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Pareto vectors of continuous linear operators

Francisco Javier García-Pacheco[1*]

*Correspondence:
garcia.pacheco@uca.es
[1]Department of Mathematics, College of Engineering, University of Cadiz, Avda. de la Universidad, Puerto Real, 11519, Cádiz, Spain

Abstract

The intersection of all zero-neighborhoods in a topological module over a topological ring is a bounded and closed submodule whose inherited topology is the trivial topology. In this manuscript, we prove that this is the smallest closed submodule and thus replaces the null submodule in the Hausdorff setting. This fact motivates to introduce a new notion in operator theory called *topological kernel*. Another new concept is also defined that of *Pareto optimal element* for a family of continuous linear operators between topological modules. It is then proved that topological kernels have a strong influence on the existence of Pareto optimal elements. This work is strongly motivated by the ongoing search for a consistent operator theory on topological modules over general topological rings.

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1 Introduction

There are many differences between Banach spaces and general topological modules. One of them, for instance, is the fact that finite-dimensional subspaces of a Banach space are always closed, while finitely spanned submodules of a topological module (even of a Hilbert C^*-module) are not necessarily closed. This is one of the reasons why some notions and results from operator theory on Banach spaces cannot directly be transported to operators on topological modules. As a consequence, operator theory on topological modules sometimes requires another approach than operator theory on Banach spaces. Fredholm and semi-Fredholm theory on the standard Hilbert module over a unital C^*-algebra is one of the examples illustrating how much the proofs and the approach in this setting differ from the situation of the classical Fredholm and semi-Fredholm theory on Banach spaces. Although Hilbert C^*-modules are also Banach spaces, semi-Fredholm theory in the sense of [9, 10, 13, 14] is very different from the classical semi-Fredholm theory, exactly due to the fact that finitely spanned submodules may behave quite differently from finite-dimensional subspaces.

The irruption of Hilbert C^*-modules boosted the development of a consistent operator theory for general topological modules over general topological rings. As mentioned above, Hilbert C^*-modules are also Banach spaces, since C^*-algebras are complex algebras, therefore a Hilbert C^*-module acquires structure of a complex Banach space. However, Hilbert C^*-modules undoubtedly present a major interest as modules over a C^*-algebra.

In [6, Theorem 2], it is proved that if M is a topological module over a topological ring R and $\mathcal{N}_0(M)$ denotes the filter of all neighborhoods of 0 in M, then $O_M := \bigcap_{V\in\mathcal{N}_0(M)} V$ is a bounded and closed submodule of M whose inherited topology is the trivial topology. As an immediate consequence, if O_M is linearly complemented in M, then any linear projection from M onto O_M is trivially continuous. This closed submodule O_M is particularly interesting in the non-Hausdorff ambience, since O_M is precisely the null submodule $\{0\}$ if and only if the module topology of M is Hausdorff [2, 18, 19]. Recall that an R-linear operator between two topological R-modules M,N is simply a map $T: M \to N$, which is additive and R-homogeneous, that is, $T(m+p) = T(m) + T(p)$ and $T(rm) = rT(m)$ for all $m,p \in M$ and all $r \in R$. When two topological modules M,N are Hausdorff over a Hausdorff topological ring R, then R-linear operators between them tend to satisfy certain properties of classical real or complex operator theory (not all of them though) [1, 12]. This manuscript is then framed in the non-classical operator theory and tires to understand as its main goal how linear operators behave, not only for general module topologies but also for seminormable module topologies.

The difference between a ring seminorm $\|\cdot\|$ and an absolute semivalue $|\cdot|$ is that the ring seminorm is submultiplicative ($\|rs\| \le \|r\|\|s\|$) and the absolute semivalue is multiplicative ($|rs| = |r||s|$). For this reason, a seminorm on a module is asked to be absolutely homogeneous ($\|rm\| = |r|\|m\|$) if the underlying ring is absolutely semivalued, and submultiplicative ($\|rm\| \le \|r\|\|m\|$) if the underlying ring is seminormed. Throughout this paper we will consider all module seminorms, ring seminorms, and absolute semivalues as nonzero, as well as all modules as left and unital ($1m = m$) and all rings as associative and unitary.

Notable subsets of a seminormed module M are the unit ball, $\mathsf{B}_M := \{m \in M : \|m\| \le 1\}$, and the unit sphere $\mathsf{S}_M := \{m \in M : \|m\| = 1\}$. Usually, the underlying seminormed ring R is required to be practical. Recall [3, 4, 6] that a topological ring R is said to be practical when the invertibles approach 0, that is, $0 \in \mathrm{cl}(\mathcal{U}(R))$, where $\mathcal{U}(R)$ stands for the multiplicative group of invertibles of R. This way, if M is a seminormed module over a practical seminormed ring R, then for every $m \in M$ there exists $r \in \mathcal{U}(R)$ such that $rm \in \mathsf{B}_M$. The following inequalities, that work for all $r \in \mathcal{U}(R)$ and all $m \in M$, are used throughout the paper:

