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# RESEARCH

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# Well-posed fixed point results and data dependence problems in controlled metric spaces

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# Abstract

The present research is aimed to analyze the existence of strict fixed points (SFPs) and fixed points of multivalued generalized contractions on the platform of controlled metric spaces (CMSs). Wardowski-type multivalued nonlinear operators have been introduced employing auxiliary functions, modifying a new contractive requirement form. Well-posedness of obtained fixed point results is also established. Moreover, data dependence result for fixed points is provided. Some supporting examples are also available for better perception. Many existing results in the literature are particular cases of the results established.

**Keywords:** Strict fixed points (SFPs); Controlled metric spaces (CMSs); Well-posedness; Data dependence result; Auxiliary functions

# 1 Introduction

The fixed point theory has significant applications in numerous branches of pure and applied mathematics, as it offers many considerable tools to find fixed points. The theory has remarkable implications since it provides a criterion for the existence of solutions for many differential and integral equations. The famous Banach contraction principle (BCP) [4] was established on metric spaces by Stephan Banach in 1922. Two directions for constructing new fixed point results use a more generalized space or modifying the contraction inequality. BCP is extended and modified in many directions in numerous ways. For example, the authors of [2, 9] considered Kannan-type contractions to prove certain fixed point results, and in [3, 17] the underlined space is changed.

It is worth mentioning that many authors used F-contractions to extend many existing results. In 2015, an F-contraction was extended in terms of nonlinear F-contractions by Klim and Wardowski [11]. The authors extended the notion of F-contractive mappings to the case of nonlinear F-contractions and proved a fixed point theorem via the dynamic processes. Following this, Wardowski [19] introduced nonlinear F-contractions by omitting one of the conditions on the F-mappings. In another paper by Wardowski [20], we can find some theorems concerning the existence of fixed points of nonlinear F-contractions and the sum of mappings of this type with a compact operator.

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In 1989, the concept of a *b*-metric space (BMS) was given by Bakhtin [3]. It was further incorporated by Czerwick [8] to develop certain fixed point results endowed by this space. Kamran et al. [17] paved a new pathway using a function  $p: \xi \times \xi \longrightarrow [1, \infty)$  and weakened the triangle inequality of a *b*-metric space. In this perspective, Mlaiki et al. [12] made another advancement by generalizing the notion of an extended *b*-metric space and declaring it a controlled metric-type space.

A data dependence problem is to estimate the distance between the sets of fixed points of two mappings. This idea is only meaningful if we are sure that there are nonempty fixed point sets of these two operators. The data dependence problem mostly deals with set-valued mappings since multivalued mappings often have larger fixed point sets than single-valued mappings. In 2021, Iqbal et al. [10] discussed data dependence, the existence of fixed points, strict fixed points, and the well-posedness of some multivalued generalized contractions in the setting of complete metric spaces using auxiliary functions. In the present paper, we extend the results of Iqbal et al. [10] by utilizing the controlled metric platform.

## 2 Preliminaries

This section is devoted to refreshing some of the crucial concepts. Let  $(\xi, d)$  be a metric space (MS), and let  $P(\xi)$  contain all subsets of  $\xi$ . We denote by  $CL(\xi)$ ,  $CB(\xi)$ , and  $K(\xi)$  the sets of nonempty closed subsets of  $\xi$ , nonempty closed bounded subsets of  $\xi$ , and nonempty compact subsets of  $\xi$ , respectively.

Let  $\Omega: \xi \longrightarrow P(\xi)$  be a multivalued mapping (MVP). An element  $\varrho \in \xi$  such that  $\varrho \in \Omega \varrho$  is called a fixed point of  $\Omega$ . The set of all fixed points of  $\Omega$  is denoted by Fix  $\Omega$ . An element  $\bar{\varrho} \in \xi$  such that  $\{\bar{\varrho}\} = \Omega \varrho$  is called a strict fixed point of  $\Omega$ . The set of strict fixed points is denoted S Fix  $\Omega$ .

**Definition 2.1** [12] Consider a nonempty set  $\xi$  and a function  $f: \xi \times \xi \longrightarrow [1, \infty)$ . The mapping  $d: \xi \times \xi \longrightarrow [0, \infty)$  is said to be a CMS if for all  $\varrho_1, \varrho_2, \varrho_3 \in \xi$ ,

- (i)  $d(\varrho_1, \varrho_2) = 0 \Leftrightarrow \varrho_1 = \varrho_2;$
- (ii)  $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1);$
- (*iii*)  $d(\varrho_1, \varrho_2) \leq \mathfrak{f}(\varrho_1, \varrho_3)d(\varrho_1, \varrho_3) + \mathfrak{f}(\varrho_3, \varrho_2)d(\varrho_3, \varrho_2).$

The pair  $(\xi, d, f)$  is called a CMS.

Berinde and Pacurar [5] defined the Hausdorff distance as follows. Let  $X, Y \in CB(\xi)$ . The mapping  $H : CB(\xi) \times CB(\xi) \longrightarrow [0, \infty)$  defined by

$$H(\mathsf{X},\mathsf{Y}) = \max\left\{\sup_{\varrho\in\mathsf{X}} D(\varrho,\mathsf{Y}), \sup_{\bar{\varrho}\in\mathsf{Y}} D(\bar{\varrho},\mathsf{X})\right\}$$

is called a Pompei–Hausdorff metric space, where  $D(\varrho, Y) = \inf\{d(\varrho, \bar{\varrho}) : \bar{\varrho} \in Y\}$ .

Following definition is due to Wardowski [19]. Let  $F: (0, \infty) \longrightarrow \mathbb{R}$  satisfy the following conditions:

- ( $\mathbb{F}$ 1) *F* is strictly increasing;
- (F2) For all sequences  $\{\Psi_{\mathfrak{s}}\} \subseteq (0, \infty)$ ,  $\lim_{\mathfrak{s} \to \infty} \Psi_{\mathfrak{s}} = 0$  iff  $\lim_{\mathfrak{s} \to \infty} F(\Psi_{\mathfrak{s}}) = -\infty$ ;
- (F3) There exists  $k \in (0, 1)$  such that  $\lim_{\Psi \to 0^+} \Psi^k F(\Psi) = 0$ .

Let us denote by  $\Delta(\mathbb{F})$  the set of all functions *F* that satisfy (F1), (F2), and (F3), Also, assume that

$$\Delta(O*) = \{F \in \Delta(\mathbb{F}) : (\mathbb{F}4) \text{ holds for } F\},\$$

where

(F4)  $F(\inf X) = \inf F(X)$  for all  $X \subset (0, \infty)$  such that  $\inf X > 0$ .

Turinici [18] replaced ( $\mathbb{F}2$ ) by

 $(\mathbb{F}2') \lim_{t \to 0^+} F(t) = -\infty.$ 

Denote by  $\Delta(0*)$  the set of functions *F* that satisfy (F1), (F2'), (F3), and (F4), A mapping  $\Omega: \xi \longrightarrow CB(\xi)$  is called a multivalued *F*-contraction if there exist  $\upsilon > 0$  and  $F \in \Delta(\mathbb{F})$  such that for all  $\varrho, \bar{\varrho} \in \xi, H(\Omega\varrho, \Omega\bar{\varrho}) > 0$  implies  $\upsilon + F(H(\Omega\varrho, \Omega\bar{\varrho})) \leq F(d(\varrho, \bar{\varrho}))$  [1].

**Definition 2.2** [13] A mapping  $\Omega : \xi \longrightarrow \xi$  is called an  $(\alpha, F)$ -contraction (or a nonlinear *F*-contraction) if there exist  $F \in \Delta(\mathbb{F})$  and a function  $\Xi : (0, \infty) \longrightarrow (0, \infty)$  that fulfill the following conditions:

- (*H*<sub>1</sub>)  $\liminf_{s \to \Psi^+} \Xi(s) > 0$  for all  $\Psi > 0$ ;
- $(H_2) \ \Xi(d(\varrho,\bar{\varrho})) + F(d(\Omega\varrho,\Omega\bar{\varrho})) \le F(d(\varrho,\bar{\varrho})) \text{ for all } \varrho, \bar{\varrho} \in \xi \text{ such that } \Omega\varrho \neq \Omega\bar{\varrho}.$

**Definition 2.3** [10] By  $\Phi$  we denote the set of functions  $\chi : (0, \infty) \longrightarrow (0, \infty)$  such that

 $\lim_{x \to \infty} \inf \chi(s) > 0 \quad \text{for all } \Psi \ge 0.$ 

# 3 Main results

The following definitions are indispensable before proving the main result.

**Definition 3.1** By  $\mathcal{P}$  we denote the set of all continuous mappings  $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$  that satisfy the following conditions:

- (i)  $\rho(1, 1, 1, \zeta + \eta, 0) \in (0, 1]$  for  $\zeta, \eta \ge 1$ ,
- (ii)  $\rho$  is subhomogeneous, that is, for all  $(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) \in (0, \infty]^5$  and  $\lambda \ge 0$ , we have  $\rho(\lambda \varrho_1, \lambda \varrho_2, \lambda \varrho_3, \lambda \varrho_4, \lambda \varrho_5) \le \lambda \rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5)$ ,
- (iii)  $\rho$  is a nondecreasing function, i.e., for  $\rho_i, \bar{\rho}_i \in \mathbb{R}^+ \rho_i$  such that  $\leq \bar{\rho}_i, i = 1, 2, 3, 4, 5$ , we have  $\rho(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \leq \rho(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, \bar{\rho}_5)$ . If  $\rho_i, \bar{\rho}_i \in \mathbb{R}^+$  are such that  $\rho_i < \bar{\rho}_i$  for i = 1, 2, 3, 4, then

 $\rho(\varrho_1,\varrho_2,\varrho_3,\varrho_4,0)<\rho(\bar\varrho_1,\bar\varrho_2,\bar\varrho_3,\bar\varrho_4,0),$ 

and

$$\rho(\varrho_1, \varrho_2, \varrho_3, 0, \varrho_4) < \rho(\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, 0, \bar{\varrho}_4).$$

Also, define  $\mathbb{P} = \{ \rho \in \mathcal{P} : \rho(1, 0, 0, \zeta, \eta) \in (0, 1] \}$ . Note that  $\mathbb{P} \subseteq \mathcal{P}$ .

# Example 3.1

- 1. Define  $\rho_1 : [0, \infty)^5 \longrightarrow [0, \infty)$  by  $\rho_1(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = g \min\{\varrho_1, \frac{1}{2}(\varrho_2, \varrho_3), \frac{1}{2}(\varrho_4, \varrho_5)\},$ where  $g \in (0, 1)$ . Then  $\rho_1 \in \mathcal{P}$ , as  $\rho_1(1, 0, 0, \zeta, \eta) = 0 \notin (0, 1]$ . Hence  $\rho_1 \notin \mathbb{P}$ .
- 2. Define  $\rho_2: [0,\infty)^5 \longrightarrow [0,\infty)$  by  $\rho_2(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \frac{\varrho_1}{2} + \frac{\varrho_2 + \varrho_3}{4}$ . Then  $\rho_2 \in \mathbb{P}$ .

