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Stability analysis for set-valued inverse mixed variational inequalities in reflexive Banach spaces

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Abstract

This work is devoted to the analysis for a new class of set-valued inverse mixed variational inequalities (SIMVIs) in reflexive Banach spaces, when both the mapping and the constraint set are perturbed simultaneously by two parameters. Several equivalence characterizations are given for SIMVIs to have nonempty and bounded solution sets. Based on the equivalence conditions, under the premise of monotone mappings, the stability result for the SIMVIs is obtained in the reflexive Banach space. Furthermore, to illustrate the results, an example of the traffic network equilibrium control problem is provided at the end of this paper. The results presented in this paper generalize and extend some known results in this area.

Mathematics Subject Classification: 47H05; 49J53

Keywords: Stability analysis; Set-valued inverse mixed variational inequality; Monotone mapping

1 Introduction

As an important part of nonlinear analysis, variational inequalities are widely applied in finance, transportation, economics, optimization, engineering science, and other fields. In the initial stage of the development of variational inequalities, researchers have mainly studied the nonemptiness and boundedness of variational inequality solution sets; see [1–3]. In recent years, the stability analysis of variational inequalities has been extensively developed because of the importance of investigating the properties of solutions to problems with perturbed data in practical applications; see [4–7]. McLinden [8] investigated the stability of a variational inequality with monotone operators and convex sets in reflexive Banach spaces and obtained a variety of results about stability involving a natural parameter. Addi et al. [9] investigated the stability of a finite semicoercive variational inequalities with respect to data perturbation by using recession analysis. He et al. [10] investigated the stability of generalized variational inequalities with either the mapping or the constraint set perturbed in reflexive Banach spaces. Fan et al. [11] studied the stability of a variational inequality where the mapping and the constraint set are perturbed simultaneously in reflexive Banach spaces. In addition, with the proposed mixed variational inequality, Zhong et al. [12] analyzed the stability of a class of Minty mixed variational inequalities in reflexive Banach spaces based on the analysis in [10] and extended the results in [10].

Inverse variational inequality (IVI) research has also made great progress, as an IVI is a special case of a variational inequality. Yang [13] considered the dynamic power price problem as defined by an IVI in finite dimensional spaces from the perspective of optimal control. He et al. [14] regarded congestion control problems in finite dimensional spaces as constrained black box inverse inequality problems and solved a class of constrained 'black box' inverse variational inequalities in finite dimensional spaces. Scrimali et al. [15] used an evolutionary IVI to study a time-dependent spatial pricing equilibrium control problem in finite dimensional spaces. Li et al. [16] used an inverse mixed variational inequality in Hilbert spaces to study the equilibrium control problem of a transportation network. Barbagallo et al. [17] proposed using an IVI in Hilbert spaces to solve an oligopolistic market equilibrium problem. Moreover, to address IVI problems, Luo [18] used the Tikhonov regularization method to study the perturbation analysis of the solution set of the regularized inverse variational inequality in finite dimensional spaces. Vuong [19] used the neural network to obtain a projection algorithm for solving the IVI in finite dimensional spaces. Xu [20] used the image space analysis to investigate an inverse variational inequality with a cone constraint. Hu and Fang [21] studied the Levitin–Polyak well-posedness of IVIs in finite dimensional spaces. Luo [22] studied the stability for the set-valued inverse variational inequality with both the mapping and the constraint set that are perturbed in a reflexive Banach space. Aussel et al. [23] studied a gap function and error bounds for the IVI in finite dimensional spaces. Jiang et al. [24] used ADMM to analyze structured IVIs to solve policy design difficulties in finite dimensional spaces. Very recently, Zhang [25] investigated error bounds of an inverse mixed quasi-variational inequality problem in Hilbert spaces. Tangkhawiwetkul [26] studied and analyzed the generalized inverse mixed variational inequality in Hilbert spaces and obtained the existence and uniqueness of the solution for the problem. For more related research works, we can see [27–29].

However, most results about IVIs are about the existence, well-posedness, and applications, there are very few studies on the stability of IVIs in infinite dimensional spaces. But the stability analysis of IVIs with perturbed parameters is very important because it can help in identifying relatively high accuracy, predicting the future changes of the equilibria as a result of the changes in the governing system, providing valuable information for designing various equilibrium systems. Moreover, most results about IVIs are discussed in finite dimensional spaces or Hilbert spaces where the mappings in IVIs are single-valued. Thus, it is worth studying the stability of a generalized inverse variational inequality, which is called a set-valued inverse mixed variational inequality (SIMVI), with the constraint set and the mapping perturbed simultaneously by different parameters in reflexive Banach spaces. To illustrate the results, some examples are provided. To the best of our knowledge, the results are new.

The paper is built up as follows. Section 2 provides a few useful definitions and lemmas. In Sect. 3, to make the SIMVI have a nonempty and bounded solution set, we offer a number of equivalent characterizations. In Sect. 4, the stability of the solutions for the SIMVI with the mapping and the constraint set perturbed simultaneously is obtained. In Sect. 5, to illustrate the results, we give an example. In Sect. 6, we give the conclusion.

2 Preliminaries

In this paper, we let E be a reflexive Banach space with its dual space E^*, and let Λ be a nonempty, convex, and closed subset of E^*. Let $\Gamma : E \to 2^{E^*}$ be a set-valued mapping and $\Phi : E^* \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex functional. We denote the set-valued inverse variational inequality by SIMVI(Λ, Γ), which means finding $w \in E$ and $w^* \in \Gamma(w) \cap \Lambda$ such that

$$\langle v - w^*, w\rangle + \Phi(v) - \Phi\bigl(w^*\bigr) \geq 0 \quad \text{for all } v \in \Lambda.$$

Note that if $E = \mathbb{R}^n$ and Γ is single-valued, the SIMVI(Λ, Γ) may be simplified to the inverse mixed variational inequality (IMVI) shown below: find $w \in \mathbb{R}^n$ such that

$$\Gamma(w) \in \Lambda, \langle \tilde{v} - \Gamma(w), w\rangle + \Phi(\tilde{v}) - \Phi\bigl(\Gamma(w)\bigr) \geq 0 \quad \text{for all } \tilde{v} \in \Lambda.$$

The work about IMVIs can be found in [4, 5, 16]. If $\Phi \equiv 0$ on R^n, then IMVI can be transformed to an inverse variational inequality (IVI): find $w \in \mathbb{R}^n$ such that

$$\Gamma(w) \in \Lambda, \langle \tilde{v} - \Gamma(w), w\rangle \geq 0 \quad \text{for all } \tilde{v} \in \Lambda.$$

We use the sign "$\to$" for strong convergence and "$\rightharpoonup$" to represent weak convergence. The barrier cone of Λ is defined by

$$\operatorname{barr}(\Lambda) := \Bigl\{ w \in E : \sup_{v \in \Lambda} \langle v, w\rangle < \infty \Bigr\}.$$

The recession cone of Λ is a closed and convex cone defined by

$$\Lambda_\infty := \bigl\{ d \in E^* : \exists t_n \downarrow 0, \exists w_n \in \Lambda, t_n w_n \rightharpoonup d \bigr\}$$

or

$$\Lambda_\infty := \bigl\{ d \in E^* : w_0 + \lambda d \in \Lambda, \text{for all } \lambda > 0, w_0 \in \Lambda \bigr\}.$$

The definition of negative polar cone of Λ is

$$\Lambda^- := \bigl\{ v \in E^* : \langle v, w\rangle \leq 0, \text{for all } w \in \Lambda \bigr\},$$

and $\operatorname{int}(\Lambda)$ represents the interior of Λ.

Assume that $\Phi : \Lambda \subset E^* \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous functional. The recession function of Φ, denoted by Φ_∞, is defined by

$$\Phi_\infty(w) := \lim_{t \to +\infty} \frac{\Phi(w_0 + tw) - \Phi(w_0)}{t},$$

where w_0 is any point in $\Phi = \{v \in E^* : \Phi(v) < +\infty\}$. Then it means that

$$\Phi_\infty(w) := \lim_{t \to +\infty} \frac{\Phi(tw)}{t}.$$

The functional $\Phi_\infty(\cdot)$ has been proved to be a proper, convex, lower semicontinuous, and weakly lower semicontinuous with the property that

$$\Phi(w+v) \leq \Phi(w) + \Phi_\infty(v) \quad \text{for all } w \in \operatorname{dom}\Phi, v \in E^*. \tag{2.1}$$

Obviously, we know that $\Phi_\infty(\cdot)$ is positively homogeneous of degree 1, i.e.,

$$\Phi_\infty(\lambda w) = \lambda \Phi_\infty(w) \quad \text{for all } w \in E, \lambda \geq 0. \tag{2.2}$$

The conjugate function $\Phi^*(w) : E \to \mathbb{R} \cup \{+\infty\}$ of Φ is defined by

$$\Phi^*(w) := \sup_{v \in \Lambda} \bigl\{ \langle v, w \rangle - \Phi(v) \bigr\},$$

where the domain of Φ^* is defined by $\operatorname{dom}\Phi^* = \{w \in E : \Phi^*(w) < +\infty\}$.

According to Proposition 2.5 in [6], we have

$$\Phi_\infty(w) \leq \liminf_{n\to\infty} \frac{\Phi(t_n w_n)}{t_n}, \tag{2.3}$$

where w_0 is any point in $\operatorname{dom}\Phi$, $\{w_n\}$ is any sequence in E converging weakly to w, and t_n is any real sequence converging to $+\infty$.

