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Necessary and sufficient conditions for discrete inequalities of Jensen–Steffensen type with applications

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Abstract

In this paper we give a necessary and sufficient condition for the discrete Jensen inequality to be satisfied for real (not necessarily nonnegative) weights. The result generalizes and completes the classical Jensen–Steffensen inequality. The validity of the strict inequality is studied. As applications, we first give the form of our result for strongly convex functions, then we study discrete quasi-arithmetic means with real (not necessarily nonnegative) weights, and finally we refine the inequality obtained.

Keywords: Convex and strongly convex functions; Discrete Jensen–Steffensen inequality; Quasi-arithmetic means; Refinement

1 Introduction

By $\mathbb{N}_+$ we denote the set of positive integers.

Let $C \subset \mathbb{R}$ be an interval with nonempty interior. Let F_C denote the set of all convex functions on C.

A fundamental consequence of the notion of a convex set and a convex function is the following remarkable statement.

Theorem 1 (*The discrete Jensen inequality, see* [9]) *Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $m \in \mathbb{N}_+$. Assume that $p_1, \dots, p_m$ are nonnegative numbers such that $\sum_{i=1}^m p_i > 0$, and assume $(s_1, \dots, s_m) \in C^m$. Then*

$$\frac{1}{\sum_{i=1}^m p_i} \sum_{i=1}^m p_i s_i \in \left[\min_{i=1,\dots,m} s_i, \max_{i=1,\dots,m} s_i \right] \subset C, \tag{1}$$

and for every function $f \in F_C$, the inequality

$$f\left(\frac{1}{\sum_{i=1}^m p_i} \sum_{i=1}^m p_i s_i \right) \le \frac{1}{\sum_{i=1}^m p_i} \sum_{i=1}^m p_i f(s_i) \tag{2}$$

holds.

If $f \in F_C$ is strictly convex, inequality (2) *is strict if the points $s_1, \dots, s_m$ are not all equal and the scalars $p_1, \dots, p_m$ are positive.*

The following statement is from Steffensen [11]. Its significance is that it contains conditions under which inclusion (1) and inequality (2) are satisfied even if the numbers $p_1, \dots, p_n$ are not all nonnegative.

Theorem 2 (*The discrete Jensen–Steffensen inequality, see* [9]) *Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $m \in \mathbb{N}_+$. Assume that $p_1, \dots p_m$ are real numbers such that $\sum_{i=1}^m p_i > 0$ and*

$$0 \le \sum_{i=1}^{l} p_i \le \sum_{i=1}^{m} p_i, \quad l = 1, \dots, m, \tag{3}$$

and assume that $(s_1, \dots, s_m) \in C^m$ is a monotonic m-tuple (either increasing or decreasing).
Then (1) *is satisfied, and for every function $f \in F_C$ inequality* (2) *holds.*

A natural question is whether condition (3) is not only sufficient but also necessary to satisfy (1) and (2). The answer is not, which is illustrated by the following example.

Example 3 Let $C := [0, 3]$, $s_1 := 3$, $s_2 := 1$, $s_3 := 0$, and $p_1 := 1$, $p_2 := -5/2$, $p_3 := 2$. Then

$$\sum_{i=1}^{3} p_i > 0, \quad \text{but } p_1 > \sum_{i=1}^{3} p_i \quad \text{and} \quad p_1 + p_2 < 0.$$

Nevertheless,

$$\frac{1}{\sum_{i=1}^3 p_i} \sum_{i=1}^{3} p_i s_i = 1 \in [0, 3],$$

and for every function $f \in F_{[0,3]}$, the inequality

$$f(1) = f\left(\frac{1}{\sum_{i=1}^3 p_i} \sum_{i=1}^{3} p_i s_i\right) \le \frac{1}{\sum_{i=1}^3 p_i} \sum_{i=1}^{3} p_i f(s_i) = 2f(3) - 5f(1) + 4f(0)$$

obviously holds.

Remark 4 From paper [3] comes the statement (adopted by some other papers on the subject) that conditions $\sum_{i=1}^m p_i > 0$ and (3) are necessary and sufficient for (1) to be satisfied (see Lemma 2 in [3]). The previous example shows that the conditions indicated are not necessary in general. The proof of the necessity statement of Lemma 2 in [3] is incorrect because it proves the satisfaction of each inequality in (3) using a different sequence $(s_1, \dots, s_m)$.

Also interesting is the question of when there is a strict inequality in the discrete Jensen–Steffensen inequality. Nice necessary and sufficient conditions for this problem are given in [1]. A more complex set of conditions is needed than for the discrete Jensen inequality.

However, the following answer can be found in some papers (see e.g. [7]): If f is strictly convex, then inequality (2) is strict unless $s_1 = \cdots = s_m$. The example below shows that this is usually not true even when $p_i \neq 0$ $(i = 1, \ldots, m)$.