3. Define  $\rho_3 : [0, \infty)^5 \longrightarrow [0, \infty)$  by  $\rho_3(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = g \min\{\frac{1}{2}(\varrho_1 + \varrho_3), \frac{1}{2}(\varrho_4 + \varrho_5)\},$ where  $g \in (0, 1)$ . Then  $\rho_3 \in \mathbb{P}$ .

**Lemma 3.1** If  $\rho \in \mathcal{P}$ ,  $\gamma, \delta \in [0, \infty)$ , and  $\zeta, \eta \in \mathbb{R}$  be such that  $\zeta, \eta \geq 1$  and

$$\begin{split} \gamma &\leq \max \Big\{ \rho(\delta, \delta, \gamma, \eta \delta + \zeta \gamma, 0), \rho(\delta, \delta, \gamma, 0, \eta \delta + \zeta \gamma), \\ \rho(\delta, \gamma, \delta, \eta \delta + \zeta \gamma, 0), \rho(\delta, \gamma, \delta, 0, \eta \delta + \zeta \gamma) \Big\}. \end{split}$$

Then  $\gamma \leq \delta$ .

Proof Without loss of generality, we can assume that

$$\gamma \le \rho(\delta, \delta, \gamma, \eta \delta + \zeta \gamma, 0). \tag{1}$$

On the contrary, suppose that  $\delta < \gamma$ . Now consider

$$\begin{split} \rho(\delta, \delta, \gamma, \eta \delta + \zeta \gamma, 0) &< \rho(\gamma, \gamma, \gamma, \eta \gamma + \zeta \gamma, 0) \\ &\leq \gamma \rho(1, 1, 1, \eta + \zeta, 0) \\ &\leq \gamma (1), \\ \rho(\delta, \delta, \gamma, \eta \delta + \zeta \gamma, 0) < \gamma, \end{split}$$

which is a contradiction to (1). Hence our supposition is wrong, so  $\gamma \leq \delta$ .

**Definition 3.2** ( $\chi F$ -contraction) Let  $F_1$ ,  $F_2$  be real-valued functions on  $(0, \infty)$ , and let  $\rho \in \mathcal{P}$  and  $\chi \in \Phi$ . The mapping  $\Omega : \xi \longrightarrow CB(\xi)$  is called a  $\chi F$ -contraction if (Ni)  $F_1(\mathfrak{c}) \leq F_2(\mathfrak{c})$  for all  $\mathfrak{c} > 0$ , (Nii)  $H(\Omega_{\mathcal{Q}}, \Omega_{\mathcal{Q}}) > 0$  implies

$$\chi(d(\varrho,\bar{\varrho})) + F_2(H(\Omega\varrho,\Omega\bar{\varrho})) \leq F_1\{\rho(d(\varrho,\bar{\varrho}),D(\varrho,\Omega\varrho),D(\bar{\varrho},\Omega\bar{\varrho}),D(\varrho,\Omega\bar{\varrho}),D(\bar{\varrho},\Omega\varrho)\}$$

for all  $\varrho, \bar{\varrho} \in \xi$ .

**Theorem 3.1** Suppose that  $(\xi, d, \mathfrak{f})$  is a complete CMS. Let  $\Omega : \xi \longrightarrow K(\xi)$  be a  $\chi F$ contraction. Suppose that  $F_1$  is nondecreasing and  $F_2$  satisfies conditions ( $\mathbb{F}2'$ ) and ( $\mathbb{F}3$ ). For  $\varrho_0 \in \xi$ , define the Picard sequence { $\varrho_{\mathfrak{s}} = \Omega^{\mathfrak{s}} \varrho_0$ } so that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\mathfrak{f}(\varrho_{i+1},\varrho_{i+2})\mathfrak{f}(\varrho_{i+1},\varrho_m)}{\mathfrak{f}(\varrho_{i},\varrho_{i+1})} < 1.$$

$$\tag{2}$$

Also, suppose

$$\lim_{s \to \infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \le 1 \quad \text{for all } \varrho \in \xi.$$
(3)

*Then* Fix  $\Omega$  *is nonempty.* 

*Proof* Let  $\varrho_0 \in \xi$  and  $\varrho_1 \in \Omega \varrho_0$ . If  $\varrho_1 \in \Omega \varrho_1$  then  $\varrho_1 \in \text{Fix }\Omega$ . Suppose  $\varrho_1 \notin \Omega \varrho_1$ , which implies  $D(\varrho_1, \Omega \varrho_1) > 0$ , and, consequently,  $H(\Omega \varrho_0, \Omega \varrho_1) > 0$ . As  $\Omega \varrho_1$  is compact, there exists  $\varrho_2 \in \Omega \varrho_1$  such that  $d(\varrho_1, \varrho_2) = D(\varrho_1, \Omega \varrho_1)$ . Now

$$F_1(d(\varrho_1, \varrho_2)) = F_1(D(\varrho_1, \Omega \varrho_1)) \leq F_1(H(\Omega \varrho_0, \Omega \varrho_1)) \leq F_2(H(\Omega \varrho_0, \Omega \varrho_1))$$
  
$$\leq F_1\{\rho(d(\varrho_0, \varrho_1), D(\varrho_0, \Omega \varrho_0), D(\varrho_1, \Omega \varrho_1), D(\varrho_0, \Omega \varrho_1), D(\varrho_1, \Omega \varrho_0)\}$$
  
$$- \chi(d(\varrho_0, \varrho_1))$$
  
$$< F_1\{\rho(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), d(\varrho_1, \varrho_1))\}.$$

As  $F_1$  is nondecreasing, we have

$$\begin{aligned} d(\varrho_1, \varrho_2) < \rho \Big( d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0 \Big) \\ & \leq \rho \Big( d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), \mathfrak{f}(\varrho_0, \varrho_1) \Big) d(\varrho_0, \varrho_1) + \mathfrak{f}(\varrho_1, \varrho_2) d(\varrho_1, \varrho_2), 0 ). \end{aligned}$$

By Lemma 3.1

$$d(\varrho_1, \varrho_2) < d(\varrho_0, \varrho_1).$$

Similarly, we get  $\rho_3 \in \Omega \rho_2$  such that  $d(\rho_2, \rho_3) = D(\rho_2, \Omega \rho_2)$  with  $D(\rho_2, \Omega \rho_2) > 0$ , and we have

$$d(\varrho_2, \varrho_3) < d(\varrho_1, \varrho_2).$$

By induction we get a sequence  $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathbb{N}}\subset\xi$  such that  $\varrho_{\mathfrak{s}+1}\in\Omega\varrho_{\mathfrak{s}}$  satisfies  $d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) = D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}})$  with  $D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}) > 0$  and

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) \text{ for all } \mathfrak{s} \in \mathbb{N}.$$

So  $\{d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\}_{\mathfrak{s}\in\mathbb{N}}$  is a decreasing sequence of real numbers. Now

$$\begin{split} \chi \left( d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) \right) + F_2 \Big( H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1}) \Big) \\ &\leq F_1 \Big\{ \rho \Big( d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}), D(\varrho_{\mathfrak{s}+1}, \Omega \varrho_{\mathfrak{s}+1}), D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1}), D(\varrho_{\mathfrak{s}+1}, \Omega \varrho_{\mathfrak{s}}) \Big) \Big\} \\ &= F_1 \Big\{ \rho \Big( d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) \Big), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+2}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+1}) \Big) \Big\} \\ &\leq F_1 \Big\{ \rho \Big( d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}) \\ &d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), 0 \Big) \Big\} \\ &< F_1 \Big\{ \rho \Big( d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}) \\ &d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), 0 \Big) \Big\} \\ &\leq F_1 \Big\{ d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), \rho \Big( 1, 1, 1, \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), 0 \Big) \Big\} \\ &\leq F_1 \Big( d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) \Big) \\ &= F_1 \Big( D(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s})} \Big) \end{split}$$

$$\leq F_1 \big( H(\Omega \varrho_{\mathfrak{s}-1}, \Omega \varrho_{\mathfrak{s}}) \big) \\ \leq F_2 \big( H(\Omega \varrho_{\mathfrak{s}-1}, \Omega \varrho_{\mathfrak{s}}) \big).$$

Hence, for all  $\mathfrak{s} \in \mathbb{N}$ , we have

$$F_2(H(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1})) < F_2(H((\Omega \varrho_{\mathfrak{s}-1}, \Omega \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})).$$

$$\tag{4}$$

As  $\chi \in \Phi$ , there exist h > 0 and  $\mathfrak{s}_0 \in \mathbb{N}$  such that  $\chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) > h$  for all  $\mathfrak{s} \ge \mathfrak{s}_0$ . Now from (4) we have

$$F_{2}(H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1})) < F_{2}(H((\Omega \varrho_{\mathfrak{s}-1}, \Omega \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})))$$

$$< F_{2}(H(\Omega \varrho_{\mathfrak{s}-2}, \Omega \varrho_{\mathfrak{s}-1})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}))$$

$$\vdots$$

$$< F_{2}(H(\Omega \varrho_{0}, \Omega \varrho_{1})) - \sum_{i=1}^{\mathfrak{s}} \chi(d(\varrho_{i}, \varrho_{i+1})))$$

$$= F_{2}(H(\Omega \varrho_{0}, \Omega \varrho_{1})) - \sum_{i=1}^{\mathfrak{s}_{0}-1} \chi(d(\varrho_{i}, \varrho_{i+1})) - \sum_{i=\mathfrak{s}_{0}}^{\mathfrak{s}} \chi(d(\varrho_{i}, \varrho_{i+1})))$$

$$< F_{2}(H(\Omega \varrho_{0}, \Omega \varrho_{1})) - (\mathfrak{s} - \mathfrak{s}_{0})h, \quad \mathfrak{s} \geq \mathfrak{s}_{0}$$

$$\implies F_{2}(H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1})) < F_{2}(H((\Omega \varrho_{0}, \Omega \varrho_{1})) - (\mathfrak{s} - \mathfrak{s}_{0})h \quad \text{for all } \mathfrak{s} \geq \mathfrak{s}_{0}.$$
(5)

Taking the limit in (5) as  $\mathfrak{s} \to \infty$ , we get  $F_2(H(\Omega_{\mathcal{Q}_{\mathfrak{s}}}, \Omega_{\mathcal{Q}_{\mathfrak{s}+1}})) \to -\infty$  and then by ( $\mathbb{F}2'$ ) we have

$$\underset{\mathfrak{s}\longrightarrow\infty}{\lim}H(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1})=0,$$

which further implies that

$$\lim_{\mathfrak{s} \to \infty} d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = \lim_{\mathfrak{s} \to \infty} D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}) \le \lim_{\mathfrak{s} \to \infty} H(\Omega \varrho_{\mathfrak{s}-1}, \Omega \varrho_{\mathfrak{s}}) = 0.$$
(6)

