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On measure of noncompactness in variable exponent Lebesgue spaces and applications to integral equations

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Abstract

A novel measure of noncompactness is defined in variable exponent Lebesgue spaces $L^{p(\cdot)}$ on an unbounded domain $\mathbb{R}^+$ and its properties are examined. Using the fixed point method, we apply that measure to study the existence theorem for nonlinear integral equations. Our results can be handily applied in studying various types of (differential, integral, functional, and partial differential) equations in $L^{p(\cdot)}$-spaces. The $L^{p(\cdot)}$-spaces are natural extensions of classical constant exponent Lebesgue spaces L_p, which allows us to bypass several restrictions that were previously discussed in the literature.

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1 Introduction

Variable exponent Lebesgue spaces $L^{p(\cdot)}$ are a proper tool to study the models with nonstandard growth condition such as elasticity theory [19], electrorheological fluids [32], fluid mechanics [14], differential equations and variational problems [8], image restoration [21], or nonlinear elastic mechanics [34]. The integral provides the model of energy in fluid dynamics [11] in the form

$$\int_{\omega} |Du(t)|^{p(t)}\,dt,$$

where the exponent is a function of the electric field and Du is a symmetric portion of the gradient of the velocity field.

It is beneficial and useful to analyze and investigate the solutions of many types of (differential, integral, functional, and partial differential) equations in various function spaces using the technique of measures of noncompactness (MNCs) related with fixed point theorems, see [3, 6, 10, 12, 23–26, 31].

That technique has lately been developed in many research papers by constructing new MNCs in numerous function spaces and applying these results to study the solutions of

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various types of problems in the considered spaces. Recall that the MNCs were demonstrated and defined in the Banach algebras $C(I)$, $BC(\mathbb{R}^+)$ [5], in the space of regular functions [9, 17, 22], in the space of all locally integrable functions $L^1_{\text{loc}}(\mathbb{R}^+)$ [28], in the space of Lebesgue integrable functions $L_1(\mathbb{R}^N)$ [7], in the classical Lebesgue spaces $L^p(\mathbb{R}^N)$ [1, 27], and in the Sobolev space $W^{k,1}(I)$ [20]. We extend and generalize the above results to the case of $L^{p(\cdot)}(\mathbb{R}^+)$-spaces.

To fulfill this gap, we construct a regular measure of noncompactness in $L^{p(\cdot)}$-spaces on unbounded domain $\mathbb{R}^+$ and apply it with the help of Darbo's fixed point approach in studying the following integral equation in the studied spaces:

$$x(t) = g(t) + f_1\bigl(t, x(t)\bigr) + \int_0^\infty K(t,s) f_2\bigl(s, x(s)\bigr)\,ds. \tag{1.1}$$

The importance of studying $L^{p(\cdot)}$-spaces is that they are logical extensions of classical constant exponent L_p-spaces, and the investigations of the solutions of the integral equations are naturally studied in these spaces. The $L^{p(\cdot)}$-spaces are not rearrangement invariant, the translation operator is not bounded, Young's convolution inequality does not hold when compared to the theory of Lp-spaces (cf. [13, Sect. 3.6]).

Moreover, that technique is simple to use when examining the solutions of various kinds of problems in $L^{p(\cdot)}(\mathbb{R}^+)$, which has a wide range of applications that can be utilized to get beyond the restrictions of Lp-spaces.

2 Notation and auxiliary facts

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. By $L^{p(\cdot)}(\mathbb{R}^+)$ we denote the space of functions $f(t)$ on $\mathbb{R}^+$ s.t.

$$I_p(f) = \int_0^\infty \bigl|f(t)\bigr|^{p(t)}\,dt < \infty,$$

where $p(t)$ is a measurable function on $\mathbb{R}^+$ with values in $[0, \infty)$ and define $p- = \operatorname{ess\,inf}_{t>0} p(t)$ and $p+ = \operatorname{ess\,sup}_{t>0} p(t)$.

The spaces $L^{p(\cdot)}(\mathbb{R}^+) = L^{p(\cdot)}$ are the Banach spaces with the norm

$$\|f\|_{p(\cdot)} = \|f\|_{L^{p(\cdot)}(\mathbb{R}^+)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\}, \tag{2.1}$$

which corresponds to the well-known Luxemburg norm in Orlicz spaces. If $p(x) = p$ is a constant function, then norm (2.1) coincides with the usual L_p-norm.

The next concepts will be useful and helpful with the framework.

The Hölder inequality is written as follows in $L^{p(\cdot)}$ [11]: If we assume that f and g are in $L^{p(\cdot)}$ and $L^{q(\cdot)}$, respectively, we have $f \cdot g \in L^1(\mathbb{R}^+)$, where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$, and

$$\int_0^\infty \bigl|f(t)g(t)\bigr|\,dt \le k_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $1 \le p(\cdot) \le \infty$ and $1 < k_{p(\cdot)} \le 4$.

Proposition 2.1 [15, *Remark* 2.1] *Suppose that the sequence* $\{x_k\} \subset L^{p(\cdot)}$ *converges in norm to* $h \in L^{p(\cdot)}$. *Then there exists a subsequence* $\{h_{k_j}\}$ *and* $g \in L^{p(\cdot)}$ *such that the subsequence converges pointwise a.e. to* h *and, for almost every* $t \in \mathbb{R}^+$, $|h_{k_j}(t)| \le g(t)$.