Example 5 Let $C := [1,2]$, $s_1 := 2$, $s_2 = s_3 := 1$, and $p_1 = p_3 := 1$, $p_2 := -1$. Then the conditions of Theorem 2 are satisfied, $s_1 \neq s_2$, but for every $f \in F_{[1,2]}$

$$f\left(\frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i s_i\right) = f(2) = \frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i f(s_i).$$

Of course, the case of equality in the previous example can be simply deduced from Theorem 1 in [1].

In this paper, we give necessary and sufficient conditions for satisfying inequality (2) in the case when $p_1, \ldots p_m$ are real numbers such that $\sum_{i=1}^{m} p_i > 0$. The result generalizes and completes the discrete Jensen–Steffensen inequality. We consider both the case where the points in the interval are ordered and the case where they are not. The validity of the strict inequality is studied. As applications, we first give the form of our result for strongly convex functions, then we study discrete quasi-arithmetic means with real (not necessarily nonnegative) weights, and finally we refine the inequality obtained.

2 Preliminary results

Let $(X, \mathcal{A})$ be a measurable space ($\mathcal{A}$ always means a σ-algebra of subsets of X). If μ is either a measure or a signed measure on $\mathcal{A}$, then the real vector space of μ-integrable real functions on X is denoted by $L(\mu)$. The integrable functions are considered to be measurable. The unit mass at $x \in X$ (the Dirac measure at x) is denoted by ε_x.

The set of all subsets of a set X is denoted by $P(X)$.

Let $C \subset \mathbb{R}$ be an interval with nonempty interior, which is denoted by C°.

Let $(X, \mathcal{A}, \mu)$ be a measure space, where μ is a finite signed measure, and let $\varphi : X \to C$ be a function such that $\varphi \in L(\mu)$. We define $F_C(\varphi)$ as the set of all functions $f \in F_C$ such that $f \circ \varphi \in L(\mu)$.

The following result is a version of the integral Jensen inequality for signed measures. It is proved in Theorem 12 of [5].

Theorem 6 *Let $(X, \mathcal{A})$ be a measurable space, and let μ be a finite signed measure on $\mathcal{A}$ such that $\mu(X) > 0$. Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $\varphi : X \to C$ be a μ-integrable function. Then*

(a) If

$$\int_{\{\varphi \geq w\}} (\varphi - w)\, d\mu \geq 0, \quad w \in C^\circ \tag{4}$$

and

$$\int_{\{\varphi < w\}} (w - \varphi)\, d\mu \geq 0, \quad w \in C^\circ \tag{5}$$

are satisfied, then

$$t_{\varphi,\mu} := \frac{1}{\mu(X)} \int_X \varphi\, d\mu \in C.$$

(*b*) *For every function* $f \in F_C(\varphi)$, *the inequality*

$$f\left(\frac{1}{\mu(X)}\int_X \varphi d\mu\right) \le \frac{1}{\mu(X)}\int_X f\circ\varphi\, d\mu \tag{6}$$

holds if and only if (4) *and* (5) *are satisfied.*

Let $C \subset \mathbb{R}$ be an interval with nonempty interior. A function $g : C \to \mathbb{R}$ is called strongly convex with modulus $d > 0$ if

$$g\big(\lambda s + (1-\lambda)t\big) \le \lambda g(s) + (1-\lambda)g(t) - d\lambda(1-\lambda)(s-t)^2$$

for all $s, t \in C$ and $\lambda \in [0,1]$.

The following statement is a special case of Lemma 2.1 in paper [10], which describes the close relationship between convex and strongly convex functions.

Lemma 7 *Let* $C \subset \mathbb{R}$ *be an interval with nonempty interior and* $d > 0$. *A function* $g : C \to \mathbb{R}$ *is strongly convex with modulus d if and only if the function* $g - d \cdot id_{\mathbb{R}}^2$ *is convex.*

The next extension of the Jensen–Steffensen inequality for strongly convex functions comes from [2].

Theorem 8 *Let* $C \subset \mathbb{R}$ *be an interval with nonempty interior, and let* $m \in \mathbb{N}_+$. *Assume that* $p_1, \dots p_m$ *are real numbers such that* $\sum_{i=1}^m p_i = 1$ *and* (3) *are satisfied. Assume that* $(s_1, \dots, s_m) \in C^m$ *is a monotonic m-tuple* (*either increasing or decreasing*). *Then*

$$\sum_{i=1}^m p_i s_i \in \Big[\min_{i=1,\dots,m} s_i, \max_{i=1,\dots,m} s_i\Big] \subset C,$$

and for every strongly convex function $g : C \to \mathbb{R}$ *with modulus* $d > 0$, *the inequality*

$$g\left(\sum_{i=1}^m p_i s_i\right) \le \sum_{i=1}^m p_i g(s_i) - d\sum_{i=1}^m p_i\left(s_i - \sum_{i=1}^m p_i s_i\right)^2 \tag{7}$$

holds.

Remark 9 (a) Of course, the previous statement can also be formulated by considering the weaker condition $\sum_{i=1}^m p_i > 0$ instead of the condition $\sum_{i=1}^m p_i = 1$.