Now by (*F*<sub>3</sub>) there exists  $k \in (0, 1)$  such that

$$\lim_{\mathfrak{s}\to\infty} \left( H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}) \right)^k F_2 \left( H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}) \right) = 0.$$
<sup>(7)</sup>

Then from (5), for all  $\mathfrak{s} \geq \mathfrak{s}_0$ , we have

$$\begin{split} \left(H(\Omega_{\mathcal{Q}_{5}},\Omega_{\mathcal{Q}_{5}+1})\right)^{k} F_{2}\left(H(\Omega_{\mathcal{Q}_{5}},\Omega_{\mathcal{Q}_{5}+1})\right) &- \left(H(\Omega_{\mathcal{Q}_{5}},\Omega_{\mathcal{Q}_{5}+1})\right)^{k} F_{2}\left(H(\Omega_{\mathcal{Q}_{0}},\Omega_{\mathcal{Q}_{1}})\right) \\ &\leq \left(H(\Omega_{\mathcal{Q}_{5}},\Omega_{\mathcal{Q}_{5}+1})\right)^{k} \left(F_{2}\left(H(\Omega_{\mathcal{Q}_{0}},\Omega_{\mathcal{Q}_{1}})\right) - (\mathfrak{s}-\mathfrak{s}_{0})h\right) \\ &- \left(H(\Omega_{\mathcal{Q}_{5}},\Omega_{\mathcal{Q}_{5}+1})\right)^{k} F_{2}\left(H(\Omega_{\mathcal{Q}_{0}},\Omega_{\mathcal{Q}_{1}})\right) \\ &= -\left(H(\Omega_{\mathcal{Q}_{5}},\Omega_{\mathcal{Q}_{5}+1})\right)^{k} (\mathfrak{s}-\mathfrak{s}_{0})h \\ &\leq 0. \end{split}$$

Taking the limit as  $\mathfrak{s} \longrightarrow \infty$  and using (6) and (7), we get that

$$0 \leq \lim_{s \to \infty} \mathfrak{s} \left( H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1}) \right)^k \leq 0$$
  
implies 
$$\lim_{s \to \infty} \mathfrak{s} \left( H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1}) \right)^k = 0.$$

By the above equation there exists  $\mathfrak{s}_1 \in \mathbb{N}$  such that  $\mathfrak{s}(H(\Omega \rho_{\mathfrak{s}}, \Omega \rho_{\mathfrak{s}+1}))^k \leq 1$  for all  $\mathfrak{s} \geq \mathfrak{s}_1$ . Thus for all  $\mathfrak{s} \geq \mathfrak{s}_1$ , we have  $H(\rho_{\mathfrak{s}}, \Omega \rho_{\mathfrak{s}+1}) \leq \frac{1}{\mathfrak{s}^{\frac{1}{k}}}$ .

Now

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}) \leq H(\Omega \varrho_{\mathfrak{s}-1}, \Omega \varrho_{\mathfrak{s}}) \leq \frac{1}{\mathfrak{s}^{\frac{1}{k}}} \quad \text{for all } \mathfrak{s} \geq \mathfrak{s}_1.$$

To prove that  $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathbb{N}}$  is a Cauchy sequencem consider  $\tau,\mathfrak{s}\in\mathbb{N}$  such that  $\tau>\mathfrak{s}>\mathfrak{s}_1$ . Then

$$\begin{aligned} d(\varrho_{\mathfrak{s}},\varrho_{\tau}) &\leq \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\tau})d(\varrho_{\mathfrak{s}+1},\varrho_{\tau}) \\ &\leq \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\mathfrak{s}+2})d(\varrho_{\mathfrak{s}+1},\varrho_{\mathfrak{s}+2}) \\ &+ \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+2},\varrho_{\tau})d(\varrho_{\mathfrak{s}+2},\varrho_{\tau}) \\ &\leq \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\mathfrak{s}+2})d(\varrho_{\mathfrak{s}+1},\varrho_{\mathfrak{s}+2}) \\ &+ \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+2},\varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+2},\varrho_{\mathfrak{s}+3})d(\varrho_{\mathfrak{s}+2},\varrho_{\mathfrak{s}+3}) \\ &+ \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+2},\varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+3},\varrho_{\tau})d(\varrho_{\mathfrak{s}+3},\varrho_{\tau}) \end{aligned}$$

$$\begin{split} & :: \\ \leq f(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-2} \left( \prod_{j=\mathfrak{s}+1}^{i} f(\varrho_{j}, \varrho_{\tau}) \right) f(\varrho_{i}, \varrho_{i+1}) d(\varrho_{i}, \varrho_{i+1}) \\ & + \left( \prod_{j=\mathfrak{s}+1}^{\tau-1} f(\varrho_{i}, \varrho_{\tau}) \right) d(\varrho_{\tau-1}, \varrho_{\tau}) \\ \leq f(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-2} \left( \prod_{j=\mathfrak{s}+1}^{i} f(\varrho_{j}, \varrho_{\tau}) \right) f(\varrho_{i}, \varrho_{i+1}) d(\varrho_{i}, \varrho_{i+1}) \\ & + \left( \prod_{j=\mathfrak{s}+1}^{\tau-1} f(\varrho_{i}, \varrho_{\tau}) \right) f(\varrho_{\tau-1}, \varrho_{\tau}) d(\varrho_{\tau-1}, \varrho_{\tau}) \\ = f(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-1} \left( \prod_{j=\mathfrak{s}+1}^{i} f(\varrho_{j}, \varrho_{\tau}) \right) f(\varrho_{i}, \varrho_{i+1}) d(\varrho_{i}, \varrho_{i+1}) \\ \leq f(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-1} \left( \prod_{j=0}^{i} f(\varrho_{j}, \varrho_{\tau}) \right) f(\varrho_{i}, \varrho_{i+1}) d(\varrho_{i}, \varrho_{i+1}). \end{split}$$

Therefore

$$d(\varrho_{\mathfrak{s}},\varrho_{\tau}) \leq \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-1} \left(\prod_{j=0}^{i} \mathfrak{f}(\varrho_{j},\varrho_{\tau})\right) \mathfrak{f}(\varrho_{i},\varrho_{i+1}) \frac{1}{i^{\frac{1}{k}}}.$$
(8)

Now

$$\sum_{i=\mathfrak{s}+1}^{\tau-1} \left( \prod_{j=0}^{i} \mathfrak{f}(\varrho_{j}, \varrho_{\tau}) \right) \mathfrak{f}(\varrho_{i}, \varrho_{i+1}) \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=\mathfrak{s}+1}^{\infty} \frac{1}{i^{\frac{1}{k}}} \left( \prod_{j=0}^{i} \mathfrak{f}(\varrho_{j}, \varrho_{\tau}) \right) \mathfrak{f}(\varrho_{i}, \varrho_{i+1})$$
$$= \sum_{i=\mathfrak{s}+1}^{\infty} U_{i} V_{i},$$

where  $U_i = \frac{1}{i\frac{1}{k}}$  and  $V_i = (\prod_{j=0}^i \mathfrak{f}(\varrho_j, \varrho_\tau))\mathfrak{f}(\varrho_i, \varrho_{i+1})$ . Since  $\frac{1}{k} > 0$ , the series  $\sum_{i=\mathfrak{s}+1}^{\infty} (\frac{1}{i\frac{1}{k}})$  converges. Also,  $\{V_i\}_i$  is increasing and bounded above, so  $\lim_{i\to\infty} \{V_i\}_i$  (which is nonzero) exists. Hence  $\{\lim_{i\to\infty} U_iV_i\}_{\mathfrak{s}}$  converges.

Consider the partial sums  $S_q = \sum_{i=0}^q (\prod_{j=0}^i \mathfrak{f}(\varrho_j, \varrho_\tau)) \mathfrak{f}(\varrho_i, \varrho_{i+1}) \frac{1}{i^{\frac{1}{k}}}$ . From (8) we have

$$d(\varrho_{\mathfrak{s}}, \varrho_{\tau}) \leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + (S_{\tau-1} - S_{\mathfrak{s}}).$$

$$\tag{9}$$

By using the ratio test and condition (2) we get that  $\lim_{s\to\infty} \{S_s\}$  exists. By taking in (9) the limit as  $s \to \infty$  we get  $\lim_{s\to\infty} d(\varrho_s, \varrho_\tau) = 0$ . Therefore  $\{\varrho_s\}$  is a Cauchy sequence, and the completeness of  $\xi$  implies that there exists  $\varrho^* \in \xi$  such that

$$\lim_{\mathfrak{s}\to\infty}\varrho_{\mathfrak{s}}=\varrho^*.$$

Now

$$F_1(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}})) \le F_2(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}})) \le \chi(d(\varrho, \bar{\varrho})) + F_2(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}}))$$
$$\le F_1\{\rho(d(\varrho, \bar{\varrho}), D(\varrho, \Omega_{\varrho}), D(\bar{\varrho}, \Omega_{\bar{\varrho}}), D(\varrho, \Omega_{\bar{\varrho}}), D(\bar{\varrho}, \Omega_{\bar{\varrho}}))\}.$$

Since  $F_1$  is a nondecreasing function, we obtain that for all  $\rho, \bar{\rho} \in \xi$ ,

 $H(\Omega_{\varrho}, \Omega_{\bar{\varrho}}) \leq \rho \left( d(\varrho, \bar{\varrho}), D(\varrho, \Omega_{\varrho}), D(\bar{\varrho}, \Omega_{\bar{\varrho}}), D(\varrho, \Omega_{\bar{\varrho}}), D(\bar{\varrho}, \Omega_{\varrho}) \right).$ 

To prove that  $\varrho^*$  is a fixed point of  $\xi$ , on the contrary, assume that  $D(\varrho^*, \Omega \varrho^*) > 0$ . Now due to the compactness of  $\Omega \varrho^*$ , there exists  $\varrho \in \Omega \varrho^*$  such that

$$\begin{split} D(\varrho^*, \Omega \varrho^*) &= d(\varrho^*, \varrho) \\ &\leq f(\varrho^*, \varrho_{\mathfrak{s}+1}) d(\varrho^*, \varrho_{\mathfrak{s}+1}) + f(\varrho_{\mathfrak{s}+1}, \varrho) d(\varrho_{\mathfrak{s}+1}, \varrho) \\ &= f(\varrho^*, \varrho_{\mathfrak{s}+1}) d(\varrho^*, \varrho_{\mathfrak{s}+1}) + f(\varrho_{\mathfrak{s}+1}, \varrho) D(\varrho_{\mathfrak{s}+1}, \Omega \varrho^*) \\ &\leq f(\varrho^*, \varrho_{\mathfrak{s}+1}) d(\varrho^*, \varrho_{\mathfrak{s}+1}) + f(\varrho_{\mathfrak{s}+1}, \varrho) H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho^*) \\ &\leq f(\varrho^*, \varrho_{\mathfrak{s}+1}) d(\varrho^*, \varrho_{\mathfrak{s}+1}) + f(\varrho_{\mathfrak{s}+1}, \varrho) \rho(d(\varrho_{\mathfrak{s}}, \varrho^*), D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}), D(\varrho^*, \Omega \varrho^*), \\ D(\varrho_{\mathfrak{s}}, \Omega \varrho^*), D(\varrho^*, \Omega \varrho_{\mathfrak{s}})) \\ &\leq f(\varrho^*, \varrho_{\mathfrak{s}+1}) d(\varrho^*, \varrho_{\mathfrak{s}+1}) + f(\varrho_{\mathfrak{s}+1}, \varrho) \rho(d(\varrho_{\mathfrak{s}}, \varrho^*), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), D(\varrho^*, \Omega \varrho^*), \\ f(\varrho_{\mathfrak{s}}, \varrho^*) d(\varrho_{\mathfrak{s}}, \varrho^*) + f(\varrho^*, \varrho) D(\varrho^*, \Omega \varrho^*), d(\varrho^*, \varrho_{\mathfrak{s}+1})). \end{split}$$