Definition 2.2 Consider that a function $f(t,x):\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ fulfills the Carathéodory conditions i.e. it is continuous in x for almost all $t\in\mathbb{R}^+$ and measurable in t for any $x\in\mathbb{R}$. The superposition (Nemytskii) operator F_f can thus be denoted for each measurable function x in the following manner:

$$F_f(x)(t)=f\bigl(t,x(t)\bigr),\quad t\in\mathbb{R}^+.$$

Lemma 2.3 [18] *If $F_f:L^{p(\cdot)}\to L^{q(\cdot)}$, then F_f is continuous and bounded, and there is a constant $b\geq 0$ and a nonnegative function $a\in L^{q(\cdot)}$ such that, for $t\in\mathbb{R}^+$ and $x\in\mathbb{R}$, the following inequality holds*:

$$\bigl|f(t,x)\bigr|\leq a(t)+b|x|^{p(t)/q(t)}. \tag{2.2}$$

On the other hand, if $f(t,x)$ satisfies (2.2), *then $F_f:L^{p(\cdot)}\to L^{q(\cdot)}$, and thus F_f is continuous and bounded.*

Next, let us assume that $(E,\|\cdot\|)$ is a Banach space that has zero element denoted by θ and that $B_r=B(r,\theta)$ denotes a ball with a radius r and a center at θ.

Let $\mathcal{N}_E$ and $\emptyset\neq\mathcal{M}_E$ be, respectively, the subfamily containing all relatively compact sets of E and the family of all nonempty and bounded subsets of E. Convex closed hull and closure of a set Y are denoted by the symbols Conv Y and $\overline{Y}$, respectively.

Definition 2.4 [4] The function $\mu:\mathcal{M}_E\to[0,\infty)$ is called a measure of noncompactness (MNC) in E if it fulfills:

(1) The family $\ker\mu=\{X\in\mathcal{M}_E:\mu(X)=0\}$ is nonempty and $\ker\mu\subset\mathcal{N}_E$.
(2) $Y\subset X\Rightarrow\mu(Y)\leq\mu(X)$.
(3) $\mu(\overline{Y})=\mu(Y)$.
(4) $\mu(ConvY)=\mu(Y)$.
(5) $\mu(\lambda Y+(1-\lambda)Y)\leq\lambda\mu(Y)+(1-\lambda)\mu(Y)$, for $\lambda\in[0,1]$.
(6) If $\emptyset\neq Y_n\subset E$ is a sequence of closed and bounded sets, $Y_n=\overline{Y}_n$ s.t. $Y_{n+1}\subset Y_n$, $n=1,2,3,\ldots$, and $\lim_{n\to\infty}\mu(Y_n)=0$, then $Y_\infty=\bigcap_{n=1}^{\infty}Y_n\neq\emptyset$.
We say that an MNC is regular if it additionally satisfies the following axioms:
(7) $\mu(Y\cup X)=\max\{\mu(Y),\mu(X)\}$.
(8) $\mu(Y+X)\leq\mu(Y)+\mu(X)$.
(9) $\mu(\lambda Y)=|\lambda|\mu(Y)$, for $\lambda\in\mathbb{R}$.
(10) $\ker\mu=\mathcal{N}_E$.

Examples of MNCs that fit all the axioms listed above are the Kuratowski and Hausdorff MNCs [4].

It will be necessary to use the following Darbo's fixed point theorem.

Theorem 2.5 [4] *Let $\emptyset\neq Q\subset E$ be a bounded, closed, and convex set and let $V:Q\to Q$ be a continuous mapping that is a contraction with respect to the MNC μ i.e. there exists $k\in[0,1)$ s.t.*

$$\mu\bigl(V(X)\bigr)\leq k\mu(X)$$

for any nonempty $X\subset E$. Then V has at least one fixed point in Q.

3 Construction of MNC in the space $L^{p(\cdot)}$

In this section, we establish a new measure of noncompactness in $L^{p(\cdot)}$ and investigate its properties. We begin by introducing an extension of the well-known Riesz–Kolmogorov compactness criterion in $L^{p(\cdot)}$.

Definition 3.1 A subset $\mathcal{F}$ of $L^{p(\cdot)}$ is called precompact if its closure is compact.

The Riesz–Kolmogorov compactness criterion in $L^{p(\cdot)}$ can be stated as follows (cf. [19, Theorem 5] and [16, Theorem 2.1]).

Lemma 3.2 *Let $\mathcal{F}$ be a subset of $L^{p(\cdot)}$, then the family $\mathcal{F}$ is precompact in $L^{p(\cdot)}$ if and only if the following conditions hold*:

(i) *$\mathcal{F}$ is bounded,*
(ii) $\forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{h<\delta}\ \forall_{f\in\mathcal{F}}\ \|f(\cdot+h)-f(\cdot)\|_{p(\cdot)}<\epsilon$,
(iii) $\forall_{\epsilon>0}\ \exists_{T>0}\ \forall_{f\in\mathcal{F}}\ \|f\|_{L^{p(\cdot)}[T,\infty)}<\epsilon$.

Theorem 3.3 *Assume that $\emptyset\neq X\subset L^{p(\cdot)}$ is a bounded set. For $x\in X$ and $\epsilon>0$, let*

$$\omega(x,\epsilon)=\sup\left\{\left\|x(\cdot+h)-x(\cdot)\right\|_{p(\cdot)}:|h|<\epsilon\right\},$$
$$\omega(X,\epsilon)=\sup\left\{\omega(x,\epsilon):x\in X\right\},$$
$$\omega(X)=\lim_{\epsilon\to 0}\omega(X,\epsilon).$$

Also, let

$$d_T(X)=\sup\left\{\|x\|_{L^{p(\cdot)}[T,\infty)}:x\in X\right\},$$
$$d(X)=\lim_{T\to\infty}d_T(X).$$

Then $\mu(X)=\omega(X)+d(X):\mathcal{M}_{L^{p(\cdot)}}\to\mathbb{R}^+$ represents an MNC in $L^{p(\cdot)}$.

