35].

Definition 2 Let $\varrho > 0$ and $\varrho \notin \{1,2,\ldots\}$, $n = [\varrho]+1$, $\psi \in AC^n[\zeta,\vartheta]$, the space of functions having $n-th$ derivatives is absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order ϱ are defined as follows:

$${}^{C}D^{\varrho}_{\zeta^+}\psi(\varkappa) = \frac{1}{\Gamma(n-\varrho)}\int_{\zeta}^{\varkappa}(\varkappa-\mathfrak{s})^{n-\varrho-1}\psi^{(n)}(\mathfrak{s})\,d\mathfrak{s}, \quad \varkappa > \zeta,$$

and

$${}^{C}D^{\varrho}_{\vartheta^-}\psi(\varkappa) = \frac{1}{\Gamma(n-\varrho)}\int_{\varkappa}^{\vartheta}(\mathfrak{s}-\varkappa)^{n-\varrho-1}\psi^{(n)}(\mathfrak{s})\,d\mathfrak{s}, \quad \varkappa < \vartheta.$$

If $\varrho = n \in \{1,2,3,\ldots\}$ and the usual derivative $\psi^{(n)}(\varkappa)$ of order n exists, then the Caputo fractional derivative ${}^{C}D^{\varrho}_{\zeta^+}\psi(\varkappa)$ coincides with $\psi^{(n)}(\varkappa)$, whereas ${}^{C}D^{\varrho}_{\vartheta^-}\psi(\varkappa)$ with exactness to a constant multiplier $(-1)^n$. For $n = 1$ and $\varrho = 0$, we have ${}^{C}D^{\varrho}_{\zeta^+}\psi(\varkappa) = {}^{C}D^{\varrho}_{\vartheta^-}\psi(\varkappa) = \psi(\varkappa)$.

The Caputo derivative is defined as the application of a fractional integral to a standard derivative of the function, whereas the Riemann–Liouville fractional derivative is obtained by differentiating the fractional integral of a function with respect to its independent variable of order n. The Caputo fractional derivative necessitates more suitable initial conditions in contrast to the conventional Riemann–Liouville fractional derivative considering fractional differential equations [14]. Al-Qurashi et al. [1] introduced a novel discrete, nonequilibrium, memristor-based Hindmarsh–Rose neuron (HRN) with the Caputo fractional difference scheme. Alsharidi et al. [3], utilizing a discrete Caputo fractional derivative, created the permanent magnet synchronous generator systems fractional-order concept. Besides, the operator of proportional derivative denoted as ${}^{P}D_{\varrho}\psi(\varkappa)$ is given by the equation [4]

$${}^{P}D_{\varrho}\psi(\mathfrak{s}) = K_1(\varrho,\mathfrak{s})\psi(\mathfrak{s}) + K_0(\varrho,\mathfrak{s})\psi'(\mathfrak{s}).$$

In this equation, K_1 and K_0 are the functions with respect to $\varrho \in [0,1]$ and $\mathfrak{s} \in \mathbb{R}$ subject to certain conditions, and also, the function ψ is differentiable with respect to $\mathfrak{s} \in \mathbb{R}$. It is connected to the extensive and growing field of conformable derivatives. The use of this operator is a natural occurrence in the field of control theory. The importance of research conducted on both the Caputo derivative and the proportional derivative has increased significantly in recent years [18, 20, 23, 27, 29, 32, 39].

In [6], Baleanu et al. gave the following definition, where they merge the concepts of Caputo derivative and proportional derivative in a novel manner, resulting in a hybrid fractional operator that can be represented as a linear combination of Caputo fractional derivative and Riemann–Liouville fractional integral.

Definition 3 Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and ψ, ψ' be locally $L_1(I)$. Then the proportional Caputo-hybrid operator may be defined as follows:

$$ {}^{C}D^{\varrho}_{\zeta^+}\psi(\mathfrak{s}) = \frac{1}{\Gamma(1-\varrho)}\int_0^{\mathfrak{s}}\big[K_1(\varrho,\tau)\psi(\tau) + K_0(\varrho,\tau)\psi'(\tau)\big](\mathfrak{s}-\tau)^{-\varrho}\,d\tau, $$

where $\varrho \in [0,1]$ and K_1 and K_0 are functions that satisfy the following conditions:

$$ \begin{aligned} &\lim_{\varrho\to 0^+} K_0(\varrho,\tau) = 0; \qquad \lim_{\varrho\to 1} K_0(\varrho,\tau) = 1; \qquad K_0(\varrho,\tau) \neq 0, \quad \varrho \in (0,1]; \\ &\lim_{\varrho\to 0} K_1(\varrho,\tau) = 0; \qquad \lim_{\varrho\to 1^-} K_1(\varrho,\tau) = 1; \qquad K_1(\varrho,\tau) \neq 0, \quad \varrho \in [0,1). \end{aligned} $$

On the other hand, Sarıkaya proposed the new definition by utilizing different K_1 and K_0 functions in the light of Definition 3 and also gave the Hermite–Hadamard inequality using this definition in [36] as follows:

Definition 4 Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and ψ, $\psi' \in L_1(I)$. The left-sided and right-sided proportional Caputo-hybrid operator of order ϱ are defined respectively as follows:

$$ {}^{PC}_{\zeta^+}D^{\varrho}_{\vartheta}\psi(\vartheta) = \frac{1}{\Gamma(1-\varrho)}\int_{\zeta}^{\vartheta}\big[K_1(\varrho,\vartheta-\tau)\psi(\tau) + K_0(\varrho,\vartheta-\tau)\psi'(\tau)\big](\vartheta-\tau)^{-\varrho}\,d\tau $$

and

$$ {}^{PC}_{\vartheta^-}D^{\varrho}_{\zeta}\psi(\zeta) = \frac{1}{\Gamma(1-\varrho)}\int_{\zeta}^{\vartheta}\big[K_1(\varrho,\tau-\zeta)\psi(\tau) + K_0(\varrho,\tau-\zeta)\psi'(\tau)\big](\tau-\zeta)^{-\varrho}\,d\tau, $$

where $\varrho \in [0,1]$ and $K_0(\varrho,\tau) = (1-\varrho)^2\tau^{1-\varrho}$ and $K_1(\varrho,\tau) = \varrho^2\tau^{\varrho}$.