Taking the limit as  $\mathfrak{s} \to \infty$  in the above inequality and using (3), we have

$$D(\varrho^*, \Omega \varrho^*) \leq (1)\rho(0, 0, D(\varrho^*, \Omega \varrho^*), 0 + \mathfrak{f}(\varrho^*, \bar{\varrho})D(\varrho^*, \Omega \varrho^*), 0).$$

Using Lemma 3.1, we get  $D(\varrho^*, \Omega \varrho^*) \leq 0$ , and hence  $D(\varrho^*, \Omega \varrho^*) = 0$ . As  $\Omega \varrho^*$  is closed, we have  $\varrho^* \in \Omega \varrho^*$ . Hence Fix  $\Omega$  is nonempty.

*Example* 3.2 Let  $\xi = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ . Define  $d : \xi \times \xi \longrightarrow \mathbb{R}^+$  and  $f : \xi \times \xi \longrightarrow [1, \infty)$  by  $d(\varrho_1, \varrho_2) = (\varrho_1 - \varrho_2)^2$  and

$$\mathfrak{f}(\varrho_1, \varrho_2) = \begin{cases} 1 & \text{if } \varrho_1 = \varrho_2 = 0, \\ \frac{1}{(\varrho_1 + \varrho_2)^4} & \text{if } \varrho_1 \neq 0 \text{ or } \varrho_2 \neq 0. \end{cases}$$

Then  $(\xi, d, \mathfrak{f})$  is a complete CMS.

Define  $F_1, F_2: (0, \infty) \longrightarrow \mathbb{R}$  by

$$F_1(\mathfrak{u}) = \begin{cases} \frac{-1}{\mathfrak{u}} & \text{if } \mathfrak{u} \in (0, 1), \\ \mathfrak{u} & \text{if } \mathfrak{u} \in [1, \infty), \end{cases}$$

and  $F_2(u) = \ln(u) + u$  for  $u \in (0, \infty)$ . Then  $F_1$  is nondecreasing,  $F_2$  satisfies (F2') and (F3), and  $F_1(u) \le F_2(u)$  for all u > 0. Now define  $\Omega : \xi \longrightarrow K(\xi)$ ,  $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ , and  $\chi : (0, \infty) \longrightarrow (0, \infty)$  by

$$\Omega \varrho = \begin{cases} \{0\} & \text{if } \varrho = 0, \\ \{0, \frac{1}{2}\} & \text{if } \varrho \neq 0, \end{cases}$$

 $\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \frac{\varrho_1}{2} + 28\varrho_5$ , and  $\chi(t) = \frac{1}{t}$ ,  $t \in (0, \infty)$ . Then  $\rho \in \mathcal{P}$  and  $\chi \in \Phi$ . Since  $H(\Omega \varrho, \Omega \bar{\varrho}) > 0$ , it follows that,

$$\chi\left(d(\varrho,\bar{\varrho})\right) + F_2\left(H(\Omega\varrho,\Omega\bar{\varrho})\right) \le F_1\left\{\rho\left(d(\varrho,\bar{\varrho}),D(\varrho,\Omega\varrho),D(\bar{\varrho},\Omega\bar{\varrho}),D(\varrho,\Omega\bar{\varrho}),D(\bar{\varrho},\Omega\varrho)\right)\right\}.$$

Note that  $\lim_{n\to\infty} f(\rho_n, \rho) \le 1$ . Hence the assumptions of Theorem 3.1 are fulfilled, and Fix  $\Omega = \{0, \frac{1}{2}\}$ .

**Theorem 3.2** Let  $(\xi, d, \mathfrak{f})$  be a complete CMS. Let  $\Omega : \xi \longrightarrow K(\xi)$  be an MVM, and let  $F_1$ ,  $F_2$  be functions satisfying  $\chi F$ -contraction. Suppose that  $F_1$  is nondecreasing and  $F_2$  satisfies condition ( $\mathbb{F}2'$ ). Also, suppose  $\lim_{k\to\infty} \mathfrak{f}(\varrho_{\tau_k}, \varrho_{s_k}) \leq 1$ . Then Fix  $\Omega$  is nonempty.

*Proof* Let  $\varrho_0 \in \xi$  and  $\varrho_1 \in \Omega \varrho_0$ . As in proof of Theorem 3.1, let  $\{\varrho_s\} \subset \xi$  be a sequence such that  $\varrho_{s+1} \in \Omega \varrho_s$ . It satisfies  $d(\varrho_s, \varrho_{s+1}) = D(\varrho_s, \Omega \varrho_s)$  with  $D(\varrho_s, \Omega \varrho_s) > 0$  and

 $d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) \quad \text{for all } \mathfrak{s} \in \mathbb{N},$ (10)

$$F_2(H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1})) < F_2(H(\Omega \varrho_0, \Omega \varrho_1)) - (\mathfrak{s} - \mathfrak{s}_0)h \quad \text{for all } \mathfrak{s} \ge \mathfrak{s}_0.$$
<sup>(11)</sup>

Taking the limit as  $\mathfrak{s} \to \infty$  in (11), we get  $F_2(H(\Omega \varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}+1})) \to -\infty$ , and by ( $\mathbb{F}^2$ )

$$\lim_{\mathfrak{s}\to\infty} H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}) = 0, \tag{12}$$

which further implies

$$\lim_{\mathfrak{s}\to\infty} d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) = \lim_{\mathfrak{s}\to\infty} D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}) \leq \lim_{\mathfrak{s}\to\infty} H(\varrho_{\mathfrak{s}-1},\Omega\varrho_{\mathfrak{s}}) = 0.$$

Also, we claim that

$$\lim_{\mathfrak{s},\tau\to\infty} d(\varrho_{\mathfrak{s}},\varrho_{\tau}) = 0.$$
(13)

If not, then there exists  $\delta > 0$  such that for all  $r \ge 0$ , there are  $\tau_k > \mathfrak{s}_k > r$  such that

 $d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}) > \delta.$ 

Moreover, there exists  $r_0 \in \mathbb{N}$  such that

$$\lambda_{r_0} = d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) < \delta \quad \text{for all } \mathfrak{s} \ge r_0.$$

There are two sub sequences  $\{\varrho_{\mathfrak{s}_k}\}$  and  $\{\varrho_{\tau_k}\}$  of  $\{\varrho_{\mathfrak{s}}\}$  such that

$$r_0 \le \mathfrak{s}_k \le \tau_k + 1 \quad \text{and} \quad d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}) > \delta \quad \text{for all } k > 0.$$
 (14)

Note that

$$d(\varrho_{\tau_k-1}, \varrho_{\mathfrak{s}_k}) \le \delta \quad \text{for all } k.$$
(15)

Also,  $\tau_k$  is the minimal index for which (15) is fulfilled.

Note that  $\mathfrak{s}_k + 2 \leq \tau_k$  for all k, because the case  $\mathfrak{s}_k + 1 \leq \mathfrak{s}_k$  is impossible due to equations (14) and (15). This shows that

$$\mathfrak{s}_k + 1 < \tau_k < \tau_k + 1$$
 for all  $k$ .

By the triangle inequality, using (14) and (15), we have

$$\begin{split} \delta < d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) &\leq \mathfrak{f}(\varrho_{\tau_k}, \varrho_{\tau_k-1}) d(\varrho_{\tau_k}, \varrho_{\tau_k-1}) + \mathfrak{f}(\varrho_{\tau_k-1}, \varrho_{\mathfrak{s}_k}) d(\varrho_{\tau_k-1}, \varrho_{\mathfrak{s}_k}) \\ &\leq \mathfrak{f}(\varrho_{\tau_k}, \varrho_{\tau_k-1}) d(\varrho_{\tau_k}, \varrho_{\tau_k-1}) + \delta \mathfrak{f}(\varrho_{\tau_k-1}, \varrho_{\mathfrak{s}_k}). \end{split}$$

Taking the limit as  $k \to \infty$ ,

$$\delta < \lim_{k \to \infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \le 0 + \delta \lim_{k \to \infty} \mathfrak{f}(\varrho_{\tau_k-1}, \varrho_{\mathfrak{s}_k})$$

$$\implies \delta < \lim_{k \to \infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \le \delta \lim_{k \to \infty} \mathfrak{f}(\varrho_{\tau_k-1}, \varrho_{\mathfrak{s}_k}) \le \delta$$

$$\implies \lim_{k \to \infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) = \delta.$$
(16)

Now using 
$$(12)$$
 and  $(16)$ , we get

$$\lim_{k \to \infty} d(\varrho_{\tau_k+1}, \varrho_{\mathfrak{s}_k+1}) = \delta.$$
(17)

Consider

$$\begin{split} \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F_1 \left( d(\varrho_{\tau_k+1}, \varrho_{\mathfrak{s}_k+1}) \right) \\ &= \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F_1 \left( D(\varrho_{\tau_k+1}, \Omega \varrho_{\mathfrak{s}_k}) \right) \\ &\leq \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F_1 \left( H(\Omega \varrho_{\tau_k}, \Omega \varrho_{\mathfrak{s}_k}) \right) \\ &\leq \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F_2 \left( H(\Omega \varrho_{\tau_k}, \Omega \varrho_{\mathfrak{s}_k}) \right) \\ &\leq F_1 \left\{ \rho \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}), D(\varrho_{\tau_k}, \Omega \varrho_{\tau_k}), D(\varrho_{\mathfrak{s}_k}, \Omega \varrho_{\mathfrak{s}_k}), D(\varrho_{\tau_k}, \Omega \varrho_{\mathfrak{s}_k}), D(\varrho_{\mathfrak{s}_k}, \Omega \varrho_{\tau_k}) \right) \right\} \\ &= F_1 \left\{ \rho \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}), d(\varrho_{\tau_k}, \varrho_{\tau_{k+1}}), d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}), d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_{k+1}}), d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_k}) \right. \\ &\left. + f(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}) d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}) f(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}) + f(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}}) d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}}) \right) \right\}. \end{split}$$

As  $F_1$  is continuous, taking the limit as  $k \rightarrow \infty$  and using (16) and (17), we obtain

$$\begin{split} \lim_{k \to \infty} \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F_1(\delta) &\leq F_1 \left\{ \rho \left( \delta, 0, 0, 0 + \delta \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), 0 + \delta \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}}) \right) \right\} \\ &\leq F_1 \left( \rho \left( \delta, 0, 0, \delta \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \delta \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}}) \right) \right\} \\ &\leq F_1 \left\{ \delta \rho \left( 1, 0, 0, \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}}) \right) \right\}. \end{split}$$

Since  $\rho \in \mathbb{P}$ , we have  $\rho(1, 0, 0, \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})) \in (0, 1]$ 

$$\implies \lim_{k \to \infty} \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F_1(\delta) \le F_1(\delta),$$

$$\implies \lim_{k \to \infty} \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) \le 0,$$

$$\implies \lim_{S \to \delta^+} \inf \chi(S) \le 0,$$

which is a contradiction, and hence (13) holds. Therefore  $\{\varrho_{\mathfrak{s}}\}$  is a Cauchy sequence, and thus there exists  $\varrho^* \in \xi$  such that  $\lim_{\mathfrak{s} \to \infty} \varrho_{\mathfrak{s}} = \varrho^*$ . The rest of the proof follows from Theorem 3.1, and we get  $\varrho^* \in \Omega \varrho^*$ .