Proof First, we begin by demonstrating the validity of axiom (1) of Definition 2.4.

Consider $X\in\mathcal{M}_{L^{p(\cdot)}}$ such that $\mu(X)=0$. Let $\eta>0$ be arbitrary. Since $\mu(X)=0$, then $\lim_{\epsilon\to 0}\omega(X,\epsilon)=0$. Therefore, for all $\eta>0$, there exists $\delta>0$ such that $\omega(X,\delta)<\eta$, and this indicates that

$$\left\|x(\cdot+h)-x(\cdot)\right\|_{p(\cdot)}<\eta$$

for all $x\in X$ and $h\in\mathbb{R}^+$ such that $|h|<\delta$. Since $\eta>0$ is arbitrary, we get

$$\lim_{h\to 0}\left\|x(\cdot+h)-x(\cdot)\right\|_{p(\cdot)}=0$$

uniformly in $x\in X$. Again, keeping in mind that $\mu(X)=0$, we have

$$\lim_{T\to\infty}d_T(X)=0,$$

and so for $\epsilon>0$, there exists $T>0$ such that

$$\|x\|_{L^{p(\cdot)}[T,\infty)}<\epsilon\quad\text{for all } x\in X.$$

Thus, from Lemma 3.2 we infer that the closure of $X \in L^{p(\cdot)}$ is compact and $\ker\mu \subset \mathcal{N}_{L^{p(\cdot)}}$. The proof of axiom (2) is clear.

Now, suppose that $X \in \mathcal{M}_{L^{p(\cdot)}}$ and $(x_n) \subset X$ such that $x_n \to x \in \bar{X}$ in $L^{p(\cdot)}$. From the definition of $\omega(X,\epsilon)$ we have

$$\left\|x_n(\cdot + h) - x(\cdot)\right\|_{p(\cdot)} \leq \omega(X,\epsilon)$$

for any $n \in \mathbb{N}$ and $|h| < \epsilon$. Letting $n \to \infty$, we get

$$\left\|x(\cdot + h) - x(\cdot)\right\|_{p(\cdot)} \leq \omega(X,\epsilon) \quad \text{for any } |h| < \epsilon,$$

then

$$\lim_{\epsilon\to 0} \omega(\bar{X},\epsilon) \leq \lim_{\epsilon\to 0} \omega(X,\epsilon)$$

implies that

$$\omega(\bar{X}) \leq \omega(X). \tag{3.1}$$

Similarly, we may demonstrate that $d(\bar{X}) \leq d(X)$, so from equation (3.1) and axiom (2) we get $\mu(\bar{X}) \leq \mu(X)$ satisfies axiom (3) of Definition 2.4.

Axioms (4) and (5) can be proved similarly by using the inequality

$$\left\|\lambda x + (1-\lambda)x\right\|_{p(\cdot)} \leq \lambda\|x\|_{p(\cdot)} + (1-\lambda)\|x\|_{p(\cdot)}.$$

To demonstrate axiom (6), let us assume that $\{X_n\}$ is a sequence of closed and nonempty sets from $\mathcal{M}_{L^{p(\cdot)}}$, where X_{n+1} is a subset of X_n for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \mu(X_n) = 0$. Now, for any $n \in \mathbb{N}$, take $x_n \in X_n$. We claim that $\mathcal{F} = \overline{\{x_n\}}$ is a compact set in $L^{p(\cdot)}$. To prove the claim, we need to check conditions (ii) and (iii) of Lemma 3.2.

Let $\epsilon > 0$ be fixed. Since $\lim_{n\to\infty} \mu(X_n) = 0$, there exists $k \in \mathbb{N}$ such that $\mu(X_k) < \epsilon$. Hence, we can find $\delta_1 > 0$ and $T_1 > 0$ such that

$$\omega(X_k,\delta_1) < \epsilon \quad \text{and} \quad d_{T_1}(X_k) < \epsilon.$$

Thus, for all $n \geq k$ and $|h| \leq \delta_1$, we get

$$\omega(X_n,\delta_1) < \epsilon \quad \text{and} \quad d_{T_1}(X_n) < \epsilon.$$

The set $\{x_1, x_2, \ldots, x_{k-1}\}$ is compact, hence there exist $\delta_2 > 0$ and $T_2 > 0$ such that

$$\omega(X_n,\delta_2) < \epsilon \quad \text{and} \quad d_{T_2}(X_n) < \epsilon$$

for all $1 \leq n \leq k$. If we choose $\delta < \min\{\delta_1,\delta_2\}$ and set $T = \max\{T_1, T_2\}$, then

$$\omega(\mathcal{F}) = \lim_{\delta\to 0} \omega(\mathcal{F},\delta) = 0 \quad \text{and} \quad d(\mathcal{F}) = \lim_{T\to\infty} d_T(\mathcal{F}) = 0.$$

Consequently, $\mu(\mathcal{F}) = 0$, indicating that $\mathcal{F}$ is compact. It follows that there exists a subsequence $\{x_{n_j}\}$ and $x_0 \in L^{p(\cdot)}$ such that $x_{n_j} \to x_0$, and since for all $x_n \in X_n$, $X_{n+1} \subset X_n$ and X_n is closed for all $n \in \mathbb{N}$, we have

$$x_0 \in \bigcap_{i=1}^{\infty} X_n = X_\infty,$$

and this completes the theorem's proof. □

We now analyze the regularity of μ.

