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Infinitely many radial solutions of superlinear elliptic problems with dependence on the gradient terms in an annulus

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Abstract

In this paper, we are concerned with elliptic problems

$$\begin{cases} -\Delta u = f(u) + g(|x|, u, \frac{x}{|x|}\cdot\nabla u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ is an annular domain, $N > 2$, $0 < R_1 < R_2 < \infty$, $4(R_2 - R_1)(N-1) \le R_1$. $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\lim_{|\xi|\to\infty} \frac{f(\xi)}{\xi} = \infty$. $g : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $|g(r, \xi_0, \xi_1)| \le C + \beta|\xi_0|$ for some $C > 0$, $0 < \beta < \frac{1}{4(R_2-R_1)^2}$. We obtain infinitely many radial solutions with prescribed nodal properties using bifurcation techniques.

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Keywords: Elliptic problems; Superlinear problem; Nodal solutions; Bifurcation; Annulus

1 Introduction

In this paper, we are concerned with the existence and multiplicity of radial nodal solutions of elliptic problems

$$\begin{cases} -\Delta u = f(u) + g(|x|, u, \frac{x}{|x|}\cdot\nabla u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ is an annular domain, $N > 2$, $0 < R_1 < R_2 < \infty$, $R_1 \ge 4(R_2 - R_1)(N-1)$.

One of the main features of the problem (1.1) is the presence of convection term $g(|x|, u, \frac{x}{|x|}\cdot\nabla u)$, depending on the function u and its gradient ∇u, which plays an important role in science and technology. For example, particles or energy are converted and transferred inside a physical system due to convection and diffusion processes. For the work related to this topic, we cite the interesting works [3, 11, 14], and references therein.

Specifically, in [21], the authors considered the existence of solutions for

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, \rho * u, \nabla(\rho * u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\rho \in L^1(\mathbb{R}^N)$. It is well known that $\nabla(\rho * u) = \rho * \nabla u$ with the convolution acting componentwise. The presence of convolution appears frequently in various applications, taking different meanings in practical problems of computer science and engineering. Concerning real-life applications, we mention, for instance, that convolution in deep learning gives the cross-correlation in signal and image processing. Specifically, the convolution is useful in signal processing to smooth out the noise in the original signal. In the related field of image processing, the result of convolution is to smooth out the rough edges in the values taken by a mapping representing mathematically the model of the image under study so that we have a blurring effect. A huge amount of technical literature is aimed at implementing concrete procedures, for example, filter operations in digital image processing that we illustrate by citing [20]. To sum up, studying elliptic problems, including gradient terms, is of great significance in both theoretical exploration and practical application.

The research on solutions of semilinear elliptic problems has been widely concerned, see [1, 2, 4–6, 8, 12, 17]. For example, the semilinear elliptic problems

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

with no gradient terms in nonlinearity, where $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ is an annular domain, $N > 2$, $0 < R_1 < R_2 < \infty$. Denote

$$f_0 := \lim_{s \to 0} \frac{f(s)}{s}, \qquad f_\infty := \lim_{s \to \infty} \frac{f(s)}{s}.$$

When $N = 1$, the existence of positive solutions of (1.3) has been obtained by Erbe and Wang [10] using fixed point techniques under $f \in C([0,\infty),[0,\infty))$ and $f_0 = 0$, $f_\infty = \infty$. Using bifurcation techniques, Ma and Thompson [18] obtained nodal solutions of (1.3) when $f \in C(\mathbb{R},\mathbb{R})$ with $f(s)s > 0$ and $f_0, f_\infty \in (0,\infty)$. When $N \geq 2$, by assuming $f(0) = 0$, $f(s) > 0$ for $s > 0$, $f_0 = 0$, $f_\infty = \infty$, Coffman and Marcus [5] showed that (1.3) had one radial solution with prescribed numbers of zeros. In [22], f is assumed to be an odd function, $f(0) = 0$ and $f_0, f_\infty \in [0,+\infty]$, by shooting method together with the Strum's comparison theorem, Naito and Tanaka demonstrated the existence of nodal solutions for (1.3). Recently, Harrabi [13] obtained the existence of nodal solutions for (1.3) by assuming $f(0) = 0$ and $f_\infty = \infty$.