Definition 2.1 [22] A set-valued mapping $\Gamma : E \to 2^{E^*}$ is said to be

(i) upper semicontinuous at $w_0 \in E$ if, for any neighborhood $N(\Gamma(w_0))$ of $\Gamma(w_0)$, there exists a neighborhood $N(w_0)$ of w_0 such that $\Gamma(w) \subset N(\Gamma(w_0))$ for all $w \in N(w_0)$;

(ii) lower semicontinuous at $w_0 \in E$ if, for any $v_0 \in \Gamma(w_0)$ and any neighborhood $N(v_0)$ of v_0, there exists a neighborhood $N(w_0)$ of w_0 such that $\Gamma(w) \bigcap N(v_0) \neq \emptyset$ for all $w \in N(w_0)$;

(iii) upper hemicontinuous iff the restriction of Γ to every line segment of E is upper semicontinuous;

(iv) monotone on E iff, for all (w, w^*), (v, v^*) in the graph Γ,

$$\bigl\langle v^* - w^*, v - w \bigr\rangle \geq 0.$$

It is evident that Γ is lower semicontinuous at $w_0 \in E$ if and only if, for any w_n with $w_n \to w_0$ and $v_0 \in \Gamma(w_0)$, there exists $v_n \in \Gamma(w_n)$ such that $v_n \to v_0$.

Lemma 2.1 [10] *Let $K \subset E$ be a nonempty closed convex set. If* $\operatorname{barr} K$ *has nonempty interior, then there does not exist $w_n \subset K$ with $\|w_n\| \to \infty$ such that $\frac{w_n}{\|w_n\|} \rightharpoonup 0$. If additionally K is a cone, then there does not exist $d_n \subset K$ with each $\|d_n\| = 1$ such that $d_n \rightharpoonup 0$.*

Lemma 2.2 [11] *Let (Z, d) be a metric space, $\alpha_0 \in Z$ be a given point. Let $L : Z \to 2^{E^*}$ be a set-valued mapping with nonempty values, and L is upper semicontinuous at α_0, then there exists a neighborhood $\mathcal{W}$ of α_0 such that $(L(\alpha))_\infty \subset (L(\alpha_0))_\infty$ for all $\alpha \in \mathcal{W}$.*

Lemma 2.3 [30] *Let K be a nonempty convex subset of a Hausdorff topological vector space X and $G : K \to 2^X$ be a set-valued mapping from K into X satisfying the following properties*:

(a) *G is a KKM mapping, i.e., for every finite subset A of K,* $\mathrm{co}(A) \subset \bigcup_{w\in A} G(w)$;
(b) *$G(w)$ is a closed set in X for every $w \in K$;*
(c) *$G(w_0)$ is compact in X for some $w_0 \in K$.*
Then $\bigcap_{w\in K} G(w) \neq \emptyset$.

3 Boundedness of solution sets

In this section, we give some characterizations about the solutions of the SIMVI(Λ, F). Theorem 3.1 is critical for demonstrating the equivalence of the nonemptiness and boundedness of the solution set. For the convenience of discussion, we let $G := \Lambda \times E$ and propose the set-valued dual inverse mixed variational inequality (for short, SDIMVI(G,Γ)), which means finding $(v, w) \in G$ such that

$$\inf_{\mu^*\in F(\mu)} \langle \mu^* - z, \mu - w\rangle + \langle z - v, \mu\rangle + \Phi(z) - \Phi(v) \geq 0 \quad \text{for all } (z,\mu) \in G, \tag{3.1}$$

which is closely related to SIMVI(Λ, Γ).

Theorem 3.1 *Assume that $\Lambda \subset E^*$ is a nonempty convex and closed set, $\Gamma : E \to 2^{E^*}$ is a set-valued mapping with nonempty values, and $\Phi : \Lambda \subset E^* \to \mathbb{R}$ is a lower semicontinuous convex functional. Then we have two conclusions as follows*:
(a) *every solution of* SIMVI(Λ, Γ) *can solve* SDIMVI(G, Γ) *when Γ is monotone*;
(b) *every solution of* SDIMVI(G, Γ) *can solve* SIMVI(Λ, Γ) *when Γ is upper hemicontinuous.*

Proof Firstly, we prove conclusion (a). Assume that w is a solution of SIMVI(Λ, Γ), then there exists $w^* \in \Gamma(w) \cap \Lambda$ such that $\langle v - w^*, w\rangle + \Phi(v) - \Phi(w^*) \geq 0$ for all $v \in \Lambda$. Because Γ is monotone, then for any $(z, \mu) \in G$ and any $\mu^* \in \Gamma(\mu)$, we have

$$\begin{aligned}
0 &\leq \langle \mu^* - w^*, \mu - w\rangle \\
&= \langle \mu^* - z + z - w^*, \mu - w\rangle \\
&= \langle \mu^* - z, \mu - w\rangle + \langle z - w^*, \mu - w\rangle \\
&= \langle \mu^* - z, \mu - w\rangle + \langle z - w^*, \mu\rangle - \langle z - w^*, w\rangle + \Phi(z) - \Phi(w^*) - (\Phi(z) - \Phi(w^*)) \\
&= \langle \mu^* - z, \mu - w\rangle + \langle z - w^*, \mu\rangle + \Phi(z) - \Phi(w^*) - \big[\langle z - w^*, u\rangle + \Phi(z) - \Phi(w^*)\big] \\
&\leq \langle \mu^* - z, \mu - w\rangle + \langle z - w^*, \mu\rangle + \Phi(z) - \Phi(w^*).
\end{aligned}$$

It follows that

$$\inf_{\mu^*\in\Gamma(\mu)} \langle \mu^* - z, \mu - w\rangle + \langle z - w^*, \mu\rangle + \Phi(z) - \Phi(w^*) \quad \text{for all } (z,\mu) \in G. \tag{3.2}$$

We replace $w^* \in \Gamma(w) \cap \Lambda$ in (3.2) with v, so there exists $(v, w) \in G$ such that

$$\inf_{\mu^*\in\Gamma(\mu)} \langle \mu^* - z, \mu - w\rangle + \langle z - v, \mu\rangle + \Phi(z) - \Phi(v) \geq 0 \quad \text{for all } (z,\mu) \in G,$$

which means w solves SDIMVI(G, Γ).

Next, we prove (b). Assume that $(v,w)\in G$ is a solution of SDIMVI(G,Γ), then we have

$$\inf_{\mu^*\in\Gamma(\mu)}\langle\mu^*-z,\mu-w\rangle+\langle z-v,\mu\rangle+\Phi(z)-\Phi(v)\geq 0\quad\text{for all }(z,\mu)\in G. \tag{3.3}$$

For any $\hat{w}\in E$, $\hat{v}\in\Lambda$ and $v\in\Lambda$, we let $w(\tau)=w+\tau(\hat{w}-w)$ and $v(\tau)=v+\tau(\hat{v}-v)\in\Lambda$ for all $\tau\in[0,1]$. Take $z=v(\tau)$, $\mu=w(\tau)$, then by virtue of (3.3), it follows that

$$\inf_{\mu^*\in\Gamma(w(\tau))}\langle\mu^*-v(\tau),w(\tau)-w\rangle+\langle v(\tau)-v,w(\tau)\rangle+\Phi\big(v(\tau)\big)-\Phi(v)\geq 0,$$

which means

$$\inf_{\mu^*\in\Gamma(w(\tau))}\langle\mu^*-v(\tau),\tau(\hat{w}-w)\rangle+\langle\tau(\hat{v}-v),w(\tau)\rangle+\Phi\big(v+\tau(\hat{v}-v)\big)-\Phi(v)\geq 0.$$

Due to Φ is convex, we know that

$$\inf_{\mu^*\in\Gamma(w(\tau))}\langle\mu^*-v(\tau),\tau(\hat{w}-w)\rangle+\langle\tau(\hat{v}-v),w(\tau)\rangle+\tau\Phi(\hat{v})-\tau\Phi(v)\geq 0.$$

Since $\tau\in[0,1]$, there is

$$\inf_{\mu^*\in\Gamma(w(\tau))}\langle\mu^*-v(\tau),\hat{w}-w\rangle+\langle\hat{v}-v,w(\tau)\rangle+\Phi(\hat{v})-\Phi(v)\geq 0,$$

and so

$$\sup_{\mu^*\in\Gamma(w(\tau))}\langle\mu^*-v(\tau),\hat{w}-w\rangle+\langle\hat{v}-v,w(\tau)\rangle+\Phi(\hat{v})-\Phi(v)\geq 0.$$

Because Γ is upper hemicontinuous and $\tau\in[0,1]$, it can be seen from (iii) of Definition 2.1 that Γ is upper semicontinuous. Let $\tau\to 0^+$, it follows from the definition of upper semicontinuity that

$$\sup_{w^*\in F(w)}\langle w^*-v,\hat{w}-w\rangle+\langle\hat{v}-v,w\rangle+\Phi(\hat{v})-\Phi(v)\geq 0\quad\text{for all }\hat{w}\in E,\hat{v}\in\Lambda.$$