Theorem 3.4 *The measure of noncompactness μ given in Theorem 3.3 is regular.*

Proof Suppose that $X, Y \in L^{p(\cdot)}$. Since for all $\epsilon > 0$, $\lambda > 0$, $T > 0$ we have

$$\omega(Y \cup X, \epsilon) \leq \max\bigl\{\omega(Y, \epsilon), \omega(X, \epsilon)\bigr\},$$

$$\omega(Y + X, \epsilon) \leq \omega(Y, \epsilon) + \omega(X, \epsilon),$$

$$\omega(\lambda Y, \epsilon) \leq \lambda\omega(Y, \epsilon)$$

and

$$\sup_{x\in Y\cup X} \|x\|_{L^{p(\cdot)}[T,\infty)} \leq \sup\Bigl\{\sup_{x\in Y} \|x\|_{L^{p(\cdot)}[T,\infty)}, \sup_{x\in X} \|x\|_{L^{p(\cdot)}[T,\infty)}\Bigr\},$$

$$\sup_{x\in Y+X} \|x\|_{L^{p(\cdot)}[T,\infty)} \leq \sup_{x\in Y} \|x\|_{L^{p(\cdot)}[T,\infty)} + \sup_{x\in X} \|x\|_{L^{p(\cdot)}[T,\infty)},$$

$$\sup_{x\in \lambda Y} \|x\|_{L^{p(\cdot)}[T,\infty)} \leq \lambda \sup_{x\in Y} \|x\|_{L^{p(\cdot)}[T,\infty)},$$

axioms (7), (8), and (9) hold. To show that (10) holds, suppose that $X \in \mathcal{N}_{L^{p(\cdot)}}$. Thus, the closure of X in $L^{p(\cdot)}$ is compact, and hence from Lemma 3.2, for any $\epsilon > 0$, there exists $T > 0$ such that $d_T(X) < \epsilon$ and also $\omega(X) < \epsilon$ uniformly in $x \in X$. From the first conclusion, for all $\eta > 0$, there exists $\delta > 0$ such that $\|x(\cdot + h) - x(\cdot)\|_{p(\cdot)} < \eta$ for any $|h| < \delta$. Then, for all $x \in X$, we have

$$\omega(x, \delta) = \sup\bigl\{\bigl\|x(\cdot + h) - x(\cdot)\bigr\|_{p(\cdot)} : |h| < \delta\bigr\} \leq \epsilon.$$

Therefore,

$$\omega(X, \delta) = \sup\bigl\{\omega(x, \delta) : x \in X\bigr\} \leq \epsilon,$$

which proves

$$\lim_{\delta\to 0} \omega(X, \delta) = 0 \tag{3.2}$$

and

$$\lim_{T\to\infty} d_T(X) = 0. \tag{3.3}$$

Now from (3.2) and (3.3), axiom (10) holds. □

4 Applications

In this section, we use the results obtained in Sect. 3 to investigate the existence of solutions for the nonlinear integral equation (1.1).

The operator H can be defined as follows:

$$x = H(x) = g + F_{f_1}(x) + Ax,$$

where

$$Ax = K_0 \circ F_{f_2}, \qquad K_0 x(t) = \int_0^\infty K(t,s)x(s)\,ds$$

and the superposition operators F_{f_i}, $i, i = 1, 2$, are the same as in Definition 2.2.

Equation (1.1) will be handled based on the following set of presumptions:

(i) $g(\cdot) \in L^{p(\cdot)}$.

(ii) For $i = 1, 2$, suppose that $f_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and there exist constants $b_i \geq 0$ and functions $a_i \in L^{q(\cdot)}$ such that

$$\left|f_i(t,0)\right| \leq a_i(t)$$

and

$$\left|f_i(t,x) - f_i(s,y)\right| \leq \left|a_i(t) - a_i(s)\right| + b_i|x-y|^{p(s)/q(s)}$$

for all $t, s \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$.

(iii) Let $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be measurable and there exist functions $g_1 \in L^{p(\cdot)}$ and $g_2 \in L^{q'(\cdot)}$, $\frac{1}{p(\cdot)} + \frac{1}{q'(\cdot)} = 1$ such that $|K(t,s)| \leq g_1(t)g_2(s)$ for all $t, s \in \mathbb{R}^+$ and

$$\left|K(t_1,s) - K(t_2,s)\right| \leq g_2(s)\left|g_1(t_1) - g_1(t_2)\right|, \quad t_1, t_2, s \in \mathbb{R}^+.$$

(iv) For $M > 1$, let r be a constant satisfying the inequalities

$$\begin{aligned}&\|g\|_{p(\cdot)} + \left(M\|a_1\|_{q(\cdot)} + b_1 M r^{(p/q)\pm}\right)\\&\quad + k_{q(\cdot)}\|g_1\|_{p(\cdot)}\|g_2\|_{q'(\cdot)}\left(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\right) \leq r\end{aligned}$$

and

$$b_1 M \cdot (2r)^{(p/q)\pm} \leq 1,$$

where the + sign occurs when $r \geq 1$ and the − sign occurs in the case $r \leq 1$.

Remark 4.1 Under assumption (iii), the linear integral operator $K_0 x(t) = \int_0^\infty K(t,s)x(s)\,ds$ maps $L^{q(\cdot)}$ into $L^{p(\cdot)}$.

Proof By utilizing assumption (iii) and the Hölder inequality, we have

$$\left|K_0 x(t)\right| \leq \int_0^\infty \left|K(t,s)\right|\left|x(s)\right|ds$$

$$
\begin{aligned}
&\leq \int_0^\infty g_1(t)g_2(s)\bigl|x(s)\bigr|\,ds \\
&\leq k_{q(\cdot)}g_1(t)\|g_2\|_{q'(\cdot)}\|x\|_{q(\cdot)}, \quad q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}.
\end{aligned}
$$

Therefore,

$$
\|K_0x\|_{p(\cdot)} \leq k_{q(\cdot)}\|g_1\|_{p(\cdot)}\|g_2\|_{q'(\cdot)}\|x\|_{q(\cdot)}, \tag{4.1}
$$

then we get our claim. □

Remark 4.2 Under assumption (ii), we have that the superposition operator $F_{f_i} : L^{p(\cdot)} \to L^{q(\cdot)}$ and is continuous.

