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Approximation of solutions to integro-differential time fractional order parabolic equations in L^p-spaces

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Abstract

In this paper we study the initial boundary value problem for a class of integro-differential time fractional order parabolic equations with a small positive parameter ε. Using the Laplace transform, Mittag-Leffler operator family, C_0-semigroup, resolvent operator, and weighted function space, we get the existence of a mild solution. For suitable indices $p \in [1, +\infty)$ and $s \in (1, +\infty)$, we first prove that the mild solution of the approximating problem converges to that of the corresponding limit problem in $L^p((0,T), L^s(\Omega))$ as $\varepsilon \to 0^+$. Then for the linear approximating problem with ε and the corresponding limit problem, we give the continuous dependence of the solutions. Finally, for a class of semilinear approximating problems and the corresponding limit problems with initial data in $L^s(\Omega)$, we prove the local existence and uniqueness of the mild solution and then give the continuous dependence on the initial data.

Mathematics Subject Classification: 26A33; 34G20; 35K57; 35K90; 41A35

Keywords: Integro-differential equation; Time fractional order parabolic equation; Limit problem; Analytic semigroup; C_0-semigroup; Mittag-Leffler function; Mittag-Leffler operator

1 Introduction

In this paper, we study the following initial boundary value problem (IBVP) for integro-differential time fractional order parabolic equations with a small positive parameter ε in a suitable L^p-space:

$$\begin{cases} D_t^\alpha u_\varepsilon(t) + \chi(\varepsilon)(-A)u_\varepsilon(t) + (-A)(k^\varepsilon * u_\varepsilon)(t) = f_\varepsilon(t), \\ u_\varepsilon(0) = u_{0,\varepsilon}, \end{cases} \tag{1.1}$$

where the kernel $k(t) : \mathbf{R} \to \mathbf{R}$ is a continuous function such that

$$k^\varepsilon(t) = \frac{1}{\varepsilon} k\left(\frac{t}{\varepsilon}\right), \qquad \left(k^\varepsilon * u_\varepsilon\right)(t) = \int_0^t k^\varepsilon(t-s) u_\varepsilon(s)\, ds, \tag{1.2}$$

$\alpha \in (0,1)$, and $D_t^\alpha u(t)$ represents the Caputo fractional derivative of u defined by

$$D_t^\alpha u(t) = \int_0^t g_{1-\alpha}(t-s)u'(s)\,ds, \qquad g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t>0, \beta>0, \tag{1.3}$$

where the Gelfand–Shilov function $g_\beta(t)$ satisfies the semigroup property $g_\alpha * g_\beta = g_{\alpha+\beta}$, while χ is a positive scalar function defined in $(0,\varepsilon_0]$ for a given real number $\varepsilon_0 > 0$ such that

$$\chi(\varepsilon) \to \chi_0 \quad \text{as } \varepsilon \to 0^+. \tag{1.4}$$

In general, A is a linear unbounded operator defined on a bounded domain $\Omega \subset \mathbf{R}^n$ with C^2-boundary, here we consider that A is a second-order differential operator in divergence form,

$$A := A(x, D_x) = \sum_{i,j=1}^n D_{x_i}\big(a_{ij}(x)D_{x_j}\big) + a_0(x), \tag{1.5}$$

endowed with vanishing Dirichlet boundary condition on $\partial\Omega$, and there are positive constants c_0, c_1, γ such that $a_0(x) \geq \gamma$, $a_0 \in C(\overline{\Omega})$, $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, and

$$c_0|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq c_1|\xi|^2, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbf{R}^n. \tag{1.6}$$

The time-fractional integro-differential equations are very popular due to their applications in real-world problems in physics, mechanics, civil and mechanical engineering. In the recent years, the research on time-fractional differential equations has attracted many authors. For fractional calculus and Mittag-Leffler function, we refer to references [15, 20]. For abstract integro-differential equations and abstract evolution equations, we refer to the monographs [19, 22] and papers [13, 14, 24]. For the fractional integro-differential equations, we refer to the references [8–10, 12]. For fractional evolution equations in Banach spaces, we refer to the PhD thesis [2]. Zhang and Li [25] considered the global well-posedness and blow-up of solutions of the Cauchy problem for a time-fractional superdiffusion equation. Li [16] discussed the regularity of mild solutions for a fractional abstract Cauchy problem with order $\alpha \in (1,2)$. Chen and Tang [5, 6] considered the homogenization of nonlinear nonlocal p-Laplacian operators with convolution kernels.

For $\alpha = 1$, Lorenzi and Messina [17] considered the approximation of solutions to linear integro-differential parabolic equation (1.1) in L^p-spaces and extended the results to the corresponding nonlinear equation [18]. For $\alpha \in (1,2)$, Zhao and Tang [26] discussed the approximation of solutions to nonlinear integro-differential time fractional wave equation (1.1) in the L^p-space.

In this paper, we first consider the linear integro-differential parabolic equation (1.1) with $\alpha \in (0,1]$, the time-fractional order evolution equation in a Banach space. Our main task is to show that, under suitable assumptions on the functions f_ε, f and the initial data $u_{0,\varepsilon}$, u_0, for suitable indices $p \in [1,+\infty)$ and $s \in (1,+\infty)$, in $L^p((0,T),L^s(\Omega))$-norm, the mild solution $u_\varepsilon(t)$ of problem (1.1) converges to the mild solution $u(t)$ of the limit problem

$$\begin{cases} D_t^\alpha u(t) + (1+\chi_0)(-A)u(t) = f(t), \quad t \in (0,T], \\ u(0) = u_0, \end{cases} \tag{1.7}$$

as $\varepsilon \to 0^+$.

In general, any order Caputo fractional derivative of $u(t)$ for $\alpha > 0$ is defined by [21]

$$D_t^\alpha u(t) = \int_0^t g_{m-\alpha}(t-s)\frac{d^m}{ds^m}u(s)\,ds, \tag{1.8}$$

where m is the smallest integer greater than or equal to α, and $g_{m-\alpha}(t)$ is defined by (1.3) with $m-\alpha$ instead of β.

Similarly, the Riemann–Liouville fractional integral is defined by

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)\,ds, \tag{1.9}$$

thus we have

$$D_t^\alpha J_t^\alpha f(t) = f(t), \qquad J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\frac{t^k}{k!}. \tag{1.10}$$

Applying the properties of Laplace transform and taking into account that $\widehat{g_\alpha}(\lambda) = \lambda^{-\alpha}$ and $\widehat{(J_t^\alpha f)}(\lambda) = \widehat{(g_\alpha * f)}(\lambda) = \widehat{g_\alpha}(\lambda)\hat{f}(\lambda) = \lambda^{-\alpha}\hat{f}(\lambda)$, the second equation in (1.10) implies that

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0)\lambda^{\alpha-1-k}. \tag{1.11}$$

For the linear time fractional order parabolic equation (1.1) with a small positive parameter ε, we now consider a semilinear approximating problem

$$\begin{cases} D_t^\alpha u_\varepsilon(t) + \chi(\varepsilon)(-A)u_\varepsilon(t) + (-A)(k^\varepsilon * u_\varepsilon)(t) = f_\varepsilon(t) + N[u_\varepsilon](t), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad x \in \Omega, \\ u_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega, \end{cases} \tag{1.12}$$

and the corresponding semilinear limit problem as $\varepsilon \to 0^+$, namely

$$\begin{cases} D_t^\alpha u(t,x) + (1+\chi_0)(-A)u(t,x) = f(t,x) + N[u](t,x), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega, \end{cases} \tag{1.13}$$

where N is a nonlinear operator admitting the following representation:

$$N(u)(t,x) = \psi\big(t, u(t,x)\big), \tag{1.14}$$

and ψ is given later.

For Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we consider the closed linear operator $A : D(A) \subset X \to Y$, the notation $[D(A)]$ represents the domain of A endowed with graph norm $\|x\|_1 = \|x\|_X + \|Ax\|_Y$, $x \in D(A)$. Recalling the classical theory of semigroups [11], the Mittag-Leffler function E_α and the nonnegative Wright function Ψ_α have the relation and property (see [2])

$$E_\alpha(-z) = \int_0^\infty \Psi_\alpha(t)e^{-zt}\,dt, \quad z \in \mathbb{C}, \tag{1.15}$$

$$\int_0^\infty \Psi_\alpha(s)s^r\,ds = \frac{\Gamma(r+1)}{\Gamma(\alpha r+1)}, \quad r > -1, \tag{1.16}$$

where the Gamma function is defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}\,dt$. In this paper we will use also the Beta function $\mathrm{B} : (0,\infty) \times (0,\infty) \to (0,\infty)$ defined by

$$\mathrm{B}(a,b) = \int_0^1 (1-s)^{a-1}s^{b-1}\,ds. \tag{1.17}$$

Using the subordination principle, we can write the Mittag-Leffler family associated to the operator A in the form

$$E_\alpha(tA) = \int_0^\infty \Psi_\alpha(s)e^{st^\alpha A}\,ds, \tag{1.18}$$

where $\{e^{tA}\}_{t\geq 0}$ is an analytic semigroup associated with the operator A defined in (1.5).

Regarding the convergence of approximating IBVP to the α-order time fractional evolutional equation $D_t^\alpha u_\varepsilon + \chi(\varepsilon)(-A)u_\varepsilon + (-A)(k^\varepsilon * u_\varepsilon) = N[u_\varepsilon]$ to the corresponding limit IBVP of the equation $D_t^\alpha u + (1+\chi_0)(-A)u = N[u]$ as scale parameter $\varepsilon \to 0^+$, for a general class of sectorial operators $A = \sum_{i,j=1}^n D_{x_i}(a_{ij}(x)D_{x_j}) + a_0(x)$ defined in equation (1.5), the novelty of this paper is to extend the results in [17, 18] from $\alpha = 1$ (classical integro-differential parabolic equation) to $\alpha \in (0,1]$ (time fractional order integro-differential parabolic equation). Also we consider a more general kernel $k(t)$ in equation (1.1) instead of the kernel included in the series representation in [17].

This paper is divided into five sections. Section 2 is devoted to showing some basic properties of the resolvent operators F_α^ε, G_α^ε defined in equations (2.2)–(2.5) and the nonlinear operator N. Our main results are stated in Sect. 3, and their proofs are given in Sect. 4. A discussion is given in Sect. 5.

2 Properties of resolvent operators

We first state some assumptions on the operator A and the initial data $u_{0,\varepsilon}(x)$ to our abstract Cauchy problems of the linear equation (1.1) and the semilinear equation (1.12), respectively. We assume that X is a given general Banach space with a norm denoted by $\|\cdot\|_X$. And approximation results we want to study in $L^p((0,T);X)$ involve the norm denoted by $\|f\|_{L^p((0,T);X)} = \int_0^T \|f(t)\|_X^p\,dt$. Furthermore, we state the following assumptions:

(H1) The operator $A : D(A) \subset X \to X$ is a closed linear operator, with $D(A)$ dense in X, and for some $\phi \in (\frac{\pi}{2}, \pi)$ there is a positive constant $C_0 = C_0(\phi)$ such that

$$\Sigma_{0,\phi} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \phi\} \subset \rho(A),$$

and the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ satisfies

$$\|R(\lambda, A)\| \leq \frac{C_0}{|\lambda|}, \quad \forall \lambda \in \Sigma_{0,\phi}.$$

(H2) The kernel $k(t) \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ is such that $\widehat{k}(\lambda)$ exists for $\mathrm{Re}(\lambda) > 0$, $\widehat{k}(\lambda)$ can be extended to $\sum_{0,\phi}$, and $\|\widehat{k}(\lambda)\| = O(\frac{1}{|\lambda|})$ as $|\lambda| \to +\infty$. For $p \in [1, +\infty)$, $\alpha \in (0, 1]$, there exist constants $\theta_0 > \frac{1}{p} + \alpha$, $k_0, r > 0$, and $r_0 > r$, $C > 0$ such that

$$\left|\widehat{k}(\lambda) - k_0\right| \leq C|\lambda|^{\theta_0}, \quad \forall \lambda \in \Sigma_{0,\phi} \cap B(0, r_0).$$

(H3) There exist constants M, $C > 0$ such that

$$\left\|\left(\lambda I - \left(1 + \chi(\varepsilon)\right)A\right)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}, \quad \forall \lambda \in \Sigma_{0,\phi}, \forall \varepsilon \in (0, \varepsilon_0).$$

(H4) The sequence $\{f_\varepsilon\}_{\varepsilon>0} \subset C([0,T], X)$ converges to f in $C([0,T], X)$ as $\varepsilon \to 0^+$.

(H5) Functions $u_{0,\varepsilon}, u_0 \in X$ and $\{u_{0,\varepsilon}\}_{\varepsilon>0}$ converges to u_0 in X as $\varepsilon \to 0^+$.

Remark 2.1 Assumptions (H1), (H2), and (H3) have been considered in [7–10, 12], with the operator $B(t) = K(t)A$ for a given function $K(t)$ and $\alpha = 1$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$, we denote a sector by

$$\Sigma_{r,\theta} = \left\{\lambda \in \mathbb{C} : |\lambda| \geq r, \left|\arg(\lambda)\right| < \theta\right\}. \tag{2.1}$$

In addition, for a given Banach space, $\rho(F_\alpha^\varepsilon)$ and $\rho(G_\alpha^\varepsilon)$ are the sets

$$\rho\left(F_\alpha^\varepsilon\right) = \left\{\lambda \in \mathbb{C} : F_\alpha^\varepsilon(\lambda) = \left(\lambda^\alpha I - \chi(\varepsilon)A - \widehat{k}(\varepsilon\lambda)A\right)^{-1} \in \mathcal{L}(X)\right\}, \tag{2.2}$$

$$\rho\left(G_\alpha^\varepsilon\right) = \left\{\lambda \in \mathbb{C} : G_\alpha^\varepsilon(\lambda) = \lambda^{\alpha-1}\left(\lambda^\alpha I - \chi(\varepsilon)A - \widehat{k}(\varepsilon\lambda)A\right)^{-1} \in \mathcal{L}(X)\right\}, \tag{2.3}$$

and $\rho(F_\alpha)$ and $\rho(G_\alpha)$ are the sets

$$\rho(F_\alpha) = \left\{\lambda \in \mathbb{C} : F_\alpha(\lambda) = \left(\lambda^\alpha I - \chi_0 A - k_0 A\right)^{-1} \in \mathcal{L}(X)\right\}, \tag{2.4}$$

$$\rho(G_\alpha) = \left\{\lambda \in \mathbb{C} : G_\alpha(\lambda) = \lambda^{\alpha-1}\left(\lambda^\alpha I - \chi_0 A - k_0 A\right)^{-1} \in \mathcal{L}(X)\right\}. \tag{2.5}$$

Next we collect some properties established in [7, 23].

Lemma 2.1 ([7]) *Under Assumptions* (H1), (H2), *and* (H3), *there exists an* $r > 0$ *such that* $\sum_{r,\phi} \subset \rho(G_\alpha^\varepsilon)$, $\sum_{r,\phi} \subset \rho(F_\alpha^\varepsilon)$, *and the operator-valued functions* $F_\alpha^\varepsilon, G_\alpha^\varepsilon : \sum_{r,\phi} \to \mathcal{L}(X)$ *are analytic. Moreover, there exists a constant* M *such that*

$$\left\|F_\alpha^\varepsilon(\lambda)\right\| \leq \frac{M}{|\lambda|^\alpha}, \qquad \left\|G_\alpha^\varepsilon(\lambda)\right\| \leq \frac{M}{|\lambda|}, \quad \forall \lambda \in \Sigma_{r,\phi}.$$

Definition 2.1 ([7, 23]) The operator families $\{S_\alpha^\varepsilon(t)\}_{t\geq 0}$ and $\{S_\alpha(t)\}_{t\geq 0}$ are defined by

$$S_\alpha^\varepsilon(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^\alpha I - \chi(\varepsilon)A - \widehat{k}(\varepsilon\lambda)A\right)^{-1} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha^\varepsilon(\lambda)\, d\lambda,$$

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{\alpha-1}\big(\lambda^\alpha I - \chi_0 A - k_0 A\big)^{-1}\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda)\,d\lambda,$$

and the operator families $\{T_\alpha^\varepsilon(t)\}_{t\ge 0}$ and $\{T_\alpha(t)\}_{t\ge 0}$ are defined by

$$T_\alpha^\varepsilon(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\big(\lambda^\alpha I - \chi(\varepsilon)A - \widehat{k}(\varepsilon\lambda)A\big)^{-1}\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} F_\alpha^\varepsilon(\lambda)\,d\lambda,$$

$$T_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\big(\lambda^\alpha I - \chi_0 A - k_0 A\big)^{-1}\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} F_\alpha(\lambda)\,d\lambda,$$

where $\Gamma_1(r,\theta) = \{te^{i\theta} : t \ge r\}$, $\Gamma_2(r,\theta) = \{re^{i\xi} : |\xi| \le \theta\}$, $\Gamma_3(r,\theta) = \{te^{-i\theta} : t \ge r\}$, $\theta \in (\pi/2,\phi)$, and $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma_i(r,\theta)$, oriented counterclockwise.