Theorem 2 *Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $\zeta, \vartheta \in I^\circ$ with $\zeta < \vartheta$, and ψ, ψ' be the convex functions on I. Then the following inequalities hold*:

$$ \begin{aligned} &\varrho^2(\vartheta-\zeta)^{\varrho}\psi\left(\frac{\zeta+\vartheta}{2}\right) + \frac{1}{2}(1-\varrho)(\vartheta-\zeta)^{1-\varrho}\psi'\left(\frac{\zeta+\vartheta}{2}\right) \\ &\quad\le \frac{\Gamma(1-\varrho)}{2(\vartheta-\zeta)^{1-\varrho}}\big[{}^{PC}_{\zeta^+}D^{\varrho}_{\vartheta}\psi(\vartheta) + {}^{PC}_{\vartheta^-}D^{\varrho}_{\zeta}\psi(\zeta)\big] \\ &\quad\le \varrho^2(\vartheta-\zeta)^{\varrho}\left[\frac{\psi(\zeta)+\psi(\vartheta)}{2}\right] + (1-\varrho)(\vartheta-\zeta)^{1-\varrho}\left[\frac{\psi'(\zeta)+\psi'(\vartheta)}{4}\right]. \end{aligned} $$

Additionally, Demir et al. [13] introduced an alternative formulation of the Hermite–Hadamard inequality utilizing the Caputo-hybrid operator in a different approach compared to the previous theorem. This can be observed in the following expression.

Theorem 3 *Let $\psi : I \subset \mathbb{R}^{+} \to \mathbb{R}$ be a differentiable function on I^{o}, the interior of the interval I, where $\zeta, \vartheta \in I^{o}$ satisfying $\zeta < \vartheta$, and ψ, ψ' be the convex functions on I. Then the following inequalities are satisfied*:

$$\begin{aligned}
&\varrho^{2}(\vartheta-\zeta)^{\varrho}2^{-\varrho}\psi\left(\frac{\zeta+\vartheta}{2}\right)+(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-2}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\\
&\quad\leq\frac{\Gamma(1-\varrho)}{2^{\varrho}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^{+}}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^{-}}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\\
&\quad\leq\varrho^{2}(\vartheta-\zeta)^{\varrho}2^{-\varrho}\left[\frac{\psi(\zeta)+\psi(\vartheta)}{2}\right]+(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-2}\left[\frac{\psi'(\zeta)+\psi'(\vartheta)}{2}\right].
\end{aligned}$$

The motivation of this paper is to investigate analogous versions of the Hermite–Hadamard-type inequalities with regard to Riemann integrals by using the proportional Caputo-hybrid operator. In line with this purpose, we initially present an identity with the help of the newly defined proportional Caputo-hybrid operator. This identity plays a crucial role in establishing various midpoint-type inequalities. Then, we give many significant inequalities by utilizing convexity, the Hölder inequality, and the power mean inequality. Also, to validate our main findings, we provide concrete examples along with graphical illustrations. These results extend and generalize the inequalities derived in previous studies by considering suitable assumptions of ϱ.

2 Main results

We rely on the following lemma to show our other main findings. We develop various integral inequalities based on this discovery that relate to the left-hand side of Hermite–Hadamard-type inequalities for proportional Caputo-hybrid operators.

Lemma 1 *Let $\psi : I \subset \mathbb{R}^{+} \to \mathbb{R}$ be a twice differentiable function on I^{o}, the interior of the interval I, where $\zeta, \vartheta \in I^{o}$ satisfying $\zeta < \vartheta$, and let $\psi, \psi', \psi'' \in L_{1}[\zeta, \vartheta]$. Then the following identity is satisfied*:

$$\begin{aligned}
&\varrho^{2}(\vartheta-\zeta)^{\varrho+1}2^{-\varrho+1}\left[\int_{0}^{\frac{1}{2}}\mathfrak{s}\psi'\big(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\big)\,d\mathfrak{s}+\int_{\frac{1}{2}}^{1}(\mathfrak{s}-1)\psi'\big(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\big)\,d\mathfrak{s}\right] \qquad (2)\\
&\qquad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\\
&\qquad\times\int_{0}^{1}\big(\mathfrak{s}^{2-2\varrho}-1\big)\left[\psi''\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\\
&\quad=-\varrho^{2}(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\\
&\qquad+\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^{+}}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^{-}}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right].
\end{aligned}$$

Proof By integration by parts, we have

$$\begin{aligned}&\int_0^1 (\mathfrak{s}-1)\psi'\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\&\quad=\frac{2}{\vartheta-\zeta}\psi\left(\frac{\zeta+\vartheta}{2}\right)-\frac{2}{\vartheta-\zeta}\int_0^1\psi\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 \left(\mathfrak{s}^{2-2\varrho}-1\right)\psi''\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\&\quad=\frac{2}{\vartheta-\zeta}\psi'\left(\frac{\zeta+\vartheta}{2}\right)-\frac{4(1-\varrho)}{\vartheta-\zeta}\int_0^1\mathfrak{s}^{1-2\varrho}\psi'\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}.\end{aligned}$$