$$\left\|r^{-1}\right\|^{-1}\|m\| \le \|rm\| \le \|r\|\|m\|. \tag{1}$$

The above inequality is simply a direct consequence of the fact that $1 \le \|1\| = \|rr^{-1}\| \le \|r\|\|r^{-1}\|$ for all $r \in \mathcal{U}(R)$. According to [5], an invertible element u of a seminormed ring R is said to be *absolutely invertible* provided that $\|u^{-1}\| = \|u\|^{-1}$. The existence of absolutely invertible elements implies that $\|1\| = 1$ (seminormed rings satisfying that $\|1\| = 1$ are called unital). Absolutely invertible elements form a subgroup of $\mathcal{U}(R)$ denoted as $\mathcal{U}_1(R)$. If $\{\|u\| : u \in \mathcal{U}_1(R)\}$ is dense in $[0,\infty)$, then R is called *hyperpractical*. Notice that hyperpractical rings satisfy that $0 \in \mathrm{cl}(\mathcal{U}_1(R))$, so in particular they are also practical.

2 Results

This section is divided into two subsections. The first subsection introduces the novel notion of topological kernel, which plays the role of the kernel of a linear operator in the classical Hausdorff setting. The second subsection deals with the study of Pareto optimal solutions to multiobjective optimization problems that involve linear operators on seminormed modules.

2.1 Topological kernel of a linear operator

Let us recall first that by $\mathcal{N}_x(X)$ we intend to denote the filter of all neighborhoods of x in a topological space X. Also, recall that

$$X_{(x)} := \bigcap \mathcal{N}_x(X) = \bigcap_{V \in \mathcal{N}_x(X)} V = \big\{ y \in X : x \in \mathrm{cl}\big(\{y\}\big) \big\}.$$

Lemma 1 *Let X be a regular topological space. For all $x, y \in X$, the following conditions are equivalent:*

1. $y \in X_{(x)}$.
2. $x \in X_{(y)}$.
3. $y \in \mathrm{cl}(\{x\})$.
4. $x \in \mathrm{cl}(\{y\})$.

As a consequence, $X_{(x)} = \mathrm{cl}(\{x\})$, hence $X_{(x)}$ is contained in any closed subset of X that contains x.

Proof It is trivial by definition that $y \in X_{(x)}$ if and only if $x \in \mathrm{cl}(\{y\})$. By switching x and y, $x \in X_{(y)}$ if and only if $y \in \mathrm{cl}(\{x\})$. Therefore, it only remains to show the equivalence of (1) and (2). Suppose that $y \in X_{(x)}$ but $x \notin X_{(y)}$. There exists $U \in \mathcal{N}_y(X)$ with $x \notin U$. Since X is regular, there exists $V \in \mathcal{N}_y(X)$ with $\mathrm{cl}(V) \subseteq U$. Then $x \in X \setminus \mathrm{cl}(V)$ so $X \setminus \mathrm{cl}(V) \in \mathcal{N}_x(X)$. By assumption, $y \in X_{(x)} \subseteq X \setminus \mathrm{cl}(V)$. This is a contradiction. As a consequence, $x \in X_{(y)}$. By swapping x and y, we obtain that $y \in X_{(x)}$ if $x \in X_{(y)}$. Finally,

$$X_{(x)} = \bigcap_{V \in \mathcal{N}_x(X)} V = \big\{ y \in X : x \in \mathrm{cl}\big(\{y\}\big) \big\} = \big\{ y \in X : y \in \mathrm{cl}\big(\{x\}\big) \big\} = \mathrm{cl}\big(\{x\}\big).$$

□

If M is a topological module over a topological ring R and $\mathcal{N}_0(M)$ denotes the filter of all neighborhoods of 0 in M, then the intersection of all 0-neighborhoods of M is commonly denoted as O_M, that is, $O_M := M_{(0)} := \bigcap_{V \in \mathcal{N}_0(M)} V$. Observe that O_M is a bounded and closed submodule of M whose inherited topology is the trivial topology [6, Theorem 2]. It is well known that M is Hausdorff if and only if $O_M = \{0\}$. Our first result assures that O_M is the smallest closed submodule of M, which is a direct consequence of Lemma 1.

Theorem 1 *If M is a topological module over a topological ring R, then O_M is contained in every closed submodule N of M.*

Proof Let N be any closed submodule of M. Since $\{0\} \subseteq N$, we have that $\mathrm{cl}(\{0\}) \subseteq N$. Finally, any topological group is regular, in particular every any topological module, therefore $O_M = \mathrm{cl}(\{0\}) \subseteq N$ in view of Lemma 1. □

Theorem 1 motivates the definition of topological kernel.