**Theorem 3.3** Let  $(\xi, d, \mathfrak{f})$  be a complete CMS, and let  $\Omega : \xi \longrightarrow C(\xi)$  be an MVM. Assume that there are  $\chi \in \Phi$ ,  $F \in \Delta(0*)$ , and a real-valued function L on  $(0, \infty)$  such that following conditions hold:

- (*G*<sub>1</sub>)  $F(\varrho) \leq L(\varrho)$  for all  $\varrho > 0$ ;
- (G<sub>2</sub>)  $H(\Omega \rho, \Omega \bar{\rho}) > 0$  implies,

$$\chi\left(d(\varrho,\bar{\varrho})\right) + L\left(H(\Omega\varrho,\Omega\bar{\varrho})\right) \le F\left\{\rho\left(d(\varrho,\bar{\varrho}),D(\varrho,\Omega\varrho),D(\bar{\varrho},\Omega\bar{\varrho}),D(\varrho,\Omega\bar{\varrho}),D(\bar{\varrho},\Omega\varrho)\right)\right\}$$

for all  $\varrho, \bar{\varrho} \in \xi$  and  $\rho \in \mathcal{P}$ . Let  $\varrho_0 \in \xi$ . Define the Picard sequence  $\{\varrho_{\mathfrak{s}} = \Omega^{\mathfrak{s}} \varrho_{\mathfrak{s}}\}$  such that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\mathfrak{f}(\varrho_{i+1}, \varrho_{i+2})\mathfrak{f}(\varrho_{i+1}, \varrho_m)}{\mathfrak{f}(\varrho_i, \varrho_{i+1})} < 1.$$
(18)

Also, suppose that  $\lim_{s\to\infty} f(\rho_s, \rho) \leq 1$  for all  $\rho \in \xi$ . Then Fix  $\Omega$  is nonempty.

*Proof* Let  $\rho_0 \in \xi$  and  $\rho_1 \in \Omega \rho_0$ . If  $\rho_1 \in \Omega \rho_1$  then  $\rho_1 \in fix \Omega$ . Suppose  $\rho_1 \notin \Omega \rho_1$ . This implies  $D(\rho_1, \Omega \rho_1) > 0$ , and, consequently,  $H(\Omega \rho_0, \Omega \rho_1) > 0$ . Due to (F4), we obtain

$$F(D(\varrho_1, \Omega \varrho_1)) = \inf_{z \in \Omega \varrho_1} F(d(\varrho_1, z)).$$
<sup>(19)</sup>

Then (19) with  $(G_1)$  and  $(G_2)$  imply that

$$\begin{split} \inf_{z \in \Omega_{\ell_1}} F(d(\varrho_1, z)) &= F(D(\varrho_1, \Omega_{\ell_1})) \\ &\leq F(H(\Omega_{\ell_0}, \Omega_{\ell_1})) \\ &\leq L(H(\Omega_{\ell_0}, \Omega_{\ell_1})) \\ &\leq F\{\rho(d(\varrho_0, \varrho_1), D(\varrho_0, \Omega_{\ell_0}), D(\varrho_1, \Omega_{\ell_1}), D(\varrho_0, \Omega_{\ell_1}), D(\varrho_1, \Omega_{\ell_0}))\} \\ &\quad -\chi(d(\varrho_0, \varrho_1)) \\ &\implies \inf_{z \in \Omega_{\ell_1}} F(d(\varrho_1, z)) < F\{\rho(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0)\}. \end{split}$$

Hence there exists  $\rho_2 \in \Omega \rho_1$  such that

$$F(d(\varrho_1, \varrho_2)) < F\{\rho(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0)\}.$$
(20)

Since *F* is a nondecreasing function, so (20) with ( $\rho_3$ ) yield that

$$d(\varrho_{1}, \varrho_{2}) < \rho \left( d(\varrho_{0}, \varrho_{1}), d(\varrho_{0}, \varrho_{1}), d(\varrho_{1}, \varrho_{2}), d(\varrho_{0}, \varrho_{2}), 0 \right)$$
  
$$\leq \rho \left( d(\varrho_{0}, \varrho_{1}), d(\varrho_{0}, \varrho_{1}), d(\varrho_{1}, \varrho_{2}), \mathfrak{f}(\varrho_{0}, \varrho_{1}) \right) d(\varrho_{0}, \varrho_{1}) + \mathfrak{f}(\varrho_{1}, \varrho_{2}) d(\varrho_{1}, \varrho_{2}), 0).$$

By Lemma 3.1

 $d(\varrho_1, \varrho_2) < d(\varrho_0, \varrho_1).$ 

Next, arguing as previously, we get  $\rho_3 \in \Omega \rho_2$  with  $D(\rho_2, \Omega \rho_2) > 0$ . By Lemma 3.1, using  $(G_1)$  and  $(G_2)$ , we have

$$d(\varrho_2, \varrho_3) < d(\varrho_1, \varrho_2).$$

By induction we have a sequence  $\{\varrho_{\mathfrak{s}}\} \subset \xi$  such that  $\varrho_{\mathfrak{s}+1} \in \Omega \varrho_{\mathfrak{s}}$  with  $D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}) > 0$  and

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) \quad \text{for all } \mathfrak{s} \in \mathbb{N}.$$

$$\tag{21}$$

Now (21) implies that  $\{d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\}_{\mathfrak{s}\in\mathbb{N}}$  is a decreasing sequence of positive real numbers. Hence from (*F*4)

$$\inf_{z \in \Omega_{\ell_{s}}} F(d(\varrho_{s}, z))$$

$$= F(D(\varrho_{s}, \Omega_{\varrho_{s}})) \leq F(H(\Omega_{\varrho_{s-1}}, \Omega_{\varrho_{s}})) \leq L(H(\Omega_{\varrho_{s-1}}, \Omega_{\varrho_{s}}))$$

$$\leq F\{\rho(d(\varrho_{s-1}, \varrho_{s}), D(\varrho_{s-1}, \Omega_{\varrho_{s-1}}), D(\varrho_{s}, \Omega_{\varrho_{s}}), D(\varrho_{s-1}, \Omega_{\varrho_{s}}), D(\varrho_{s}, \Omega_{\varrho_{s-1}}))\}$$

$$- \chi(d(\varrho_{s-1}, \varrho_{s}), d(\varrho_{s-1}, \varrho_{s}), d(\varrho_{s}, \varrho_{s+1}), f(\varrho_{s-1}, \varrho_{s})d(\varrho_{s-1}, \varrho_{s}) + f(\varrho_{s}, \varrho_{s+1})$$

$$d(\varrho_{s}, \varrho_{s+1}), 0)) - \chi(d(\varrho_{s-1}, \varrho_{s}))$$

$$\leq F\{\rho(d(\varrho_{s-1}, \varrho_{s}), d(\varrho_{s-1}, \varrho_{s}), d(\varrho_{s-1}, \varrho_{s}), f(\varrho_{s-1}, \varrho_{s})d(\varrho_{s-1}, \varrho_{s}) + f(\varrho_{s}, \varrho_{s+1})$$

$$d(\varrho_{s-1}, \varrho_{s}), 0)) - \chi(d(\varrho_{s-1}, \varrho_{s}), f(\varrho_{s}, \varrho_{s+1}), 0)) - \chi(d(\varrho_{s-1}, \varrho_{s}))$$

$$\leq F\{d(\varrho_{s-1}, \varrho_{s}), \rho(1, 1, 1, f(\varrho_{s-1}, \varrho_{s})) + f(\varrho_{s}, \varrho_{s+1}), 0)) - \chi(d(\varrho_{s-1}, \varrho_{s}))$$

$$\Rightarrow \inf_{z \in \Omega_{\varrho_{s}}} F(d(\varrho_{s}, z)) \leq F(d(\varrho_{s-1}, \varrho_{s})) - \chi(d(\varrho_{s-1}, \varrho_{s})) \quad \text{for all } s \in \mathbb{N}.$$
(22)

Since  $\xi \in \phi$ , there exist h > 0 and  $\mathfrak{s}_0 \in \mathbb{N}$  such that  $\chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) < h$  for all  $\mathfrak{s} \ge \mathfrak{s}_0$ . From (22)

$$F(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) \leq F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))$$

$$\leq F(d(\varrho_{\mathfrak{s}-2}, \varrho_{\mathfrak{s}-1})) - \chi(d(\varrho_{\mathfrak{s}-2}, \varrho_{\mathfrak{s}-1})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))$$

$$\vdots$$

$$\leq F(d(\varrho_{0}, \varrho_{1})) - \sum_{i=1}^{\mathfrak{s}-1} \chi(d(\varrho_{i-1}, \varrho_{i}))$$

$$= F(d(\varrho_{0}, \varrho_{1})) - \sum_{i=1}^{\mathfrak{s}_{0}-1} \chi(d(\varrho_{i-1}, \varrho_{i})) - \sum_{i=\mathfrak{s}_{0}}^{\mathfrak{s}-1} \chi(d(\varrho_{i-1}, \varrho_{i}))$$

$$= F(d(\varrho_{0}, \varrho_{1})) - (\mathfrak{s} - \mathfrak{s}_{0})h, \quad \mathfrak{s} \geq \mathfrak{s}_{0}.$$
(23)

Taking the limit as  $\mathfrak{s} \to \infty$  in (23), we get  $F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) \to -\infty$ , and from ( $\mathbb{F}2'$ )

$$\lim_{\mathfrak{s}\to\infty} d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}) = 0.$$
<sup>(24)</sup>

Now by ( $\mathbb{F}$ 3) there exists 0 < k < 1 such that

$$\lim_{s \to \infty} \left( d(\varrho_{s-1}, \varrho_s) \right)^k F\left( d(\varrho_{s-1}, \varrho_s) \right) = 0.$$
(25)