Using the Laplace transform, we give the definitions of L^q-mild solutions to the equations (1.1), (1.7), (1.12), and (1.13), respectively.

Definition 2.2 Let $T > 0$, $\varepsilon \in (0,\varepsilon_0]$, $u_{0,\varepsilon}, u_0 \in L^q(\Omega)$. Functions $u_\varepsilon \in C([0,T],L^q(\Omega))$ and $u \in C([0,T],L^q(\Omega))$ are called L^q-mild solutions of equations (1.1) and (1.7) in $[0,T]$ if u_ε and u respectively satisfy the following equations:

$$u_\varepsilon(t) = S_\alpha^\varepsilon(t)u_{0,\varepsilon} + \int_0^t T_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds, \tag{2.6}$$

$$u(t) = S_\alpha(t)u_0 + \int_0^t T_\alpha(t-s)f(s)\,ds. \tag{2.7}$$

Definition 2.3 Let $T > 0$, $\varepsilon \in (0,\varepsilon_0]$, $u_{0,\varepsilon}, u_0 \in L^q(\Omega)$. Functions $u_\varepsilon \in C([0,T],L^q(\Omega))$ and $u \in C([0,T],L^q(\Omega))$ are called L^q-mild solutions of equations (1.12) and (1.13) in $[0,T]$ if u_ε and u respectively satisfy the following equations:

$$u_\varepsilon(t) = S_\alpha^\varepsilon(t)u_{0,\varepsilon} + \int_0^t T_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds + \int_0^t T_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)\,ds, \tag{2.8}$$

$$u(t) = S_\alpha(t)u_0 + \int_0^t T_\alpha(t-s)f(s)\,ds + \int_0^t T_\alpha(t-s)Nu(s)\,ds. \tag{2.9}$$

Lemma 2.2 ([7]) *Under Assumptions* (H1), (H2), *and* (H3), *the operator-valued function $S_\alpha^\varepsilon(t)$ defined in Definition 2.1 is* (i) *exponentially bounded in $\mathcal{L}(X)$;* (ii) *exponentially bounded in $\mathcal{L}(D[A])$;* (iii) *strongly continuous on $[0,\infty)$ and uniformly continuous on $(0,\infty)$ in $\mathcal{L}(X)$;* (iv) *strongly continuous on $[0,\infty)$ in $\mathcal{L}(D[A])$. The operator function $T_\alpha^\varepsilon(t)$ defined in Definition 2.1 is* (v) *exponentially bounded in $\mathcal{L}(X)$;* (vi) *uniformly continuous on $(0,\infty)$ in $\mathcal{L}(X)$.*

Lemma 2.3 *Let $1 \le q \le r < \infty$ be such that $\frac{n}{2}(\frac{1}{q} - \frac{1}{r}) < 1$. If the operator A is defined in equation* (1.5) *and satisfies Assumption* (H1), *then for every $\varphi \in L^q(\Omega)$ we have*

$$\big\|S_\alpha(t)\varphi\big\|_{L^r(\Omega)} \le Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_{L^q(\Omega)}, \tag{2.10}$$

$$\big\|T_\alpha(t)\varphi\big\|_{L^r(\Omega)} \le Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})+\alpha-1}\|\varphi\|_{L^q(\Omega)}, \tag{2.11}$$

where C is a positive constant. Moreover, the operator family $\{S_\alpha(t)\}_{t\ge 0}$ is strongly continuously in $L^r(\Omega)$.

Proof The proof of estimate (2.10) can be found in [18]. In order to compare the operator families $\{S_\alpha(t)\}_{t\geq 0}$ and $\{T_\alpha(t)\}_{t\geq 0}$, here we give the proof of estimate (2.10) again. From Theorem 3.1 [23], we get

$$S_\alpha(t)\varphi = \int_0^\infty \Psi_\alpha(s)e^{(k_0+\chi_0)st^\alpha A}\varphi\,ds, \tag{2.12}$$

$$T_\alpha(t)\varphi = t^{\alpha-1}\int_0^\infty \alpha s\Psi_\alpha(s)e^{(k_0+\chi_0)st^\alpha A}\varphi\,ds. \tag{2.13}$$

From the proof of Lemma 2.1 [1] and using equation (1.16) of the Wright function Ψ_α, we have

$$\begin{aligned}
\|S_\alpha(t)\varphi\|_{L^r(\Omega)} &\leq \int_0^\infty \Psi_\alpha(s)\|e^{(k_0+\chi_0)st^\alpha A}\varphi\|_{L^r(\Omega)}\,ds \\
&\leq Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})}\int_0^\infty \Psi_\alpha(s)s^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\,ds\|\varphi\|_{L^q(\Omega)} \\
&= Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_{L^q(\Omega)}\frac{\Gamma(-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})+1)}{\Gamma(-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})+1)} \\
&= Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_{L^q(\Omega)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|T_\alpha(t)\varphi\|_{L^r(\Omega)} &\leq t^{\alpha-1}\int_0^\infty \alpha s\Psi_\alpha(s)\|e^{(k_0+\chi_0)st^\alpha A}\varphi\|_{L^r(\Omega)}\,ds \\
&\leq Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})+\alpha-1}\int_0^\infty \Psi_\alpha(s)s^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})+1}\,ds\|\varphi\|_{L^q(\Omega)} \\
&= Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})+\alpha-1}\|\varphi\|_{L^q(\Omega)}\frac{\Gamma(-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})+2)}{\Gamma(-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})+\alpha+1)} \\
&= Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})+\alpha-1}\|\varphi\|_{L^q(\Omega)}.
\end{aligned}$$

Now, for $0\leq t_1\leq t_2$, we have

$$\|S_\alpha(t_2)\varphi - S_\alpha(t_1)\varphi\|_{L^r} \leq \int_0^\infty \Psi_\alpha(s)\|e^{(k_0+\chi_0)st_1^\alpha A}[e^{(k_0+\chi_0)s(t_2^\alpha-t_1^\alpha)A}\varphi-\varphi]\|_{L^r}\,ds,$$

thus, $\forall s>0$, as $t_2\to t_1^+$, one gets

$$\|e^{(k_0+\chi_0)st_1^\alpha A}[e^{(k_0+\chi_0)s(t_2^\alpha-t_1^\alpha)A}\varphi-\varphi]\|_{L^r} \leq C\|e^{(k_0+\chi_0)s(t_2^\alpha-t_1^\alpha)A}\varphi-\varphi\|_{L^r}\to 0.$$

Now Lebesgue's dominated convergence theorem and the fact $\int_0^\infty \Psi_\alpha(s)\,ds = 1$ imply that

$$\lim_{t_2\to t_1^+}\int_0^\infty \Psi_\alpha(s)\|e^{(k_0+\chi_0)Ast_1^\alpha}[e^{(k_0+\chi_0)As(t_2^\alpha-t_1^\alpha)}\varphi-\varphi]\|_{L^r}\,ds = 0,$$

therefore,

$$\lim_{t_2\to t_1^+}\|S_\alpha(t_2)\varphi - S_\alpha(t_1)\varphi\|_{L^r(\Omega)} = 0,$$

so the operator family $\{S_\alpha(t)\}_{t\geq 0}$ is strongly continuously in $L^r(\Omega)$. This completes the proof of Lemma 2.3. □