Definition 1 (Topological kernel) Let M, N be topological modules over a topological ring R. The topological kernel of an R-linear operator $T : M \to N$ is defined as

$$\ker_{\mathrm{t}}(T) := T^{-1}(O_N) = \left\{ m \in M : T(m) \in \bigcap_{V \in \mathcal{N}_0(M)} V \right\}.$$

Instead of taking the pre-image of 0, take the pre-image of the intersection of all neighborhoods of 0. Notice that if N is Hausdorff, then $\ker_{\mathrm{t}}(T) = T^{-1}(O_N) = T^{-1}(\{0\}) = \ker(T)$. In case M, N are seminormed modules, then $O_N = \{n \in N : \|n\| = 0\}$ in view of [6, Theorem 2], hence $\ker_{\mathrm{t}}(T) = \{m \in M : \|T(m)\| = 0\}$.

The notion of topological kernel introduced in Definition 1 is not the first generalization of the classical concept of kernel of a linear operator. For instance, there exists the notion of *generalized kernel* [7, 11, 17, 21] which consists in $\bigcup_{n\in\mathbb{N}} \ker(T^n)$ for $T : X \to Y$ a continuous linear operator between real or complex topological vector spaces X, Y.

Theorem 2 *Let M, N be topological modules over a topological ring R. Consider an R-linear operator $T : M \to N$. Then $\ker_{\mathrm{t}}(T)$ is a submodule of M. Even more, if T is continuous, then $\ker_{\mathrm{t}}(T)$ is closed and contains O_M.*

Proof In view of [6, Theorem 2], O_N is a closed submodule of N, so $\ker_{\mathrm{t}}(T) := T^{-1}(O_N)$ is a submodule of M. Suppose that T is continuous. Then $\ker_{\mathrm{t}}(T)$ is closed in M because O_N is a closed submodule of N. On the other hand, the continuity of T allows that T^{-1} preserve neighborhoods of 0, hence

$$\left\{ T^{-1}(V) : V \in \mathcal{N}_0(N) \right\} \subseteq \mathcal{N}_0(M).$$

Therefore,

$$O_M = \bigcap_{W \in \mathcal{N}_0(M)} W \subseteq \bigcap_{V \in \mathcal{N}_0(N)} T^{-1}(V) = T^{-1}\left(\bigcap_{V \in \mathcal{N}_0(N)} V \right) = T^{-1}(O_N) = \ker_{\mathrm{t}}(T). \qquad \square$$

Notice that, under the settings of the previous theorem, the fact that $\ker_{\mathrm{t}}(T)$ contains O_M if T is continuous can be directly inferred from Lemma 1 together with the fact that $\ker_{\mathrm{t}}(T)$ is closed.

2.2 Pareto optimal elements for a family of linear operators

We refer the reader to the Appendix for a review on multiobjective optimization problems and proper references. Commonly studied multiobjective optimization problems in bioengineering and physics involve linear operators on seminormed modules over seminormed rings.

Definition 2 (Optimal elements) Let M, N be seminormed modules over a seminormed ring R. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. An element $m_0 \in \mathcal{R}$ is said to be optimal for the family $\mathcal{F}$ on $\mathcal{R}$ provided that $\|T_i(m_0)\| \geq \|T_i(m)\|$ and $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $i, j \in I$ and all $m \in \mathcal{R}$. The set of optimal elements of $\mathcal{F}$ on $\mathcal{R}$ is denoted by $\mathrm{sol}_{\mathcal{R}}(\mathcal{F})$.

If $\mathcal{R} := M$, then we will simply write $\mathrm{sol}(\mathcal{F})$. The need to consider Pareto optimal solutions is justified by plenty of multi-objective optimization problems for which there are no optimal solutions [15, Theorem 2].

Proposition 3 *Let M, N be seminormed modules over a seminormed ring R. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Then $\mathrm{sol}(\mathcal{F}) = \varnothing$ in any of the following situations*:

1. *There exist $i_0 \in I$ and $j_0 \in J$ such that $\ker_{\mathrm{t}}(S_{j_0}) \subseteq \ker_{\mathrm{t}}(T_{i_0}) \neq M$.*
2. *There is a sequence $(m_n)_{n\in\mathbb{N}}$ in M and some $i_0 \in I$ such that $\|T_{i_0}(m_n)\| \to \infty$.*
3. *R is practical and there exist $i_0 \in I$ and $m_0 \in M$ such that $\|T_{i_0}(m_0)\| > 0$.*

Proof The proof is itemized according to the statement of the proposition.

1. If $m_0 \in \mathrm{sol}(\mathcal{F})$, then $\|S_{j_0}(m_0)\| \le \|S_{j_0}(0)\| = 0$, which implies that $m_0 \in \ker_{\mathrm{t}}(S_{j_0}) \subseteq \ker_{\mathrm{t}}(T_{i_0})$. Next, if we consider any $m \in M \setminus \ker_{\mathrm{t}}(T_{i_0})$, then we reach the contradiction that $0 = \|T_{i_0}(m_0)\| \ge \|T_{i_0}(m)\| > 0$.
2. Let $m \in M$ be arbitrary. Then there is some positive integer n such that $\|T_{i_0}(m)\| < \|T_{i_0}(m_n)\|$, hence $m \notin \mathrm{sol}(\mathcal{F})$. This shows that $\mathrm{sol}(\mathcal{F}) = \varnothing$.
3. Next, suppose that R is practical and there exist $i_0 \in I$ and $m_0 \in M$ such that $\|T_{i_0}(m_0)\| > 0$. Take a sequence $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(R)$ of invertibles converging to 0. Following (1), we have that