By utilizing a change of variable, multiplying the outcomes by $\varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho-1}$ and $(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}$, and merging them side by side, we attain the following result:

$$\begin{aligned}&\varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho-1}\int_0^1(\mathfrak{s}-1)\psi'\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\&\qquad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\int_0^1\left(\mathfrak{s}^{2-2\varrho}-1\right)\psi''\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\&\quad=\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho}\psi\left(\frac{\zeta+\vartheta}{2}\right)+(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-2}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\\&\qquad-\frac{2^{1-\varrho}}{(\vartheta-\zeta)^{1-\varrho}}\int_{\frac{\zeta+\vartheta}{2}}^{\vartheta}\left[\varrho^2\left(\tau-\frac{\zeta+\vartheta}{2}\right)^{\varrho}\psi(\tau)+(1-\varrho)^2\left(\tau-\frac{\zeta+\vartheta}{2}\right)^{1-\varrho}\psi'(\tau)\right]\\&\qquad\times\left(\tau-\frac{\zeta+\vartheta}{2}\right)^{-\varrho}d\tau.\end{aligned}\tag{3}$$

Another result derived using similar methods is presented here:

$$\begin{aligned}&\varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho-1}\int_0^1(\mathfrak{s}-1)\psi'\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\&\qquad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\int_0^1\left(\mathfrak{s}^{2-2\varrho}-1\right)\psi''\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\&\quad=-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-2}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\\&\qquad+\frac{2^{1-\varrho}}{(\vartheta-\zeta)^{1-\varrho}}\int_{\zeta}^{\frac{\zeta+\vartheta}{2}}\left[\varrho^2\left(\frac{\zeta+\vartheta}{2}-\tau\right)^{\varrho}\psi(\tau)+(1-\varrho)^2\left(\frac{\zeta+\vartheta}{2}-\tau\right)^{1-\varrho}\psi'(\tau)\right]\\&\qquad\times\left(\frac{\zeta+\vartheta}{2}-\tau\right)^{-\varrho}d\tau.\end{aligned}\tag{4}$$

Subtracting (4) from (3), we get

$$\begin{aligned}&\varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho-1}\int_0^1(\mathfrak{s}-1)\left[\psi'\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)-\psi'\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\\&\qquad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\end{aligned}$$

$$\times \int_0^1 \left(\mathfrak{s}^{2-2\varrho}-1\right)\left[\psi''\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}$$
$$= -\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)$$
$$+\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^+}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right].$$

Therefore, with the utilization of the equality

$$\int_0^1 (\mathfrak{s}-1)\left[\psi'\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)-\psi'\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}$$
$$=4\left[\int_0^{\frac{1}{2}}\mathfrak{s}\psi'\left(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}+\int_{\frac{1}{2}}^1(\mathfrak{s}-1)\psi'\left(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}\right],$$

we reach the proof's conclusion. □

Remark 1 By considering the limit as $\varrho\to 1$ in Lemma 1, it can be deduced that

$$\frac{1}{\vartheta-\zeta}\int_\zeta^\vartheta \psi(\varkappa)\,d\varkappa-\psi\left(\frac{\zeta+\vartheta}{2}\right)$$
$$=(\vartheta-\zeta)\left[\int_0^{\frac{1}{2}}\mathfrak{s}\psi'\left(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}+\int_{\frac{1}{2}}^1(\mathfrak{s}-1)\psi'\left(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}\right],$$

as demonstrated by Kırmacı [25].

Corollary 1 *When we consider the limiting case of ϱ approaching 0 in Lemma 1, we find that*

$$\frac{(\vartheta-\zeta)^2}{8}\left(\int_0^1\left(\mathfrak{s}^2-1\right)\left[\psi''\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\right)$$
$$=-\frac{(\vartheta-\zeta)}{2}\psi'\left(\frac{\zeta+\vartheta}{2}\right)+\left[\psi(\vartheta)-\psi(\zeta)\right]$$
$$+\frac{2}{\vartheta-\zeta}\left(\int_\zeta^{\frac{\zeta+\vartheta}{2}}\psi(\varkappa)\,d\varkappa-\int_{\frac{\zeta+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\,d\varkappa\right).$$

Moreover, by selecting $\varrho=\frac{1}{2}$, equality (2) takes the form

$$\frac{1}{\vartheta-\zeta}\left\{-\psi\left(\frac{\zeta+\vartheta}{2}\right)-\psi'\left(\frac{\zeta+\vartheta}{2}\right)+\frac{1}{\vartheta-\zeta}\left[\int_\zeta^\vartheta\psi(\varkappa)\,d\varkappa+\psi(\vartheta)-\psi(\zeta)\right]\right\}$$
$$=\int_0^{\frac{1}{2}}\mathfrak{s}\psi'\left(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}+\int_{\frac{1}{2}}^1(\mathfrak{s}-1)\psi'\left(\mathfrak{s}\zeta+(1-\mathfrak{s})\vartheta\right)d\mathfrak{s}$$
$$+\frac{1}{4}\int_0^1(\mathfrak{s}-1)\left[\psi''\left(\frac{1+\mathfrak{s}}{2}\zeta+\frac{1-\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1-\mathfrak{s}}{2}\zeta+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}.$$