When the nonlinearity depends on the gradient terms [9], Ehrnstrom discussed the existence problem of positive radial solutions for the elliptic equation with linear gradient terms

$$-\Delta u = f\big(|x|, u\big) + g\big(|x|\big)x \cdot \nabla u, \qquad x \in \Omega_1 \tag{1.4}$$

in an exterior domain $\Omega_1 = \{x \in \mathbb{R}^N : |x| > r_0\}$. Equation (1.4) was also studied by Vrodljak [24] in the special case $g(r) \equiv \frac{1}{2}$. Recently, Li [16], Xu and Wei [26] were concerned with the elliptic problems with general gradient terms:

$$\begin{cases} -\Delta u = f(|x|, u, \frac{x}{|x|} \cdot \nabla u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

where $\Omega := \{x \in \mathbb{R}^n, a < |x| < b\}$, $0 < a < b < \infty$ is an annular domain in $\mathbb{R}^N$, $N > 2$. Under the suitable conditions, Li [16], Xu and Wei [26] obtained the existence of radial solutions for (1.5). However, to our knowledge, when $N > 2$, the existence results of nodal solutions for elliptic problems in annular domains with gradient terms have not been studied. In fact, the presence of gradient terms determines the loss of the variational structure. This brings serious technical difficulty since we cannot use variational methods. Inspired by the above, in this paper, we are concerned with the existence of nodal solutions for nonhomogeneous elliptic problems (1.1) using the bifurcation technique.

Letting $u = u(r)$, $r = |x|$, (1.1) is rewritten as

$$\begin{cases} -u''(r) - \frac{N-1}{r} u'(r) = f(u(r)) + g(r, u(r), u'(r)), & r \in (R_1, R_2), \\ u(R_1) = u(R_2) = 0 \end{cases} \tag{1.6}$$

or

$$\begin{cases} -(r^{N-1} u'(r))' = r^{N-1}(f(u(r)) + g(r, u(r), u'(r))), & r \in (R_1, R_2), \\ u(R_1) = u(R_2) = 0. \end{cases} \tag{1.7}$$

In this paper, we make the following assumptions:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(\xi)\xi \geq 0$ for $\xi \in \mathbb{R}$ and satisfies

$$\lim_{|\xi| \to \infty} \frac{f(\xi)}{\xi} = \infty \quad \bigl(f \text{ is superlinear as } |\xi| \to \infty\bigr).$$

(H2) $g : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies

$$\bigl|g(r, \xi_0, \xi_1)\bigr| \leq C + \beta|\xi_0| \quad \text{for some } C > 0, 0 < \beta < \frac{1}{4(R_2 - R_1)^2}.$$

It is worth noting that although to obtain the nodal solutions for the elliptic problems, we do not require the growth conditions of f at zero like in [13, 18, 22], in fact, when the nonlinearity includes nonhomogeneous terms, we can only obtain infinitely many "large" solutions having specified nodal properties.

For any integer $i \geq 0$, let $C^i[R_1, R_2]$ denote the standard Banach space of real valued, i-times continuously differentiable functions defined on $[R_1, R_2]$, with the norm $|u|_i = \sum_{j=0}^{i} |u^{(j)}|_\infty$, where $|\cdot|_\infty$ denotes the usual sup-norm on $C^0[R_1, R_2]$. Let

$$E = \bigl\{u \in C^1[R_1, R_2] : u(R_1) = u(R_2) = 0\bigr\}, \qquad X = E \cap C^2[R_1, R_2], \qquad Y = C^0[R_1, R_2].$$

To state our results, we first recall some standard notation to describe the nodal properties of solutions. From now on, ν will denote an element of $\{\pm\}$, that is, either $\nu = +$ or $\nu = -$. For each integer $k \geq 1$, let $S_{k,\nu}$ denote the set $u \in E$ such that:

(i) u has only simple zeros in (R_1, R_2) and exactly $k-1$ simple zeros.

(ii) u is positive near R_1 if $u \in S_{k,+}$, and u is negative near R_1 if $u \in S_{k,-}$.

$S_{k,\nu} = S_{k,+} \cup S_{k,-}$, $S_{k,-} = -S_{k,+}$, $S_{k,+}$ and $S_{k,-}$ are disjoint and open in E. Our main result is the following theorem.

Theorem 1.1 *There exists an integer $k_0 \geq 1$ such that for all integers $k \geq k_0$, and each ν, the problem* (1.1) *has at least one radial solution $u_{k,\nu} \in S_{k,\nu}$.*

Remark 1.2 In [15], Kajikiya and Ko only obtained the existence of positive radial solutions for (1.3) under $f(0) < 0$. In fact, our conditions (H1)-(H2) allow for $f(0) + g(r, 0, \xi_1) < 0$. Therefore, our main results also include the research on the nodal solutions of semipositone problems.