(b) As a special case, the discrete Jensen inequality for strongly convex functions is formulated and proved in [8].

We need the following lemma.

Lemma 10 *Let* $C \subset \mathbb{R}$ *be an interval with nonempty interior, and let* $m \in \mathbb{N}_+$. *Assume that* $p_1, \dots p_m$ *are real numbers and* $(s_1, \dots, s_m) \in C^m$ *such that* $\sum_{i=1}^m p_i = 1$ *and* $\sum_{i=1}^m p_i s_i \in C$.

Then inequality (7) holds for every strongly convex function $g : C \to \mathbb{R}$ with modulus $d > 0$ if and only if the inequality

$$f\left(\sum_{i=1}^{m} p_i s_i\right) \le \sum_{i=1}^{m} p_i f(s_i) \tag{8}$$

holds for every convex function $f : C \to \mathbb{R}$.

Proof Assume that (8) holds for every convex function $f : C \to \mathbb{R}$, and let $g : C \to \mathbb{R}$ be a strongly convex function with modulus d. By Lemma 7, the function $g - d \cdot id_{\mathbb{R}}^2$ is convex, and hence it satisfies (8), that is,

$$g\left(\sum_{i=1}^{m} p_i s_i\right) - d \cdot \left(\sum_{i=1}^{m} p_i s_i\right)^2 \le \sum_{i=1}^{m} p_i\bigl(g(s_i) - d \cdot s_i^2\bigr),$$

which is exactly (7).

Conversely, assume that (7) holds for every strongly convex function $g : C \to \mathbb{R}$ with modulus d, and let $f : C \to \mathbb{R}$ be a convex function. Also, because of Lemma 7, the function $f + d \cdot id_{\mathbb{R}}^2$ is a strongly convex function with modulus d, and therefore (7) yields that

$$f\left(\sum_{i=1}^{m} p_i s_i\right) + d \cdot \left(\sum_{i=1}^{m} p_i s_i\right)^2 \le \sum_{i=1}^{m} p_i\bigl(f(s_i) + d \cdot s_i^2\bigr) - d\sum_{i=1}^{m} p_i\left(s_i - \sum_{i=1}^{m} p_i s_i\right)^2,$$

which gives (8) by an elementary calculation.

The proof is complete. □

3 Main results

We now formulate Theorem 6 for discrete signed measures. First, we assume that the points in the interval are in monotonic order, because this is the basis of the proof of the general case, and we need this special case.

Theorem 11 *Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $m \in \mathbb{N}_+$. Assume $\mathbf{p} := (p_1, \dots, p_m)$ are real numbers such that $\sum_{i=1}^{m} p_i > 0$, and $\mathbf{s} := (s_1, \dots, s_m) \in C^m$ is a monotonic m-tuple.*

Suppose that $\mathbf{s}$ is decreasing.

(a) If

$$\sum_{i=1}^{l} p_i s_i \ge s_{l+1} \sum_{i=1}^{l} p_i, \quad l = 1, \dots, m-1, \tag{9}$$

and

$$s_{m-l} \sum_{i=m+1-l}^{m} p_i \ge \sum_{i=m+1-l}^{m} p_i s_i, \quad l = 1, \dots, m-1, \tag{10}$$

are satisfied, then

$$t_{\mathbf{s},\mathbf{p}} := \frac{1}{\sum_{i=1}^{m} p_i} \sum_{i=1}^{m} p_i s_i \in [s_m, s_1] \subset C. \tag{11}$$

(*b*) *For every function* $f \in F_C$, *the inequality*

$$f\left(\frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i s_i\right) \leq \frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i f(s_i) \tag{12}$$

holds if and only if (9) *and* (10) *are satisfied.*

Suppose that $\mathbf{s}$ *is increasing.*

(*c*) *If*

$$\sum_{i=1}^{l} p_i s_i \leq s_{l+1}\sum_{i=1}^{l} p_i, \quad l = 1, \dots, m-1, \tag{13}$$

and

$$s_{m-l}\sum_{i=m+1-l}^{m} p_i \leq \sum_{i=m+1-l}^{m} p_i s_i, \quad l = 1, \dots, m-1, \tag{14}$$

are satisfied, then

$$t_{\mathbf{s},\mathbf{p}} \in [s_1, s_m] \subset C.$$

(*d*) *For every function* $f \in F_C$, *inequality* (12) *holds if and only if* (13) *and* (14) *are satisfied.*

Proof We first consider the case where $s_1 \geq s_2 \geq \cdots \geq s_m$.

If $m = 1$, then the statement is obvious, and hence we can suppose that $m \geq 2$.

Let $X := \{1, \dots, m\}$, and let the set function $\mu : P(X) \to \mathbb{R}$ be defined by

$$\mu := \sum_{i \in X} p_i \varepsilon_i.$$

Then $(X, P(X), \mu)$ is a measure space with the finite signed measure μ.