Lemma 2.4 *There exists a positive constant C depending only on* α, T, M, θ, r *such that*

$$\left\|S_\alpha^\varepsilon(t)\right\|_{\mathcal{L}(X)} \leq C, \quad \forall t \in [0,T], \tag{2.14}$$

$$\left\|T_\alpha^\varepsilon(t)\right\|_{\mathcal{L}(X)} \leq C, \quad \forall t \in [0,T]. \tag{2.15}$$

Proof From (i) and (v) in Lemma 2.2, for any $\lambda \in \Sigma_{r,\varphi}$, $t \in [0,T]$, we have

$$\begin{aligned}
&\left\|S_\alpha^\varepsilon(t)\right\|_{\mathcal{L}(X)} \leq Ce^{rT} \leq C,\\
&\left\|T_\alpha^\varepsilon(t)\right\|_{\mathcal{L}(X)} 1 \leq Ce^{rT} \leq C.
\end{aligned}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5 *Under Assumptions* (H1)–(H3), *for* $p \in [1,+\infty)$, $\alpha + \frac{1}{p} > 1$, $\theta_0 \in (\alpha + \frac{1}{p}, +\infty)$, *and* $\varepsilon_1 \in (0, \min\{\varepsilon_0, \frac{r_0}{r}\})$, *there exists a positive constant C depending only on* $\alpha, T, M, \theta, r, \varepsilon_0, \theta_0, p, r_0$ *such that*

$$\left\|S_\alpha^\varepsilon - S_\alpha\right\|_{L^p((0,T),\mathcal{L}(X))} \leq C\left(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \left|\chi(\varepsilon) - \chi_0\right|\right), \quad \forall \varepsilon \in (0,\varepsilon_1], \tag{2.16}$$

$$\left\|T_\alpha^\varepsilon - T_\alpha\right\|_{L^p((0,T),\mathcal{L}(X))} \leq C\left(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \left|\chi(\varepsilon) - \chi_0\right|\right), \quad \forall \varepsilon \in (0,\varepsilon_1]. \tag{2.17}$$

Proof From Lemma 2.1, we have

$$\begin{aligned}
&\left\|\left(\lambda^\alpha I - \left(\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda)\right)A\right)^{-1} - \left(\lambda^\alpha I - (k_0 + \chi_0)A\right)^{-1}\right\|_{\mathcal{L}(X)}\\
&\quad \leq \left\|\left\{\lambda^\alpha I - \left(\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda)\right)A\right\}^{-1}\left\{\left(\chi(\varepsilon) - \chi_0\right) + \left(\widehat{k}(\varepsilon\lambda) - k_0\right)\right\}\right\|_{\mathcal{L}(X)}\\
&\qquad \times \left\|A\left(\lambda^\alpha I - (k_0 + \chi_0)A\right)^{-1}\right\|_{\mathcal{L}(X)}\\
&\quad \leq \frac{M}{|\lambda|^\alpha}\left(\left|\chi(\varepsilon) - \chi_0\right| + \left|\widehat{k}(\varepsilon\lambda) - k_0\right|\right)\left\|A\left(\lambda^\alpha - (k_0 + \chi_0)A\right)^{-1}\right\|_{\mathcal{L}(X)}\\
&\quad \leq \frac{M}{|\lambda|^\alpha}\left(\left|\chi(\varepsilon) - \chi_0\right| + \left|\widehat{k}(\varepsilon\lambda) - k_0\right|\right)\left\|I - \lambda^\alpha\left(\lambda^\alpha - (k_0 + \chi_0)A\right)^{-1}\right\|_{\mathcal{L}(X)}\\
&\quad \leq \frac{M(1 + C_0)}{|\lambda|^\alpha(k_0 + \chi_0)}\left(\left|\chi(\varepsilon) - \chi_0\right| + \left|\widehat{k}(\varepsilon\lambda) - k_0\right|\right),
\end{aligned}$$

then

$$\begin{aligned}
&\left\|S_\alpha^\varepsilon(t) - S_\alpha(t)\right\|_{\mathcal{L}(X)}\\
&\quad \leq \frac{1}{2\pi}\left\|\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{\alpha-1}\left\{\left(\lambda^\alpha I - \left(\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda)\right)A\right)^{-1} - \left(\lambda^\alpha - (k_0 + \chi_0)A\right)^{-1}\right\}d\lambda\right\|_{\mathcal{L}(X)}\\
&\quad \leq \frac{M(1 + C_0)}{2\pi(k_0 + \chi_0)}\int_{\Gamma_{r,\theta}} \frac{e^{t\operatorname{Re}\lambda}}{|\lambda|}\left(\left|\chi(\varepsilon) - \chi_0\right| + \left|\widehat{k}(\varepsilon\lambda) - k_0\right|\right)d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| T_\alpha^\varepsilon(t) - T_\alpha(t)\right\|_{\mathcal{L}(X)} \\
&\le \frac{1}{2\pi}\left\|\int_{\Gamma_{r,\theta}} e^{\lambda t}\left\{\left[\lambda^\alpha I - \left(\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda)\right)A\right]^{-1} - \left[\lambda^\alpha - (k_0+\chi_0)A\right]^{-1}\right\} d\lambda\right\|_{\mathcal{L}(X)} \\
&\le \frac{M(1+C_0)}{2\pi(k_0+\chi_0)}\int_{\Gamma_{r,\theta}} \frac{e^{t\operatorname{Re}\lambda}}{|\lambda|^\alpha}\left(\left|\chi(\varepsilon)-\chi_0\right| + \left|\widehat{k}(\varepsilon\lambda)-k_0\right|\right) d\lambda.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left\| S_\alpha^\varepsilon - S_\alpha\right\|_{L^p((0,T),X)} \\
&\le \frac{M(1+C_0)}{2\pi(k_0+\chi_0)}\int_{\Gamma_{r,\theta}} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|\lambda|}\left(\int_0^{+\infty} e^{tp\operatorname{Re}(\lambda)}\, dt\right)^{\frac{1}{p}} d\lambda \\
&= p^{-\frac{1}{p}}\frac{M(1+C_0)}{2\pi(k_0+\chi_0)}\sum_{j=1}^{3}\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}(\lambda)|^{\frac{1}{p}}|\lambda|} d\lambda
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&\left\| T_\alpha^\varepsilon - T_\alpha\right\|_{L^p((0,T),X)} \\
&\le \frac{M(1+C_0)}{2\pi(k_0+\chi_0)}\int_{\Gamma_{r,\theta}} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|\lambda|^\alpha}\left(\int_0^{+\infty} e^{\operatorname{Re}(\lambda)}\, dt\right)^{\frac{1}{p}} d\lambda \\
&= p^{-\frac{1}{p}}\frac{M(1+C_0)}{2\pi(k_0+\chi_0)}\sum_{j=1}^{3}\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}(\lambda)|^{\frac{1}{p}}|\lambda|^\alpha} d\lambda.
\end{aligned} \tag{2.19}$$

We first estimate the integrals over $\Gamma_j(r,\theta)$, $j = 1, 3$. From Assumption (H2), there exists a positive constant C such that for $\Gamma_1(r,\theta) = \{te^{i\theta} : t \ge r\}$ and $\Gamma_3(r,\theta) = \{te^{-i\theta} : t \ge r\}$, we have

$$\begin{aligned}
\int_{\Gamma_j(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|} d\lambda &\le \cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}}\int_{\varepsilon r}^{+\infty} \frac{|\widehat{k}(se^{i\theta})-k_0|}{s^{1+\frac{1}{p}}} ds \\
&\le C\cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}}\left(\int_0^{r_0} s^{\theta_0-1-\frac{1}{p}}\, ds + \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}}\, ds\right) \\
&= C\varepsilon^{\frac{1}{p}},
\end{aligned} \tag{2.20}$$

where $I_1 = \int_0^{r_0} s^{\theta_0-1-\frac{1}{p}}\, ds$ converges for $\theta_0 > \frac{1}{p}$, and $I_2 = \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}}\, ds$ converges for $p \ge 1$.