$$\|r_n\|^{-1}\left\|T_{i_0}(m_0)\right\| \le \left\|r_n^{-1}T_{i_0}(m_0)\right\| = \left\|T_{i_0}\left(r_n^{-1}m_0\right)\right\| \to \infty$$

as $n \to \infty$, since $\|r_n\|^{-1} \to \infty$ as $n \to \infty$. As a consequence, $\mathrm{sol}(\mathcal{F}) = \varnothing$ in view of Proposition 3(2) by taking $m_n := r_n^{-1}m_0$ for all $n \in \mathbb{N}$. □

Proposition 3 motivates the notion of Pareto optimal element for a family of linear operators. The notion of Pareto optimal solution is proper from optimization theory and serves to partially solve a multi-objective optimization problem when it lacks optimal solutions. The notion of Pareto optimal element for a family of linear operators is novel of this work and is motivated by the one of Pareto optimal solution.

Definition 3 (Pareto optimal elements) Let M, N be seminormed modules over a seminormed ring R. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. An element $m_0 \in \mathcal{R}$ is said to be Pareto optimal for the family $\mathcal{F}$ on $\mathcal{R}$ provided that the following Pareto condition holds: If $m \in \mathcal{R}$ satisfies that there exists $i \in I$ with $\|T_i(m)\| > \|T_i(m_0)\|$ or exists $j \in J$ with $\|S_j(m)\| < \|S_j(m_0)\|$, then there exists $i' \in I$ with $\|T_{i'}(m)\| < \|T_{i'}(m_0)\|$ or exists $j' \in J$ with $\|S_{j'}(m)\| > \|S_{j'}(m_0)\|$. The set of Pareto optimal elements of $\mathcal{F}$ on $\mathcal{R}$ is denoted by $\mathrm{Par}_{\mathcal{R}}(\mathcal{F})$.

If $\mathcal{R} := M$, then we simply write $\mathrm{Par}(\mathcal{F})$. Next remark highlights an important observation related to Pareto optimal elements.

Remark 1 Let M, N be seminormed modules over a seminormed ring R. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. Suppose that $m_0 \in \mathrm{Par}_{\mathcal{R}}(\mathcal{F})$. If $m \in \mathcal{R}$ satisfies that $\|T_i(m)\| \ge \|T_i(m_0)\|$ for all $i \in I$ and $\|S_j(m)\| \le \|S_j(m_0)\|$ for all $j \in J$, then $\|T_i(m)\| = \|T_i(m_0)\|$ for all $i \in I$ and $\|S_j(m)\| = \|S_j(m_0)\|$ for all $j \in J$, hence $m \in \mathrm{Par}_{\mathcal{R}}(\mathcal{F})$.

The following theorem establishes a necessary condition for the existence of Pareto optimal elements.

Theorem 4 *Let R be a practical seminormed ring and M,N seminormed R-modules. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Suppose the following*:

- $\{\|T_i(m)\| : i \in I\}$ *is bounded below for every* $m \in \bigcap_{j\in J} \ker_{\mathrm{t}}(S_j)$.
- $\mathrm{Par}(\mathcal{F}) \neq \varnothing$ *and* $\{\|T_i(m)\| : i \in I\}$ *is bounded above for some* $m \in \mathrm{Par}(\mathcal{F})$.

Then $\bigcap_{j\in J} \ker_{\mathrm{t}}(S_j) \subseteq \bigcup_{i\in I} \ker_{\mathrm{t}}(T_i)$.

Proof Fix $m_0 \in \mathrm{Par}(\mathcal{F})$ such that $\{T_i(m_0) : i \in I\}$ is bounded above. Let us assume that, on the contrary, there exists $m_1 \in \bigcap_{j\in J} \ker_{\mathrm{t}}(S_j) \setminus \bigcup_{i\in I} \ker_{\mathrm{t}}(T_i)$. In particular, $\|S_j(m_1)\| = 0$ for all $j \in J$ and $\|T_i(m_1)\| > 0$ for all $i \in I$. Let $L > 0$ be a lower bound for $\{T_i(m_1) : i \in I\}$ and $K > 0$ an upper bound for $\{T_i(m_0) : i \in I\}$. Since R is practical, we can find a sequence $(r_k)_{k\in\mathbb{N}} \subseteq R$ such that $(\|r_k\|)_{k\in\mathbb{N}}$ converges to 0. By bearing in mind again (1), for every $i \in I$ and each $k \in \mathbb{N}$, we have that

$$\|r_k\|^{-1}L \leq \|r_k\|^{-1}\|T_i(m_1)\| \leq \|r_k^{-1}T_i(m_1)\| = \|T_i(r_k^{-1}m_1)\|.$$