Since $\hat{w}\in E$ was chosen arbitrarily, we take $\hat{w}=w-ru$ for any $r\in\mathbb{R}$ and any $u\in E$, we know that there exists $w^*\in\Gamma(w)$ such that

$$\langle w^*-v,-ru\rangle+\langle\hat{v}-v,w\rangle+\Phi(\hat{v})-\Phi(v)\geq 0\quad\text{for all }\hat{v}\in\Lambda,$$

which means

$$r\langle w^*-v,u\rangle\leq\langle\hat{v}-v,w\rangle+\Phi(\hat{v})-\Phi(v)\quad\text{for all }\hat{v}\in\Lambda. \tag{3.4}$$

If $\hat{v}\in\Lambda$ is fixed, then $\langle\hat{v}-v,w\rangle+\Phi(\hat{v})-\Phi(v)$ is a constant; therefore, we get for any $r\in\mathbb{R}$

$$r\langle w^*-v,u\rangle\leq\text{constant}.$$

As a result, we can deduce that $w^* = v \in \Lambda$. Owing to $u \in E$ was chosen arbitrarily, from (3.4), there exists $w^* \in \Gamma(w) \cap \Lambda$ such that

$$\langle \hat{v} - w^*, w \rangle + \Phi(\hat{v}) - \Phi(w^*) \geq 0 \quad \text{for all } \hat{v} \in \Lambda.$$

Thus, we conclude that w solves the SIMVI(Λ, Γ). □

Remark 3.1 When $\Phi \equiv 0$, based on the same conditions, Luo [22] obtained the corresponding result of Theorem 3.1. Thus, we note that Theorem 3.1 extends the results in Theorem 3.1 in [22].

Theorem 3.2 *Assume that $\Lambda \subset E^*$ is a nonempty, convex, and closed set, $\Gamma : E \to 2^{E^*}$ is a set-valued mapping with nonempty values, and $\Phi : \Lambda \subset E^* \to \mathbb{R}$ is a convex and lower semicontinuous functional,* $\operatorname{int}\{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\} \neq \emptyset$ *and* $\operatorname{int}(\operatorname{dom}\Phi^*) \neq \emptyset$. *Consider the following assertions*:

(a) $\Lambda_\infty \cap \{d \in E^* : \langle d, w \rangle + \Phi_\infty(d) \leq 0$ *for all* $w \in \{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\}\} = \{0\}$;

(b) *$D_1 \subset E$ and $D_2 \subset \Lambda$ are two bounded sets, where $D := D_2 \times D_1 \subset G$, such that for any $w \in E/D_1$, $v \in \Lambda/D_2$, there exist some $\bar{\mu} \in D_1$, $\bar{z} \in D_2$ such that*

$$\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - \bar{z}, \mu - w \rangle + \langle \bar{z} - v, \bar{\mu} \rangle + \Phi(\bar{z}) - \Phi(v) < 0; \tag{3.5}$$

(c) *The solution set of* SIMVI(Λ, Γ) *is nonempty and bounded*;

(d) $\operatorname{int}\{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\} \cap \operatorname{int}(-\operatorname{dom}\Phi^*) \neq \emptyset$.

Then (a) ⇒ (b) *if* $\operatorname{int}(\operatorname{barr}\Lambda) \neq \emptyset$; (b) ⇒ (c) *if F is upper hemicontinuous and monotone*; (c) ⇒ (d); (d) ⇒ (a).

Proof (a) ⇒ (b): If not, we suppose that (b) does not hold, then we can choose a sequence $\{(v_n, w_n)\} \subset G$, satisfying for any n, $\|v_n\| \geq n$, $\|w_n\| \geq n$, and

$$\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - w_n \rangle + \langle z - v_n, \mu \rangle + \Phi(z) - \Phi(v_n) \geq 0 \tag{3.6}$$

for any $(z, \mu) \in G$, where $\|z\| < n$, $\|\mu\| < n$. Without losing the generality, we let $d_n = \frac{v_n}{\|v_n\|}$, and so d_n weakly converges d_0 as $n \to \infty$. By the definition of the recession cone, we know that $d_n \in \Lambda_\infty$. Because Λ_∞ is closed, we obtain $d_0 \in \Lambda_\infty$. Since $\operatorname{int}(\operatorname{barr}\Lambda) \neq \emptyset$ and from Lemma 2.1, it can be seen that $d_0 \neq 0$. Now, we let $\mu = \tilde{\mu} \in \{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\}$, $\mu^* = \tilde{\mu}^* \in \Gamma(\tilde{\mu}) \cap \Lambda$, and $z = \tilde{\mu}^*$ in (3.6), there exists a large number $M_1 > 0$, for $n \in [M_1, +\infty)$ one has

$$\langle \tilde{\mu}^* - \tilde{\mu}^*, \tilde{\mu} - w_n \rangle + \langle \tilde{\mu}^* - v_n, \tilde{\mu} \rangle + \Phi(\tilde{\mu}^*) - \Phi(v_n) \geq 0,$$

and so

$$\langle \tilde{\mu}^* - v_n, \tilde{\mu} \rangle + \Phi(\tilde{\mu}^*) - \Phi(v_n) \geq 0.$$

Multiplying both sides by $\frac{1}{\|v_n\|}$, we get

$$\frac{1}{\|v_n\|} \langle \tilde{\mu}^* - v_n, \tilde{\mu} \rangle + \frac{\Phi(\tilde{\mu}^*) - \Phi(v_n)}{\|v_n\|} \geq 0,$$

it shows that

$$\left\langle \frac{\tilde{\mu}^*}{\|v_n\|}, \tilde{\mu} \right\rangle + \frac{\Phi(\tilde{\mu}^*)}{\|v_n\|} \geq \left\langle \frac{v_n}{\|v_n\|}, \tilde{\mu} \right\rangle + \frac{\Phi(v_n)}{\|v_n\|}.$$

Then we have

$$\liminf_{n\to\infty} \left[\left\langle \frac{\tilde{\mu}^*}{\|v_n\|}, \tilde{\mu} \right\rangle + \frac{\Phi(\tilde{\mu}^*)}{\|v_n\|} \right] \geq \liminf_{n\to\infty} \left[\left\langle \frac{v_n}{\|v_n\|}, \tilde{\mu} \right\rangle + \frac{\Phi(v_n)}{\|v_n\|} \right],$$

it implies that

$$\liminf_{n\to\infty} \left[\left\langle \frac{\tilde{\mu}^*}{\|v_n\|}, \tilde{\mu} \right\rangle + \frac{\Phi(\tilde{\mu}^*)}{\|v_n\|} \right] \geq \liminf_{n\to\infty} \left\langle \frac{v_n}{\|v_n\|}, \tilde{\mu} \right\rangle + \liminf_{n\to\infty} \frac{\Phi(v_n)}{\|v_n\|}.$$

By (2.3), we get

$$0 \geq \langle d_0, \tilde{\mu} \rangle + \Phi_\infty(d_0),$$

which implies $d_0 \in \{d \in E^* : \langle d, w \rangle + \Phi_\infty(d) \leq 0 \quad \text{for all } w \in \{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\}\}$, and so $0 \neq d_0 \in \Lambda_\infty \cap \{d \in E^* : \langle d, w \rangle + \Phi_\infty(d) \leq 0 \quad \text{for all } w \in \{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\}\}$, which contradicts (a).

(b) $\Rightarrow$ (c): We define $H : G \to 2^G$ as follows:

$$H(z,\mu) := \left\{ (v,w) \in G : \inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - w \rangle + \langle z - v, \mu \rangle + \Phi(z) - \Phi(v) \geq 0 \right\}$$
$$\text{for all } (z,\mu) \in G.$$

Let $\{(v_n, w_n)\} \subset H(z,\mu)$ with $(v_n, w_n) \to (v_0, w_0)$, then

$$\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - w_n \rangle + \langle z - v_n, \mu \rangle + \Phi(z) - \Phi(v_n) \geq 0.$$

It turns out that

$$\liminf_{n\to\infty} \left[\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - w_n \rangle + \langle z - v_n, \mu \rangle + \Phi(z) \right] \geq \liminf_{n\to\infty} \Phi(v_n),$$

because Φ is a lower semicontinuous functional, we have

$$\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - w_0 \rangle + \langle z - v_0, \mu \rangle + \Phi(z) \geq \Phi(v_0).$$

We deduce that $(v_0, w_0) \in H(z,\mu)$, which means $H(z,\mu)$ is closed.