Similarly, there exists a positive constant C such that

$$\begin{aligned}
&\int_{\Gamma_j(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|^\alpha} d\lambda \\
&\le \cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}+\alpha-1}\int_{\varepsilon r}^{+\infty} \frac{|\widehat{k}(se^{i\theta})-k_0|}{s^{\alpha+\frac{1}{p}}} ds \\
&\le C\cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}+\alpha-1}\left(\int_0^{r_0} s^{\theta_0-1-\frac{1}{p}-\alpha}\, ds + \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}-\alpha}\, ds\right)
\end{aligned} \tag{2.21}$$

$$= C\varepsilon^{\frac{1}{p}+\alpha-1},$$

where $I_3 = \int_0^{r_0} s^{\theta_0-1-\frac{1}{p}-\alpha}\,ds$ converges for $\theta_0 > \frac{1}{p}+\alpha$ and $I_4 = \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}-\alpha}\,ds$ converges for $p \ge 1$ and $\alpha \in (0,1)$.

For the function $\chi(\varepsilon)$, we can also get

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|}\,d\lambda = \int_r^{+\infty} \frac{|\chi(\varepsilon)-\chi|}{s^{1+\frac{1}{p}}\cos^{\frac{1}{p}}(\theta)}\,ds \le C\big|\chi(\varepsilon)-\chi_0\big|r^{-\frac{1}{p}} \tag{2.22}$$

and

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}\,d\lambda = \int_r^{+\infty} \frac{|\chi(\varepsilon)-\chi|}{s^{\alpha+\frac{1}{p}}\cos^{\frac{1}{p}}(\theta)}\,ds \le C\big|\chi(\varepsilon)-\chi_0\big|r^{1-\alpha-\frac{1}{p}}, \tag{2.23}$$

where C is a positive constant.

From (2.20)–(2.23), we get the estimates of the integrals over $\Gamma_j(r,\theta)$ $(j=1,3)$:

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|+|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|}\,d\lambda \le C\varepsilon^{\frac{1}{p}} + C\big|\chi(\varepsilon)-\chi_0\big|r^{-\frac{1}{p}}, \tag{2.24}$$

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|+|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}\,d\lambda \le C\varepsilon^{\frac{1}{p}+\alpha-1} + C\big|\chi(\varepsilon)-\chi_0\big|r^{1-\alpha-\frac{1}{p}}. \tag{2.25}$$

We now estimate the integrals over $\Gamma_2(r,\theta)$ in (2.18) and (2.19). Choosing $\varepsilon \in (0,\varepsilon_1]$ with $\varepsilon_1 \le \min\{\varepsilon_0, \frac{r_0}{r}\}$ and making use of assumptions (iv) and (v) in Lemma 2.2, we have

$$\begin{aligned}
&\int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|+|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|}\,d\lambda \\
&\quad= \int_{\Gamma_2(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|}\,d\lambda + \int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|}\,d\lambda \\
&\quad\le r^{-\frac{1}{p}}\left[\int_{-\theta}^{\theta} \frac{|\widehat{k}(\varepsilon re^{i\xi})-k_0|}{\cos^{\frac{1}{p}}(\xi)}\,d\xi + \int_{-\theta}^{\theta} \frac{|\chi(\varepsilon)-\chi_0|}{\cos^{\frac{1}{p}}(\xi)}\,d\xi\right] \\
&\quad\le Cr^{-\frac{1}{p}}\left[(\varepsilon r)^{\theta_0}\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)\,d\xi + \big|\chi(\varepsilon)-\chi_0\big|\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)\,d\xi\right] \\
&\quad\le C\big(\varepsilon^{\theta_0} + \big|\chi(\varepsilon)-\chi_0\big|\big),
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
&\int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|+|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}\,d\lambda \\
&\quad= \int_{\Gamma_2(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}\,d\lambda + \int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|\operatorname{Re}\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}\,d\lambda \\
&\quad\le r^{1-\alpha+\frac{1}{p}}\left[\int_{-\theta}^{\theta} \frac{|\widehat{k}(\varepsilon re^{i\xi})-k_0|}{\cos^{\frac{1}{p}}(\xi)}\,d\xi + \int_{-\theta}^{\theta} \frac{|\chi(\varepsilon)-\chi_0|}{\cos^{\frac{1}{p}}(\xi)}\,d\xi\right] \\
&\quad\le Cr^{1-\alpha+\frac{1}{p}}(\varepsilon r)^{\theta_0}\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)\,d\xi + \big|\chi(\varepsilon)-\chi_0\big|\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)\,d\xi
\end{aligned} \tag{2.27}$$

$$\leq C\big(\varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\big).$$

From (2.18), (2.24), and (2.26), we get the estimate (2.16); and from (2.19), (2.25), and (2.27), we get the estimate (2.17). This completes the proof of Lemma 2.5. □

Lemma 2.6 *If f_ε and f satisfy Assumption* (H4), *and* $\frac{1}{p} + \alpha > 1$ ($p \geq 1$), *then*

$$\begin{aligned}&\big\|T_\alpha^\varepsilon * f_\varepsilon - T_\alpha * f\big\|_{L^r((0,T),X)}\\&\quad\leq C\|f_\varepsilon - f\|_{C([0,T],X)} + C\big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\big)\|f\|_{C([0,T],X)},\end{aligned}$$

where C is a positive constant depending only on α, T, M, θ, r, ε_0, θ_0, p, r_0, and

$$\frac{1}{r} = \begin{cases}\frac{1}{p} + \frac{1}{q} - 1, & \text{if } \frac{1}{p} + \frac{1}{q} \geq 1, p \geq 1, q \geq 1,\\ \frac{1}{p}, & \text{if } \frac{1}{p} + \frac{1}{q} < 1, p > 1, q > 1,\end{cases}$$

the convolution is defined by $(T * f)(t) = \int_0^t T(t-s)f(s)\,ds$, and we have

$$T_\alpha^\varepsilon * f_\varepsilon \to T_\alpha * f \quad \text{in } L^r\big((0,T);X\big) \text{ as } \varepsilon \to 0^+. \tag{2.28}$$

Proof From the identity

$$T_\alpha^\varepsilon * f_\varepsilon - T_\alpha * f = T_\alpha^\varepsilon * (f_\varepsilon - f) + \big(T_\alpha^\varepsilon - T_\alpha\big) * f,$$

we have

$$\begin{aligned}&\big\|T_\alpha^\varepsilon * f_\varepsilon - T_\alpha * f\big\|_{L^r((0,T),X)}\\&\quad= \big\|T_\alpha^\varepsilon * (f_\varepsilon - f) + \big(T_\alpha^\varepsilon - T_\alpha\big) * f\big\|_{L^r((0,T),X)}\\&\quad\leq \big\|T_\alpha^\varepsilon * (f_\varepsilon - f)\big\|_{L^r((0,T),X)} + \big\|\big(T_\alpha^\varepsilon - T_\alpha\big) * f\big\|_{L^r((0,T),X)}.\end{aligned} \tag{2.29}$$

For $\frac{1}{p} + \frac{1}{q} \geq 1$ ($p \geq 1$, $q \geq 1$), we can use Young's inequality for convolution, together with Minkowski's inequality and Lemmas 2.4 and 2.5, to obtain

$$\begin{aligned}&\big\|T_\alpha^\varepsilon * f_\varepsilon - T_\alpha * f\big\|_{L^r((0,T),X)}\\&\quad\leq \big\|T_\alpha^\varepsilon\big\|_{L^p((0,T),X)}\|f_\varepsilon - f\|_{L^q((0,T),X)} + \big\|T_\alpha^\varepsilon - T_\alpha\big\|_{L^p((0,T),X)}\|f\|_{L^q((0,T),X)}\\&\quad\leq C\big\{\|f_\varepsilon - f\|_{C([0,T],X)} + \big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\big)\|f\|_{C([0,T],X)}\big\}.\end{aligned}$$

For $\frac{1}{p} + \frac{1}{q} < 1$ ($p > 1, q > 1$), we first choose $r = p$ in (2.29) and then use Young's inequality for convolution, together with Lemmas 2.4 and 2.5, to get

$$\begin{aligned}&\big\|T_\alpha^\varepsilon * f_\varepsilon - T_\alpha * f\big\|_{L^p((0,T),X)}\\&\quad\leq \big\|T_\alpha^\varepsilon\big\|_{L^p((0,T),X)}\|f_\varepsilon - f\|_{L^1((0,T),X)} + \big\|T_\alpha^\varepsilon - T_\alpha\big\|_{L^p((0,T),X)}\|f\|_{L^1((0,T),X)}\\&\quad\leq C\big\{\|f_\varepsilon - f\|_{C([0,T],X)} + \big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\big)\|f\|_{C([0,T],X)}\big\}.\end{aligned}$$

Thus the estimate (2.28) holds true for $\frac{1}{p} + \frac{1}{q} < 1$ ($p > 1$, $q > 1$).