Remark 1.3 Our research is inspired by [23], a study of nodal solutions for the fourth order problem

$$\begin{cases} u^{(4)} = g(u(x)) + p(x, u^{(0)}(x), \dots, u^{(3)}(x)), & x \in (0,1), \\ u(0) = u(1) = u^{(2)}(0) = u^{(2)}(1) = 0. \end{cases} \tag{1.8}$$

In fact, due to equivalent problem (1.6), we are dealing with contains the $\frac{N-1}{r}u'(r)$ to obtain the bound of the norm of u, we have to limit the size of the annulus. The proof of [23] relies heavily on some lemmas of [19], which introduce some properties of fourth-order problems. However, for second-order problems in the annular domain, similar lemmas as in [19] are difficult to establish. All these bring great difficulties to considering our elliptic problems, some of the proof methods in [23] are no longer applicable. Therefore, we adopt some new proof methods; see the proof of Lemmas 2.4–2.5 and Lemmas 3.4–3.6.

2 Preliminaries

By (H1), we know that

$$f(\xi)\xi \geq 0, \quad \xi \in \mathbb{R}. \tag{2.1}$$

For any $u \in X$, we define $e(u) : [R_1, R_2] \to \mathbb{R}$ by $e(u)(r) = g(r, u(r), u'(r))$, $r \in [R_1, R_2]$. It follows from (H2) that

$$\left|e(u)(r)\right| \leq C + \beta\left|u(r)\right|, \quad r \in [R_1, R_2]. \tag{2.2}$$

For any $s \in \mathbb{R}$, let $F(s) = \int_0^s f(\xi)\, d\xi \geq 0$, and for any $s \geq 0$ let

$$\gamma(s) = \max\left\{\left|f(\xi)\right| : |\xi| \leq s\right\}, \qquad \Gamma(s) = \max\left\{\left|F(\xi)\right| : |\xi| \leq s\right\}.$$

We now consider the problem

$$\begin{cases} -(r^{N-1}u'(r))' = \lambda r^{N-1}u + \alpha r^{N-1}(f(u) + e(u)), & r \in [R_1, R_2], \\ u(R_1) = u(R_2) = 0, \end{cases} \tag{2.3}$$

where $\alpha \in [0,1]$ is an arbitrary fixed number, $\lambda \in \mathbb{R}$. In the following lemmas, $(\lambda, u) \in \mathbb{R} \times X$ will be an arbitrary solution for (2.3), while $R \geq 0$ will be an arbitrary number. Also, $c_1, c_2, \dots,$ will be constants, $\zeta_1, \zeta_2, \dots,$ will be continuous functions (from $[0,+\infty)$ to $[0,+\infty)$ unless stated otherwise), and these will depend only on f and g, not on (λ, u) or α.

Lemma 2.1 *For $u \in X$, we have*

$$|u|_\infty \leq (R_2 - R_1)\left|u'\right|_\infty \leq (R_2 - R_1)^2\left|u''\right|_\infty. \tag{2.4}$$

Proof By $u(R_1) = u(R_2) = 0$ and Rolle's theorem, for any $u \in X$, each of the functions u, u' has a zero in $[R_1, R_2]$, then there exists $\tau \in (R_1, R_2)$ such that $u'(\tau) = 0$. So, the repeated application of mean value theorem shows that

$$\begin{aligned} \left|u(r)\right| &= \left|\int_{R_1}^{r} u'(s)\,ds\right| \leq \int_{R_1}^{R_2} \left|u'(s)\right|ds \leq (R_2 - R_1)\left|u'\right|_\infty, \\ \left|u'(r)\right| &= \left|\int_{\tau}^{r} u''(s)\,ds\right| \leq \int_{R_1}^{R_2} \left|u''(s)\right|ds \leq (R_2 - R_1)\left|u''\right|_\infty. \end{aligned}$$

Therefore,

$$|u|_\infty \leq (R_2 - R_1)\left|u'\right|_\infty \leq (R_2 - R_1)^2\left|u''\right|_\infty. \qquad \square$$

Lemma 2.2 *There exists ζ_1 such that if*

$$0 \leq \lambda \leq R, \qquad |u|_\infty \leq R, \tag{2.5}$$

then $|u|_1 \leq \zeta_1(R)$.

Proof By Rolle's theorem, we know that there exists $\tau \in (R_1, R_2)$ such that $u'(\tau) = 0$. By (2.2)-(2.5), we have

$$\begin{aligned} r^{N-1}u'(r) &= \lambda \int_r^\tau \xi^{N-1}u(\xi)\,d\xi + \int_r^\tau \alpha\xi^{N-1}\bigl(f(u) + e(u)\bigr)\,d\xi \\ &\leq \lambda \int_{R_1}^{R_2} \xi^{N-1}|u|_\infty\,d\xi + \int_{R_1}^{R_2} \xi^{N-1}\bigl(\left|f(u)\right| + \left|e(u)\right|\bigr)\,d\xi \\ &\leq (R_2 - R_1)R_2^{N-1}R^2 + (R_2 - R_1)R_2^{N-1}\bigl(\gamma(R) + C + \beta R\bigr). \end{aligned} \tag{2.6}$$