Theorem 4 *Let $\psi: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\zeta,\vartheta\in I^o$ satisfying $\zeta<\vartheta$, and let $\psi,\psi',\psi''\in L_1[\zeta,\vartheta]$. If $|\psi'|^q$ and $|\psi''|^q$*

are convex on $[\zeta,\vartheta]$ *for* $q\geq 1$, *then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^+}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\right. \\
&\qquad\left.-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right| \\
&\quad\leq \varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho-2}\left(\frac{1+2^{\frac{1}{q}}}{3^{\frac{1}{q}}}\right)\left(\left|\psi'(\zeta)\right|+\left|\psi'(\vartheta)\right|\right) \\
&\qquad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\left\{\left(\frac{2-2\varrho}{3-2\varrho}\right)^{\frac{q-1}{q}}\left(\left[\frac{|\psi''(\zeta)|^q}{2}\left(\frac{3}{2}-\frac{1}{3-2\varrho}-\frac{1}{4-2\varrho}\right)\right.\right.\right. \\
&\qquad\left.+\frac{|\psi''(\vartheta)|^q}{2}\left(\frac{1}{2}-\frac{1}{3-2\varrho}+\frac{1}{4-2\varrho}\right)\right]^{\frac{1}{q}}+\left[\frac{|\psi''(\zeta)|^q}{2}\left(\frac{1}{2}-\frac{1}{3-2\varrho}+\frac{1}{4-2\varrho}\right)\right. \\
&\qquad\left.\left.\left.+\frac{|\psi''(\vartheta)|^q}{2}\left(\frac{3}{2}-\frac{1}{3-2\varrho}-\frac{1}{4-2\varrho}\right)\right]^{\frac{1}{q}}\right)\right\}.
\end{aligned}\tag{5}$$

Proof To begin with, consider the case where $q=1$. By employing the convexity of $|\psi'|$ and $|\psi''|$, it follows from Lemma 1 that

$$\begin{aligned}
&\left|\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^+}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\right. \\
&\qquad\left.-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right| \\
&\quad\leq \varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho+1}\left[\int_0^{\frac{1}{2}}\mathfrak{s}\left(\mathfrak{s}\left|\psi'(\zeta)\right|+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|\right)d\mathfrak{s}\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1(1-\mathfrak{s})\left(\mathfrak{s}\left|\psi'(\zeta)\right|+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|\right)d\mathfrak{s}\right] \\
&\qquad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right) \\
&\qquad\times\left(\frac{1-\mathfrak{s}}{2}\left|\psi''(\zeta)\right|+\frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|+\frac{1+\mathfrak{s}}{2}\left|\psi''(\zeta)\right|+\frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|\right)d\mathfrak{s}.
\end{aligned}\tag{6}$$

Then, because of

$$\int_0^{\frac{1}{2}}\mathfrak{s}^2\,d\mathfrak{s}=\int_{\frac{1}{2}}^1(1-\mathfrak{s})^2\,d\mathfrak{s}=\frac{1}{24},\qquad \int_0^{\frac{1}{2}}\mathfrak{s}(1-\mathfrak{s})\,d\mathfrak{s}=\int_{\frac{1}{2}}^1(1-\mathfrak{s})\mathfrak{s}\,d\mathfrak{s}=\frac{1}{12}$$

and

$$\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right)d\mathfrak{s}=\frac{2-2\varrho}{3-2\varrho},$$

we have that the expression on the right-hand side of inequality (6) is

$$\frac{\varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho+1}}{4}\left(\frac{|\psi'(\zeta)|+|\psi'(\vartheta)|}{2}\right)$$
$$+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-2}\left(\frac{2-2\varrho}{3-2\varrho}\right)\left(\frac{|\psi''(\zeta)|+|\psi''(\vartheta)|}{2}\right).$$

Furthermore, for $q>1$, by utilizing Lemma 1, the power mean inequality, and considering the convexity of $|\psi'|^q$ and $|\psi''|^q$, we can conclude

$$\begin{aligned}
&\left|\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^+}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\right.\\
&\qquad\left.-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right|\\
&\le\varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho+1}\left\{\left(\int_0^{\frac12}\mathfrak{s}\,d\mathfrak{s}\right)^{\frac1p}\left(\int_0^{\frac12}\mathfrak{s}\left[\mathfrak{s}\left|\psi'(\zeta)\right|^q+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac1q}\right.\\
&\qquad\left.+\left(\int_{\frac12}^1(1-\mathfrak{s})\,d\mathfrak{s}\right)^{\frac1p}\left(\int_{\frac12}^1(1-\mathfrak{s})\left[\mathfrak{s}\left|\psi'(\zeta)\right|^q+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac1q}\right\}\\
&\qquad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\left\{\left(\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right)d\mathfrak{s}\right)^{\frac1p}\right.\\
&\qquad\times\left(\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right)\left[\frac{1+\mathfrak{s}}{2}\left|\psi''(\zeta)\right|^q+\frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac1q}\\
&\qquad+\left(\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right)d\mathfrak{s}\right)^{\frac1p}\\
&\qquad\left.\times\left(\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right)\left[\frac{1-\mathfrak{s}}{2}\left|\psi''(\zeta)\right|^q+\frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac1q}\right\}.
\end{aligned}$$

Thus, since

$$\int_0^{\frac12}\mathfrak{s}\,d\mathfrak{s}=\int_{\frac12}^1(1-\mathfrak{s})\,d\mathfrak{s}=\frac18,$$
$$\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right)(1+\mathfrak{s})\,d\mathfrak{s}=\frac32-\frac{1}{3-2\varrho}-\frac{1}{4-2\varrho},$$

and

$$\int_0^1\left(1-\mathfrak{s}^{2-2\varrho}\right)(1-\mathfrak{s})\,d\mathfrak{s}=\frac12-\frac{1}{3-2\varrho}+\frac{1}{4-2\varrho},$$

inequality (5) that we aimed to prove holds true. □

Remark 2 By taking the limit as $\varrho \to 1$ and setting $q = 1$ in Theorem 4, we find that

$$\left|\frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta}\psi(\varkappa)\,d\varkappa-\psi\left(\frac{\zeta+\vartheta}{2}\right)\right|\leq\frac{(\vartheta-\zeta)}{8}\left(\left|\psi'(\zeta)\right|+\left|\psi'(\vartheta)\right|\right),$$

which was proved by Kırmacı [25].