Thus from (23) for all  $\mathfrak{s} \geq \mathfrak{s}_0$ , we have

$$(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}))^{k}F(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})) - (d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}))^{k}F(d(\varrho_{0},\varrho_{1}))$$

$$\leq (d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}))^{k}(F(d(\varrho_{0},\varrho_{1})) - (\mathfrak{s}-\mathfrak{s}_{0})h) - (d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}))^{k}F(d(\varrho_{0},\varrho_{1}))$$

$$= -(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}))^{k}(\mathfrak{s}-\mathfrak{s}_{0})h \leq 0.$$

$$(26)$$

Taking the limit as  $\mathfrak{s} \to \infty$  in (26) and using (24) and (25), we get

$$0 \le -\lim_{s \to \infty} \mathfrak{s} \left( d(\varrho_{s-1}, \varrho_s) \right)^k \le 0 \tag{27}$$

$$\implies \lim_{\mathfrak{s} \to \infty} \mathfrak{s} \left( d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) \right)^k = 0$$
(28)

Note that by (28) there exists  $\mathfrak{s}_1 \in \mathbb{N}$  such that  $\mathfrak{s}(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k \leq 1$  for all  $\mathfrak{s} \geq \mathfrak{s}_1$ . We get

$$d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) \leq \frac{1}{\mathfrak{s}^{rac{1}{k}}} \quad \text{for all } \mathfrak{s} \geq \mathfrak{s}_1.$$

Now to prove that  $\{\varrho_s\}_{s\in\mathbb{N}}$  is a Cauchy sequence, consider  $\tau, s \in \mathbb{N}$  such that  $\tau > \mathfrak{s} > \mathfrak{s}_1$ . The rest of the proof follows from Theorem 3.1, and by using (18) with ratio test we deduce that  $\{\varrho_s\}$  is a Cauchy sequence, and thus there exists  $\varrho^* \in \xi$  such that

$$\lim_{\mathfrak{s}\to\infty}\varrho_{\mathfrak{s}}=\varrho^*.$$

Now

$$F(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}})) \leq L(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}})) \leq \chi(d(\varrho, \bar{\varrho})) + L(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}}))$$
$$\leq F\{\rho(d(\varrho, \bar{\varrho}), D(\varrho, \Omega_{\varrho}), D(\bar{\varrho}, \Omega_{\bar{\varrho}}), D(\varrho, \Omega_{\bar{\varrho}}), D(\bar{\varrho}, \Omega_{\varrho}))\}.$$

Since F is a nondecreasing function, we get

 $H(\Omega \varrho, \Omega \bar{\varrho})) \le \rho \left( d(\varrho, \bar{\varrho}), D(\varrho, \Omega \varrho), D(\bar{\varrho}, \Omega \bar{\varrho}), D(\varrho, \Omega \bar{\varrho}), D(\bar{\varrho}, \Omega \varrho) \right) \quad \text{for all } \varrho, \bar{\varrho} \in \xi.$ 

Let  $\varrho^*$  be a fixed point of  $\xi$ . On the contrary, we have  $D(\varrho^*, \Omega \varrho^*) > 0$ . Then by following the proof of Theorem 3.1,  $D(\varrho^*, \Omega \varrho^*) = 0$ . Since  $\Omega \varrho^*$  is closed,  $\varrho^* \in \Omega \varrho^*$ . Hence Fix  $\Omega$  is nonempty.

**Theorem 3.4** Let  $(\xi, d, \mathfrak{f})$  be a complete CMS, and let  $\Omega : \xi \longrightarrow C(\xi)$  be a multivalued mapping. Suppose there exist  $\chi \in \phi$ ,  $\rho \in \mathbb{P}$ , and a nondecreasing continuous real-valued function  $F : (0, \infty) \longrightarrow \mathbb{R}$  that satisfy ( $\mathbb{F}2'$ ). Moreover, *L* be a real-valued function on  $(0, \infty)$  such that the following conditions hold:

- (G<sub>1</sub>)  $F(\varrho) \leq L(\varrho)$  for all  $\varrho > 0$ ;
- (G<sub>2</sub>)  $H(\Omega \rho, \Omega \bar{\rho}) > 0$  implies

 $\chi\left(d(\varrho,\bar{\varrho})\right) + L\left(H(\Omega\varrho,\Omega\bar{\varrho})\right) \le F\left\{\rho\left(d(\varrho,\bar{\varrho}),D(\varrho,\Omega\varrho),D(\bar{\varrho},\Omega\bar{\varrho}),D(\varrho,\Omega\bar{\varrho}),D(\bar{\varrho},\Omega\varrho)\right)\right\}$ 

for all  $\varrho, \bar{\varrho} \in \xi$ .

Also, suppose

$$\lim_{\mathfrak{s}\to\infty}\mathfrak{f}(\varrho_{\mathfrak{s}},\varrho)\leq 1 \quad \text{for all } \varrho\in\xi.$$

*Then* Fix  $\Omega$  *is non-empty.* 

*Proof* Let  $\rho_0 \in \xi$  be an arbitrary point, and let  $\rho_1 \in \Omega \rho_0$ . As in proof of Theorem 3.1, we get a sequence  $\{\rho_s\} \subset \xi$  such that  $\rho_{s+1} \in \Omega \rho_s$  with  $D(\rho_s, \Omega \rho_{s+1}) > 0$ ,

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}),$$

and

$$F(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})) \le F(d(\varrho_0,\varrho_1)) - (\mathfrak{s}-\mathfrak{s}_0)h \quad \text{for all } \mathfrak{s} \ge \mathfrak{s}_0.$$
<sup>(29)</sup>

Taking the limit as  $\mathfrak{s} \to \infty$  in (29), we get  $F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) \to -\infty$ , and by ( $\mathbb{F}2'$ )

$$\lim_{\mathfrak{s}\to\infty}d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})=0.$$

Now we claim that

$$\lim_{\mathfrak{s},\tau\to\infty} d(\varrho_{\mathfrak{s}},\varrho_{\tau}) = 0.$$
(30)

If (30) does not hold, then there exists  $\delta > 0$  such that for all  $r \ge 0$ , we have  $\tau_k > \mathfrak{s}_k > r$ ,

$$d(\varrho_{\mathfrak{s}}, \varrho_{\tau}) < \delta.$$

Also, there exists  $r_0 \in \mathbb{N}$  such that

$$\lambda_{r_0} = d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) < \delta \quad \text{for all } \mathfrak{s} \ge r_0.$$

There exist two subsequences  $\{\varrho_{\tau_k}\}$  and  $\{\varrho_{\mathfrak{s}_k}\}$  of  $\{\varrho_{\mathfrak{s}}\}$ . Then following the proof of Theorem 3.2, we get  $\lim_{k\to\infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) = \delta$  and also

$$\lim_{k \to \infty} d(\varrho_{\tau_{k+1}}, \varrho_{\mathfrak{s}_{k+1}}) = \delta.$$
(31)

By the monotonicity of *F*, using  $(G_1)$  and  $(G_2)$ , we get

$$\begin{split} \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) &+ F \left( d(\varrho_{\tau_{k+1}}, \Omega \varrho_{\mathfrak{s}_{k+1}}) \right) \\ &= \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F \left( D(\varrho_{\tau_{k+1}}, \Omega \varrho_{\mathfrak{s}_k}) \right) \\ &\leq \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F \left( H(\Omega \varrho_{\tau_k}, \Omega \varrho_{\mathfrak{s}_k}) \right) \\ &\leq \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + L \left( H(\Omega \varrho_{\tau_k}, \Omega \varrho_{\mathfrak{s}_k}) \right) \\ &\leq F \left\{ \rho \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}), d(\varrho_{\tau_k}, \varrho_{\tau_{k+1}}), d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}), \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_k}) d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_k}) \right) \end{split}$$

$$+ \mathfrak{f}(\varrho_{\mathfrak{s}_{k}},\varrho_{\tau_{k}})d(\varrho_{\mathfrak{s}_{k}},\varrho_{\tau_{k}}),\mathfrak{f}(\varrho_{\mathfrak{s}_{k}},\varrho_{\mathfrak{s}_{k+1}})d(\varrho_{\mathfrak{s}_{k}},\varrho_{\mathfrak{s}_{k+1}}) + \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}},\varrho_{\tau_{k+1}})d(\varrho_{\mathfrak{s}_{k+1}},\varrho_{\tau_{k+1}}))\Big\}.$$
(32)

By the continuity of *F*, taking the limit as  $k \rightarrow \infty$  and using (31) and (32), we have

$$\lim_{k \to \infty} \chi \left( d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \right) + F(\delta) \leq F \left\{ \rho \left( \delta, 0, 0, \delta \lim_{k \to \infty} \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \delta \lim_{k \to \infty} \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}}) \right) \right\}$$
$$\leq F \left\{ \delta \rho \left( 1, 0, 0, \delta V \lim_{k \to \infty} \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \delta \lim_{k \to \infty} \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}}) \right) \right\}.$$

Since  $\rho \in \mathbb{P}$ , we have  $\rho(1, 0, 0, \lim_{k \to \infty} \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \lim_{k \to \infty} \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})) \in (0, 1]$ . Hence

$$\lim_{s \to s^+} \inf \chi(s) \le 0,$$

which is a contradiction to definition of  $\Phi$ . Therefore (30) is fulfilled and ensures that  $\{\varrho_{\mathfrak{s}}\}$  is a Cauchy sequence. Hence there exists  $\varrho^* \in \xi$  such that

$$\lim_{\mathfrak{s}\to\infty}\varrho_{\mathfrak{s}}=\varrho^*.$$

By following the proof of Theorem 3.3 we get  $\rho^* \in \Omega \rho^*$ .

## 4 Data dependence

For a metric space  $(\xi, d)$  and mappings  $\Omega_1, \Omega_2 : \xi \to P(\xi)$ , the fixed points sets Fix  $\Omega_1$ and Fix  $\Omega_2$  are nonempty. The problem of finding the Pomeiu–Hausdroff distance H between Fix  $\Omega_1$  and Fix  $\Omega_2$  under the condition that for s > 0,  $H(\Omega_1 \rho, \Omega_2 \rho) < s$  for all  $\rho \in \xi$ , is addressed by many authors. See, for example, [6, 7, 16]. In this section, we give a data dependence result of the established result.

**Definition 4.1** Let  $(\xi, d)$  be an MS, and let  $\Omega: \xi \longrightarrow CL(\xi)$  be a multivalued operator. Suppose that for all  $\varrho \in \xi$  and  $\overline{\varrho} \in \Omega \varrho$ , there exists sequence  $\{\varrho_s\}_{s \in \mathbb{N}}$  such that

- (i)  $\varrho_0 = \varrho$  and  $\varrho_1 = \bar{\varrho}$ ,
- (ii)  $\varrho_{\mathfrak{s}+1} = \Omega \varrho_{\mathfrak{s}}$  for all  $\mathfrak{s} \in \mathbb{N}$ , and
- (iii) the sequence  $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathbb{N}}$  is convergent, and the fixed point of  $\Omega$  is its limit.