Thus,

$$\left|u'\right|_\infty \leq (R_2 - R_1)\left(\frac{R_2^{N-1}}{R_1^{N-1}}\right)\bigl[R^2 + \gamma(R) + C + \beta R\bigr],$$

consequently,

$$\begin{aligned} |u|_1 &= |u|_\infty + \left|u'\right|_\infty \\ &\leq \bigl[(R_2 - R_1) + 4(R_2 - R_1)^2\bigr]\left(\frac{R_2^{N-1}}{R_1^{N-1}}\right)\bigl[R^2 + \gamma(R) + C + \beta R\bigr] =: \zeta_1(R). \end{aligned} \qquad \square$$

Lemma 2.3 *For any* $r_0, r_1 \in [R_1, R_2]$, *we have*

$$
\begin{aligned}
& u'(r_1)^2 + \lambda u(r_1)^2 + 2\alpha F\bigl(u(r_1)\bigr) \\
&\quad = u'(r_0)^2 + \lambda u(r_0)^2 + 2\alpha F\bigl(u(r_0)\bigr) - 2\int_{r_0}^{r_1} \frac{N-1}{\xi} u'(\xi)^2 \, d\xi \\
&\qquad - 2\alpha \int_{r_0}^{r_1} e(u)(\xi) u'(\xi) \, d\xi.
\end{aligned}
$$

Proof (2.3) is equivalent to

$$
-u'' = \frac{N-1}{r} u' + \lambda u + \alpha\bigl(f(u) + e(u)\bigr), \quad r \in [R_1, R_2]. \tag{2.7}
$$

Multiplying (2.7) by u' and integrating, we have

$$
\begin{aligned}
-\int_{r_0}^{r_1} u''(\xi) u'(\xi) \, d\xi = {} & \int_{r_0}^{r_1} \frac{N-1}{\xi} u'(\xi) u'(\xi) \, d\xi + \lambda \int_{r_0}^{r_1} u(\xi) u'(\xi) \, d\xi \\
& + \alpha \int_{r_0}^{r_1} \bigl[f\bigl(u(\xi)\bigr) + e(u)(\xi)\bigr] u'(\xi) \, d\xi,
\end{aligned}
$$

then we have

$$
\begin{aligned}
& -\frac{1}{2} u'(r_1)^2 + \frac{1}{2} u'(r_0)^2 \\
&\quad = \int_{r_0}^{r_1} \frac{N-1}{\xi} u'(\xi)^2 \, d\xi + \frac{\lambda}{2} u(r_1)^2 - \frac{\lambda}{2} u(r_0)^2 + \alpha F\bigl(u(r_1)\bigr) - \alpha F\bigl(u(r_0)\bigr) \\
&\qquad + \alpha \int_{r_0}^{r_1} e(u)(\xi) u'(\xi) \, d\xi.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& u'(r_1)^2 + \lambda u(r_1)^2 + 2\alpha F\bigl(u(r_1)\bigr) \\
&\quad = u'(r_0)^2 + \lambda u(r_0)^2 + 2\alpha F\bigl(u(r_0)\bigr) - 2\int_{r_0}^{r_1} \frac{N-1}{\xi} u'(\xi)^2 \, d\xi \\
&\qquad - 2\alpha \int_{r_0}^{r_1} e(u)(\xi) u'(\xi) \, d\xi.
\end{aligned} \tag{2.8}
$$

□

Lemma 2.4 *There exists an increasing function* ζ_2 *such that if* $0 \le \lambda \le R$, *and* $|u(r_0)| + |u'(r_0)| \le R$, *for some* $r_0 \in [R_1, R_2]$, *then* $|u|_\infty \le \zeta_2(R)$.

Proof Choose $r_1 \in [R_1, R_2]$ satisfying $|u'|_\infty = |u'(r_1)|$. By Lemma 2.3, we have

$$
\begin{aligned}
\bigl|u'(r_1)\bigr|^2 \le {} & u'(r_0)^2 + \lambda u(r_0)^2 + 2F\bigl(u(r_0)\bigr) + 2\left|\int_{R_1}^{R_2} \frac{N-1}{\xi} u'(\xi)^2 \, d\xi\right| \\
& + 2\left|\int_{r_0}^{r_1} e(u)(\xi) u'(\xi) \, d\xi\right|.
\end{aligned} \tag{2.9}
$$

Therefore,

$$\begin{aligned}\left|u'\right|_\infty^2 &\le R^2 + R^3 + 2\Gamma(R) + 2(R_2 - R_1)\frac{N-1}{R_1}\left|u'\right|_\infty^2 \\ &\quad + 2\bigl(C + \beta(R_2 - R_1)\left|u'\right|_\infty\bigr)\left|u'\right|_\infty(R_2 - R_1).\end{aligned}$$