Corollary 2 *As ϱ approaches 0 and for $q \geq 1$ in Theorem 4, we obtain*

$$\begin{aligned}&\left|\psi(\vartheta)-\psi(\zeta)+\frac{2}{\vartheta-\zeta}\left(\int_{\zeta}^{\frac{\zeta+\vartheta}{2}}\psi(\varkappa)\,d\varkappa-\int_{\frac{\zeta+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\,d\varkappa\right)-\frac{(\vartheta-\zeta)}{2}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right|\\&\quad\leq\frac{(\vartheta-\zeta)^2}{8}\left(\frac{2}{3}\right)^{\frac{q-1}{q}}\left[\left(\frac{11}{12}\left|\psi''(\zeta)\right|^q+\frac{5}{12}\left|\psi''(\vartheta)\right|^q\right)^{1/q}\right.\\&\qquad\left.+\left(\frac{5}{12}\left|\psi''(\zeta)\right|^q+\frac{11}{12}\left|\psi''(\vartheta)\right|^q\right)^{1/q}\right].\end{aligned}$$

Additionally, as ϱ tends to 1 and for $q \geq 1$, the inequality stated in Theorem 4 simplifies to

$$\begin{aligned}&\left|\frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta}\psi(\varkappa)\,d\varkappa-\psi\left(\frac{\zeta+\vartheta}{2}\right)\right|\\&\quad\leq\frac{(\vartheta-\zeta)}{8}\left(\frac{1+2^{\frac{1}{q}}}{3^{\frac{1}{q}}}\right)\left(\left|\psi'(\zeta)\right|+\left|\psi'(\vartheta)\right|\right).\end{aligned}$$

Now, as an illustration of our theorem's applicability, we present an example.

Example 1 If we take the function $\psi(\varkappa) = 2\varkappa^3$ defined on the interval $[0, 2]$, we can evaluate the right-hand side of inequality (5) in the following manner:

$$\begin{aligned}&12\varrho^2\left(\frac{1+2^{\frac{1}{q}}}{3^{\frac{1}{q}}}\right)+\frac{(1-\varrho)}{2}\left(\frac{2-2\varrho}{3-2\varrho}\right)^{\frac{q-1}{q}}\\&\quad\times\frac{24}{2^{\frac{1}{q}}}\left[\left(\frac{1}{2}-\frac{1}{3-2\varrho}+\frac{1}{4-2\varrho}\right)^{\frac{1}{q}}+\left(\frac{3}{2}-\frac{1}{3-2\varrho}-\frac{1}{4-2\varrho}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Furthermore, we show that

$$\begin{aligned}&\left|\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}_{\zeta^+}^{PC}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}_{\vartheta^-}^{PC}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\right.\\&\qquad\left.-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right|\\&\quad=4\varrho^2+\frac{6(1-\varrho)^2}{2-\varrho}.\end{aligned}$$

Figure 1 clearly demonstrates that the left-hand side of inequality (5) is consistently situated below the right-hand side of this inequality for all $\varrho \in (0, 1)$ and $q \geq 1$.

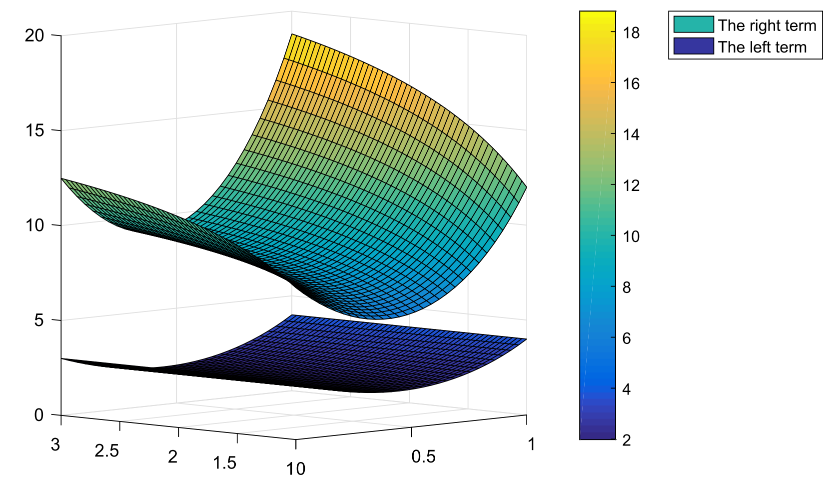

Figure 1 The graph of both sides of inequality (5) depending on $\varrho \in (0,1)$ and $q \in [1,3]$

Theorem 5 *Let $\psi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\zeta, \vartheta \in I^o$ satisfying $\zeta < \vartheta$, and let $\psi, \psi', \psi'' \in L_1[\zeta, \vartheta]$. If $|\psi'|^q$ and $|\psi''|^q$ are convex on $[\zeta, \vartheta]$ for $q > 1$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^+}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\right. \\
&\quad\left.-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right| \\
&\le \frac{\varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho-\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{|\psi'(\zeta)|^q+3|\psi'(\vartheta)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\psi'(\zeta)|^q+|\psi'(\vartheta)|^q}{8}\right)^{\frac{1}{q}}\right] \\
&\quad+(1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3}\left(\frac{(2-2\varrho)p}{(2-2\varrho)p+1}\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\frac{|\psi''(\zeta)|^q+3|\psi''(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\psi''(\zeta)|^q+|\psi''(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof Using absolute value in Lemma 1 and applying the well-known Hölder's inequality and the convexity of $|\psi'|^q$, $|\psi''|^q$, we get