Since $L^q((0,T),X) \hookrightarrow L^1((0,T),X)$, we easily get the convergence (2.28). This completes the proof of Lemma 2.6. □

Lemma 2.7 *If $u_{0,\varepsilon}$ and u_0 satisfy Assumption* (H5), *then for $p \geq 1$ and for any $\varepsilon \in (0,\varepsilon_1]$, we have*

$$\left\|S_\alpha^\varepsilon u_{0,\varepsilon} - S_\alpha u_0\right\|_{L^p((0,T),X)} \leq C\left\{\|u_{0,\varepsilon} - u_0\| + \left(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \left|\chi(\varepsilon) - \chi_0\right|\right)\|u_0\|\right\}, \tag{2.30}$$

for some positive constant C depending only on α, T, M, θ, r, ε_0, θ_0, p, r_0. Moreover,

$$S_\alpha^\varepsilon u_{0,\varepsilon} \to S_\alpha u_0 \quad \text{in } L^p\big((0,T),X\big) \text{ as } \varepsilon \to 0^+. \tag{2.31}$$

Proof Due to the identity $S_\alpha^\varepsilon u_{0,\varepsilon} - S_\alpha u_0 = S_\alpha^\varepsilon(u_{0,\varepsilon} - u_0) + (S_\alpha^\varepsilon - S_\alpha)u_0$, from Minkowski's inequality and Lemmas 2.5 and 2.6, we obtain that

$$\begin{aligned}
&\left\|S_\alpha^\varepsilon u_{0,\varepsilon} - S_\alpha u_0\right\|_{L^p((0,T),X)} \\
&\quad = \left\|S_\alpha^\varepsilon(u_{0,\varepsilon} - u_0) + \left(S_\alpha^\varepsilon - S_\alpha\right)u_0\right\|_{L^p((0,T),X)} \\
&\quad \leq \left\|S_\alpha^\varepsilon(u_{0,\varepsilon} - u_0)\right\|_{L^p((0,T),X)} + \left\|\left(S_\alpha^\varepsilon - S_\alpha\right)u_0\right\|_{L^p((0,T),X)} \\
&\quad \leq \left\|S_\alpha^\varepsilon\right\|_{L^p((0,T),X)}\|u_{0,\varepsilon} - u_0\| + \left\|S_\alpha^\varepsilon - S_\alpha\right\|_{L^p((0,T),X)}\|u_0\| \\
&\quad \leq C\left\{\|u_{0,\varepsilon} - u_0\| + \left(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \left|\chi_0 - \chi(\varepsilon)\right|\right)\|u_0\|\right\}.
\end{aligned} \tag{2.32}$$

This completes the proof of Lemma 2.7. □

3 Statement of the problems and main results

Now we consider the existence of solutions u_ε and u to problem (1.1) with a small parameter ε and the limit problem (1.7) as $\varepsilon \to 0^+$, respectively, and then we focus on the convergence of u_ε to u in the space X_ρ, where

$$X_\rho = \left\{u \in C\big((0,T],L^{\rho q}(\Omega)\big) : \sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)}\left\|u(t)\right\|_{L^{\rho q}(\Omega)} < \infty\right\} \tag{3.1}$$

is endowed with the norm

$$\|u\|_{X_\rho} = \sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)}\left\|u(t)\right\|_{L^{\rho q}(\Omega)}. \tag{3.2}$$

Define the space

$$Y_\rho = \left\{u \in L^p\big((0,T),L^{\rho q}(\Omega)\big) : \left\|t^{\frac{\alpha n}{2\rho q}(\rho-1)}u(t,x)\right\|_{L^p((0,T),L^{\rho q}(\Omega))} < \infty\right\} \tag{3.3}$$

endowed with the norm

$$\|u\|_{Y_\rho} = \left\|t^{\frac{\alpha n}{2\rho q}(\rho-1)}u(t,x)\right\|_{L^p((0,T),L^{\rho q}(\Omega))}. \tag{3.4}$$

We now state our main results.

Theorem 3.1 *Suppose the operator A defined in equation (1.5) satisfies Assumptions* (H1), (H3), *the coefficient χ satisfies the convergence condition* (1.4), *the convolution kernel k satisfies Assumption* (H2), *and $\rho \geq 1$, $q \geq 1$, $q > \frac{\alpha n}{2}(\rho - 1)$, $\alpha \in (\frac{1}{\rho}, 1]$. Then*

(i) *There exist constants $T > 0$ and $R > 0$ such that the problem* (1.1) *admits an L^q-mild solution $u_\varepsilon : [0, T] \to L^q(\Omega)$ in the ball $\mathbf{B}_R$ which is unique in $C([0, T], L^q(\Omega)) \cap X_\rho$, for any $(f_\varepsilon, u_{0,\varepsilon}) \in \mathrm{C}([0, T], L^{\rho q}(\Omega)) \times L^{\rho q}(\Omega)$, $\varepsilon \in [0, \varepsilon_0]$.*

(ii) *For constants $T > 0$ and $R > 0$, in the ball $\mathbf{B}_R$ the problem* (1.7) *admits an L^q-mild solution $u : [0, T] \to L^q(\Omega)$ which is unique in $C([0, T], L^q(\Omega)) \cap X_\rho$, for any $(f, u_0) \in \mathrm{C}([0, T], L^q(\Omega)) \times L^q(\Omega)$.*

(iii) *The solutions u_ε, u depend continuously on the initial data, that is, if u_ε, v_ε and u, v are solutions of problems* (1.1) *and* (1.7) *starting from $u_{0,\varepsilon}$, $v_{0,\varepsilon}$ and u_0, v_0, respectively, then*

$$\sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)} \big\| u_\varepsilon(t) - v_\varepsilon(t) \big\|_{L^{\rho q}(\Omega)} \leq C \| u_{0,\varepsilon} - v_{0,\varepsilon} \|_{L^{\rho q}(\Omega)}, \tag{3.5}$$

$$\sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)} \big\| u(t) - v(t) \big\|_{L^{\rho q}(\Omega)} \leq C \| u_0 - v_0 \|_{L^q(\Omega)}. \tag{3.6}$$

Theorem 3.2 *Suppose the operator A defined in equation (1.5) satisfies Assumptions* (H1), (H3), *the nonlinear operator $N : [0, T] \times X \to X$ defined by equation* (1.14) *is continuous with respect to t, and there exists a constant $M > 0$ such that*

$$\big\| N(t, x) \big\| \leq M \big(1 + \| x \|^{\rho-1} \big), \quad \forall x, y \in X, \tag{3.7}$$

$$\big\| N(t, x) - N(t, y) \big\| \leq M \big(1 + \| x \|^{\rho-1} + \| y \|^{\rho-1} \big) \| x - y \|, \quad \forall x, y \in X. \tag{3.8}$$

Assume that the coefficient χ satisfies the convergence condition (1.4), *the convolution kernel k satisfies Assumption* (H2), *and $\rho \geq 1$, $q \geq 1$, $\alpha > \rho\beta$, $\alpha \in (\frac{1}{\rho}, 1]$, $q > \frac{\alpha n(\rho-1)}{2}$. Then*

(i) *There exist constants $T > 0$ and $R > 0$ such that the problem* (1.12) *admits an L^q-mild solution $u_\varepsilon : [0, T] \to L^q(\Omega)$ in the ball $\mathbf{B}_R$ which is unique in $\mathrm{C}([0, T], L^q(\Omega)) \cap X_\rho$ for any $(f_\varepsilon, u_{0,\varepsilon}) \in \mathrm{C}([0, T], L^{\rho q}(\Omega)) \times L^{\rho q}(\Omega)$, $\varepsilon \in [0, \varepsilon_0]$.*

(ii) *For constants $T > 0$ and $R > 0$, in the ball $\mathbf{B}_R$ the problem* (1.13) *admits an L^q-mild solution $u : [0, T] \to L^q(\Omega)$ which is unique in $\mathrm{C}([0, T], L^q(\Omega)) \cap X_\rho$ for any $(f, u_0) \in \mathrm{C}([0, T], L^q(\Omega)) \times L^q(\Omega)$.*

(iii) *The solutions u_ε and u depend continuously on the initial data, that is, if u_ε, v_ε and u, v are solutions of the problem* (1.12) *and the problem* (1.13) *starting from $u_{0,\varepsilon}$, $v_{0,\varepsilon}$ and u_0, v_0, respectively, then we have*

$$\sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)} \big\| u_\varepsilon(t) - v_\varepsilon(t) \big\|_{L^{\rho q}(\Omega)} \leq C \| u_{0,\varepsilon} - v_{0,\varepsilon} \|_{L^{\rho q}(\Omega)}, \tag{3.9}$$

$$\sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)} \big\| u(t) - v(t) \big\|_{L^{\rho q}(\Omega)} \leq C \| u_0 - v_0 \|_{L^q(\Omega)}. \tag{3.10}$$

In the following, we denote $\beta = \frac{\alpha n}{2\rho q}(\rho - 1)$.