We can then find $k_1 \in \mathbb{N}$ such that $\|r_{k_1}\| < \frac{L}{K}$. This way

$$\|T_i(r_{k_1}^{-1}m_1)\| \geq \|r_{k_1}\|^{-1}L > K \geq \|T_i(m_0)\|$$

for all $i \in I$. Since $m_0 \in \mathrm{Par}(\mathcal{F})$, there must exist $j_0 \in J$ such that

$$\|S_{j_0}(m_0)\| < \|S_{j_0}(r_{k_1}^{-1}m_1)\| \leq \|r_{k_1}^{-1}\|\|S_{j_0}(m_1)\| = 0,$$

which is a contradiction. □

The reader may trivially notice that, under the settings of Theorem 4, if I is finite, then $\{\|T_i(m)\| : i \in I\}$ is bounded below for every $m \in \bigcap_{j\in J} \ker_{\mathrm{t}}(S_j)$ and bounded above for every $m \in \mathrm{Par}(\mathcal{F})$. In order to prove our next theorem, a technical lemma is needed first.

Lemma 2 *Let R be a seminormed ring and M a seminormed R-module. Then $\|um\| = \|u\|\|m\|$ for all $u \in \mathcal{U}_1(R)$ and all $m \in M$.*

Proof Simply observe that $\|m\| = \|u^{-1}um\| \leq \|u^{-1}\|\|um\| = \|u\|^{-1}\|um\|$, so $\|u\|\|m\| \leq \|um\| \leq \|u\|\|m\|$. □

Theorem 5 *Let R be a unital seminormed ring and M,N seminormed R-modules. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Then* $\mathrm{Par}(\mathcal{F}) = \mathcal{U}_1(R)\mathrm{Par}(\mathcal{F})$.

Proof Since $1 \in \mathcal{U}_1(R)$, we trivially have that $\mathrm{Par}(\mathcal{F}) \subseteq \mathcal{U}(R)\mathrm{Par}(\mathcal{F})$. Conversely, fix arbitrary elements $u \in \mathcal{U}_1(R)$ and $m \in \mathrm{Par}(\mathcal{F})$. We have to prove that $um \in \mathrm{Par}(\mathcal{F})$. Suppose not. There exists $m_0 \in M$ satisfying one of the following conditions: either there exists $i_0 \in I$ such that $\|T_{i_0}(m_0)\| > \|T_{i_0}(um)\|$, $\|T_i(m_0)\| \geq \|T_i(um)\|$ for all $i \in I \setminus \{i_0\}$, and $\|S_j(m_0)\| \leq \|S_j(um)\|$ for all $j \in J$; or there exists $j_0 \in J$ such that $\|S_{j_0}(m_0)\| < \|S_{j_0}(um)\|$, $\|S_j(m_0)\| \leq \|S_j(um)\|$ for all $j \in J \setminus \{j_0\}$, and $\|T_i(m_0)\| \geq \|T_i(um)\|$ for all $i \in I$. We may assume without any loss of generality that the first condition above is satisfied. Then Lemma 2 assures that $\|T_{i_0}(u^{-1}m_0)\| = \|u^{-1}\|\|T_{i_0}(m_0)\| > \|u^{-1}\|\|T_{i_0}(um)\| = \|T_{i_0}(m)\|$. Since $m \in \mathrm{Par}(\mathcal{F})$, there must exist $i \in I \setminus \{i_0\}$ or $j \in J$ such that either $\|T_i(m)\| > \|T_i(u^{-1}m_0)\|$ or $\|S_j(m)\| < \|S_j(u^{-1}m_0)\|$. Each condition implies that $\|T_i(um)\| > \|T_i(m_0)\|$ or $\|S_j(um)\| < \|S_j(m_0)\|$, respectively, reaching then a contradiction with the first condition. □

The following theorem relates Pareto optimal elements when a family of linear operators suffers a slight perturbation.

Theorem 6 *Let R be a seminormed ring and M,N seminormed R-modules. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Fix $j' \in J$ and consider the family $\mathcal{F}'$ of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J \setminus \{j'\}$. Take $t \geq 0$ and consider $\mathcal{R}' := \{m \in M : \|S_{j'}(m)\| \leq t\}$. Then $\mathrm{Par}(\mathcal{F}) \cap \{m \in M : \|S_{j'}(m)\| = t\} \subseteq \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}')$.*

Proof Fix an arbitrary $m \in \mathrm{Par}(\mathcal{F})$ such that $\|S_{j'}(m)\| = t$. Suppose to the contrary that $m \notin \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}')$. There exists $m_0 \in \mathcal{R}'$ satisfying one of the following conditions: either there exists $i_0 \in I$ such that $\|T_{i_0}(m_0)\| > \|T_{i_0}(m)\|$, $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I \setminus \{i_0\}$, and $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $j \in J \setminus \{j'\}$; or there exists $j_0 \in J \setminus \{j'\}$ such that $\|S_{j_0}(m_0)\| < \|S_{j_0}(m)\|$, $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $j \in J \setminus \{j_0, j'\}$, and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$. If any one of the two previous conditions is satisfied, then, by bearing in mind that $m \in \mathrm{Par}(\mathcal{F})$, the only possibility left is that $\|S_{j'}(m)\| < \|S_{j'}(m_0)\|$. However, this means the contradiction that $t = \|S_{j'}(m)\| < \|S_{j'}(m_0)\| \leq t$, since $m_0 \in \mathcal{R}'$. □

The following final remark analyzes whether it is possible to reverse the inclusion provided by Theorem 6 or not.