$$\begin{aligned}
&\left|\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}^{PC}_{\zeta^+}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\right. \\
&\quad\left.-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right| \\
&\le \varrho^2(\vartheta-\zeta)^{\varrho+1}2^{-\varrho+1}\left[\left(\int_0^{\frac{1}{2}}\mathfrak{s}^p\,d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\mathfrak{s}\left|\psi'(\zeta)\right|^q+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^1(1-\mathfrak{s})^p\,d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[\mathfrak{s}\left|\psi'(\zeta)\right|^q+(1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{8}$$

$$+ (1-\varrho)(\vartheta-\zeta)^{2-\varrho}2^{\varrho-3} \times \left[\left(\int_0^1 \left[1-\mathfrak{s}^{2-2\varrho}\right]^p d\mathfrak{s}\right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1+\mathfrak{s}}{2}\left|\psi''(\zeta)\right|^q + \frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right] d\mathfrak{s}\right)^{\frac{1}{q}} + \left(\int_0^1 \left[1-\mathfrak{s}^{2-2\varrho}\right]^p d\mathfrak{s}\right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1-\mathfrak{s}}{2}\left|\psi''(\zeta)\right|^q + \frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right] d\mathfrak{s}\right)^{\frac{1}{q}}\right]. \tag{9}$$

The integrals in the inequality mentioned above can be assessed by

$$\int_0^{\frac{1}{2}} \mathfrak{s}^p \, d\mathfrak{s} = \int_{\frac{1}{2}}^1 (1-\mathfrak{s})^p \, d\mathfrak{s} = \frac{1}{2^{p+1}(p+1)},$$

$$\int_0^{\frac{1}{2}} \left[\mathfrak{s}\left|\psi'(\zeta)\right|^q + (1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right] d\mathfrak{s} = \frac{|\psi'(\zeta)|^q + 3|\psi'(\vartheta)|^q}{8},$$

$$\int_{\frac{1}{2}}^1 \left[\mathfrak{s}\left|\psi'(\zeta)\right|^q + (1-\mathfrak{s})\left|\psi'(\vartheta)\right|^q\right] d\mathfrak{s} = \frac{3|\psi'(\zeta)|^q + |\psi'(\vartheta)|^q}{8}$$

$$\int_0^1 \left[\frac{1-\mathfrak{s}}{2}\left|\psi''(\zeta)\right|^q + \frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right] d\mathfrak{s} = \frac{|\psi''(\zeta)|^q + 3|\psi''(\vartheta)|^q}{4},$$

$$\int_0^1 \left[\frac{1+\mathfrak{s}}{2}\left|\psi''(\zeta)\right|^q + \frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right] d\mathfrak{s} = \frac{3|\psi''(\zeta)|^q + |\psi''(\vartheta)|^q}{4}.$$

Also, using the property that is $(A-B)^p \leq A^p - B^p$ for $A > B \geq 0$ and $p \geq 1$, we have

$$\int_0^1 \left[1-\mathfrak{s}^{2-2\varrho}\right]^p d\mathfrak{s} \leq \int_0^1 \left[1-\mathfrak{s}^{(2-2\varrho)p}\right] d\mathfrak{s} = \frac{(2-2\varrho)p}{(2-2\varrho)p+1}.$$

Hence, by substituting the computed integral results into inequality (8), the desired outcome can be attained. □

Remark 3 In the special case where ϱ approaches 1 in Theorem 5, we obtain

$$\left|\frac{1}{\vartheta-\zeta}\int_\zeta^\vartheta \psi(\varkappa)\,d\varkappa - \psi\left(\frac{\zeta+\vartheta}{2}\right)\right| \leq \frac{(\vartheta-\zeta)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}} \left\{\left[\left|\psi'(\zeta)\right|^q + 3\left|\psi'(\vartheta)\right|^q\right]^{\frac{1}{q}} + \left[3\left|\psi'(\zeta)\right|^q + \left|\psi'(\vartheta)\right|^q\right]^{\frac{1}{q}}\right\},$$

which was proved by Kırmacı in [25].

Corollary 3 *In the particular case when ϱ tends to 0 in Theorem 5, we obtain*

$$\left|\psi(\vartheta) - \psi(\zeta) + \frac{2}{\vartheta-\zeta}\left(\int_\zeta^{\frac{\zeta+\vartheta}{2}} \psi(\varkappa)\,d\varkappa - \int_{\frac{\zeta+\vartheta}{2}}^\vartheta \psi(\varkappa)\,d\varkappa\right) - \frac{(\vartheta-\zeta)}{2}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right| \leq \frac{(\vartheta-\zeta)^2}{8}\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}} \left[\left(\frac{|\psi''(\zeta)|^q + 3|\psi''(\vartheta)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|\psi''(\zeta)|^q + |\psi''(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right].$$

In addition, by choosing $\varrho = \frac{1}{2}$, we get

$$\begin{aligned}&\left|\frac{1}{\vartheta-\zeta}\left[\int_{\zeta}^{\vartheta}\psi(\varkappa)\,d\varkappa+\psi(\vartheta)-\psi(\zeta)\right]-\left[\psi\left(\frac{\zeta+\vartheta}{2}\right)+\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right]\right|\\&\quad\leq\frac{(\vartheta-\zeta)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\psi'(\zeta)|^q+3|\psi'(\vartheta)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\psi'(\zeta)|^q+|\psi'(\vartheta)|^q}{8}\right)^{\frac{1}{q}}\right]\\&\qquad+\frac{\vartheta-\zeta}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\psi''(\zeta)|^q+3|\psi''(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\psi''(\zeta)|^q+|\psi''(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Now, we give an example that proves the validity of the inequality established in Theorem 5 in order to illustrate it.