Step 1. Next we will demonstrate that H is a KKM mapping. In fact, by contradiction, suppose that there exist $\gamma_1, \gamma_2, \ldots, \gamma_n \in [0,1]$, $\sum_{i=1}^n \gamma_i = 1$, and

$$(\tilde{v}, \tilde{w}) = \gamma_1(z_1, \mu_1) + \gamma_2(z_2, \mu_2) + \cdots + \gamma_n(z_n, \mu_n) \in \operatorname{co}\left\{(z_1, \mu_1), (z_2, \mu_2), \ldots, (z_n, \mu_n)\right\}$$

for any finite set $\{(z_1, \mu_1), (z_2, \mu_2), \ldots, (z_n, \mu_n)\} \in G$ such that

$$(\tilde{v}, \tilde{w}) \notin \bigcup H_{i \in \{1,2,\ldots,n\}}(z_i, \mu_i).$$

Then, for any $i = 1, 2, \dots, n$,

$$\inf_{\mu_i^* \in \Gamma(\mu_i)} \langle \mu_i^* - z_i, \mu_i - \tilde{w} \rangle + \langle z_i - \tilde{v}, \mu_i \rangle + \Phi(z_i) - \Phi(\tilde{v}) < 0.$$

Due to Γ is monotone, there is $\mu_i^* \in \Gamma(\mu_i)$ such that for any $u^* \in \Gamma(\tilde{u})$ and $i = 1, 2, \dots, n$,

$$\begin{aligned} 0 &> \langle \mu_i^* - w^* + w^* - z_i, \mu_i - \tilde{w} \rangle + \langle z_i - \tilde{v}, \mu_i \rangle + \Phi(z_i) - \Phi(\tilde{v}) \\ &\geq \langle w^* - z_i, \mu_i \rangle - \langle w^* - z_i, \tilde{w} \rangle + \langle z_i - \tilde{v}, \mu_i \rangle + \Phi(z_i) - \Phi(\tilde{v}) \\ &= \langle w^* - \tilde{v}, \mu_i \rangle - \langle w^* - z_i, \tilde{w} \rangle + \Phi(z_i) - \Phi(\tilde{v}). \end{aligned}$$

Since Φ is convex, then we get

$$\begin{aligned} 0 &> \left\langle w^* - \tilde{v}, \sum_{i=1}^n \gamma_i \mu_i \right\rangle - \left\langle w^* - \sum_{i=1}^n \gamma_i z_i, \tilde{w} \right\rangle + \Phi\left(\sum_{i=1}^n \gamma_i z_i \right) - \Phi(\tilde{v}) \\ &= \langle w^* - \tilde{v}, \tilde{w} \rangle - \langle w^* - \tilde{v}, \tilde{w} \rangle + \Phi(\tilde{v}) - \Phi(\tilde{v}) \\ &= 0, \end{aligned}$$

which is contradiction. Therefore, H is the KKM mapping.

Step 2. We can suppose that D is a bounded, convex, and closed subset(if not, we consider replacing D with the closed convex hull of D). Let $\{(z_1, \mu_1), (z_2, \mu_2), \dots, (z_m, \mu_m)\}$ be the definite number of points in G, and let $N := \operatorname{co}(D \cup \{(z_1, \mu_1), (z_2, \mu_2), \dots, (z_m, \mu_m)\})$. N is weakly compact convex. Next, we consider the set-valued mapping $\widetilde{H}$, defined by $\widetilde{H}(z, \mu) := H(z, \mu) \cap N$ for any $(z, \mu) \in N$.

Firstly, we prove that $H(z, \mu)$ is a convex set for any $(z, \mu) \in N$. Let $\lambda \in [0, 1]$, for arbitrary $w_1, w_2 \in E$ and $v_1, v_2 \in \Lambda$, there is

$$\begin{aligned} &\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - [\lambda w_1 + (1 - \lambda) w_2] \rangle + \langle z - [\lambda v_1 + (1 - \lambda) v_2], \mu \rangle + \Phi(z) \\ &\quad - \Phi(\lambda v_1 + (1 - \lambda) v_2) \\ &= \inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \lambda(\mu - w_1) \rangle + \langle \mu^* - z, (1 - \lambda)(\mu - w_2) \rangle + \langle \lambda(z - v_1), x \rangle \\ &\quad + \langle (1 - \lambda)(z - v_2), \mu \rangle + \lambda \Phi(z) + (1 - \lambda) \Phi(z) - \Phi(\lambda v_1 + (1 - \lambda) v_2) \\ &\geq \lambda \Big[\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - w_1 \rangle + \langle z - v_1, \mu \rangle + \Phi(z) - \Phi(v_1) \Big] \\ &\quad + (1 - \lambda) \Big[\inf_{\mu^* \in \Gamma(\mu)} \langle \mu^* - z, \mu - w_2 \rangle + \langle z - v_2, \mu \rangle + \Phi(z) - \Phi(v_2) \Big] \\ &\geq 0, \end{aligned}$$

so $H(z, \mu)$ is convex. It is easy to know that each $\widetilde{H}(z, \mu)$ is a weakly compact convex subset of N. Obviously, $\widetilde{H}(z, \mu)$ is a closed set for any $(z, \mu) \in N$.

Now we prove $\widetilde{H}$ is a KKM mapping. On the contrary, suppose that there are $\eta_1, \eta_2, \dots, \eta_n \in [0, 1]$, $\sum_{i=1}^n \eta_i = 1$, and

$$(t, s) = \eta_1(z_1, \mu_1) + \eta_2(z_2, \mu_2) + \cdots + \eta_n(z_n, \mu_n) \in \operatorname{co}\{(z_1, \mu_1), (z_2, \mu_2), \dots, (z_n, \mu_n)\}$$

for any finite set $\{(z_1,\mu_1),(z_2,\mu_2),\dots,(z_n,\mu_n)\}\in N$ such that

$$(t,s)\notin\bigcup\widetilde{H}_{i\in\{1,2,\dots,n\}}(z_i,\mu_i).$$

Then, for any $i=1,2,\dots,n$,

$$\inf_{\mu_i^*\in\Gamma(\mu_i)}\langle\mu_i^* z_i,\mu_i-s\rangle+\langle z_i-t,\mu_i\rangle+\Phi(z_i)-\Phi(t)<0.$$

Because Γ is monotone, there exists $\mu_i^*\in\Gamma(\mu_i)$ such that for any $w^*\in\Gamma(s)$ and $i=1,2,\dots,n$,

$$\begin{aligned}0&>\langle\mu_i^*-w^*+w^*-v_i,\mu_i-s\rangle+\langle z_i-t,\mu_i\rangle+\Phi(z_i)-\Phi(t)\\&\geq\langle\mu_i^*-w^*,\mu_i-s\rangle+\langle w^*-z_i,\mu_i-s\rangle+\langle z_i-t,\mu_i\rangle+\Phi(z_i)-\Phi(t)\\&\geq\langle w^*-z_i,\mu_i\rangle-\langle w^*-z_i,s\rangle+\langle z_i-t,\mu_i\rangle+\Phi(z_i)-\Phi(t)\\&=\langle w^*-t,\mu_i\rangle-\langle w^*-z_i,s\rangle+\Phi(z_i)-\Phi(t).\end{aligned}$$

Since Φ is convex, then we get

$$\begin{aligned}0&>\left\langle w^*-t,\sum_{i=1}^n\eta_i\mu_i\right\rangle-\left\langle w^*-\sum_{i=1}^n\eta_i z_i,s\right\rangle+\Phi\left(\sum_{i=1}^n\eta_i z_i\right)-\Phi(t)\\&=\langle w^*-t,s\rangle-\langle w^*-t,s\rangle+\Phi(t)-\Phi(t)\\&=0,\end{aligned}$$

which leads to a contradiction. Therefore, $\widetilde{H}$ is a KKM mapping.

Step 3. Since $\widetilde{H}(z,\mu)$ is a weakly compact closed set for any $(z,\mu)\in N$ and $\widetilde{H}$ is also a KKM mapping, so by Lemma 2.3 we have

$$\emptyset\neq\bigcap_{(z,x)\in N}\widetilde{H}(z,\mu).$$

Furthermore, if there exists some $(v_0,w_0)\in\bigcap_{(z,\mu)\in N}\widetilde{H}(z,\mu)$ but $(v_0,w_0)\notin D$, then from (3.5) we know that for some $(\bar{z},\bar{\mu})\in D_2\times D_1\subset G$, there is

$$\inf_{\mu^*\in\Gamma(\mu)}\langle\mu^*-\bar{z},\bar{\mu}-w_0\rangle+\langle\bar{z}-v_0,\bar{\mu}\rangle+\Phi(\bar{\mu})-\Phi(v_0)<0,$$

therefore, $(v_0,w_0)\notin H(z,\mu)$, and so $(v_0,w_0)\notin\widetilde{H}(z,\mu)$, it is a contradiction. Then

$$\emptyset\neq\bigcap_{(z,\mu)\in N}\widetilde{H}(z,\mu)\subset D.\tag{3.7}$$

Let $(v,w)\in\bigcap_{(z,\mu)\in N}\widetilde{H}(z,\mu)$, it can be seen from (3.7) that $(v,w)\in D$, then we have

$$\bigcap_{(z,\mu)\in N}\widetilde{H}(z,\mu)\subset\bigcap_{i=1}^m\widetilde{H}(z_i,\mu_i),$$

and hence $(v,w) \in \bigcap_{i=1}^{m}(H(z_i,\mu_i)\cap D)$. It can be known that $\{H(z,\mu)\cap D : (z,\mu)\in G\}$ has the property of finite intersection. For each $(z,\mu)\in G$, it follows from the weak compactness of $H(z,\mu)\cap D$ that $\bigcap_{(z,\mu)\in G}(H(z,\mu)\cap D)$ is nonempty, which coincides with the solution set of SDIMVI(G,Γ). Thus, according to Theorem 3.1, we can obtain that the solution set of SIMVI(Λ,Γ) is nonempty and bounded.