Proof From assumption (ii), for $i = 1, 2$, we have

$$
\begin{aligned}
&\bigl|f_i(t,x)\bigr| \leq \bigl|f_i(t,x) - f_i(t,0)\bigr| + \bigl|f_i(t,0)\bigr| \leq b_i|x|^{p(t)/q(t)} + \bigl|f_i(t,0)\bigr| \\
&\quad\Longrightarrow \quad \bigl|f_i(t,x)\bigr| \leq a_i(t) + b_i|x|^{p(t)/q(t)}.
\end{aligned}
$$

Since $a_i \in L^{q(\cdot)}$, $i = 1, 2$, and by utilizing Lemma 2.3, we get our claim. □

Proposition 4.3 [8] *Since $p(t) \leq q(t)$, $t \in \mathbb{R}^+$, the operator $F_{f_1} : L^{p(\cdot)} \to L^{p(\cdot)}$ is well defined, bounded, and continuous and*

$$
\bigl\|x^{p(\cdot)/q(\cdot)}\bigr\|_{q(\cdot)} \leq \|x\|_{p(\cdot)}^{(p/q)\pm}.
$$

Theorem 4.4 *Let assumptions* (*i*)–(*iv*) *be satisfied, then equation* (1.1) *has at least one solution $x \in L^{p(\cdot)}(\mathbb{R}^+)$.*

Proof We will carry out the proof in a number of steps.

Step I. We will show that the operator H is well defined on $L^{p(\cdot)}$ and is continuous.

At the outset, according to (ii) and Remark 4.2, we have that the operators $F_{f_i} : L^{p(\cdot)} \to L^{q(\cdot)}$, where $i = 1, 2$, and they are continuous. Furthermore, according to Proposition 4.3, the operator $F_{f_1} : L^{p(\cdot)} \to L^{p(\cdot)}$ and is continuous.

Now, we will prove that property for the operator $A = K_0 \circ F_{f_2}$.

Considering our underlying presumptions and Remark 4.2 with the Hölder inequality, we have

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \bigl|K(t,s)f_2\bigl(s,x(s)\bigr)\bigr|\,ds\,dt \\
&\quad\leq \int_0^\infty \int_0^\infty \bigl|K(t,s)\bigr|\bigl(a_2(s) + b_2\bigl|x(s)\bigr|^{p(s)/q(s)}\bigr)\,ds\,dt \\
&\quad\leq \int_0^\infty k_{q(\cdot)}\bigl\|K(t,\cdot)\bigr\|_{q'(\cdot)}\bigl\|a_2 + b_2|x|^{p(\cdot)/q(\cdot)}\bigr\|_{q(\cdot)}\,dt \\
&\quad\leq k_{q(\cdot)}\bigl(\|a_2\|_{q(\cdot)} + b_2\bigl\|x^{p(\cdot)/q(\cdot)}\bigr\|_{q(\cdot)}\bigr)\int_0^\infty \bigl\|g_1(t)g_2(\cdot)\bigr\|_{q'(\cdot)}\,dt \\
&\quad\leq k_{q(\cdot)}\bigl(\|a_2\|_{q(\cdot)} + b_2\|x\|_{p(\cdot)}^{(p/q)\pm}\bigr)k_{p(\cdot)}\|g_1 \cdot \|g_2\|_{q'(\cdot)}\|_{p(\cdot)}\|1\|_{p'(\cdot)}
\end{aligned}
$$

$$= k_{q(\cdot)}k_{p(\cdot)}\|g_1\|_{p(\cdot)}\|g_2\|_{q'(\cdot)}\|1\|_{p'(\cdot)}\big(\|a_2\|_{q(\cdot)} + b_2\|x\|_{p(\cdot)}^{(p/q)\pm}\big), \tag{4.2}$$

where $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. Then $A(L^{p(\cdot)}(\mathbb{R}^+)) \subset L_1(\mathbb{R}^+)$.

We will now demonstrate that $A(L^{p(\cdot)}) \subset L^{p(\cdot)}$. Using Remark 4.1 and Remark 4.2 again, we derive that, for almost every $t \in \mathbb{R}^+$,

$$\begin{aligned}
&\left\| \int_0^\infty K(\cdot,s)f_2\big(s,x(s)\big)\,ds \right\|_{p(\cdot)} \\
&\quad\le \left\| \int_0^\infty K(\cdot,s)\big(a_2(s) + b_2\big|x(s)\big|^{p(s)/q(s)}\big)\,ds \right\|_{p(\cdot)} \\
&\quad\le \|k_{q(\cdot)}\|K(t,\cdot)\|_{q'(\cdot)}\|a_2 + b_2|x|^{p(\cdot)/q(\cdot)}\|_{q(\cdot)}\|_{p(\cdot)} \\
&\quad\le k_{q(\cdot)}\big(\|a_2\|_{q(\cdot)} + b_2\big\|x^{p(\cdot)/q(\cdot)}\big\|_{q(\cdot)}\big)\|g_1\cdot\|g_2\|_{q'(\cdot)}\|_{p(\cdot)} \\
&\quad= k_{q(\cdot)}\|g_1\|_{p(\cdot)}\|g_2\|_{q'(\cdot)}\big(\|a_2\|_{q(\cdot)} + b_2\|x\|_{p(\cdot)}^{(p/q)\pm}\big).
\end{aligned} \tag{4.3}$$

Then $A(L^{p(\cdot)}) \subset L^{p(\cdot)}$.