Let the function $\varphi : X \to C$ be defined by

$$\varphi(i) := s_i.$$

Then $\varphi \in L(\mu)$ and $F_C(\varphi) = F_C$.

Condition (9) implies $t_{\mathbf{s},\mathbf{p}} \geq s_m$, while condition (10) yields $t_{\mathbf{s},\mathbf{p}} \leq s_1$.

Now, by Theorem 6, it is enough to show that (9) is equivalent to (4), and (10) is equivalent to (5).

We prove the first equivalence, the second can be treated in a similar way.

(i) Suppose $s_1 > s_2 > \cdots > s_m$.

Assume that (4) holds, that is,

$$\int_{\{\varphi \geq w\}} (\varphi - w)\, d\mu = \sum_{\{i \in \{1,\dots,m\} \mid s_i \geq w\}} p_i(s_i - w) \geq 0, \quad w \in C^{\circ}. \tag{15}$$

If $s_{l+1} \le w < s_l$ for some $l \in \{1, \dots, m-1\}$, then we get from (15) that

$$\sum_{\{i \in \{1,\dots,m\} \mid s_i \ge w\}} p_i(s_i - w) = \sum_{i=1}^{l} p_i(s_i - w) = \sum_{i=1}^{l} p_i s_i - w \sum_{i=1}^{l} p_i \ge 0,$$

especially

$$\sum_{i=1}^{l} p_i s_i - s_{l+1} \sum_{i=1}^{l} p_i \ge 0.$$

It now follows that (9) is satisfied.

Now assume conversely that (9) holds.

If $w \ge s_1$, then either $\{\varphi \ge w\} = \emptyset$ or $\{\varphi \ge w\} = \{s_1\}$, and hence (15) is obvious.

If $s_{l+1} \le w < s_l$ for some $l \in \{1, \dots, m-1\}$, then

$$\begin{aligned}\int_{\{\varphi \ge w\}} (\varphi - w)\, d\mu &= \sum_{i=1}^{l} p_i(s_i - w) \\ &\ge \min\left(\sum_{i=1}^{l} p_i(s_i - s_l), \sum_{i=1}^{l} p_i(s_i - s_{l+1})\right) \\ &= \min\left(\sum_{i=1}^{l-1} p_i(s_i - s_l), \sum_{i=1}^{l} p_i(s_i - s_{l+1})\right). \end{aligned} \tag{16}$$

By (9), expression (16) is nonnegative.

If $w < s_m$, then by $\sum_{i=1}^{m} p_i > 0$ and (9),

$$\begin{aligned}\int_{\{\varphi \ge w\}} (\varphi - w)\, d\mu &= \sum_{i=1}^{m} p_i(s_i - w) = \sum_{i=1}^{m} p_i s_i - w \sum_{i=1}^{m} p_i \\ &\ge \sum_{i=1}^{m} p_i s_i - s_m \sum_{i=1}^{m} p_i = \sum_{i=1}^{m-1} p_i s_i - s_m \sum_{i=1}^{m-1} p_i \ge 0.\end{aligned}$$

(ii) Now we consider the general case.

If either $s_{l+k+2} < s_{l+1+k} = \cdots = s_{l+1} < s_l$ for some $l \in \{1, \dots, m-3\}$ and $k \in \{1, \dots, m-l-2\}$ or $s_m = \cdots = s_{l+1} < s_l$ for some $l \in \{1, \dots, m-2\}$ (in this case $k = m-l$, then

$$\sum_{i=1}^{j} p_i s_i \ge s_{j+1} \sum_{i=1}^{j} p_i, \quad j = l, \dots, l+k,$$

is equivalent to

$$\sum_{i=1}^{l} p_i s_i \ge s_{l+1} \sum_{i=1}^{l} p_i,$$

and therefore (i) can be applied to the different elements of $(s_i)_{i=1}^{m}$.

We now turn to the case $s_m \ge s_{m-1} \ge \cdots \ge s_1$.

Define

$$q_i := p_{m-i+1} \quad \text{and} \quad t_i := s_{m-i+1}, \quad i = 1, \dots, m,$$

and apply the first part by using $q_1, \dots q_m$ instead of $p_1, \dots p_m$ and $t_1, \dots t_m$ instead of $s_1, \dots s_m$.

The proof is complete. □

Remark 12 Assume that $p_1, \dots p_m$ are real numbers such that $\sum_{i=1}^m p_i > 0$ and (3) are satisfied.

(a) If $\mathbf{s}$ is decreasing, it follows from Theorem 2 and Theorem 11 that conditions (9) and (10) must also be satisfied. This can also be easily shown directly by induction on l. If $\mathbf{s}$ is increasing, an analogous remark applies.

(b) We can see that Theorem 11 is a generalization of the discrete Jensen–Steffensen inequality, and since it contains necessary and sufficient conditions, it is the best result of this kind.