(c) $\Rightarrow$ (d): If (d) does not hold, then $\operatorname{int}(\{w\in E : \Gamma(w)\cap\Lambda\neq\emptyset\})\cap\operatorname{int}(-\operatorname{dom}\Phi^*)=\emptyset$. There exists $w\in E$ and $\Gamma(w)\cap\Lambda\neq\emptyset$, but $w\notin\operatorname{int}(-\operatorname{dom}\Phi^*)$, then we have

$$\sup_{v\in\Lambda}\bigl\{\langle v,w\rangle-\Phi(v)\bigr\}\geq+\infty. \tag{3.8}$$

From (c) we know that the solution set of SIMVI(Λ,Γ) is nonempty, then there exists $w\in E$, $w^*\in\Gamma(w)\cap\Lambda$ such that

$$\bigl\langle\tilde{v}-w^*,w\bigr\rangle+\Phi(\tilde{v})-\Phi\bigl(w^*\bigr)\geq 0\quad\text{for all }\tilde{v}\in\Lambda.$$

Let $d_0\in\Lambda_\infty$, according to the definition of recession of cone, there is $w^*+\lambda d_0\in\Lambda$, where $\lambda>0$. Due to the arbitrariness of $\tilde{v}\in\Lambda$, taking $\tilde{v}=w^*+td\in\Lambda$, we have

$$\bigl\langle w^*+td-w^*,u\bigr\rangle+\Phi\bigl(w^*+td\bigr)-\Phi\bigl(w^*\bigr)\geq 0$$

and

$$\bigl\langle w^*+td,-w\bigr\rangle-\Phi\bigl(w^*+td\bigr)\leq\bigl\langle -w^*,w\bigr\rangle-\Phi\bigl(w^*\bigr).$$

Since $-\Phi(w^*)<+\infty$. Thus, $\langle -w^*,w\rangle-\Phi(w^*)<+\infty$, and so

$$\bigl\langle w^*+td,-w\bigr\rangle-\Phi\bigl(w^*+td\bigr)<+\infty,$$

which contradicts (3.8). The proof is complete.

(d) $\Rightarrow$ (a): If (a) does not hold, then $\Lambda_\infty\cap\{d\in E^* : \langle d,w\rangle+\Phi_\infty(d)\leq 0$ for all $w\in\{w\in E : \Gamma(w)\cap\Lambda\neq\emptyset\}\}\neq\{0\}$. We know that there exists a sequence $\{d_n\}\subset\Lambda_\infty\cap\{d\in E^* : \langle d,w\rangle+\Phi_\infty(d)\leq 0$ for all $w\in\{w\in E : \Gamma(w)\cap\Lambda\neq\emptyset\}\}$. Without losing the generality, we let $d_n=\frac{v_n}{\|v_n\|}$, and so d_n weakly converges d_0 as $n\to\infty$. Since Λ_∞ is a closed and convex cone, so $d_0\in\Lambda_\infty$. According to Lemma 2.1, it follows that $d_0\neq 0$.

For any $\tilde{w}\in\{w\in E : \Gamma(w)\cap\Lambda\neq\emptyset\}$, we have

$$\langle d_n,\tilde{w}\rangle+\Phi_\infty(d_n)\leq 0.$$

Combining with $d_n\rightharpoonup d_0$ and the weak lower semicontinuity of $\Phi_\infty(\cdot)$, it follows that

$$\langle d_0,-\tilde{w}\rangle-\Phi_\infty(d_0)\geq 0. \tag{3.9}$$

Due to $\operatorname{int}(\{w\in E : \Gamma(w))\cap\Lambda\neq\emptyset\}\cap(\operatorname{int}(-\operatorname{dom}\Phi^*)\neq\emptyset$, then there exists $\xi\in\operatorname{int}(\{w\in E : \Gamma(w))\cap\Lambda\neq\emptyset\}\cap\operatorname{int}(-\operatorname{dom}\Phi^*)$. Next we prove that

$$\langle d_0,-\xi\rangle-\Phi_\infty(d_0)=0. \tag{3.10}$$

Indeed, if (3.10) does not hold, then

$$\langle d_0, -\xi\rangle - \Phi_\infty(d_0) > 0, \tag{e}$$

as (3.9) holds.

By $\xi \in \operatorname{int}(-\operatorname{dom}\Phi^*)$, we obtain

$$\Phi^*(-\xi) = \sup_{v\in\Lambda} \langle v, -\xi\rangle - \Phi(v) < +\infty. \tag{3.11}$$

Let $w_0 \in \Lambda$. $d_0 \in \Lambda_\infty$ implies that $w_0 + td_0 \in \Lambda$ for all $t > 0$. It follows from (3.11) that

$$\langle w_0 + td_0, -\xi\rangle - \Phi(w_0 + td_0) < +\infty.$$

From (2.1) and (2.2), we have

$$\langle w_0 + td_0, -\xi\rangle - \Phi(w_0) - t\Phi_\infty(d_0) < +\infty,$$

which immediately implies that

$$\langle w_0, -\xi\rangle - \Phi(w_0) + t\big(\langle d_0, -\xi\rangle - \Phi_\infty(d_0)\big) < +\infty.$$

It is known that $w_0 \in \Lambda$, and from (3.11) we can deduced that $\langle w_0, -\xi\rangle - \Phi(w_0) < +\infty$. Thus

$$t\big(\langle d_0, -\xi\rangle - \Phi_\infty(d_0)\big) < +\infty. \tag{3.12}$$

According to (e), and letting $t \to +\infty$, we get $t(\langle d_0, -\xi\rangle - \Phi_\infty(d_0))$ has no upper bound, which contradicts (3.12). Therefore, (3.10) is proved. Since $\xi \in \operatorname{int}\{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\} \cap \operatorname{int}(-\operatorname{dom}\Phi^*)$, for any $w_1 \in E$, there exists $t \in (0,1)$ such that $\xi + (1-t)w_1 \in \operatorname{int}\{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\} \cap \operatorname{int}(-\operatorname{dom}\Phi^*)$. From (3.9), we know that

$$\big\langle d_0, -\big(\xi + (1-t)w_1\big)\big\rangle - \Phi_\infty(d_0) \geq 0,$$

because $\langle d_0, -\xi\rangle - \Phi_\infty(d_0) = 0$, then we have $\langle d_0, -w_1\rangle \geq 0$.

For any $w_1 \in E$, there exists $t \in (0,1)$ such that $\xi - (1-t)w_1 \in \operatorname{int}\{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\} \cap \operatorname{int}(-\operatorname{dom}\Phi^*)$. By (3.9), we get

$$\big\langle d_0, -\big(\xi - (1-t)w_1\big)\big\rangle - \Phi_\infty(d_0) \geq 0.$$

By $\langle d_0, -\xi\rangle - \Phi_\infty(d_0) = 0$, we can obtain $\langle d_0, w_1\rangle \geq 0$. Then it can be deduced that $\langle d_0, w_1\rangle = 0$, which contradicts $d_0 \neq 0$. Thus $\Lambda_\infty \cap \{d \in E^* : \langle d, w\rangle + \Phi_\infty \leq 0 \quad \text{for all } w \in \{w \in E : \Gamma(w) \cap \Lambda \neq \emptyset\}\} = \{0\}$ is verified. □

4 Stability for the SIMVI

Our goal in this section is to establish the stability of the solutions for SIMVI(Λ, Γ) with monotone and upper hemicontinuous mappings. The following Theorem 4.1 is of great help in obtaining the stability results.

Theorem 4.1 *Let $\alpha_0 \in Z_1$ and $\beta_0 \in Z_2$ be given points, (Z_1, d_1) and (Z_2, d_2) be metric spaces, $\Gamma : E \times Z_2 \to 2^{E^*}$ be a lower semicontinuous set-valued mapping for the second variable on Z_2, $\Phi : L(\alpha) \subset E^* \to \mathbb{R}$ be a convex and lower semicontinuous functional, and $L : Z_1 \to 2^{E^*}$ be a continuous set-valued mapping. Assume there is a neighborhood $\mathcal{W} \times \mathcal{U}$ of (α_0, β_0) such that $\Gamma(w, \beta)$ has nonempty, closed values for every $w \in E$ and $\beta \in \mathcal{U}$, and $L(\alpha)$ has nonempty, convex, closed values for any $\alpha \in \mathcal{W}$. If*

$$\begin{aligned}&\big(L(\alpha_0)\big)_\infty \cap \big\{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \textit{ for all } w \in \big\{w \in E : \Gamma(w, \beta_0) \cap L(\alpha_0)\big\} \ne \emptyset\big\}\\ &\quad = \{0\},\end{aligned}$$

then there exists a neighborhood $\bar{\mathcal{W}} \times \bar{\mathcal{U}}$ of (α_0, β_0) with $\bar{\mathcal{W}} \times \bar{\mathcal{U}} \subset \mathcal{W} \times \mathcal{U}$, such that for any $(\alpha, \beta) \in \bar{\mathcal{W}} \times \bar{\mathcal{U}}$,

$$\begin{aligned}&\big(L(\alpha)\big)_\infty \cap \big\{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0, \textit{for all } w \in \big\{w \in E : \Gamma(w, \beta) \cap L(\alpha)\big\} \ne \emptyset\big\}\\ &\quad = \{0\}.\end{aligned}$$

Proof Suppose the conclusion is not true, then for any neighborhood $\bar{\mathcal{W}} \times \bar{\mathcal{U}}$ of (α_0, β_0), there is $(\alpha, \beta) \in \bar{\mathcal{W}} \times \bar{\mathcal{U}}$ such that $(L(\alpha))_\infty \cap \{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \quad \text{for all } w \in \{w \in E : \Gamma(w, \beta) \cap L(\alpha)\} \ne \emptyset\} \ne \{0\}$ holds. Then, we can select a sequence (α_n, β_n) in $Z_1 \times Z_2$ with (α_n, β_n) converging to (α_0, β_0) such that for any n, $(L(\alpha_n))_\infty \cap \{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \quad \text{for all } w \in \{w \in E : \Gamma(w, \beta_n) \cap L(\alpha_n)\} \ne \emptyset\} \ne \{0\}$. Thus, there exists a sequence $\{d_n\}$ such that for any n, $\|d_n\| = 1$ and $d_n \in (L(\alpha_n))_\infty \cap \{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \quad \text{for all } w \in \{w \in E : \Gamma(w, \beta_n) \cap L(\alpha_n)\} \ne \emptyset\}$. Without losing the generality, we can suppose that $d_n \rightharpoonup d_0$ as $n \to \infty$. Furthermore, according to Lemma 2.1 it can be known that $d_0 \ne 0$. Since $d_n \in (L(\alpha_n))_\infty$ and L on Z_1 is upper semicontinuous, then from Lemma 2.2 we have $d_n \in (L(\alpha_0))_\infty$ for sufficiently large n. Because $(L(\alpha_0))_\infty$ is a closed cone, so $d_0 \in (L(\alpha_0))_\infty$.