Remark 2 Let R be a seminormed ring and M,N seminormed R-modules. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J$. Fix $j' \in J$ and consider the family $\mathcal{F}'$ of R-linear operators $T_i, S_j : M \to N$, $i \in I$, $j \in J \setminus \{j'\}$. Take $t \geq 0$ and consider $\mathcal{R}' := \{m \in M : \|S_{j'}(m)\| \leq t\}$. It cannot be assured that $\mathrm{Par}_{\mathcal{R}'}(\mathcal{F}') \subseteq \mathrm{Par}(\mathcal{F})$. Indeed, fix an arbitrary $m \in \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}')$. Let us assume, on the contrary, that $m \notin \mathrm{Par}(\mathcal{F})$. There exists $m_0 \in M$ satisfying one of the following conditions: either there exists $i_0 \in I$ such that $\|T_{i_0}(m_0)\| > \|T_{i_0}(m)\|$, $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I \setminus \{i_0\}$, and $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $j \in J$; or there exists $j_0 \in J$ such that $\|S_{j_0}(m_0)\| < \|S_{j_0}(m)\|$, $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $j \in J \setminus \{j_0\}$, and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$. If the first condition holds, then $m_0 \in \mathcal{R}'$, meaning a contradiction with the fact that $m \in \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}')$. So, let us assume that the second condition holds. If $j_0 \neq j'$, then again $m_0 \in \mathcal{R}'$, reaching a contradiction with $m \in \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}')$. Therefore, we will assume that $j_0 = j'$. In this situation, we have that $\|S_{j'}(m_0)\| < \|S_{j'}(m)\|$, $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $j \in J \setminus \{j'\}$, and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$. Since $m \in \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}')$, we actually have that $\|S_{j'}(m_0)\| < \|S_{j'}(m)\|$, $\|S_j(m_0)\| = \|S_j(m)\|$ for all $j \in J \setminus \{j'\}$, and $\|T_i(m_0)\| = \|T_i(m)\|$ for all $i \in I$. At this stage we cannot advance further.

If we rely on hyperpractical rings and restrict J to a singleton, then we can reverse the inclusion provided by Theorem 6.

Theorem 7 *Let R be a hyperpractical seminormed ring and M, N seminormed R-modules. Let $\mathcal{F}$ be a family of R-linear operators $T_i, S : M \to N$, $i \in I$. Consider the family $\mathcal{F}'$ of R-linear operators $T_i : M \to N$, $i \in I$. Take $t \geq 0$ and consider $\mathcal{R}' := \{m \in M : \|S(m)\| \leq t\}$. Then $\operatorname{Par}(\mathcal{F}) \cap \{m \in M : \|S(m)\| = t\} = \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$.*

Proof In accordance with Theorem 6, $\operatorname{Par}(\mathcal{F}) \cap \{m \in M : \|S(m)\| = t\} \subseteq \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$. Fix an arbitrary $m \in \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$. We will show first that $\|S(m)\| = t$. Indeed, let us assume, on the contrary, that $\|S(m)\| < t$. By hypothesis, R is hyperpractical, therefore, there exists $u \in \mathcal{U}_1(R)$ with $1 < \|u\| < \frac{t}{\|S(m)\|}$, meaning, by Lemma 2, that $\|S(m)\| < \|S(um)\| < t$ (in case $\|S(m)\| = 0$, then it suffices to take any $u \in \mathcal{U}_1(R)$). Then $um \in \mathcal{R}'$ and $\|T_i(um)\| = \|u\|\|T_i(m)\| > \|T_i(m)\|$ for all $i \in I$, meaning a contradiction with the fact that $m \in \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$. As a consequence, $\|S(m)\| = t$. Let us prove now that $m \in \operatorname{Par}(\mathcal{F})$. Assume this is not the case. There exists $m_0 \in M$ satisfying one of the following conditions: either there exists $i_0 \in I$ such that $\|T_{i_0}(m_0)\| > \|T_{i_0}(m)\|$, $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I \setminus \{i_0\}$, and $\|S(m_0)\| \leq \|S(m)\|$; or $\|S(m_0)\| < \|S(m)\|$ and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$. If the first condition holds, then $m_0 \in \mathcal{R}'$, meaning a contradiction with the fact that $m \in \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$. So, let us assume that the second condition holds. In this situation, we have that $\|S(m_0)\| < \|S(m)\|$ and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$. Since $m \in \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$, we actually have that $\|S(m_0)\| < \|S(m)\|$ and $\|T_i(m_0)\| = \|T_i(m)\|$ for all $i \in I$. This shows that $m_0 \in \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$, implying the contradiction that $t = \|S(m_0)\| < \|S(m)\| = t$. □