We shall now demonstrate that the operator A is continuous between indicated spaces. To accomplish this, let us select the sequence $(x_n) \subset L^{p(\cdot)}$ such that

$$\lim_{n\to\infty} \|x_n - x\|_{p(\cdot)} = 0.$$

We will demonstrate that $A(x_n) \to A(x)$ in $L^{p(\cdot)}$ as $n \to \infty$. To accomplish this, it is sufficient to demonstrate that any sequence (x_n) has a subsequence denoted by (x_{n_k}) such that $A(x_{n_k}) \to A(x)$ in $L^{p(\cdot)}$ as $k \to \infty$.

Take $(x_{n_k}) \subset (x_n)$. It follows by Proposition 2.1 that there is a subsequence (x_{n_k}) such that

$$x_{n_k} \to x \quad \text{almost everywhere in } \mathbb{R}^+ \tag{4.4}$$

and there exists $h \in L^{p(\cdot)}$ satisfying

$$\big|x_{n_k}(t)\big| \le h(t) \quad \text{almost every } t \in \mathbb{R}^+ \text{ for all } k \in \mathbb{N}. \tag{4.5}$$

Since the function $K(t,s)f_2(s,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, from (4.4), we deduce that

$$K(t,s)f_2\big(s,x_{n_k}(s)\big) \to K(t,s)f_2\big(s,x(s)\big) \quad \text{for almost every } t,s \in \mathbb{R}^+.$$

Now, based on presumption (ii) with inequality (4.5), we have for almost every $t,s \in \mathbb{R}^+$,

$$\big|K(t,s)f_2\big(s,x_{n_k}(s)\big)\big| \le K(t,s)\big(a_2(s) + b_2\big|h(s)\big|^{p(s)/q(s)}\big).$$

As before, we can show that for almost every $t \in \mathbb{R}^+$ the function $t \to K(t,s)(a_2(s) + b_2|h(s)|^{p(\cdot)/q(\cdot)})$ lies in $L_1(\mathbb{R}^+)$.

Consequently, using the Lebesgue dominated theorem, we obtain

$$\int_0^\infty K(t,s)f_2\big(s,x_{n_k}(s)\big)\,ds \to \int_0^\infty K(t,s)f_2\big(s,x(s)\big)\,ds$$

for almost every $t \in \mathbb{R}^+$. Moreover, by using Remark 4.1 and (4.5), we have

$$\left\| \int_0^\infty K(t,s) f_2\big(s, x_{n_k}(s)\big)\, ds \right\|_{p(\cdot)} \le k_{q(\cdot)} \|g_1\|_{p(\cdot)} \|g_2\|_{q'(\cdot)} \big(\|a_2\|_{q(\cdot)} + b_2 \|h\|_{p(\cdot)}^{(p/q)\pm} \big).$$

Consequently, the Lebesgue dominated theorem implies that

$$\left\| A(x_{n_k}) - A(x) \right\|_{p(\cdot)} \to 0,$$

then we obtain that $\|A(x_n) - A(x)\|_{p(\cdot)} \to 0$,

which implies that A maps continuously $L^{p(\cdot)} \to L^{p(\cdot)}$. Finally, by assumption (i), we have that the operator H maps $L^{p(\cdot)} \to L^{p(\cdot)}$ and is continuous.

Step II. We will construct an invariant set B_r, where r is as in assumption (iv). We shall first show that the operator H is bounded in $L^{p(\cdot)}$ i.e.

$$\left\| H(x) \right\|_{p(\cdot)} \le \|g\|_{p(\cdot)} + \left\| F_{f_1}(x) \right\|_{p(\cdot)} + \left\| A(x) \right\|_{p(\cdot)}.$$

To do this, we use assumptions (ii) and (iii) and the following facts: (i) the $L^{p(\cdot)}$-norm is order preserving, (ii) the triangle inequality, (iii) the embedding $L^{q(\cdot)} \subset L^{p(\cdot)}$ since $p(\cdot) \le q(\cdot)$ (see [11, Theorem 2.45 and Corollary 2.48]), which implies that there exists $M > 1$ such that, for all $a_1 \in L^{q(\cdot)}$, $\|a_1\|_{p(\cdot)} \le M \|a_1\|_{q(\cdot)}$. Then we have

$$\begin{aligned}
\left\| H(x) \right\|_{p(\cdot)} &\le \|g\|_{p(\cdot)} + \left\| a_1 + b_1 |x|^{p(\cdot)/q(\cdot)} \right\|_{p(\cdot)} \\
&\quad + k_{q(\cdot)} \|g_1\|_{p(\cdot)} \|g_2\|_{q'(\cdot)} \left\| F_{f_2}(x) \right\|_{q(\cdot)} \\
&\le \|g\|_{p(\cdot)} + \big(\|a_1\|_{p(\cdot)} + b_1 \left\| x^{p(\cdot)/q(\cdot)} \right\|_{p(\cdot)} \big) \\
&\quad + k_{q(\cdot)} \|g_1\|_{p(\cdot)} \|g_2\|_{q'(\cdot)} \left\| a_2 + b_2 |x|^{p(\cdot)/q(\cdot)} \right\|_{q(\cdot)} \\
&\le \|g\|_{p(\cdot)} + \big(M \|a_1\|_{q(\cdot)} + b_1 M \left\| x^{p(\cdot)/q(\cdot)} \right\|_{q(\cdot)} \big) \\
&\quad + k_{q(\cdot)} \|g_1\|_{p(\cdot)} \|g_2\|_{q'(\cdot)} \big(\|a_2\|_{q(\cdot)} + b_2 \left\| x^{p(\cdot)/q(\cdot)} \right\|_{q(\cdot)} \big) \\
&\le \|g\|_{p(\cdot)} + \big(M \|a_1\|_{q(\cdot)} + b_1 M \|x\|_{p(\cdot)}^{(p/q)\pm} \big) \\
&\quad + k_{q(\cdot)} \|g_1\|_{p(\cdot)} \|g_2\|_{q'(\cdot)} \big(\|a_2\|_{q(\cdot)} + b_2 \|x\|_{p(\cdot)}^{(p/q)\pm} \big).
\end{aligned}$$