Example 2 Taking into account the function ψ defined in Example 1, the expression on the right-hand side of inequality (7) can be evaluated as follows:

$$24\left(\frac{1+3^{\frac{p-1}{p}}}{4^{\frac{p-1}{p}}}\right)\left[\varrho^2\frac{1}{(p+1)^{\frac{1}{p}}}+\frac{(1-\varrho)}{2}\left(\frac{(2-2\varrho)p}{(2-2\varrho)p+1}\right)^{\frac{1}{p}}\right].$$

On the other hand, we know that

$$\begin{aligned}&\left|\frac{\Gamma(1-\varrho)}{2^{\varrho-1}(\vartheta-\zeta)^{-\varrho+1}}\left[{}_{\zeta^+}^{PC}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)+{}_{\vartheta^-}^{PC}D^{\varrho}_{(\frac{\zeta+\vartheta}{2})}\psi\left(\frac{\zeta+\vartheta}{2}\right)\right]\right.\\&\qquad\left.-\varrho^2(\vartheta-\zeta)^{\varrho}2^{-\varrho+1}\psi\left(\frac{\zeta+\vartheta}{2}\right)-(1-\varrho)(\vartheta-\zeta)^{1-\varrho}2^{\varrho-1}\psi'\left(\frac{\zeta+\vartheta}{2}\right)\right|\\&\quad=4\varrho^2+\frac{6(1-\varrho)^2}{2-\varrho}.\end{aligned}$$

Therefore, it can be seen from Fig. 2 that the left-hand side of inequality (7) is consistently lower than the right-hand side for all values of $\varrho\in(0,1)$ and $p>1$.

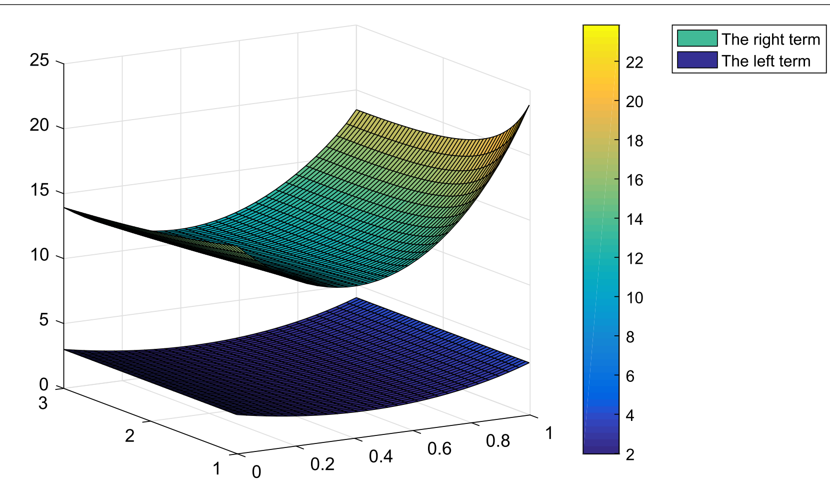

Figure 2 The graph of both sides of inequality (7) depending on $\varrho\in(0,1)$ and $p\in(1,3]$

3 Conclusion

Numerous research works have been conducted to optimize the bounds with the aid of various fractional integral operators in light of recent advancements in the field of fractional analysis. One of these operators is the proportional Caputo hybrid operator. The primary goal of this study is to obtain new and general inequalities that establish a connection between inequality theory and fractional analysis by utilizing this operator since this topic is important and has numerous implications in modeling real-world natural events. In this work, we develop a new technique concerning the left-hand side of Hermite–Hadamard-type inequalities for the current operator with respect to functions whose derivatives in absolute value at certain powers are convex functions. Moreover, we demonstrate how the results develop upon and enhance a great deal of prior research in the context of integral inequalities. Afterward, we provide concrete examples together with the corresponding graphs to provide greater understanding of the newly found inequalities. Therefore, it is expected that these theoretical studies will pave the way to further investigation of novel approaches for the proportional Caputo-hybrid operator as well as in many areas of application. In future work, one can investigate different inequalities, such as Grüss-type inequalities, Chebyshev-type inequalities, or Simpson-type inequalities, via this operator. Moreover, one can concentrate on how these inequalities are used in the real world.

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Competing interests
The authors declare no competing interests.

Author contributions
İ.D and T.T. wrote the main manuscript text and İ.D. prepared Figs. 1–2. All authors reviewed manuscript.

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References

1. Al-Qurashi, M., Asif, Q.U., Chu, Y.M., Rashid, S., Elagan, S.K.: Complexity analysis and discrete fractional difference implementation of the Hindmarsh-Rose neuron system. Results Phys. **51**, 106627 (2023)
2. Alp, N., Sarıkaya, M.Z., Kunt, M., İşcan, İ.: *q*-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. J. King Saud Univ., Sci. **30**(2), 193–203 (2018)
3. Alsharidi, A.K., Rashid, S., Elagan, S.K.: Short-memory discrete fractional difference equation wind turbine model and its inferential control of a chaotic permanent magnet synchronous transformer in time-scale analysis. AIMS Math. **8**(8), 19097–19120 (2023)
4. Anderson, D.R., Ulness, D.J.: Newly defined conformable derivatives. Adv. Dyn. Syst. Appl. **10**(2), 109–137 (2015)
5. Atangana, A., Baleanu, D.: New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm. Sci. **20**(2), 763–769 (2016)
6. Balenau, D., Ferrandez, A., Akgül, A.: On a fractional operator combining proportional and classical differintegrals. Mathematics **8**(3), 360 (2020)
7. Budak, H., Ertugral, F., Sarıkaya, M.Z.: New generalization of Hermite-Hadamard type inequalities via generalized fractional integrals. An. Univ. Craiova, Ser. Mat. Inform. **47**(2), 369–386 (2020)
8. Budak, H., Tunç, T., Sarıkaya, M.Z.: Fractional Hermite-Hadamard-type inequalities for interval-valued functions. Proc. Am. Math. Soc. **148**(2), 705–718 (2020)
9. Budak, H., Yıldırım, S.K., Sarıkaya, M.Z., Yıldırım, H.: Some parameterized Simpson-, midpoint-and trapezoid-type inequalities for generalized fractional integrals. J. Inequal. Appl. **2022**(1), 40 (2022)
10. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. **1**(2), 73–85 (2015)
11. Chu, Y.M., Rashid, S., Asif, Q.U., Abdalbagi, M.: On configuring new chaotic behaviours for a variable fractional-order memristor-based circuit in terms of Mittag-Leffler kernel. Results Phys. **53**, 106939 (2023)
12. Chu, Y.M., Rashid, S., Karim, S., Khalid, A., Elagan, S.K.: Deterministic-stochastic analysis of fractional differential equations malnutrition model with random perturbations and crossover effects. Sci. Rep. **13**(1), 14824 (2023)