3 Applications

Theorem 6 together with Theorem 7 allows to reformulate the following common multi-objective optimization problem into a single-optimization problem that frequently arises in physics and bioengineering [16, 20]:

$$\begin{cases} \max \|A\psi\|, \\ \min \|\psi\|, \end{cases} \tag{2}$$

where $A \in \mathbb{K}^{m\times n}$ is a real or complex matrix ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\psi \in \mathbb{K}^n$. The reformulation provided by Theorem 7 is precisely the following:

$$\begin{cases} \max \|A\psi\|, \\ \|\psi\| = 1, \end{cases} \tag{3}$$

which essentially consists of finding all normalized vectors $\psi \in \mathbb{K}^n$ at which A attains its matrix norm.

4 Discussion

Let M be a topological module over a topological ring R. A subset $A \subseteq X$ is said to be bounded if for each 0-neighborhood U in M there is an invertible $u \in \mathcal{U}(R)$ such that $A \subseteq uU$. The following proposition shows that seminorm-boundedness and boundedness coincide when the underlying ring is practical.

Proposition 8 *Let R be a seminormed ring and M a seminormed R-module. If $A \subseteq M$ is bounded, then A is seminorm-bounded. Conversely, if R is absolutely semivalued and practical and $A \subseteq M$ is seminorm-bounded, then it is bounded.*

Proof Assume first that $A \subseteq M$ is bounded. There is $u \in \mathcal{U}(R)$ with $A \subseteq u\mathsf{B}_M \subseteq \mathsf{B}_M(0, \|u\|)$, which means that A is seminorm-bounded. Conversely, assume that R is absolutely semivalued and practical and $A \subseteq M$ is seminorm-bounded. Fix any arbitrary 0-neighborhood V in M. Take $t > 0$ with $\mathsf{B}_M(0,t) \subseteq V$. Since A is seminorm-bounded, there exists $s > 0$ with $A \subseteq \mathsf{B}_M(0,s)$. Since R is practical, we can find an invertible $v \in R$ with $|u| \leq \frac{t}{s}$. Then

$$A \subseteq \mathsf{B}_M(0,s) \subseteq \mathsf{B}_M\big(0, |u|^{-1}t\big) = u^{-1}\mathsf{B}_M(0,t) \subseteq u^{-1}V. \qquad \square$$

Lemma 3 *Let R be a topological ring and M,N topological R-modules. Consider an R-linear operator $T : M \to N$. Let $B \subseteq M$ bounded. Then*:

1. *If T is continuous, then $T(B)$ is bounded in N.*
2. *If there exists a 0-neighborhood $V \subseteq M$ such that $T(V)$ is bounded, then T is continuous.*

Proof Suppose first that T is continuous. We will show that $T(B)$ is bounded in N. Let $U \in \mathcal{N}_0(N)$. Then $T^{-1}(U) \in \mathcal{N}_0(M) \in \mathcal{N}_0(M)$. Since $B \subseteq M$ is bounded, there exists $u \in \mathcal{U}(R)$ with $B \subseteq uT^{-1}(U)$. Thus, $T(B) \subseteq uU$. This implies that $T(B)$ is bounded. Next, let us assume that T is R-linear and that there exists a 0-neighborhood $V \subseteq M$ such that $T(V)$ is bounded. We will prove that T is continuous. Fix any arbitrary 0-neighborhood $W \subseteq N$. By hypothesis, there exists an invertible $u \in \mathcal{U}(R)$ in such a way that $T(V) \subseteq uW$, meaning that $u^{-1}V \subseteq T^{-1}(W)$, so $T^{-1}(W)$ is a neighborhood of 0 in M. This is sufficient to assure that T is continuous. $\square$

An R-linear operator $T : M \to N$ between seminormed modules M,N over a seminormed ring R is said to be bounded provided that $\|T\| := \sup\{\|T(m)\| : \|m\| \leq 1\} < \infty$. The set of supporting vectors of a bounded R-linear operator $T : M \to N$ is defined as

$$\operatorname{suppv}(T) := \arg\max_{\|m\|\leq 1} \big\|T(m)\big\| = \big\{m \in \mathsf{B}_M : \big\|T(m)\big\| = \|T\|\big\}.$$

In order to assure that bounded operators coincide with continuous operators, it is precise to require certain properties from the underlying ring R, such as the practical property [3, 6]. If R is absolutely semivalued, then it is a trivial observation that

$$\mathsf{S}_R\operatorname{suppv}(T) = \operatorname{suppv}(T).$$

If, in addition, N is a seminormed (R,S)-bimodule for S, another absolutely semivalued ring, then $\operatorname{suppv}(T) = \operatorname{suppv}(Ts)$ for all $s \in \mathsf{S}_S$.