Thus, $H : L^{p(\cdot)} \to L^{p(\cdot)}$. It follows from our assumption (iv) that the following inequality has a positive solution r such that

$$\begin{aligned}
\left\| H(x) \right\|_{p(\cdot)} &\le \|g\|_{p(\cdot)} + \big(M \|a_1\|_{q(\cdot)} + b_1 M r^{(p/q)\pm} \big) \\
&\quad + k_{q(\cdot)} \|g_1\|_{p(\cdot)} \|g_2\|_{q'(\cdot)} \big(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm} \big) \le r,
\end{aligned}$$

which implies that $H : \overline{B}_r \to \overline{B}_r$ is continuous.

Step III. It is obvious that $\overline{B}_r$ is a bounded, closed, and convex set in $L^{p(\cdot)}$.

Step IV. We will demonstrate that H is a contraction in terms of the MNC μ.

Assume that $\emptyset \neq X \subset \overline{B}_r$ and fixed arbitrary constant $\epsilon > 0$. Then, for arbitrary $x \in X$ and for $t, h \in \mathbb{R}$ and $|h| \leq \epsilon$, we have

$$\begin{aligned}
&\bigl|(Hx)(t+h) - (Hx)(t)\bigr| \\
&\quad \leq \bigl|g(t+h) - g(t)\bigr| + \bigl|F_{f_1}(x)(t+h) - F_{f_1}(x)(t)\bigr| + \bigl|A(x)(t+h) - A(x)(t)\bigr| \\
&\quad \leq \bigl|g(t+h) - g(t)\bigr| + \bigl|f_1\bigl(t+h, x(t+h)\bigr) - f_1\bigl(t, x(t)\bigr)\bigr| \\
&\qquad + \int_0^\infty \bigl|K(t+h,s) - K(t,s)\bigr|\bigl|f_2\bigl(s, x(s)\bigr)\bigr|\, ds \\
&\quad \leq \bigl|g(t+h) - g(t)\bigr| + \bigl(\bigl|a_1(t+h) - a_1(t)\bigr| + b_1\bigl|x(t+h) - x(t)\bigr|^{p(t)/q(t)}\bigr) \\
&\qquad + \int_0^\infty \bigl|g_1(t+h) - g_1(t)\bigr|\bigl|g_2(s)\bigr|\bigl|a_2(s) + b_2\bigl|x(s)\bigr|^{p(s)/q(s)}\bigr|\, ds \\
&\quad \leq \bigl|g(t+h) - g(t)\bigr| + \bigl(\bigl|a_1(t+h) - a_1(t)\bigr| + b_1\bigl|x(t+h) - x(t)\bigr|^{p(t)/q(t)}\bigr) \\
&\qquad + k_{q(\cdot)}\bigl|g_1(t+h) - g_1(t)\bigr| \cdot \|g_2\|_{q'(\cdot)}\bigl(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\bigr).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\bigl\|(Hx)(\cdot+h) - (Hx)(\cdot)\bigr\|_{p(\cdot)} \\
&\quad \leq \bigl\|g(\cdot+h) - g(\cdot)\bigr\|_{p(\cdot)} \\
&\qquad + \bigl\|a_1(\cdot+h) - a_1(\cdot)\bigr\|_{p(\cdot)} + b_1\bigl\|\bigl|x(\cdot+h) - x(\cdot)\bigr|^{p(\cdot)/q(\cdot)}\bigr\|_{p(\cdot)} \\
&\qquad + \bigl\|k_{q(\cdot)}\bigl|g_1(\cdot+h) - g_1(\cdot)\bigr| \cdot \|g_2\|_{q'(\cdot)}\bigl(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\bigr)\bigr\|_{p(\cdot)} \\
&\quad \leq \bigl\|g(\cdot+h) - g(\cdot)\bigr\|_{p(\cdot)} \\
&\qquad + \bigl(M\bigl\|a_1(\cdot+h) - a_1(\cdot)\bigr\|_{q(\cdot)} + Mb_1\bigl\|\bigl|x(\cdot+h) - x(\cdot)\bigr|^{p(\cdot)/q(\cdot)}\bigr\|_{q(\cdot)}\bigr) \\
&\qquad + k_{q(\cdot)}\bigl\|g_1(\cdot+h) - g_1(\cdot)\bigr\|_{p(\cdot)}\|g_2\|_{q'(\cdot)}\bigl(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\bigr) \\
&\quad \leq \bigl\|g(\cdot+h) - g(\cdot)\bigr\|_{p(\cdot)} \\
&\qquad + \bigl(M\bigl\|a_1(\cdot+h) - a_1(\cdot)\bigr\|_{q(\cdot)} + Mb_1\bigl\|x(\cdot+h) - x(\cdot)\bigr\|_{p(\cdot)}^{(p/q)\pm}\bigr) \\
&\qquad + k_{q(\cdot)}\bigl\|g_1(\cdot+h) - g_1(\cdot)\bigr\|_{p(\cdot)}\|g_2\|_{q'(\cdot)}\bigl(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\bigr) \\
&\quad \leq \bigl\|g(\cdot+h) - g(\cdot)\bigr\|_{p(\cdot)} \\
&\qquad + \bigl(M\bigl\|a_1(\cdot+h) - a_1(\cdot)\bigr\|_{q(\cdot)} + Mb_1(2r)^{(p/q)\pm}\bigl\|x(\cdot+h) - x(\cdot)\bigr\|_{p(\cdot)}\bigr) \\
&\qquad + k_{q(\cdot)}\bigl\|g_1(\cdot+h) - g_1(\cdot)\bigr\|_{p(\cdot)}\|g_2\|_{q'(\cdot)}\bigl(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\bigr),
\end{aligned}$$