Let

$$K(R) = R^2 + R^3 + 2\Gamma(R),$$

thus

$$\left[1 - 2(R_2 - R_1)\frac{N-1}{R_1} - 2\beta(R_2 - R_1)^2\right]\left|u'\right|_\infty^2 - 2C(R_2 - R_1)\left|u'\right|_\infty - K(R) \le 0,$$

since

$$1 - 2(R_2 - R_1)\frac{N-1}{R_1} - 2\beta(R_2 - R_1)^2 > 0,$$

let

$$\hat{A} := (R_2 - R_1), \qquad \hat{B} := (R_2 - R_1)^2,$$

consequently,

$$\begin{aligned}&\left|u'\right|_\infty \le \hat{\zeta}_2(R) := \frac{2C\hat{A} + \sqrt{4C^2\hat{A}^2 + (4 - 8\hat{A}\frac{N-1}{R_1} - 8\beta\hat{B})K(R)}}{2 - 4\hat{A}\frac{N-1}{R_1} - 4\beta\hat{B}}, \\ &|u|_\infty \le \hat{A}\hat{\zeta}_2(R) := \zeta_2(R).\end{aligned}$$

□

By (H1), we can choose $c_1 \ge (R_2 - R_1)$ such that

$$|\xi| \ge c_1 \quad \Rightarrow \quad \left|f(\xi)\right| \ge C + \beta|\xi|. \tag{2.10}$$

We also define

$$\zeta_3(\xi) = \begin{cases} \zeta_2(\xi + \xi^2) + c_1 & \text{for } \xi \ge c_1, \\ \zeta_3(c_1) & \text{for } \xi < c_1. \end{cases}$$

Lemma 2.5 *If $R \ge c_1$, $0 \le \lambda \le R$ and $|u|_\infty \ge \zeta_3(R)$, then for any $r_0 \in [R_1, R_2]$ with $|u(r_0)| \le R$, we have $|u'(r_0)| \ge R^2$.*

Proof Suppose that for some $R \ge c_1$, there exists $r_0 \in (R_1, R_2)$ such that $|u(r_0)| \le R$ and $|u'(r_0)| < R^2$. We will show that it is impossible if $|u|_\infty \ge \zeta_3(R)$.

Now,

$$\left|u(r_0)\right| + \left|u'(r_0)\right| < R + R^2, \tag{2.11}$$

Combining this with $\lambda \le R < R + R^2$ and using Lemma 2.4, we conclude that

$$|u|_\infty \le \zeta_2\bigl(R + R^2\bigr).$$

However, it is impossible if $|u|_\infty \ge \zeta_3(R)$.

Finally, if $r_0 \in \{R_1, R_2\}$ and $|u'(r_0)| < R^2$, then by continuity, there exists $r^* \in (R_1, R_2)$ such that $|u(r^*)| \le R$ and $|u'(r^*)| < R^2$, which contradicts the result just proved. □

3 Auxiliary problem

We now consider the problem

$$-\bigl(r^{N-1}u'(r)\bigr)' = \lambda r^{N-1}u + \theta\left(\frac{|u|_\infty}{\zeta_3(\lambda)}\right) r^{N-1}\bigl(f(u) + e(u)\bigr), \quad u \in X, \tag{3.1}$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is increasing, C^∞ function with $\theta(s) = 0$, $s \le 1$ and $\theta(s) = 1$, $s \ge 2$ (we have replaced α in (2.3) with the function $\theta(\frac{|u|_\infty}{\zeta_3(\lambda)})$. The nonlinearity in (3.1) is a continuous function of $(\lambda, u) \in \mathbb{R} \times X$ and is zero for $\lambda \in \mathbb{R}$, $|u|_\infty \le \zeta_3(\lambda)$. So, (3.1) becomes a linear eigenvalue problem in this region, and overall the problem can be regarded as a bifurcation (from $u = 0$) problem.

Regarding the linear problem, define the operator $L : X \to Y$ by $Lu = -(r^{N-1}u'(r))'$, $u \in X$. By [25](Chapter IV, Sect. 27), we know that the eigenvalue problem

$$\begin{cases} -(r^{N-1}u'(r))' = \mu r^{N-1}u, & r \in [R_1, R_2], \\ u(R_1) = u(R_2) = 0, \end{cases} \tag{3.2}$$

has a strictly increasing sequence of eigenvalues

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$$

with $\lim_{k\to\infty} \mu_k = \infty$. Each eigenvalue μ_k, $k \ge 1$, has a corresponding eigenfunction ϕ_k having exactly $k - 1$ zeros in (R_1, R_2). The next Lemmas now follow immediately:

Lemma 3.1 *The set of solutions* (λ, u) *of* (3.1) *with* $|u|_\infty \le \zeta_3(\lambda)$ *is*

$$\bigl\{(\lambda, 0) : \lambda \in \mathbb{R}\bigr\} \cup \left\{(\lambda_k, t\phi_k) : k \ge 1, |t| \le \frac{\zeta_3(\lambda)}{|\phi_k|_\infty}\right\}.$$

Lemma 3.2 *For each* $k \ge 1$ *and* ν, *there exists a connected set* $\mathcal{C}_{k,\nu} \subset \mathbb{R} \times E$ *of nontrivial solution for* (3.1) *such that* $\mathcal{C}_{k,\nu} \cup (\mu_k, 0)$ *is closed and connected and*:

(i) *there exists a neighbourhood* N_k *of* $(\mu_k, 0)$ *in* $\mathbb{R} \times E$ *such that* $N_k \cap \mathcal{C}_{k,\nu} \subset \mathbb{R} \times S_{k,\nu}$,

(ii) *either* $\mathcal{C}_{k,\nu} \cap \mathcal{C}_{k',\nu'} \ne \emptyset$ *for some* $(k', \nu') \ne (k, \nu)$, *or* $\mathcal{C}_{k,\nu}$ *meets infinity in* $\mathbb{R} \times E$ (*that is, there exists a sequence* $(\lambda_n, u_n) \in \mathcal{C}_{k,\nu}$, $n = 1, 2, \ldots$, *such that* $|\lambda_n| + |u_n|_\infty \to \infty$).

Proof Since $L^{-1} : Y \to X$ exists and is bounded, (3.1) can be rewritten in the following form

$$u = \lambda L^{-1}r^{N-1}u + \theta\left(\frac{|u|_\infty}{\zeta_3(\lambda)}\right) L^{-1}r^{N-1}\bigl(f(u) + e(u)\bigr). \tag{3.3}$$

Since L^{-1} can be regarded as a compact operator from Y to E, it is clear that finding a solution (λ, u) for (3.1) in $R \times X$ is equivalent to finding a solution for (3.3) in $\mathbb{R} \times E$. Now, by the similar method used in the proof of [7], we may deduce the desired result. □

In the following section, we will rely on the preservation of nodal properties for "large" solutions encapsulated in the following Lemmas.

Lemma 3.3 *If* (λ, u) *is a solution for* (3.1) *with* $\lambda \geq 0$ *and* $|u|_\infty \geq \zeta_3(\lambda)$, *then* $u \in S_{k,\nu}$ *for some* $k \geq 1$ *and* ν.

Proof If $u \notin S_{k,\nu}$ for any $k \geq 1$ and ν, then u must have a double zero, but this contradicts Lemma 2.5. □

In view of Lemmas 3.1 and 3.3, in the following Lemmas, we suppose that (λ, u) is an arbitrary nontrivial solution for (3.1) with $\lambda \geq 0$ and $u \in S_{k,\nu}$, for some $k \geq 1$ and ν.

Lemma 3.4 *There exists an integer* $k_0 \geq 1$ *such that if* $\lambda = 0$ *and* $\zeta_3(0) \leq |u|_\infty \leq 2\zeta_3(0)$, *then* $k < k_0$.

Proof Let r_1, r_2 be consecutive zeros of u. Then there exists $r_3 \in (r_1, r_2)$ such that $u'(r_3) = 0$, and hence, by Lemma 2.5 with $R = c_1$,

$$\left|u(r_3)\right| \geq R = c_1 \geq (R_2 - R_1).$$

Hence

$$\begin{aligned}\left|u'\right|_\infty (r_2 - r_1) &= \left|u'\right|_\infty (r_3 - r_1) + \left|u'\right|_\infty (r_2 - r_3) \\ &\geq \left|u(r_3) - u(r_1)\right| + \left|u(r_2) - u(r_3)\right| \geq 2(R_2 - R_1).\end{aligned}$$

So,

$$|r_2 - r_1| \geq \frac{2(R_2 - R_1)}{|u'|_\infty}.$$

By Lemma 2.2, we have

$$|u|_\infty + \left|u'\right|_\infty \leq \zeta_1\bigl(2\zeta_3(0)\bigr),$$

so

$$|r_2 - r_1| \geq \frac{2(R_2 - R_1)}{|u'|_\infty} \geq \frac{2(R_2 - R_1)}{\zeta_1(2\zeta_3(0)) - |u|_\infty} \geq \frac{2(R_2 - R_1))}{\zeta_1(2\zeta_3(0)) - \zeta_3(0)}.$$

Thus,

$$R_2 - R_1 = (R_2 - r_{k-1}) + \cdots + (r_1 - R_1) \geq k \frac{2(R_2 - R_1)}{\zeta_1(2\zeta_3(0)) - \zeta_3(0)} > (R_2 - R_1)$$

if $k > [\frac{\zeta_1(2\zeta_3(0)) - \zeta_3(0)}{2}] + 1 =: k_0$. Contradiction! □