Then  $\Omega$  is said to be a multivalued weakly Picard operator (MWP operator). The sequence of successive approximations is defined as a sequence  $\{\varrho_s\}_{s\in\mathbb{N}}$  that satisfies conditions (ii) and (ii) of Definition 4.1.

The main result of this section is as follows.

**Theorem 4.1** Let  $(\xi, d)$  be a CMS, let  $\Omega_1, \Omega_2 : \xi \longrightarrow K(\xi)$  be multivalued mappings, and let  $\chi \in \Phi$ . Let  $F_1$  be a real-valued nondecreasing function on  $(0, \infty)$ , and let  $F_2$  be a realvalued function on  $(0, \infty)$  satisfying ( $\mathbb{F}2'$ ) and ( $\mathbb{F}3$ ) such that  $\chi F$ -contraction is satisfied for  $\Omega_i$ , where  $i \in \{1, 2\}$ , and there exists  $\lambda > 0$  such that  $H(\Omega_1(\varrho), \Omega_2(\varrho)) \le \lambda$  for all  $\varrho \in \xi$ . For  $\varrho_0 \in \xi$ , define a Picard sequence { $\varrho_s = \Omega^s \varrho_0$ } such that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\mathfrak{f}(\varrho_{i+1}, \varrho_{i+2})\mathfrak{f}(\varrho_{i+1}, \varrho_m)}{\mathfrak{f}(\varrho_i, \varrho_{i+1})} < 1.$$
(33)

Also, suppose that  $\lim_{s\to\infty} \mathfrak{f}(\varrho_s, \varrho) \leq 1$  for all  $\varrho \in \xi$ . Then

- (*a*) Fix  $\Omega_i \in CL(\xi)$  for  $i \in \{1, 2\}$ ,
- (b)  $\Omega_1$ ,  $\Omega_2$  are MWP operators, and

$$H(\text{Fix }\Omega_1, \text{Fix }\Omega_2) \le \frac{\lambda}{1 - \max\{\rho_1(1, 1, 1, \zeta + \eta, 0), \rho_2(1, 1, 1, \zeta + \eta, 0)\}},$$

where  $\zeta, \eta \geq 1$ .

*Proof* (*a*) Using Theorem 3.1, we have that Fix  $\Omega_i$  is not empty for  $i \in \{1, 2\}$ . Now we prove that for  $i \in \{1, 2\}$ , the fixed point set of  $\Omega_i$  is closed. Consider a sequence  $\{\varrho_s\}$  in Fix  $\Omega_i$  such that  $\varrho_s \longrightarrow \varrho$  as  $s \longrightarrow \infty$ . Now

$$F_{1}(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}}) \leq F_{2}(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}}) \leq \chi(d(\varrho, \bar{\varrho}) + F_{2}(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}})))$$
$$\leq F_{1}(\rho(d(\varrho, \bar{\varrho}), D(\varrho, \Omega_{\varrho}), D(\bar{\varrho}, \Omega_{\bar{\varrho}}), D(\varrho, \Omega_{\bar{\varrho}}), D(\bar{\varrho}, \Omega_{\varrho}))).$$

Since  $F_1$  is a nondecreasing function, we have that for all  $\rho, \bar{\rho} \in \xi$ ,

$$H(\Omega_{\varrho}, \Omega_{\bar{\varrho}}) \le \rho \left( d(\varrho, \bar{\varrho}), D(\varrho, \Omega_{\varrho}), D(\bar{\varrho}, \Omega_{\bar{\varrho}}), D(\varrho, \Omega_{\bar{\varrho}}), D(\bar{\varrho}, \Omega_{\varrho}) \right).$$
(34)

Assume that  $D(\bar{\varrho}, \Omega \bar{\varrho}) > 0$ . Then there exists  $\varrho \in \Omega \bar{\varrho}$  such that

$$\begin{split} D(\bar{\varrho},\Omega\bar{\varrho}) &= d(\bar{\varrho},\varrho) \\ &\leq \mathfrak{f}(\bar{\varrho},\varrho_{\mathfrak{s}+1})d(\bar{\varrho},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)d(\varrho_{\mathfrak{s}+1},\varrho) \\ &= \mathfrak{f}(\bar{\varrho},\varrho_{\mathfrak{s}+1})d(\bar{\varrho},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)D(\varrho_{\mathfrak{s}+1},\Omega\bar{\varrho}) \\ &\leq \mathfrak{f}(\bar{\varrho},\varrho_{\mathfrak{s}+1})d(\bar{\varrho},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)H(\Omega\varrho_{\mathfrak{s}},\Omega\bar{\varrho}) \\ &\leq \mathfrak{f}(\bar{\varrho},\varrho_{\mathfrak{s}+1})d(\bar{\varrho},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)\rho(d(\varrho_{\mathfrak{s}},\bar{\varrho}),D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}),D(\bar{\varrho},\Omega\bar{\varrho}),D(\varrho_{\mathfrak{s}},\Omega\bar{\varrho}), \\ &\quad D(\bar{\varrho},\Omega\varrho_{\mathfrak{s}})) \\ &\leq \mathfrak{f}(\bar{\varrho},\varrho_{\mathfrak{s}+1})d(\bar{\varrho},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)\rho(d(\varrho_{\mathfrak{s}},\bar{\varrho}),d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}),D(\bar{\varrho},\Omega\bar{\varrho}), \\ &\quad \mathfrak{f}(\varrho_{\mathfrak{s}},\bar{\varrho})d(\varrho_{\mathfrak{s}},\bar{\varrho}) + \mathfrak{f}(\bar{\varrho},\varrho)D(\bar{\varrho},\Omega\bar{\varrho}),d(\bar{\varrho},\varrho_{\mathfrak{s}+1})). \end{split}$$

Taking the limit as  $\mathfrak{s} \longrightarrow \infty$  in the above inequality, we get

$$D(\bar{\varrho}, \Omega\bar{\varrho}) \leq (1)\rho(0, 0, D(\bar{\varrho}, \Omega\bar{\varrho}), 0 + \mathfrak{f}(\bar{\varrho}, \varrho_1)D(\bar{\varrho}, \Omega\bar{\varrho}), 0).$$

Using Lemma 3.1,  $D(\bar{\varrho}, \Omega \bar{\varrho}) \leq 0$ , and hence  $D(\bar{\varrho}, \Omega \bar{\varrho}) = 0$ . As  $\Omega \bar{\varrho}$  is closed,  $\bar{\varrho} \in \Omega \bar{\varrho}$ .

(b) Using Theorem 3.1, we get that  $\Omega_1$ ,  $\Omega_2$  are *MWP* operators. So we have to prove that

$$H(\text{Fix }\Omega_1, \text{Fix }\Omega_2) \leq \frac{\lambda}{1 - \max\{\rho_1(1, 1, 1, \zeta + \eta, 0), \rho_2(1, 1, 1, \zeta + \eta, 0)\}}.$$

Suppose q > 1 and  $\varrho_0 \in Fix \Omega_2$ . Then there exists  $\varrho_1 \in \Omega_2(\varrho_0)$  such that  $d(\varrho_0, \varrho_1) = D(\varrho_0, \Omega_2(\varrho_0))$  and  $d(\varrho_1, \varrho_2) \leq qH(\Omega_1(\varrho_0), \Omega_2(\varrho_0))$ . Now there exists  $\varrho_2 \in \Omega_2(\varrho_1)$  such that  $d(\varrho_0, \varrho_1) = D(\varrho_0, \Omega_2(\varrho_0))$  and  $d(\varrho_1, \varrho_2) \leq qH(\Omega_2(\varrho_0), \Omega_2(\varrho_1))$ . Also, we get  $d(\varrho_1, \varrho_2) \leq d(\varrho_0, \varrho_1)$  and

$$d(\varrho_1, \varrho_2) \leq \mathsf{q} H(\Omega_2(\varrho_0), \Omega_2(\varrho_1))$$

$$\leq \mathsf{q}\rho(d(\varrho_{0},\varrho_{1}), D(\varrho_{0}, \Omega(\varrho_{0})), D(\varrho_{1}, \Omega(\varrho_{1})), D(\varrho_{0}, \Omega(\varrho_{1})), D(\varrho_{1}, \Omega(\varrho_{0}))$$

$$\leq \mathsf{q}\rho(d(\varrho_{0},\varrho_{1}), d(\varrho_{0},\varrho_{1}), d(\varrho_{1},\varrho_{2}), d(\varrho_{0},\varrho_{2}), d(\varrho_{1},\varrho_{1}))$$

$$\leq \mathsf{q}\rho(d(\varrho_{0},\varrho_{1}), d(\varrho_{0},\varrho_{1}), d(\varrho_{1},\varrho_{2}), \mathfrak{f}(\varrho_{0},\varrho_{1})) d(\varrho_{0},\varrho_{1}) + \mathfrak{f}(\varrho_{1},\varrho_{2})$$

$$d(\varrho_{1},\varrho_{2}), 0)$$

$$< \mathsf{q}\rho(d(\varrho_{0},\varrho_{1}), d(\varrho_{0},\varrho_{1}), d(\varrho_{0},\varrho_{1}), \mathfrak{f}(\varrho_{0},\varrho_{1})) d(\varrho_{0},\varrho_{1}) + \mathfrak{f}(\varrho_{1},\varrho_{2})$$

$$d(\varrho_{0},\varrho_{1}), 0)$$

$$\leq \mathsf{q}d(\varrho_{0},\varrho_{1})\rho(1, 1, 1, \mathfrak{f}(\varrho_{0},\varrho_{1})) + \mathfrak{f}(\varrho_{1},\varrho_{2}), 0).$$

Hence we get a sequence of successive approximations of  $\Omega$  starting from  $\rho_0$  that satisfies

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) \leq \left(\mathsf{q}\rho_{1}(1, 1, 1, \zeta + \eta, 0)\right)^{\mathfrak{s}} d(\varrho_{0}, \varrho_{1}) \quad \text{for all } \mathfrak{s} \in \mathbb{N}$$
  
$$\Rightarrow \quad d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+m}) \leq \frac{(\mathsf{q}\rho_{1}(1, 1, 1, \zeta + \eta, 0))^{\mathfrak{s}}}{1 - \mathsf{q}\rho_{1}(1, 1, 1, \zeta + \eta, 0)} d(\varrho_{0}, \varrho_{1}) \quad \text{for all } \mathfrak{s} \in \mathbb{N}.$$
(35)