For any given $\bar{w} \in \{w \in E : \Gamma(w, \beta_0) \cap L(\alpha_0) \ne \emptyset\}$, there is $\delta_0 \in E^*$ satisfying $\delta_0 \in \Gamma(\bar{w}, \beta_0) \cap L(\alpha_0)$. Next we prove that for any n, there is a sequence $\{\bar{\delta_n}\}$ such that $\bar{\delta_n} \to \delta_0$, $\bar{\delta}_n \in \Gamma(\bar{u}, \beta_n) \cap L(\alpha_n)$. If not, we prove the contrary conclusion holds then for any sequence $\{\delta_n\}$ such that $\delta_n \to \delta_0$. However, $\delta_n \notin \Gamma(\bar{w}, \beta_n) \cap L(\alpha_n)$. Since $\beta_n \to \beta_0$ and $\delta_0 \in \Gamma(\bar{w}, \beta_0)$, and Γ on Z_2 is lower semicontinuous, so there is a sequence $\{\tilde{\delta}_n\}$ such that $\{\tilde{\delta}_n\} \to \delta_0$ and $\tilde{\delta}_n \in \Gamma(\bar{w}, \beta_n)$ for any n. Therefore, $\tilde{\delta}_n \notin L(\alpha_n)$. Because Γ and L have nonempty closed values and L is lower semicontinuous, $\delta_0 \notin L(\alpha_0)$. Therefore, we get a contradiction. Now we know that there exists $\bar{\delta}_n \in \Gamma(\bar{w}, \beta_n) \cap L(\alpha_n)$, it means that $\Gamma(\bar{w}, \beta_n) \cap L(\alpha_n) \ne \emptyset$. Moreover, $\bar{w}$ is fixed, then we get $\bar{w} \in \{w \in E : \Gamma(w, \beta_n) \cap L(\alpha_n) \ne \emptyset\}$. From $d_n \in \{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \quad \text{for all } w \in \{w \in E : \Gamma(w, \beta_n) \cap L(\alpha_n) \ne \emptyset\}\}$, we can obtain

$$\langle d_n, \bar{w}\rangle + \Phi_\infty(d_n) \le 0.$$

Combining with $d_n \rightharpoonup d_0$ and the weak lower semicontinuity of $\Phi_\infty(\cdot)$, it follows that

$$\langle d_0, \bar{w}\rangle + \Phi_\infty(d_0) \le 0.$$

Then we have

$$d_0 \in \big\{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0, \text{for all } w \in \big\{w \in E : \Gamma(w, \beta_0) \cap L(\alpha_0) \ne \emptyset\big\}\big\},$$

and so

$$d_0 \in \big(L(\alpha_0)\big)_\infty \cap \Big\{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0, \text{for all } w \in \big\{w \in E : \Gamma(w, \beta_0) \cap L(\alpha_0) \neq \emptyset\big\}\Big\}$$

with $d_0 \neq 0$, which contradicts the assumption and ends the proof of the theorem. □

Remark 4.1 If $\Phi \equiv 0$, then $\Phi_\infty \equiv 0$. Consequently, Theorem 4.1 reduces to Theorem 4.1 of [22]. Thus, Theorem 4.1 is a generalization of Theorem 4.1 in [22].

If K is bounded, we know $K_\infty = \{0\}$, so from Theorem 4.1 we have the following result.

Corollary 4.1 *Let $\alpha_0 \in Z_1$ and $\beta_0 \in Z_2$ be given points, (Z_1, d_1) and (Z_2, d_2) be metric spaces, $\Gamma : E \times Z_2 \to 2^{E^*}$ be a set-valued mapping with nonempty value, $\Phi : L(\alpha) \subset E^* \to \mathbb{R}$ be a convex and lower semicontinuous functional, and $L : Z_1 \to 2^{E^*}$ be a set-valued mapping with nonempty bounded value. Then, for any $(\alpha, \beta) \in Z_1 \times Z_2$,*

$$\big(L(\alpha)\big)_\infty \cap \Big\{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \text{ for all } w \in \big\{w \in E : \Gamma(w, \beta) \cap L(\alpha) \neq \emptyset\big\}\Big\} = \{0\}.$$

Theorem 4.2 *Let $(\alpha, \beta) \in \bar{\mathcal{W}} \times \bar{\mathcal{U}}$, $S(\alpha, \beta)$ and $S(\alpha_0, \beta_0)$ represent the solution sets of* SIMVI$(L(\alpha), \Gamma(\cdot, \beta))$ *and* SIMVI$(L(\alpha_0), \Gamma(\cdot, \beta_0))$, *respectively. If the conditions in Theorem 4.1 hold and*

(a) *for every $\beta \in \mathcal{U}$, the mapping $w \mapsto \Gamma(w, \beta)$ is monotone and upper hemicontinuous on E;*
(b) *the solution set of* SIMVI$(L(\alpha_0), \Gamma(\cdot, \beta_0))$ *is nonempty and bounded;*
(c) int$\{w \in E : \Gamma(w, \beta) \cap L(\alpha) \neq \emptyset\} \neq \emptyset$ *and* int$(-\operatorname{dom}\Phi^*) \neq \emptyset$.

Then

(A) *there exists a neighborhood $\bar{\mathcal{W}} \times \bar{\mathcal{U}}$ of (α_0, β_0) with $\bar{\mathcal{W}} \times \bar{\mathcal{U}} \subset \mathcal{W} \times \mathcal{U}$ such that for every $(\alpha, \beta) \in \bar{\mathcal{W}} \times \bar{\mathcal{U}}$, the solution set of* SIMVI$(L(\alpha), \Gamma(\cdot, \beta))$ *is nonempty and bounded;*
(B) *if Φ is continuous on $L(\alpha)$ with $\alpha \in \bar{\mathcal{W}}$ and $\bigcup_{\alpha \in \bar{\mathcal{W}}} L(\alpha)$ is compact, then $\limsup_{(\alpha,\beta)\to(\alpha_0,\beta_0)} S(\alpha, \beta) \subset S(\alpha_0, \beta_0)$.*

Proof

(A) By Theorem 3.2, it follows from condition (b) in Theorem 4.2 that

$$\big(L(\alpha_0)\big)_\infty \cap \{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \text{ for all } w \in \big\{w \in E : \Gamma(w, \beta_0) \cap L(\alpha_0) \neq \emptyset\big\} = \{0\}.$$

Next, applying Theorem 4.1, we know that there is a neighborhood $\bar{\mathcal{W}} \times \bar{\mathcal{U}}$ of (α_0, β_0) with $\bar{\mathcal{W}} \times \bar{\mathcal{U}} \subset \mathcal{W} \times \mathcal{U}$ such that for any $(\alpha, \beta) \in \bar{\mathcal{W}} \times \bar{\mathcal{U}}$,

$$\big(L(\alpha)\big)_\infty \cap \{d \in E^* : \langle d, w\rangle + \Phi_\infty(d) \le 0 \text{ for all } w \in \big\{w \in E : \Gamma(w, \beta) \cap L(\alpha) \neq \emptyset\big\} = \{0\}.$$

Utilizing Theorem 3.2 once more, we can now derive the solution set of SIMVI($L(\alpha), \Gamma(\cdot,\beta)$) is nonempty and bounded for any $(\alpha,\beta) \in \bar{\mathcal{W}} \times \bar{\mathcal{U}}$.