where

$$\begin{aligned}
\bigl\|x(\cdot+h) - x(\cdot)\bigr\|_{p(\cdot)}^{(p/q)\pm} &= \bigl\|x(\cdot+h) - x(\cdot)\bigr\|_{p(\cdot)}^{(p/q)\pm-1}\bigl\|x(\cdot+h) - x(\cdot)\bigr\|_{p(\cdot)} \\
&\leq (2r)^{(p/q)\pm}\bigl\|x(\cdot+h) - x(\cdot)\bigr\|_{p(\cdot)}.
\end{aligned}$$

Thus we obtain

$$\omega(HX,\epsilon) \le \omega(g,\epsilon) + \big(M\omega(a_1,\epsilon) + b_1 M \cdot (2r)^{(p/q)\pm}\omega(x,\epsilon)\big) \\ + k_{q(\cdot)}\|g_2\|_{q'(\cdot)}\big(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\big)\omega(g_1,\epsilon).$$

Also we have $\omega(g,\epsilon) \to 0$, $\omega(a_1,\epsilon) \to 0$, and $\omega(g_1,\epsilon) \to 0$ as $\epsilon \to 0$. Then we obtain

$$\omega(HX) \le b_1 M \cdot (2r)^{(p/q)\pm} \cdot \omega(X). \tag{4.6}$$

Next, let us fix an arbitrary number $T > 0$. Then, using our hypotheses, for an arbitrary function $x \in X$ we have

$$\|HX\|_{L^{p(\cdot)}[T,\infty)} \le \|g\|_{L^{p(\cdot)}[T,\infty)} \\ + \big(M\|a_1\|_{L^{q(\cdot)}[T,\infty)} + b_1 M(2r)^{(p/q)\pm}\|x\|_{L^{p(\cdot)}[T,\infty)}\big) \\ + k_{q(\cdot)} \cdot \|g_1\|_{L^{p(\cdot)}[T,\infty)}\|g_2\|_{q'(\cdot)}\big(\|a_2\|_{q(\cdot)} + b_2 r^{(p/q)\pm}\big).$$

Also we have $\|g\|_{L^{p(\cdot)}[T,\infty)} \to 0$, $\|a_1\|_{L^{q(\cdot)}[T,\infty)} \to 0$, and $\|g_1\|_{L^{p(\cdot)}[T,\infty)} \to 0$ as $T \to \infty$, we have

$$d(HX) \le b_1 M \cdot (2r)^{(p/q)\pm} \cdot d(X). \tag{4.7}$$

Combining (4.6) and (4.7), we have

$$\mu(HX) \le b_1 M \cdot (2r)^{(p/q)\pm} \cdot \mu(X).$$

From the above inequality and by recalling all the above established properties, we can apply Theorem 2.5, which finishes the proof. □

5 Particular cases and examples

Let us recall some examples and particular cases of our outcomes that were covered and studied in earlier publications.

Example The tools of the weak MNC related to the fixed point theorem for contractions in the space $L^1(0,1)$ were utilized for studying the existence of integrable solutions of the Hammerstein and Urysohn integral equations (cf. [2])

$$x(t) = g(t) + \int_0^1 k(t,s)f\big(s,x(s)\big)\,ds,$$
$$x(t) = g(t) + \int_0^1 u\big(t,s,x(s)\big)\,ds.$$

Example The authors in [28] constructed a new MNC in the space $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ of all real functions locally integrable on $\mathbb{R}^+$, and they utilized that method together with a family of MNCs to investigate the existence of solutions of the nonlinear Volterra integral equation

$$x(s) = f\left(s, \int_0^s u\big(s,t,x(t)\big)\,dt\right).$$

Example The authors in [1] examined a set of nonlinear functional integral equations using a new MNC on $L_p(\mathbb{R}^N)(1 \leq p < \infty)$ together with Darbo's fixed point theorem

$$x(s) = g\bigl(s, x(s)\bigr) + \int_{\mathbb{R}^N} k(s,t)(Qx)(t)\,dt,$$

where Q is an operator mapping the space $L_p(\mathbb{R}^N)$ into itself.

Example The Urysohn integral equations were examined in Orlicz spaces $L_\varphi(I)$ in [29, 30]

$$x(t) = g(t) + \int_I u\bigl(t, s, x(s)\bigr)\,ds, \quad t \in I.$$

The case of Hammerstein integral equations were also discussed in Orlicz spaces in [33].

Example The existence and uniqueness of the solutions of Volterra integral equations with Carathéodory functions having diverse growth behaviors

$$x(s) = g\bigl(s, x(s)\bigr) + \lambda \int_a^b u\bigl(s, t, x(t)\bigr)\,dt, \quad s \in (a,b), \lambda \in \mathbb{R}$$

were studied in [8] in $L^{p(\cdot)}(a,b)$ by using degree theory and fixed point approach.

Example In [20], the authors presented a new MNC in the Sobolev spaces $W^{k,1}(I)$ and used it to investigate the existence of solutions of the integral-differential equation

$$x(t) = p(t) + q(t)x(t) + \int_I k(t,s) g\left(t, x(s), \frac{\partial u}{\partial t_1}(s), \dots, \frac{\partial u}{\partial t_n}(s), Tu(s)\right) ds.$$

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