Theorem 3.3 *Suppose the operator A defined in equation (1.5) satisfies Assumptions* (H1), (H3), *the coefficient χ satisfies the convergence condition* (1.4), *the convolution kernel k satisfies Assumption* (H2), *and $\rho \geq 1$, $p, q \geq 1$, $\alpha \in (\frac{1}{\rho}, 1]$, $\frac{1}{p} + \alpha \geq 1$. If $f_\varepsilon, f \in C([0, T]; L^{\rho q}(\Omega))$*

and $u_{0,\varepsilon}, u_0 \in L^{\rho q}(\Omega)$ are such that

$$\|f_\varepsilon - f\|_{C([0,T],L^{\rho q}(\Omega))} \to 0 \quad \text{and} \quad \|u_{0,\varepsilon} - u_0\|_{L^{\rho q}(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0^+, \tag{3.11}$$

then the mild solution u_ε of the approximating problem (1.1) *converges in the space Y_ρ to the mild solution u of the limit problem* (1.7) *as $\varepsilon \to 0^+$. More precisely, there exists a positive constant C such that*

$$\begin{aligned} \|u_\varepsilon - u\|_{Y_\rho} \le C\Big\{&\|u_{0,\varepsilon} - u_0\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon - f\|_{C([0,T],L^{\rho q}(\Omega))} \\ &+ \Big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\Big)\|u_0\|_{L^{\rho q}(\Omega)} \\ &+ \Big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\Big)\|f\|_{C([0,T],L^{\rho q}(\Omega))}\Big\}. \end{aligned} \tag{3.12}$$

Theorem 3.4 *Suppose the operator A defined in equation* (1.5) *satisfies Assumptions* (H1), (H3), *the nonlinear operator $N : [0,T] \times X \to X$ defined by equation* (1.14) *is continuous with respect to t and satisfies* (3.7) *and* (3.8). *Assume that the coefficient χ satisfies the convergence condition* (1.4), *the convolution kernel k satisfies Assumption* (H2), *and $\rho \ge 1$, $p,q \ge 1$, $q > \frac{\alpha n(\rho-1)}{2}$, $\alpha \in (\frac{1}{\rho}, 1]$, $\alpha > \rho\beta$, $\frac{1}{p} + \alpha \ge 1$. If $f_\varepsilon, f \in C([0,T], L^{\rho q}(\Omega))$ and $u_{0,\varepsilon}, u_0 \in L^{\rho q}(\Omega)$ are such that*

$$\|f_\varepsilon - f\|_{C([0,T],L^{\rho q}(\Omega))} \to 0 \quad \text{and} \quad \|u_{0,\varepsilon} - u_0\|_{L^{\rho q}(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0^+, \tag{3.13}$$

then the mild solution u_ε of the approximating problem (1.12) *converges in the space Y_ρ to the mild solution u of the limit problem* (1.13) *as $\varepsilon \to 0^+$. More precisely, there exists a positive constant C such that*

$$\begin{aligned} \|u_\varepsilon - u\|_{Y_\rho} \le C\Big\{&\|u_{0,\varepsilon} - u_0\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon - f\|_{C([0,T],L^{\rho q}(\Omega))} \\ &+ \Big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\Big)\|u_0\|_{L^{\rho q}(\Omega)} \\ &+ \Big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\Big)\Big(\|u_0\|_{L^{\rho q}(\Omega)} + \|u_0\|^{\rho}_{L^{\rho q}(\Omega)} \\ &+ \|f\|_{C([0,T],L^{\rho q}(\Omega))} + \big\|f(t)\big\|^{\rho}_{C([0,T],L^{\rho q}(\Omega))}\Big)\Big\}. \end{aligned} \tag{3.14}$$

4 Proofs of Theorems 3.1–3.4

Proof of Theorem 3.1. For the approximating problem (1.1) with a small positive parameter ε and the limit problem (1.7) as $\varepsilon \to 0^+$, due to Definition 2.3, we can transform them into the integral equations (2.6) and (2.7) by the operator families $\{S_\alpha(t)\}_{t\ge0}$, $\{T_\alpha(t)\}_{t\ge0}$ and the Laplace transform. Direct computation then implies that

$$\begin{aligned} &\rho\beta = \rho\frac{\alpha n}{2\rho q}(\rho - 1) = \frac{\alpha n}{2q}(\rho - 1) < 1, \\ &\frac{n}{2}\left(\frac{1}{q} - \frac{1}{\rho q}\right) = \frac{n(\rho-1)}{2\rho q} = \frac{\beta}{\alpha} < \frac{1}{\rho\alpha} < 1, \end{aligned}$$

from which, together with $u_{0,\varepsilon} \in L^{\rho q}(\Omega)$ and Lemma 2.2 (i), we obtain

$$\begin{aligned}
\|u_\varepsilon\|_{X_\rho} &\leq \left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon}\right\|_{X_\rho} + \left\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds\right\|_{X_\rho} \\
&= \sup_{t\in(0,T]} t^\beta \left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon}\right\|_{L^{\rho q}(\Omega)} + \sup_{t\in(0,T]} t^\beta \left\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds\right\|_{L^{\rho q}(\Omega)} \\
&\leq \sup_{t\in(0,T]} Ce^{rt}t^\beta \|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \sup_{t\in(0,T]} t^\beta \int_0^t \left\|\mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\right\|_{L^{\rho q}(\Omega)}\,ds \\
&\leq Ce^{rT}T^\beta \|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + C\sup_{t\in(0,T]} t^\beta \int_0^t e^{r(t-s)}\left\|f_\varepsilon(s)\right\|_{L^{\rho q}(\Omega)}\,ds \\
&\leq Ce^{rT}T^\beta \|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + C\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))} \sup_{t\in(0,T]} e^{rt}t^\beta \int_0^t e^{-rs}\,ds \\
&\leq Ce^{rT}T^\beta \|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \frac{C}{r}T^\beta\left(e^{rT}-1\right)\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))} \leq R.
\end{aligned}$$

Next, we will prove that u_ε is an L^q-mild solution. Indeed, as $t \to 0^+$, we have

$$\begin{aligned}
&\left\|u_\varepsilon(t) - u_{0,\varepsilon}\right\|_{L^q(\Omega)} \\
&\quad\leq \left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\right\|_{L^q(\Omega)} + \left\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\right\|_{L^q(\Omega)} \\
&\quad\leq C\left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\right\|_{L^{\rho q}(\Omega)} + \int_0^t \left\|\mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\right\|_{L^q(\Omega)}\,ds \\
&\quad\leq C\left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\right\|_{L^{\rho q}(\Omega)} + C\int_0^t e^{r(t-s)}\left\|f_\varepsilon(s)\right\|_{L^q(\Omega)}\,ds \\
&\quad\leq C\left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\right\|_{L^{\rho q}(\Omega)} + C\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))}e^{rt}\int_0^t e^{-rs}\,ds \\
&\quad= C\left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\right\|_{L^{\rho q}(\Omega)} + \frac{C}{r}\left(e^{rt}-1\right)\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))} \to 0.
\end{aligned}$$