13. Demir, İ., Tunç, T., Sarıkaya, M.Z.: A new approach to Hermite-Hadamard-type inequality with proportional Caputo-hybrid operator (submitted)
14. Diethelm, K.: The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Springer, Berlin (2010)
15. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998)
16. Dragomir, S.S., Pearce, C.E.M.: Selected topics on the Hermite-Hadamard inequalities and applications. RGMIA Monographs, Victoria University
17. Ertuğral, F., Sarıkaya, M.Z., Budak, H.: On Hermite-Hadamard type inequalities associated with the generalized fractional integrals. Filomat **36**(12), 3983–3995 (2022)
18. Günerhan, H., Dutta, H., Dokuyucu, M.A., Adel, W.: Analysis of a fractional HIV model with Caputo and constant proportional Caputo operators. Chaos Solitons Fractals **139**, 110053 (2020)
19. Hadamard, J.: Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considree par Riemann. J. Math. Pures Appl. **58**, 171–215 (1893)
20. Hajaj, R., Odibat, Z.: Numerical solutions of fractional epidemic models with generalized Caputo-type derivatives. Phys. Scr. **98**(4), 045206 (2023)
21. Hezenci, F., Kara, H., Budak, H.: Conformable fractional versions of Hermite-Hadamard-type inequalities for twice-differentiable functions. Bound. Value Probl. **2023**, 48 (2023)
22. Iqbal, M., Iqbal, B.M., Nazeer, K.: Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals. Bull. Korean Math. Soc. **52**(3), 707–716 (2015)
23. Jarad, F., Abdeljawad, T., Baleanu, D.: On the generalized fractional derivatives and their Caputo modification. J. Nonlinear Sci. Appl. **10**(5), 2607–2619 (2017)
24. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
25. Kırmacı, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula. Appl. Math. Comput. **147**(5), 137–146 (2004)
26. Mohammed, P.O., Brevik, I.: A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. Symmetry **12**(4), 610 (2020)
27. Rahman, G., Nisar, K.S., Abdeljawad, T.: Certain Hadamard proportional fractional integral inequalities. Mathematics **8**(4), 504 (2020)
28. Rashid, S., Abouelmagd, E., Khalid, A., Farooq, F.B., Chu, Y.M.: Some recent developments on dynamical Planck constant over two $\hbar$-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels. Fractals **30**(02), 2240110 (2022)
29. Rashid, S., Abouelmagd, E.I., Sultana, S., Chu, Y.M.: New developments in weighted n-fold type inequalities via discrete generalized $\widehat{\hbar}$-proportional fractional operators. Fractals **30**(02), 2240056 (2022)
30. Rashid, S., Karim, S., Akgül, A., Bariq, A., Elagan, S.K.: Novel insights for a nonlinear deterministic-stochastic class of fractional-order Lassa fever model with varying kernels. Sci. Rep. **13**(1), 15320 (2023)
31. Rashid, S., Khalid, A., Bazighifan, O., Oros, G.I.: New modifications of integral inequalities via p-convexity pertaining to fractional calculus and their applications. Mathematics **9**(15), 1753 (2021)
32. Rashid, S., Noor, M.A., Noor, K.I.: Caputo fractional derivatives and inequalities via preinvex stochastic processes. Filomat **37**(19), 6569–6584 (2023)
33. Rashid, S., Sobia, S., Hammouch, Z., Jarad, F., Hamed, Y.S.: Novel aspects of discrete dynamical type inequalities within fractional operators having generalized $\hbar$-discrete Mittag-Leffler kernels and application. Chaos Solitons Fractals **151**, 111204 (2021)
34. Sabzikar, F., Meerschaert, M.M., Chen, J.: Tempered fractional calculus. J. Comput. Phys. **293**, 14–28 (2015)
35. Samko, S., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, London (1993)
36. Sarıkaya, M.Z.: On Hermite-Hadamard type inequalities for proportional Caputo-hybrid operator. Konuralp J. Math. **11**(1), 31–39 (2023)
37. Sarıkaya, M.Z., Set, E., Yaldız, H., Basak, N.: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9–10), 2403–2407 (2013)
38. Sarıkaya, M.Z., Yıldırım, H.: On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. Miskolc Math. Notes **17**(2), 1049–1059 (2017)
39. Zhou, S.S., Rashid, S., Set, E., Ahmad, A.G., Hamed, Y.S.: On more general inequalities for weighted generalized proportional Hadamard fractional integral operator with applications. AIMS Math. **6**(9), 9154–9176 (2021)

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