Taking the limit as  $\mathfrak{s} \to \infty$ , we conclude that  $\{\varrho_{\mathfrak{s}}\}$  is a Cauchy sequence in  $(\xi, d)$  and thus converges to some  $\mathsf{v} \in \xi$ . Using the proof of Theorem 3.1, we have  $\mathsf{v} \in \operatorname{Fix} \Omega_2$ . Taking the limit as  $m \to \infty$ , we get

$$d(\varrho_{\mathfrak{s}},\mathsf{v}) \leq \frac{(\mathsf{q}\rho_{1}(1,1,1,\zeta+\eta,0))^{\mathfrak{s}}}{1-\mathsf{q}\rho_{1}(1,1,1,\zeta+\eta,0)}d(\varrho_{0},\varrho_{1}) \quad \text{for all } \mathfrak{s} \in \mathbb{N}.$$

Letting  $\mathfrak{s} = 0$ ,

$$d(\varrho_0, \mathsf{v}) \leq \frac{1}{1 - \mathsf{q}\rho_1(1, 1, 1, \zeta + \eta, 0)} d(\varrho_0, \varrho_1) \leq \frac{\mathsf{q}\lambda}{1 - \mathsf{q}\rho_1(1, 1, 1, \zeta + \eta, 0)}.$$

Interchange the role of  $\Omega_1$  and  $\Omega_2$ , for each  $v_0 \in Fix \Omega_1$ , we get

$$d(\mathsf{v}_0,c) \leq \frac{1}{1 - \mathsf{q}\rho_2(1,1,1,\zeta+\eta,0)} d(\mathsf{v}_0,\mathsf{v}_1) \leq \frac{\mathsf{q}\lambda}{1 - \mathsf{q}\rho_2(1,1,1,\zeta+\eta,0)}.$$

So

$$H(\operatorname{Fix} \Omega_1, \operatorname{Fix} \Omega_2) \leq \frac{q\lambda}{1 - \max(q\rho_1(1, 1, 1, \zeta + \eta, 0), q\rho_2(1, 1, 1, \zeta + \eta, 0))}.$$

By taking the limit as  $q \rightarrow 1$  the result is proved.

# 5 Strict fixed point and well-posedness

**Definition 5.1** Consider an MS  $(\xi, d)$ ,  $\mathbb{B} \in P(\xi)$ , and a multivalued mapping  $\Omega \colon \mathbb{B} \longrightarrow C(\xi)$ . The fixed point problem is said to be well posed for  $\Omega$  with respect to *D* if

(a) Fix  $\Omega = \{\varrho^*\}$ ,

(b) if  $\rho_{\mathfrak{s}} \in \mathbb{B}$ ,  $\mathfrak{s} \in \mathbb{N}$  and  $D(\rho_{\mathfrak{s}}, \Omega \rho_{\mathfrak{s}}) \longrightarrow 0$  as  $\mathfrak{s} \longrightarrow \infty$ , then  $\rho_{\mathfrak{s}} \longrightarrow \rho^* \in \operatorname{Fix} \Omega$  as  $\mathfrak{s} \longrightarrow \infty$  [14, 15].

**Definition 5.2** Consider an MS  $(\xi, d)$ ,  $\mathbb{B} \in P(\xi)$ , and a multivalued mapping  $\Omega \colon \mathbb{B} \longrightarrow C(\xi)$ . The fixed point problem is said to be well posed for  $\Omega$  with respect to *H* if

(a)  $S \operatorname{Fix} \Omega = \{\varrho^*\}$ , (b) if  $\varrho_{\mathfrak{s}} \in \mathbb{B}$ ,  $\mathfrak{s} \in \mathbb{N}$ , and  $H(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}) \longrightarrow 0$  as  $\mathfrak{s} \longrightarrow \infty$ , then  $\varrho_{\mathfrak{s}} \longrightarrow \varrho^* \in S \operatorname{Fix} \Omega$  as  $\mathfrak{s} \longrightarrow \infty$  [14, 15].

**Theorem 5.1** Let  $(\xi, d, \mathfrak{f})$  be a complete CMS, let  $\Omega : \xi \longrightarrow K(\xi)$  be a multivalued mapping, and let  $F_1$ ,  $F_2$  be functions satisfying a  $\chi F$ -contraction. Suppose  $F_1$  is nondecreasing,  $F_2$  satisfies condition ( $\mathbb{F}2'$ ) with  $\rho(1,0,0,1,1) \in (0,1)$ , and  $S \operatorname{Fix} \Omega \neq \phi$ . Also, suppose  $\lim_{s \to \infty} \mathfrak{f}(\varrho_s, \varrho) \leq 1$  for all  $\varrho \in \xi$ . Then

- (a) Fix  $\Omega = S$  Fix  $\Omega = \{\varrho^*\}$ ;
- (b) The fixed point problem is well posed for the multivalued mapping Ω with respect to H.

*Proof* (a) Using Theorem 3.2, we conclude that Fix  $\Omega \neq \phi$ . Now we prove that Fix  $\Omega = \{\varrho^*\}$ . Using (Ni) and (Nii), we have

$$F_1(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}})) \le F_2(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}})) \le \chi(d(\varrho, \bar{\varrho})) + F_2(H(\Omega_{\varrho}, \Omega_{\bar{\varrho}}))$$
$$\le F_1\{\rho(d(\varrho, \bar{\varrho}), D(\varrho, \Omega_{\varrho}), D(\bar{\varrho}, \Omega_{\bar{\varrho}}), D(\varrho, \Omega_{\bar{\varrho}}), D(\varrho, \Omega_{\bar{\varrho}}), D(\bar{\varrho}, \Omega_{\varrho}))\}.$$

Since  $F_1$  is a nondecreasing function, we obtain that for all  $\rho, \bar{\rho} \in \xi$ ,

$$H(\Omega \varrho, \Omega \bar{\varrho}) \leq \rho \big( d(\varrho, \bar{\varrho}), D(\varrho, \Omega \varrho), D(\bar{\varrho}, \Omega \bar{\varrho}), D(\varrho, \Omega \bar{\varrho}), D(\bar{\varrho}, \Omega \varrho) \big).$$

Let  $v \in Fix \Omega$  with  $v \neq \varrho^*$ . Then  $D(\varrho^*, \Omega v) > 0$ . Now we have

$$D(\varrho^*, \Omega \mathsf{v}) = H(\Omega \varrho^*, \Omega \mathsf{v})$$
  

$$\leq \rho(d(\varrho^*, \mathsf{v}), D(\varrho^*, \Omega \varrho^*), D(\mathsf{v}, \Omega \mathsf{v}), D(\varrho^*, \Omega \mathsf{v}), D(\mathsf{v}, \Omega \varrho^*))$$
  

$$\leq \rho(d(\varrho^*, \mathsf{v}), 0, 0, d(\varrho^*, \mathsf{v}), d(\mathsf{v}, \varrho^*))$$
  

$$\leq d(\varrho^*, \mathsf{v})\rho(1, 0, 0, 1, 1).$$

As  $\rho(1, 0, 0, 1, 1) \in (0, 1)$ , we have

$$d(\varrho^*, \mathsf{v}) = D(\varrho^*, \Omega \mathsf{v}) < d(\varrho^*, \mathsf{v}),$$

which is a contradiction, and hence  $d(\varrho^*, v) = 0$  and  $\varrho^* = v$ . (b) Let  $\varrho_{\mathfrak{s}} \in \mathbb{B}$  and  $\mathfrak{s} \in \mathbb{N}$  be such that

$$\lim_{\mathfrak{s}\to\infty} D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}) = 0.$$
(36)

Now we claim that

$$\lim_{\mathfrak{s}\to\infty}d(\varrho_{\mathfrak{s}},\varrho^*)=0,$$

where  $\varrho^* \in Fix \Omega$ . If the above equation is not true, then for every  $\mathfrak{s} \in \mathbb{N}$ , there exists  $\epsilon > 0$  such that

$$d(\varrho_{\mathfrak{s}}, \varrho^*) > \epsilon.$$

But (36) implies that there exists  $\mathfrak{s}_{\epsilon} \in \mathbb{N} - \{0\}$  such that

$$\lim_{\mathfrak{s}\to\infty} D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}) < \epsilon$$

for each  $s > s_{\epsilon}$ . Hence for each  $s > s_{\epsilon}$ , we obtain

$$d(\varrho_{\mathfrak{s}}, \varrho^*) = D(\varrho_{\mathfrak{s}}, \Omega \varrho^*).$$

The compactness of  $\Omega \varrho^*$  implies that there exists  $\varrho \in \Omega \varrho^*$  such that

$$\begin{split} d(\varrho_{\mathfrak{s}},\varrho^{*}) &= D(\varrho_{\mathfrak{s}},\Omega\varrho^{*}) = d(\varrho_{\mathfrak{s}},\varrho) \\ &\leq \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)d(\varrho_{\mathfrak{s}+1},\varrho) \\ &= \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)D(\varrho_{\mathfrak{s}+1},\Omega\varrho^{*}) \\ &\leq \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)H(\Omega\varrho_{\mathfrak{s}},\Omega\varrho^{*}) \\ &< \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)\rho(d(\varrho_{\mathfrak{s}},\varrho^{*}),D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}),D(\varrho^{*},\Omega\varrho^{*}), \\ &D(\varrho_{\mathfrak{s}},\Omega\varrho^{*}),D(\varrho^{*},\Omega\varrho_{\mathfrak{s}})) \\ &\leq \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1},\varrho)\rho(d(\varrho_{\mathfrak{s}},\varrho^{*}),D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}),d(\varrho^{*},\varrho^{*}), \\ &d(\varrho_{\mathfrak{s}},\varrho^{*}),\mathfrak{f}(\varrho^{*},\varrho_{\mathfrak{s}})d(\varrho^{*},\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}})). \end{split}$$

As  $\lim_{s\to\infty} f(\varrho_s, \varrho) \leq 1$  and  $\rho(1, 0, 0, 1, 1) \in (0, 1)$ , taking the limit as  $\mathfrak{s} \to \infty$ , we get  $d(\varrho_s, \varrho^*) \to 0$  as  $\mathfrak{s} \to \infty$ , which is a contradiction. Hence the fixed point problem is well posed for the multivalued mapping  $\Omega$  with respect to *D*. Also, Fix  $\Omega = \mathsf{SFix} \Omega$ , and hence the fixed point problem is well posed with respect to *H*.

### 6 Conclusion

In this research, we have established some fixed and strict fixed point results on controlled metric spaces. We followed the scheme of Iqbal et al. [10] and used the platform of controlled metric setting, and hence results given in [10] are particular cases of those given in the present paper. We have also provided the well-posedness of the theorems. The data dependence problem of fixed points of the considered mappings is also established. Many nontrivial examples are provided for authentication purposes.

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#### Availability of data and materials

The data used to support the findings of this study are available from the corresponding author upon request.

#### Declarations

#### **Competing interests**

The authors declare no competing interests.

#### Author contributions

Conceptualization and methodology, D.S., S.B., A.D., and A.S.; validation and formal analysis, S.B., H.A., S.M., and W.K.; investigation and data curation, D.S., S.B., A.D., A.S., and H.A.; writing—original draft preparation, D.S., and S.B.; funding acquisition, S.M., and W.K. All authors have read and agreed to the published version of the manuscript. All authors read and approved the final manuscript

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