Corollary 1 *Let M be a seminormed module over a seminormed ring R. Let $T : M \to M$ be an R-linear operator. Consider the families $\mathcal{F} := \{T, I_M\}$ and $\mathcal{F}' := \{T\}$. Let $\mathcal{R}' := \{m \in M : \|m\| \leq 1\}$. Then*:

1. $\operatorname{suppv}(T) = \operatorname{Par}_{\mathcal{R}'}(\mathcal{F}')$.
2. $\operatorname{Par}(\mathcal{F}) \cap \mathsf{S}_M \subseteq \operatorname{suppv}(T)$.
3. *If R is hyperpractical, then* $\operatorname{Par}(\mathcal{F}) \cap \mathsf{S}_M = \operatorname{suppv}(T)$.
4. *If M is normed, then* $0 \in \operatorname{Par}(\mathcal{F})$.

Proof Since $\mathcal{F}'$ is a singleton, $\mathrm{Par}_{\mathcal{R}'}(\mathcal{F}') = \mathrm{sol}_{\mathcal{R}'}(\mathcal{F}') = \arg\max_{\|m\|\le 1} \|T(m)\| = \mathrm{suppv}(T)$. Thus, by applying Theorem 6, $\mathrm{Par}(\mathcal{F}) \cap \mathsf{S}_M \subseteq \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}') = \mathrm{suppv}(T)$. If R is hyperpractical, then $\mathrm{Par}(\mathcal{F}) \cap \mathsf{S}_M = \mathrm{Par}_{\mathcal{R}'}(\mathcal{F}') = \mathrm{suppv}(T)$ in view of Theorem 7. Finally, if M is normed and $m \in M$ is so that $\|T(m)\| > \|T(0)\| = 0$, then $m \neq 0$, so $\|m\| > 0 = \|0\|$, which proves that $0 \in \mathrm{Par}(\mathcal{F})$. □

5 Conclusion

In the category of modules over a ring, the null object is the null submodule. However, when modules are endowed with a module topology that is not necessarily Hausdorff, then the null submodule does not necessarily behave as the smallest closed submodule. This observation is the key fact motivating the main results of this manuscript, because it allows to consider topological kernels. The study of continuous linear operators over non-Hausdorff spaces necessarily involves dealing with topological kernels. On the other hand, topological kernels serve to study the feasibility of multiobjective optimization of linear operators by providing necessary conditions for the existence of Pareto optimal solutions. Finally, future development of this trend includes a first isomorphism theorem involving topological kernels.

Appendix: Multiobjective optimization problem

Multiobjective optimization problems (MOPs) appear quite often in all areas of Pure, Experimental, Medical, and Social Sciences [8, 16, 20]. By means of MOPs, many real-life situations can be modeled accurately.

Problem 1 (MOP) *Let X be a nonempty set and A a totally ordered set. Let $f_i, g_j : X \to A$, $i \in I$, $j \in J$, be functions. Let $\mathcal{R}$ be a nonempty subset of X. Solve*:

$$\begin{cases} \max f_i(x), & i \in I, \\ \min g_j(x), & j \in J, \\ x \in \mathcal{R}. \end{cases} \tag{A.1}$$

The functions f_i, g_j, $i \in I$, $j \in J$, are called *objective functions*. The set $\mathcal{R}$ is called *feasible region*, *region of constrains/restrictions*, or *set of feasible solutions*, and it is often denoted as fea(A.1). The set of optimal solutions of a MOP is defined as those feasible solutions that optimize all the objective functions at once.

Definition 4 (Optimal solution) The set of optimal solutions of (A.1) is defined as $\mathrm{sol}(\text{A.1}) := \{x \in \mathcal{R} : \forall i \in I \forall j \in J \forall y \in \mathcal{R}, f_i(x) \ge f_i(y) \text{ and } g_j(x) \le g_j(y)\}$.

Note that

$$\mathrm{sol}(\text{A.1}) = \bigcap_{i\in I} \arg\max_{\mathcal{R}}(f_i) \cap \bigcap_{j\in J} \arg\min_{\mathcal{R}}(g_j).$$

The above intersection is commonly empty, hence the Pareto optimal solutions come into play.

Definition 5 (Pareto optimal solution) The set of Pareto optimal solutions of (A.1) is defined as the set of feasible solutions satisfying the Pareto condition $\mathrm{Par}(A.1) := \{x \in \mathcal{R}$: If $y \in \mathcal{R}$ satisfies that there exists $i \in I$ with $f_i(y) > f_i(x)$ or exists $j \in J$ with $g_j(y) < g_j(x)$, then there exists $i' \in I$ with $f_{i'}(y) < f_{i'}(x)$ or exists $j' \in J$ with $g_{j'}(x) < g_{j'}(y)\}$.

It is trivial that $\mathrm{sol}(A.1) \subseteq \mathrm{Par}(A.1) \subseteq \mathrm{fea}(A.1)$. In fact, if $\mathrm{sol}(A.1) \neq \varnothing$, then $\mathrm{sol}(A.1) = \mathrm{Par}(A.1)$.

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Data availability
Not applicable.

Declarations

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The authors declare no competing interests.

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