(B) For any sequence $\{(\alpha_n,\beta_n)\} \subset \bar{\mathcal{W}} \times \bar{\mathcal{U}}$ with $(\alpha_n,\beta_n) \to (\alpha_0,\beta_0)$ when $n \to \infty$, we will verify that

$$\limsup_{n\to\infty} S(\alpha_n,\beta_n) \subset S(\alpha_0,\beta_0).$$

Let $w_0 \in \limsup_{n\to\infty} S(\alpha_n,\beta_n)$, then there exists a sequence $\{w_n\} \subset S(\alpha_n,\beta_n)$ such that $w_n \to w_0$ as $n \to \infty$. From Theorem 3.1 and Theorem 4.2 (A), it can be known that the solution set of SDIMVI($G(\alpha_n), \Gamma(\cdot,\beta_n)$) is nonempty and bounded, where $G(\alpha_n) := L(\alpha_n) \times E$. So there exists $v_n \in L(\alpha_n)$ such that $v_n \to v_0$ as $n \to \infty$ and

$$\begin{aligned}&\inf_{\mu^*\in\Gamma(\mu,\beta_n)} \langle \mu^* - z, \mu - w_n\rangle + \langle z - v_n, \mu\rangle + \Phi(z) - \Phi(v_n)\\ &\quad \geq 0, \quad \text{for all } (z,\mu) \in L(\alpha_n) \times E. \end{aligned}\tag{4.1}$$

On the other side, for any $z_0 \in L(\alpha_0)$ and ${\mu_0}^* \in \Gamma(\mu,\beta_0)$, L is lower semicontinuous on Z_1, and $\alpha_n \to \alpha_0$, then there exists $z_n \in L(\alpha_n)$ such that $z_n \to z_0$ as $n \to \infty$. Since Γ is lower semicontinuous on Z_2 and $\beta_n \to \beta_0$, there exists ${\mu_n}^* \in \Gamma(\mu,\beta_n)$ such that ${\mu_n}^* \to {\mu_0}^*$ as $n \to \infty$. According to $v_n \in L(\alpha_n)$ and upper semicontinuity of L, we get $v_0 \in L(\alpha_0)$. Therefore, (4.1) means that

$$\langle {\mu_n}^* - z_n, \mu - w_n\rangle + \langle z_n - v_n, \mu\rangle + \Phi(z_n) - \Phi(v_n) \geq 0 \quad \text{for all } \mu \in E.$$

Since Φ is continuous on $L(\alpha)$, letting $n \to +\infty$, we obtain $(v_0, w_0) \in L(\alpha_0) \times E$ and

$$\begin{aligned}&\inf_{{\mu_0}^*\in\Gamma(\mu,\beta_0)} \langle {\mu_0}^* - z_0, \mu - w_0\rangle + \langle z_0 - v_0, w\rangle + \Phi(z_0) - \Phi(v_0)\\ &\quad \geq 0 \quad \text{for all } z_0 \in L(\alpha_0), \mu \in E.\end{aligned}$$

Using Theorem 3.1 once more, there exist $w_0 \in E$ and ${w_0}^* \in \Gamma(w_0,\beta_0) \cap L(\alpha_0)$ such that

$$\langle \tilde{v} - {w_0}^*, w_0\rangle + \Phi(\tilde{v}) - \Phi({w_0}^*) \geq 0, \quad \text{for all } \tilde{v} \in L(\alpha_0),$$

so $w_0 \in S(\alpha_0,\beta_0)$. □

Next, we will give Examples 4.1, 4.2, 4.3, 4.4 to demonstrate the necessity of the conditions in Theorem 4.2.

Example 4.1 Let $Z_1 = Z_2 = [-1,1]$, $\alpha_0 = \beta_0 = 0$, $\Phi(w) = \frac{1}{2}w$

$$L(\alpha) = [0,2],$$

$$\Gamma(w,\beta) = \begin{cases} \{0\}, & \beta \neq 0, \\ \{2\}, & \beta = 0. \end{cases}$$

It is easy to note that $L(\alpha)$ is continuous, $\Gamma(\cdot,\beta)$ is both monotone and upper hemicontinuous on $[0,2]$, and $\Gamma(w,\cdot)$ is not lower semicontinuous at $\beta = 0$. By calculation, we get $S(0,0) = (-\infty,-\frac{1}{2}]$ and $S(0,\beta) = [-\frac{1}{2},+\infty)$ for any $\beta \neq 0$. Consequently, $\limsup_{\beta\to 0} S(0,\beta) = [-\frac{1}{2},+\infty) \not\subset S(0,0)$.

Example 4.2 Let $Z_1 = Z_2 = [-1,1]$, $\alpha_0 = \beta_0 = 0$, $\Phi(w) = \frac{1}{2}w$

$$L(\alpha) = [-1,1],$$

$$\Gamma(w,\beta) = \begin{cases} \{w\}, & \beta = 0, \\ \{e^w\}, & \beta \neq 0, w \geq 0, \\ \{\ln w\}, & \beta \neq 0, w > 0. \end{cases}$$

Obviously, $L(\alpha)$ is continuous at $\alpha = 0$, $\Gamma(\cdot,\beta)$ is not monotone on $[-1,1]$, and $\Gamma(w,\cdot)$ is not lower semicontinuous at $\beta = 0$. By calculation, we get $S(0,0) = \{-\frac{1}{2}\}$ and $S(0,\beta) = \{-\frac{1}{2},\frac{1}{e}\}$ for any $\beta \neq 0$. Consequently, $\limsup_{\beta\to 0} S(0,\beta) = \{-\frac{1}{2},\frac{1}{e}\} \not\subset S(0,0)$.

Example 4.3 Let $Z_1 = Z_2 = [-1,1]$, $\alpha_0 = \beta_0 = 0$, $\Phi(w) = \frac{1}{2}w$

$$L(\alpha) = \begin{cases} [-2,0], & \alpha \neq 0, \\ [-3,0], & \alpha = 0, \end{cases}$$

$$\Gamma(w,\beta) \equiv \{-2\}.$$

Note that $L(\alpha)$ is not lower semicontinuous when $\alpha = 0$, but it is upper semicontinuous. Moreover, $\Gamma(\cdot,\beta)$ is not only monotone on $[-3,0]$, but also upper hemicontinuous. And $\Gamma(w,\cdot)$ is lower semicontinuous at $\beta = 0$. By calculations, we get $S(0,0) = \{-\frac{1}{2}\}$ and $S(\alpha,0) = [-\frac{1}{2},+\infty)$ for any $\alpha \neq 0$. Consequently, $\limsup_{\alpha\to 0} S(\alpha,0) = [-\frac{1}{2},+\infty) \not\subset S(0,0)$.

Example 4.4 Let $Z_1 = Z_2 = [-1,1]$, $\alpha_0 = \beta_0 = 0$, $\Phi(w) = \frac{1}{2}w$

$$L(\alpha) = \begin{cases} [-1,0], & \alpha \neq 0, \\ [1,2], & \alpha = 0, \end{cases}$$

$$\Gamma(w,\beta) = \begin{cases} [0,2], & \beta \neq 0, \\ [0,1], & \beta = 0. \end{cases}$$

At $\alpha = 0$, it is obvious that $L(\alpha)$ is neither lower semicontinuous nor upper semicontinuous. Moreover, $\Gamma(\cdot,\beta)$ is not monotone on $[-1,2]$, and $\Gamma(w,\cdot)$ is lower semicontinuous at $\beta = 0$. By calculation, we obtain $S(0,0) = [-\frac{1}{2},+\infty)$ and $S(\alpha,\beta) = (-\infty,-\frac{1}{2}]$ for any $\alpha \neq 0$ and $\beta \neq 0$. Consequently, $\limsup_{\alpha\to 0,\beta\to 0} S(\alpha,\beta) = (-\infty,-\frac{1}{2}] \not\subset S(0,0)$.

5 An example

In this section, we will give an example about the stability of IMVI in the traffic network equilibrium control problem.

5.1 The traffic network equilibrium control problem

As an application of our main results, we shall give an example similar to Example 2.2 in [16]. We describe it simply.

Let a network have n parallel links linking a basic origin-destination pair, each link represent a feasible path and x_i denote the flow on each link i, t_i be the user cost associated with traversing the link i, d represent the travel demand of customers traveling between origin-destination pairs. We denote

$$\Omega = \left\{ x \middle| x \geq 0, \sum_{i=1}^{n} x_i = d \right\}.$$

Now from traffic management authorities' point of view, we consider the traffic network equilibrium control problem. Assume that the total loss of vehicle accidents and road damage is determined by the flow of all network links, that is,

$$f(x) = \sum_{i=1}^{n} f_i(x_i),$$

where $f_i : R_+ \to R$ is a convex and continuous function. The goal of the traffic management authorities is to control the traffic flow x_i within predetermined intervals by adjusting the link toll y_i and to minimize the total loss of vehicle accidents and road damage in the network. For a given adjustment of $y \in R^n$, we know that the resultant traffic network equilibrium flow $x(y)$ is a solution of the following parametric variational inequality:

$$\langle z - x, t + y \rangle \leq 0 \quad \text{for all } z \in \Omega. \tag{5.1}$$

As a control approach, traffic management authorities could change link tolls to reduce loss and prevent traffic jams. We regard (5.1) as a 'black-box' procedure that returns a value of x at the point y. Consequently, the path flow $x(y)$ can be revealed. Assume that the desired link flows are constrained with the feasible link flow set $K = \{x | 0 \leq x \leq b\}$. Thus, by Example 2.2 in [16], the problem faced by the authority can be interpreted as follows:

$$\begin{aligned} &\min f\left[x(y)\right], \\ &\text{s.t.} \quad x(y) \in K, \end{aligned}$$

where $x(y)$ is a solution of (5.1). Based on the KKT theory, from [16] we know the above problem can be transformed as an inverse mixed variational inequality as follows: find $y \in C$ such that $x(y) \in K$ and

$$\left\langle z - x(y), y \right\rangle + f(z) - f\left[x(y)\right] \geq 0, \quad \text{for all } z \in K, \tag{5.2}$$

where $C = \{y \in R^n | \exists x(y) \in K, \langle z - x(y), y \rangle \leq 0\}$. We use the interval $[a, b]$ to represent the fluctuation range of gasoline prices, where a represents the lowest oil price and b represents the highest oil price. However, with the large fluctuation of gasoline prices, it will affect the traffic of vehicles on the road, which means that the feasible link flow set K and the traffic network equilibrium flow $x(y)$ will be influenced by a parameter q, where $q \in [a,b]$. Hence, we can easily see that K should be a set-valued mapping of q and x should be a set-valued mapping of y and q. Then (5.2) will be transformed as follows:

$$\langle z - x(y,q), y\rangle + f(z) - f\bigl[x(y,q)\bigr] \geq 0, \quad \text{for all } z \in K(q). \tag{5.3}$$

So, the traffic network equilibrium control problem influenced by holidays and weekdays will lead to a stability problem for a class of inverse mixed variational inequality.