Remark 13 Assume $\sum_{i=1}^m p_i > 0$. Conditions (9) and (10), although weaker than condition (3), are also only sufficient, but not necessary, to satisfy (11). Indeed, let us start again from Example 3: Let $C := [0, 3]$ and $s_1 := 3$, $s_2 := 1$. Then $p_1 + p_2 + p_3 > 0$ and (11) are equivalent to

$$p_1 + p_2 + p_3 > 0, \qquad 3p_1 + p_2 \geq 0, \qquad 2p_2 + 3p_3 \geq 0.$$

These conditions are satisfied, for example, if $p_1 := -2$, $p_2 := 8$, $p_3 := -1$, but neither (9) nor (10) hold since in this case the latter conditions imply that $p_1, p_3 \geq 0$.

We now turn to the study of the strict inequality in (12).

Theorem 14 *Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $m \in \mathbb{N}_+$, $m \geq 3$. Assume that $\mathbf{p} := (p_1, \dots p_m)$ are real numbers, and $\mathbf{s} := (s_1, \dots, s_m) \in C^m$ is a decreasing m-tuple. If either*

$$p_1 > 0, \qquad \sum_{i=2}^m p_i > 0, \quad \textit{and} \quad s_1 > s_2 \tag{17}$$

or

$$\sum_{i=1}^{m-1} p_i > 0, \qquad p_m > 0, \quad \textit{and} \quad s_m < s_{m-1}, \tag{18}$$

and (9) and (10) are satisfied, then for every strictly convex function $f \in F_C$, inequality (12) is strict.

Proof The conditions of Theorem 11(a) are satisfied, and therefore inequality (12) holds for every $f \in F_C$.

Assume that (17) is satisfied.

We first show that $(p_2,\dots p_m)$ and $(s_2,\dots,s_m)$ satisfy conditions analogous to (9) or (10), that is,

$$\sum_{i=2}^{l} p_i s_i \geq s_{l+1}\sum_{i=2}^{l} p_i,\quad l=2,\dots,m-1, \tag{19}$$

and

$$s_{m-l}\sum_{i=m+1-l}^{m} p_i \leq \sum_{i=m+1-l}^{m} p_i s_i,\quad l=2,\dots,m-1. \tag{20}$$

Since $\mathbf{p}$ and $\mathbf{s}$ satisfy (10), (20) is also obviously true.

By using $p_1>0$ and $s_1>s_{l+1}$, it comes from (9) that

$$\begin{aligned}\sum_{i=2}^{l} p_i s_i &= \sum_{i=1}^{l} p_i s_i - p_1 s_1 \geq s_{l+1}\sum_{i=1}^{l} p_i - p_1 s_1\\ &= s_{l+1}\sum_{i=2}^{l} p_i - p_1(s_1-s_{l+1}) \geq s_{l+1}\sum_{i=2}^{l} p_i,\quad l=2,\dots,m-1.\end{aligned}$$

It follows from the above that Theorem 11 (a–b) can be applied to $(p_2,\dots p_m)$ and $(s_2,\dots,s_m)$, and it gives that

$$\frac{1}{\sum_{i=2}^{m} p_i}\sum_{i=2}^{m} p_i s_i \in [s_m,s_2], \tag{21}$$

and for every $f\in F_C$, the inequality

$$f\left(\frac{1}{\sum_{i=2}^{m} p_i}\sum_{i=2}^{m} p_i s_i\right) \leq \frac{1}{\sum_{i=2}^{m} p_i}\sum_{i=2}^{m} p_i f(s_i) \tag{22}$$

holds.

Since

$$\frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i s_i = \frac{p_1}{\sum_{i=1}^{m} p_i}s_1 + \frac{\sum_{i=2}^{m} p_i}{\sum_{i=1}^{m} p_i}\left(\frac{1}{\sum_{i=2}^{m} p_i}\sum_{i=2}^{m} p_i s_i\right),$$

we obtain from the discrete Jensen inequality by using the conditions in (17), (21), and (22) that

$$\begin{aligned}f\left(\frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i s_i\right) &< \frac{p_1}{\sum_{i=1}^{m} p_i}f(s_1) + \frac{1}{\sum_{i=1}^{m} p_i}f\left(\frac{1}{\sum_{i=2}^{m} p_i}\sum_{i=2}^{m} p_i s_i\right)\\ &\leq \frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i f(s_i).\end{aligned}$$

We can prove similarly if (18) is satisfied.

The proof is complete. □

Remark 15 The previous result contains the condition for strict inequality given in the discrete Jensen inequality, and from it we can derive conditions for strict inequality for the discrete Jensen–Steffensen inequality. Of course, conditions (17) and (18) for the discrete Jensen–Steffensen inequality are not as sharp as the conditions in Theorem 1 of paper [1]. Their importance lies in the fact that we can say at least sufficient conditions for the strict inequality in Theorem 11. It may be an interesting problem to give necessary and sufficient conditions for strict inequality in Theorem 11, but this does not seem to be an easy task.