Now let

$$V_R(u) = \{r \in [R_1, R_2] : |u(r)| \geq R\}, \qquad W_R(u) = \{r \in [R_1, R_2] : |u(r)| < R\}.$$

Lemma 3.5 *Suppose that $R \geq c_1$, $0 \leq \lambda \leq R$ and $|u|_\infty \geq \zeta_3(R)$. Then $W_R(u)$ consists of exactly $k+1$ intervals, each of length less than $\frac{2}{R}$, and $V_R(u)$ consists of exactly k intervals.*

Proof Lemma 2.5 implies that $|u'(r)| \geq R^2$ for all $x \in W_R(u)$. For any interval $I \subset W_R$, $u'(r) \geq R^2$, $r \in I$, we claim that the length of I is less than $\frac{2}{R}$.

In fact, for $x, y \in I$ with $x > y$, we have

$$u(x) - u(y) = \int_x^y u'(s)\,ds \geq R^2(x - y).$$

Thus,

$$x - y \leq \frac{R - (-R)}{R^2} = \frac{2}{R},$$

which implies

$$|I| \leq \frac{2}{R}.$$

The case $u'(r) \leq -R^2$, $r \in I$ can be treated by the similar method. □

Lemma 3.6 *There exists ζ_4, satisfying $\lim_{R\to\infty} \zeta_4(R) = 0$, and $c_2 \geq c_1$ such that, for any $R \geq c_2$, if either*

(a) $0 \leq \lambda \leq R$ *and* $|u|_\infty = 2\zeta_3(R)$, *or*

(b) $\lambda = R$ *and* $\zeta_3(R) \leq |u|_\infty \leq 2\zeta_3(R)$.

Then the length of each interval of $V_R(u)$ is less than $\zeta_4(R)$.

Proof Define $H = H(R)$ by

$$H(R) := \min\left\{R, \min\left\{\frac{f(\xi)}{\xi} : |\xi| \geq R\right\} - \left(\frac{C}{R} + \beta\right)\right\},$$

and let

$$\zeta_4(R) := \frac{2}{R_1^{N-1} m^* \mathcal{M} H(R)},$$

where m^*, $\mathcal{M}$ are defined in detail below.

By (H1), $\lim_{R\to\infty} H(R) = \infty$, so $\lim_{R\to\infty} \zeta_4(R) = 0$, and we may choose $c_2 \geq c_1$ sufficiently large that $H(R) > 0$ for all $R \geq c_2$.

Choose r_0, r_2 such that $u(r_0) = u(r_2) = R$ and $u > R$ on (r_0, r_2), that is, $I := [r_0, r_2]$ is an interval of $V_R(u)$ (the case of intervals on which $u < -R$ is similar). By (3.1) and the construction of H, if either (a) or (b) holds then $-(r^{N-1}u'(r))' \geq r^{N-1}H(R)u(r) > 0$ for $r \in I$. Now let $v(r) = u(r) - R$, suppose now that $r_2 - r_0 < \zeta_4(R)$, we have

$$\begin{cases} -(r^{N-1}v'(r))' \geq r^{N-1}H(R)(v(r) + R), & r \in (r_0, r_2), \\ v(r_0) = v(r_2) = 0. \end{cases} \tag{3.4}$$

Set

$$\phi(r)=\frac{1}{N-2}\left[\frac{1}{r_0^{N-2}}-\frac{1}{r^{N-2}}\right],\qquad \psi(r)=\frac{1}{N-2}\left[\frac{1}{r^{N-2}}-\frac{1}{r_2^{N-2}}\right],\quad r\in[r_0,r_2],\tag{3.5}$$

then

$$G(r,s)=\begin{cases}\frac{1}{\rho}\phi(r)\psi(s), & r_0\le r\le s\le r_2,\\ \frac{1}{\rho}\phi(s)\psi(r), & r_0\le s\le r\le r_2,\end{cases}\tag{3.6}$$

where

$$\rho:=\frac{1}{N-2}\left[\frac{1}{r_0^{N-1}}-\frac{1}{r_2^{N-1}}\right].\tag{3.7}$$

The Green function $G(r,s)$ satisfies $G(r,s)>0$, $\forall r,s\in(r_0,r_2)$.