Corollary 5.1 *Let $q_0 \in [a,b]$, $x(y,\cdot) : [a,b] \to R$ be a lower semicontinuous mapping on $[a,b]$, and $K : [a,b] \to R$ be a continuous set-valued mapping with nonempty, convex, closed bounded value, where $y \in R^n$. If there is a neighborhood $\mathcal{U}$ of q_0 such that $x(y,q)$ has nonempty, closed values for any $q \in \mathcal{U}$. If the following conditions hold*:

- (a) *for every $q \in \mathcal{U}$, the mapping $y \mapsto x(q,y)$ is monotone and upper hemicontinuous on R*;
- (b) *the solution set of* $\mathrm{SIMVI}(K(q_0), x(\cdot,q_0))$ *is nonempty and bounded*;
- (c) $\mathrm{int}\{(y,q) \in R^n \times [a,b] : x(y,q) \cap K(q)\} \bigcap \mathrm{int}(-\operatorname{dom} f^*) \neq \emptyset$;
- (d) *if f is continuous and convex on $K(q)$ with $q \in \mathcal{U}$ and $\cup_{q\in\mathcal{U}} K(q)$ is compact.*

Then $\limsup_{q\to q_0} S(q) \subset S(q_0)$, where $q \in \mathcal{U}$, $S(q)$ and $S(q_0)$ represent solution sets of $\mathrm{SIMVI}(K(q), x(\cdot,q))$ *and* $\mathrm{SIMVI}(K(q_0), x(\cdot,q_0)))$, *respectively.*

Proof Since $K(q)$ is a bounded set for any $q \in \mathcal{U}$, then from Corollary 4.1 we know that the conclusions of Theorem 4.1 hold. Then, using Theorem 3.2, there exists a neighborhood $\bar{\mathcal{U}}$ of q_0 with $\bar{\mathcal{U}} \subset \mathcal{U}$ such that for every $q \in \bar{\mathcal{U}}$, the solution set of $\mathrm{SIMVI}(K(q), x(\cdot,q))$ is nonempty and bounded. By using Theorem 4.2, we can obtain $\limsup_{q\to q_0} S(q) \subset S(q_0)$, where $q \in \bar{\mathcal{U}}$. □

6 Conclusion

In this paper, we introduced a new class of set-valued inverse mixed variational inequalities (SIMVI) in reflexive Banach spaces. We gave some equivalent characterizations such that the solution set of $\mathrm{SIMVI}(\Lambda, \Gamma)$ is nonempty and bounded in Theorem 3.2. The stability of SIMVI was obtained in Theorem 4.2 by using equivalent conditions when the mapping and the constraint set are perturbed simultaneously by different parameters in reflexive Banach spaces. We also gave some examples to show the conditions were necessary in Theorem 4.2. At the end of the paper, an example of the traffic network equilibrium control problem was provided to illustrate the application of the stability of IMVI. For further research, we can apply the theorem of set-valued analysis and inverse variational inequalities to study the stability of set-valued inverse quasi-mixed variational inequalities in Banach spaces.

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The authors declare no competing interests.

Author contributions
X.L.Q. and W.L. wrote primarily sections two, three, and four of the manuscript text, and C.K.X. and X.P.L. wrote sections one and five of the manuscript. All authors reviewed the manuscript.

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References

1. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, Berlin (2003)
2. Xiao, Y.B., Liu, M.T., Chen, T., Huang, N.J.: Stability analysis for evolutionary variational-hemivariational inequalities with constraint sets. Sci. China Math. **65**(7), 1469–1484 (2022)
3. Zhong, R.Y., Huang, N.J.: Strict feasibility for generalized mixed variational inequality in reflexive Banach spaces. J. Optim. Theory Appl. **152**(3), 696–709 (2012)
4. Ju, X.X., Li, C., He, X., Gang, F.: A proximal neurodynamic model for solving inverse mixed variational inequalities. Neural Netw. **138**, 1–9 (2021)
5. Chen, J.W., Ju, X.X., Kobis, E., Liou, Y.C.: Tikhonov type regularization methods for inverse mixed variational inequalities. Optimization **69**(2), 401–413 (2020)
6. Baiocchi, C., Buttazzo, G., Gastaldi, F., Tomarelli, F.: General existence theorems for unilateral problems in continuum mechanics. Arch. Ration. Mech. Anal. **100**, 149–189 (1988)
7. Adly, S., Ernst, E., Théra, M.: Stability of the solution set of non-coercive variational inequalities. Commun. Contemp. Math. **4**(1), 145–160 (2002)
8. McLinden, L.: Stable monotone variational inequalities. Math. Program. **48**, 303–338 (1990)
9. Addi, K., Adly, S., Goeleven, D., Saoud, H.: A sensitivity analysis of a class of semi-coercive variational inequalities using recession tools. J. Glob. Optim. **40**, 7–27 (2008)
10. He, Y.R.: Stable pseudomonotone variational inequality in reflexive Banach spaces. J. Math. Anal. Appl. **330**(1), 352–363 (2007)
11. Fan, J.H., Zhong, R.Y.: Stability analysis for variational inequality in reflexive Banach spaces. Nonlinear Anal. **69**(8), 2566–2574 (2008)
12. Zhong, R.Y., Huang, N.J.: Stability analysis for Minty mixed variational inequality in reflexive Banach spaces. J. Optim. Theory Appl. **147**(3), 454–472 (2010)
13. Yang, J.F.: Dynamic power price problem: an inverse variational inequality approach. J. Ind. Manag. Optim. **4**(4), 673–684 (2008)
14. He, B.S., He, X.Z., Liu, H.X.: Solving a class of constrained 'black-box' inverse variational inequalities. Eur. J. Oper. Res. **204**(3), 391–401 (2010)
15. Scrimali, L.: An inverse variational inequality approach to the evolutionary spatial price equilibrium problem. Optim. Eng. **13**(3), 375–387 (2012)
16. Li, X., Li, X.S., Huang, N.J.: A generalized f-projection algorithm for inverse mixed variational inequalities. Optim. Lett. **8**(3), 1063–1076 (2014)
17. Barbagallo, A., Mauro, P.: Inverse variational inequality approach and applications. Numer. Funct. Anal. Optim. **35**(7–9), 851–867 (2014)
18. Luo, X.P.: Tikhonov regularization methods for inverse variational inequalities. Optim. Lett. **8**(3), 877–887 (2014)
19. Vuong, P.T., He, X.Z., Thong, D.V.: Global exponential stability of a neural network for inverse variational inequalities. J. Optim. Theory Appl. **190**(3), 915–930 (2021)
20. Xu, Y.D.: Nonlinear separation approach to inverse variational inequalities. Optimization **56**(7), 1315–1335 (2016)
21. Hu, R., Fang, Y.P.: Levitin-Polyak well-posedness by perturbations of inverse variational inequalities. Optim. Lett. **7**(2), 343–359 (2013)
22. Luo, X.P.: Stability analysis of set-valued inverse variational inequalities in reflexive Banach spaces. J. Fixed Point Theory Appl. **23**(3) (2021)
23. Aussel, D., Gupta, R., Mehra, A.: Gap functions and error bounds for inverse quasi-variational inequality problems. J. Math. Anal. Appl. **407**(2), 270–280 (2013)
24. Jiang, Y.N., Cai, X.J., Han, D.R.: Solving policy design problems: alternating direction method of multipliers-based methods for structured inverse variational inequalities. Eur. J. Oper. Res. **280**(2), 417–427 (2020)

25. Zhang, Y.F., Yu, G.L.: Error bounds for inverse mixed quasi-variational inequality via generalized residual gap functions. Asia-Pac. J. Oper. Res. **39**(2) (2022)
26. Tangkhawiwetkul, J.: A neural network for solving the generalized inverse mixed variational inequality problem in Hilbert spaces. AIMS Math. **8**(3), 7258–7276 (2023)
27. Li, W., Wang, X., Huang, N.J.: Differential inverse variational inequalities in finite dimensional spaces. Acta Math. Sci. **35**(2), 407–422 (2015)
28. Zou, X.J., Gong, D.W., Wang, L.P., Chen, Z.Y.: A novel method to solve inverse variational inequality problems based on neural networks. Neurocomputing **173**(3), 1163–1168 (2016)
29. Zhu, X.Y., Li, W., Luo, X.P.: Stability for a class of differential set-valued inverse variational inequalities in finite dimensional spaces. Axioms **11**(9), 475 (2022)
30. Zalinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, River Edge (2002)

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