Conditions (17) and (18) are more related to the conditions of the discrete Jensen–Steffensen inequality than to the conditions of Theorem 11. It would be interesting if we could formulate conditions for the strict inequality that are related to conditions (9) and (10). The following statement illustrates this for $m = 3$.

Proposition 16 *Let $C \subset \mathbb{R}$ be an interval with nonempty interior. Assume that $\mathbf{p} := (p_1, p_2, p_3)$ are real numbers such that $p_1 \neq 0, p_3 \neq 0$, and $\sum_{i=1}^{3} p_i > 0$, and $\mathbf{s} := (s_1, s_2, s_3) \in C^3$ is a decreasing 3-tuple. If*

$$p_1 s_1 \geq p_1 s_2, \qquad p_1 s_1 + p_2 s_2 > s_3 (p_1 + p_2) \tag{23}$$

and

$$p_3 s_2 \geq p_3 s_3, \qquad s_1 (p_2 + p_3) > p_2 s_2 + p_3 s_3 \tag{24}$$

are satisfied, then for every strictly convex function $f \in F_C$, the inequality

$$f\left(\frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i s_i\right) < \frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i f(s_i)$$

holds.

Proof For $m = 3$, conditions (23) and (24) imply conditions (9) and (10), respectively, and therefore Theorem 11(a) gives that

$$f\left(\frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i s_i\right) \leq \frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i f(s_i) \tag{25}$$

for every $f \in F_C$.

From the second inequality in either (23) or (24) it follows that $s_3 < s_1$.

Assume $p_1 < 0$ and $p_3 < 0$. Then it comes from the first inequalities in (23) and (24) that $s_1 = s_2 = s_3$, which contradicts $s_3 < s_1$. It follows that either $p_1 > 0$ or $p_3 > 0$.

(i) Consider the case $p_2 > 0$.

Suppose $p_1 > 0$. If $p_3 > 0$ too, then the result follows from the discrete Jensen inequality. If $p_3 < 0$, then $s_2 = s_3$, and hence

$$s_1 (p_2 + p_3) > p_2 s_2 + p_3 s_3 = s_2 (p_2 + p_3),$$

and this yields that $p_2 + p_3 > 0$. We can also apply the discrete Jensen inequality by $s_2 < s_1$.

The case $p_3 > 0$ can be handled similarly.

(ii) Assume $p_2 \leq 0$ and equality is satisfied in (25).

It is easy to check that

$$\frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i s_i = \frac{p_1 s_1 + p_2 s_2 - s_3(p_1 + p_2)}{(p_1 + p_2 + p_3)(s_1 - s_3)} \cdot s_1 + \frac{s_1(p_2 + p_3) - (p_2 s_2 + p_3 s_3)}{(p_1 + p_2 + p_3)(s_1 - s_3)} \cdot s_3 = \alpha \cdot s_1 + \beta \cdot s_3.$$

By the second inequalities in (23) and (24), $\alpha > 0$ and $\beta > 0$. Since $\alpha + \beta = 1$, we obtain from the discrete Jensen inequality that

$$\frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i f(s_i) = f\left(\frac{1}{\sum_{i=1}^{3} p_i} \sum_{i=1}^{3} p_i s_i\right) < \alpha \cdot f(s_1) + \beta \cdot f(s_3).$$

By a simple calculation we obtain from the previous inequality that

$$p_2 f(s_2)(s_1 - s_3) < p_2 f(s_1)(s_2 - s_3) + p_2 f(s_3)(s_1 - s_2),$$

and therefore $p_2 < 0$, and thus

$$f(s_2) > \frac{s_2 - s_3}{s_1 - s_3} f(s_1) + \frac{s_1 - s_2}{s_1 - s_3} f(s_3),$$

which contradicts the discrete Jensen inequality.

The proof is complete. □

Remark 17 (a) Conditions (23) and (24) may be fulfilled even if none of the conditions (17) and (18) are fulfilled. Really, let

$$s_1 := 2, \qquad s_2 := 1, \qquad s_3 := 0, \qquad p_1 = p_3 := 3, \qquad p_2 := -4.$$

Then it is easy to check that (23) and (24) are satisfied, but (17) and (18) are not satisfied.

(b) It is worth noting that the method of proof of Proposition 16 is different from that of Theorem 14.

If points in the interval are not ordered, Theorem 11 can be formulated as follows.

Corollary 18 *Let $X := \{1, \dots, m\}$ for some $m \in \mathbb{N}_+$. Let $C \subset \mathbb{R}$ be an interval with nonempty interior. Assume that $p_1, \dots p_m$ are real numbers such that $\sum_{i=1}^{m} p_i > 0$, and $\mathbf{s} := (s_1, \dots, s_m) \in C^m$. Let $u_1 > u_2 > \cdots > u_o$ be the different elements of $\mathbf{s}$ in decreasing order* $(1 \leq o \leq m)$.