In fact,

$$v(r)\ge d_0G(r,r)|v|_\infty,\tag{3.8}$$

where

$$d_0:=\frac{\rho}{\phi(r_2)\psi(r_0)}.\tag{3.9}$$

Let

$$m^*:=\min_{r_0+\frac{r_2-r_0}{4}\le r,s\le r_2-\frac{r_2-r_0}{4}}G(r,s),\tag{3.10}$$

$$\phi^*\psi^*:=\min_{s\in(r_0+\frac{r_2-r_0}{4},r_2-\frac{r_2-r_0}{4})}\left[\frac{1}{r_0^{N-2}}-\frac{1}{s^{N-2}}\right]\left[\frac{1}{s^{N-2}}-\frac{1}{r_2^{N-2}}\right].\tag{3.11}$$

By (3.4)-(3.11), we have

$$\begin{aligned}
v(r)&\ge\int_{r_0}^{r_2}s^{N-1}G(r,s)H(R)\big(v(s)+R\big)\,ds\\
&\ge\int_{r_0+\frac{r_2-r_0}{4}}^{r_2-\frac{r_2-r_0}{4}}s^{N-1}G(r,s)H(R)\big(d_0G(s,s)|v|_\infty+R\big)\,ds\\
&\ge m^*\int_{r_0+\frac{r_2-r_0}{4}}^{r_2-\frac{r_2-r_0}{4}}s^{N-1}H(R)\left(\frac{\phi(s)\psi(s)}{\phi(r_2)\psi(r_0)}|v|_\infty\right)ds\\
&\ge m^*\int_{r_0+\frac{r_2-r_0}{4}}^{r_2-\frac{r_2-r_0}{4}}s^{N-1}H(R)\left(\frac{[\frac{1}{r_0^{N-2}}-\frac{1}{s^{N-2}}][\frac{1}{s^{N-2}}-\frac{1}{r_2^{N-2}}]}{(\frac{r_2^{N-2}-r_0^{N-2}}{(r_0^{N-2})^2})^2}|v|_\infty\right)ds\\
&\ge m^*R_1^{N-1}\int_{r_0+\frac{r_2-r_0}{4}}^{r_2-\frac{r_2-r_0}{4}}H(R)\left(\frac{\phi^*\psi^*}{(\frac{r_2^{N-2}-r_0^{N-2}}{(r_0^{N-2})^2})^2}|v|_\infty\right)ds\\
&\ge m^*R_1^{N-1}\frac{\phi^*\psi^*}{(\frac{R_2^{N-2}-R_1^{N-2}}{(R_1^{N-2})^2})^2}\int_{r_0+\frac{r_2-r_0}{4}}^{r_2-\frac{r_2-r_0}{4}}H(R)|v|_\infty\,ds.
\end{aligned}$$

Set

$$\mathcal{M} := \frac{\phi^* \psi^*}{(\frac{R_2^{N-2} - R_1^{N-2}}{(R_1^{N-2})^2})^2},$$

Thus,

$$r_2 - r_0 \leq \frac{2}{R_1^{N-1} m^* \mathcal{M} H(R)}.$$

Contradiction. □

Now choose an arbitrary integer $k \geq k_0$ and ν, and choose $\Lambda > \max\{c_2, \mu_k\}$ such that

$$\frac{2(k+1)}{\Lambda} + k\zeta_4(\Lambda) < R_2 - R_1. \tag{3.12}$$

Let

$$\mathcal{B} = \big\{(\lambda, u) : 0 \leq \lambda \leq \Lambda, \zeta_3(\lambda) \leq |u|_\infty \leq 2\zeta_3(\Lambda)\big\}.$$

$$\Omega_1 = \big\{(\lambda, u) : 0 \leq \lambda \leq \Lambda, \leq |u|_\infty = \zeta_3(\lambda)\big\}.$$

$$\Omega_2 = \big\{(0, u) : 2\zeta_3(0) \leq |u|_\infty \leq \zeta_3(\Lambda)\big\}.$$

It follows from Lemma 3.1 that $\mathcal{C}_{k,\nu}$ "enters" $\mathcal{B}$ through the set Ω_1, while from Lemma 3.3, $\mathcal{C}_{k,\nu} \cap \mathcal{B} \subset \mathbb{R} \times S_{k,\nu}$. Thus, by Lemma 2.1 and Lemma 3.2, $\mathcal{C}_{k,\nu}$ must "leave" $\mathcal{B}$, and since $\mathcal{C}_{k,\nu}$ is connected, it must intersect $\partial\mathcal{B}$. However, Lemmas 3.4–3.6 (together with (3.12)) show that the only portion of $\partial\mathcal{B}$ (other than Ω_1), which $\mathcal{C}_{k,\nu}$ can intersect, is Ω_2. Thus, there exists a point $(0, u_{k,\nu}) \in \mathcal{C}_{k,\nu} \cap \Omega_2$, and clearly $u_{k,\nu}$ provides the desired solution for (1.1).

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Zhu and Ma wrote the main manuscript text, and Su checked and revised the manuscript. All authors reviewed the manuscript.

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