(*a*) *If*

$$\sum_{\{i \in X \mid s_i \geq u_l\}} p_i s_i \geq u_{l+1} \sum_{\{i \in X \mid s_i \geq u_l\}} p_i, \quad l = 1, \dots, o-1, \tag{26}$$

and

$$u_{o-l} \sum_{\{i \in X \mid s_i < u_{o-l}\}} p_i \geq \sum_{\{i \in X \mid s_i < u_{o-l}\}} p_i s_i, \quad l = 1, \dots, o-1, \tag{27}$$

are satisfied, then

$$t_{\mathbf{s},\mathbf{p}} := \frac{1}{\sum_{i=1}^m p_i} \sum_{i=1}^m p_i s_i \in \left[\min_{i=1,\dots,m} s_i, \max_{i=1,\dots,m} s_i \right] \subset C.$$

(*b*) *For every function* $f \in F_C$, *inequality* (12) *holds if and only if* (26) *and* (27) *are satisfied.*

Proof Let

$$q_j := \sum_{\{i \in X | s_i = u_j\}} p_i, \quad j = 1, \dots, o.$$

Then

$$\sum_{\{i \in X | s_i \geq u_l\}} p_i s_i = \sum_{j=1}^{l} q_j u_j, \qquad \sum_{\{i \in X | s_i \geq u_l\}} p_i = \sum_{j=1}^{l} q_j,$$

$$\sum_{\{i \in X | s_i < u_{o-l}\}} p_i s_i = \sum_{j=o+1-l}^{o} q_j u_j, \qquad \sum_{\{i \in X | s_i < u_{o-l}\}} p_i = \sum_{j=o+1-l}^{o} q_j,$$

and

$$\frac{1}{\sum_{i=1}^m p_i} \sum_{i=1}^m p_i s_i = \frac{1}{\sum_{i=1}^m p_i} \sum_{j=1}^{o} q_j u_j, \qquad \frac{1}{\sum_{i=1}^m p_i} \sum_{i=1}^m p_i f(s_i) = \frac{1}{\sum_{i=1}^m p_i} \sum_{j=1}^{o} q_j f(u_j),$$

and therefore the result is an immediate consequence of Theorem 11 (a–b).

The proof is complete. □

4 Applications

In this section, the statements are formulated only for decreasing sequences for clarity, but the results are valid for increasing or general sequences with appropriate reformulation.

The conditions of Theorem 8, as for the Jensen–Steffensen inequality for convex functions, are only sufficient but not necessary for the inequality to be fulfilled. Using our main result, we can generalize Bakula's result considerably, obtaining necessary and sufficient conditions for the strongly convex case as well.

Theorem 19 *Let* $C \subset \mathbb{R}$ *be an interval with nonempty interior, and let* $m \in \mathbb{N}_+$. *Assume that* $\mathbf{p} := (p_1, \dots p_m)$ *are real numbers such that* $\sum_{i=1}^m p_i = 1$, *and* $\mathbf{s} := (s_1, \dots, s_m) \in C^m$ *is a decreasing m-tuple. Then inequality* (7) *holds for every strongly convex function* $g : C \to \mathbb{R}$ *with modulus* $d > 0$ *if and only if* (9) *and* (10) *are satisfied.*

Proof It follows from Theorem 11 (a–b) by applying Lemma 10.

The proof is complete. □

The second application concerns quasi-arithmetic means.

Let $C \subset \mathbb{R}$ be an interval, and let $z : C \to \mathbb{R}$ be a continuous and strictly monotone function. If $\mathbf{p} := (p_1, \dots p_m)$ are nonnegative numbers such that $\sum_{i=1}^m p_i > 0$ and $\mathbf{s} := (s_1, \dots, s_m) \in$

C^m, then the weighted quasi-arithmetic mean is defined by

$$A_z(\mathbf{s},\mathbf{p}) := z^{-1}\left(\frac{1}{\sum_{i=1}^m p_i}\sum_{i=1}^m p_i z(s_i)\right). \tag{28}$$

The quantity (28) is called mean because

$$A_z(\mathbf{s},\mathbf{p}) \in \left[\min_{i=1,\dots,m} s_i, \max_{i=1,\dots,m} s_i\right].$$

This property is equivalent to

$$\sum_{i=1}^m p_i z(s_i) \in \left[\min_{i=1,\dots,m} z(s_i), \max_{i=1,\dots,m} z(s_i)\right]$$

regardless of whether the numbers $p_1,\dots p_m$ are nonnegative or not.

In the following statement we compare quasi-arithmetic means with real (not necessarily nonnegative) weights.

Theorem 20 *Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $m \in \mathbb{N}_+$. Assume that $\mathbf{p} := (p_1,\dots p_m)$ are real numbers such that $\sum_{i=1}^m p_i > 0$, $\mathbf{s} := (s_1,\dots,s_m) \in C^m$ is a decreasing m-tuple, and $h, g : C \to \mathbb{R}$ are continuous and strictly monotone functions. Assume further that*

$$\sum_{i=1}^l p_i z(s_i) \geq z(s_{l+1})\sum_{i=1}^l p_i, \quad l = 1,\dots,m-1,$$

and

$$z(s_{m-l})\sum_{i=m+1-l}^m p_i \geq \sum_{i=m+1-l}^m p_i z(s_i), \quad l = 1,\dots,m-1,$$

are satisfied, where the function z is either h or g. If

(*a*) *either $h \circ g^{-1}$ is convex and h is increasing or $h \circ g^{-1}$ is concave and h is decreasing,*

(*b*) *either $g \circ h^{-1}$ is convex and g is decreasing or $g \circ h^{-1}$ is concave and g is increasing,*

then

$$A_g(\mathbf{s},\mathbf{p}) \leq A_h(\mathbf{s},\mathbf{p}). \tag{29}$$

Proof It follows from Theorem 11(a) that

$$A_z(\mathbf{s},\mathbf{p}) \in \left[\min_{i=1,\dots,m} z(s_i), \max_{i=1,\dots,m} z(s_i)\right]$$

for both $z = g$ and $z = h$, so the expressions on the left- and right-hand sides of (29) are well defined.

We only prove the case where $h \circ g^{-1}$ is convex and h is increasing, the others can be treated similarly.

Since $h \circ g^{-1}$ is convex, Theorem 11(b) implies that

$$h \circ g^{-1}\left(\frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i g(s_i)\right) \le \frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i h(s_i).$$

By applying h^{-1} to this inequality, we obtain (29).
The proof is complete. □

Finally, a refinement of the generalized discrete Jensen–Steffensen inequality is given. This is interesting because while there are many refinements for the discrete Jensen inequality (see e.g. the book [6] and the references therein), there are relatively few for the discrete Jensen–Steffensen inequality (see [2, 4], and [12]).

Theorem 21 *Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $k, m \in \mathbb{N}_+$, $k < m$. Assume that $\mathbf{p} := (p_1, \dots p_m)$ are real numbers such that $\sum_{i=1}^{k} p_i > 0$, and $\sum_{i=k+1}^{m} p_i > 0$. Assume that $\mathbf{s} := (s_1, \dots, s_m) \in C^m$ is a decreasing m-tuple.*

If

$$\sum_{i=1}^{l} p_i s_i \ge s_{l+1}\sum_{i=1}^{l} p_i, \quad l = 1, \dots, k-1,$$

$$s_{k-l}\sum_{i=k+1-l}^{k} p_i \ge \sum_{i=k+1-l}^{k} p_i s_i, \quad l = 1, \dots, k-1,$$

$$\sum_{i=k+1}^{l} p_i s_i \ge s_{l+1}\sum_{i=k+1}^{l} p_i, \quad l = k+1, \dots, m-1,$$

and

$$s_{m-l}\sum_{i=m+1-l}^{m} p_i \ge \sum_{i=m+1-l}^{m} p_i s_i, \quad l = k+1, \dots, m-1$$

are satisfied, then for every function $f \in F_C$, the inequalities

$$f\left(\frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i s_i\right) \le \frac{\sum_{i=1}^{k} p_i}{\sum_{i=1}^{m} p_i} f\left(\frac{1}{\sum_{i=1}^{k} p_i}\sum_{i=1}^{k} p_i s_i\right) + \frac{\sum_{i=k+1}^{m} p_i}{\sum_{i=1}^{m} p_i} f\left(\frac{1}{\sum_{i=k+1}^{m} p_i}\sum_{i=k+1}^{m} p_i s_i\right) \le \frac{1}{\sum_{i=1}^{m} p_i}\sum_{i=1}^{m} p_i f(s_i) \quad (30)$$

hold.

Proof We can apply Theorem 11(a), which implies that

$$\frac{1}{\sum_{i=1}^{k} p_i}\sum_{i=1}^{k} p_i s_i \in [s_k, s_1] \subset C$$

and

$$\frac{1}{\sum_{i=k+1}^{m} p_i} \sum_{i=k+1}^{m} p_i s_i \in [s_m, s_{k+1}] \subset C.$$

Since

$$\frac{\sum_{i=1}^{k} p_i}{\sum_{i=1}^{m} p_i} > 0, \qquad \frac{\sum_{i=k+1}^{m} p_i}{\sum_{i=1}^{m} p_i} > 0 \quad \text{and} \quad \frac{\sum_{i=1}^{k} p_i + \sum_{i=k+1}^{m} p_i}{\sum_{i=1}^{m} p_i} = 1,$$

from the previous inequalities and the discrete Jensen inequality it follows

$$\frac{1}{\sum_{i=1}^{m} p_i} \sum_{i=1}^{m} p_i s_i \in C$$

and the first inequality in (30).

The second inequality in (30) is obtained by applying Theorem 11(b) to both members of the sum.

The proof is complete. □

Remark 22 (a) The previous theorem can be extended analogously by splitting the set $\{1, \ldots, m\}$ into more than two blocks.

(b) The refinement in Theorem 21 shows a technique for obtaining refinements of the generalized discrete Jensen–Steffensen inequality from refinements to the discrete Jensen inequality.

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