Therefore, $u_\varepsilon \in C([0,T],L^q(\Omega)) \cap C((0,T],L^{\rho q}(\Omega))$ is an L^q-mild solution of the approximating problem (1.1). In order to obtain the continuous dependence on the initial data of the solutions, we assume that u_ε and v_ε are solutions of the problem (1.1) starting from $u_{0,\varepsilon}, v_{0,\varepsilon} \in L^{\rho q}(\Omega)$, respectively. For $T > 0$, we have

$$\begin{aligned}
t^\beta\left\|u_\varepsilon(t) - v_\varepsilon(t)\right\|_{L^{\rho q}(\Omega)} &\leq t^\beta\left\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - S_\alpha^\varepsilon(t)v_{0,\varepsilon}\right\|_{L^{\rho q}(\Omega)} \\
&\leq Ce^{rT}T^\beta\|u_{0,\varepsilon} - v_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} \leq C\|u_{0,\varepsilon} - v_{0,\varepsilon}\|_{L^{\rho q}(\Omega)}.
\end{aligned}$$

Next, we prove the existence of a solution to the linear limit problem (1.7). Using Lemma 2.3 and the integral equation (2.9), we have

$$\begin{aligned}
\|u\|_{X_\rho} &\leq \left\|S_\alpha(t)u_0\right\|_{X_\rho} + \left\|\int_0^t \mathrm{T}_\alpha(t-s)f(s)\,ds\right\|_{X_\rho} \\
&= \sup_{t\in(0,T]} t^\beta\left\|S_\alpha(t)u_0\right\|_{L^{\rho q}(\Omega)} + \sup_{t\in(0,T]} t^\beta\left\|\int_0^t \mathrm{T}_\alpha(t-s)f(s)\,ds\right\|_{L^{\rho q}(\Omega)}
\end{aligned}$$

$$\leq \sup_{t\in(0,T]} t^{\beta}t^{-\beta}\|u_0\|_{L^q(\Omega)} + \sup_{t\in(0,T]} t^{\beta}\int_0^t \big\|\mathrm{T}_\alpha(t-s)f(s)\big\|_{L^{\rho q}(\Omega)}\,ds$$

$$\leq C\|u_0\|_{L^q(\Omega)} + C\sup_{t\in(0,T]} t^{\beta}\int_0^t (t-s)^{-\beta+\alpha-1}\big\|f(s)\big\|_{L^q(\Omega)}\,ds$$

$$\leq C\|u_0\|_{L^q(\Omega)} + C\|f\|_{C([0,T],L^q(\Omega))}\sup_{t\in(0,T]} t^{\beta}\int_0^t (t-s)^{\alpha-1-\beta}\,ds$$

$$\leq C\|u_0\|_{L^q(\Omega)} + CT^{\alpha}\|f\|_{C([0,T];L^q(\Omega))} \leq R.$$

The contraction principle implies the existence of the mild solution u in the ball $\mathbf{B}_R$. We will prove that u is an L^q-mild solution and is unique in X_ρ. Indeed, as $t\to 0^+$, we have

$$\begin{aligned}\big\|u(t)-u_0\big\|_{L^q(\Omega)} &\leq \big\|S_\alpha(t)u_0-u_0\big\|_{L^q(\Omega)} + \int_0^t \big\|\mathrm{T}_\alpha(t-s)f(s)\big\|_{L^q(\Omega)}\,ds\\ &\leq \big\|S_\alpha(t)u_0-u_0\big\|_{L^q(\Omega)} + \int_0^t (t-s)^{\alpha-1}\big\|f(s)\big\|_{L^q(\Omega)}\,ds\\ &\leq \big\|S_\alpha(t)u_0-u_0\big\|_{L^q(\Omega)} + C\|f\|_{C([0,T],L^q(\Omega))}\int_0^t (t-s)^{\alpha-1}\,ds\\ &= \big\|S_\alpha(t)u_0-u_0\big\|_{L^q(\Omega)} + \frac{C}{\alpha}t^{\alpha}\|f\|_{C([0,T],L^q(\Omega))} \to 0.\end{aligned}$$

Therefore, $u\in \mathrm{C}([0,T],L^q(\Omega))\cap \mathrm{C}((0,T],L^{\rho q}(\Omega))$ is an L^q-mild solution of the linear limit problem (1.7). In order to obtain the continuous dependence on the initial data of the solutions, we assume that u and v are solutions of the problem (1.7) starting from $u_0, v_0 \in L^q(\Omega)$, respectively. For $T>0$, we have

$$\begin{aligned}t^{\beta}\big\|u(t)-v(t)\big\|_{L^{\rho q}} &\leq t^{\beta}\big\|S_\alpha(t)u_0-S_\alpha(t)v_0\big\|_{L^{\rho q}(\Omega)}\\ &\leq Ct^{\beta}t^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{\rho q})}\|u_0-v_0\|_{L^q(\Omega)} = C\|u_0-v_0\|_{L^q(\Omega)},\end{aligned}$$

finishing the proof of Theorem 3.1. □

Proof of Theorem 3.2. For $\rho\beta<1$, $\frac{n}{2}(\frac{1}{q}-\frac{1}{\rho q})<1$, we can choose T such that

$$\begin{aligned}&Ce^{rT}T^{\beta}\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \frac{CT^{\beta}(e^{rT}-1)}{r}\big[\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))}+M\big]\\ &\quad+\frac{CMR^{\rho}e^{rT}T^{1-\rho\beta+\beta}}{1-\rho\beta} \leq R,\\ &C\|u_0\|_{L^q(\Omega)} + \frac{CT^{\alpha}}{\alpha-\beta}\|f\|_{C([0,T];L^q(\Omega))}\\ &\quad+CMT^{\alpha+\beta}\left[\frac{1}{\alpha}+R^{\rho}T^{-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta)\right] \leq R,\\ &\frac{CMe^{rT}T^{\beta}\Gamma(1-\beta)}{r^{1-\beta}} + \frac{2CMR^{\rho-1}e^{rT}T^{\beta}\Gamma(1-\rho\beta)}{r^{1-\rho\beta}} \leq \frac{1}{2},\\ &MCT^{\alpha}\mathrm{B}(\alpha,1-\beta) + 2R^{\rho-1}T^{\alpha+\beta-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta) \leq \frac{1}{2},\end{aligned}$$

where $\Gamma(\alpha)$ and $\mathrm{B}(\alpha,\beta)$ are the Gamma and Beta functions, respectively. Let $\mathbf{B}_R$ be the closed ball in X_ρ centered at the origin with radius R. Then, for the nonlinear approximating problem (1.12), define an operator $\Lambda : \mathbf{B}_R \to \mathbf{B}_R$ by

$$\Lambda u_\varepsilon(t) = \mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon} + \int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds + \int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)\,ds. \tag{4.1}$$

By the assumptions on the nonlinear operator N, due to Lemma 2.2 and for $u_\varepsilon \in X_\rho$, we have

$$\begin{aligned}
&\|\Lambda u_\varepsilon\|_{X_\rho} \\
&\quad\le \big\|\mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon}\big\|_{X_\rho} + \bigg\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds\bigg\|_{X_\rho} + \bigg\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)\,ds\bigg\|_{X_\rho} \\
&\quad\le \sup_{t\in(0,T]} t^\beta \big\|\mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon}\big\|_{L^{\rho q}(\Omega)} + \sup_{t\in(0,T]} t^\beta \int_0^t \big\|\mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\qquad + \sup_{t\in[0,T]} t^\beta \int_0^t \big\|\mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\le \sup_{t\in(0,T]} t^\beta Ce^{rt}\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \sup_{t\in(0,T]} \int_0^t t^\beta \big\|\mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\qquad + CM \sup_{t\in(0,T]} t^\beta \int_0^t e^{r(t-s)}\big(1 + \|u_\varepsilon(s)\|_{L^{\rho q}(\Omega)}^\rho\big)\,ds,
\end{aligned}$$

thus,

$$\begin{aligned}
&\|\Lambda u_\varepsilon\|_{X_\rho} \\
&\quad\le Ce^{rT}T^\beta\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + CT^\beta\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))} \sup_{t\in(0,T]} e^{rt}\int_0^t e^{-rs}\,ds \\
&\qquad + CM \sup_{t\in(0,T]} e^{rt}t^\beta \int_0^t e^{-rs}\,ds \\
&\qquad + CM \sup_{t\in(0,T]} e^{rt}t^\beta \int_0^t e^{-rs}s^{-\rho\beta}s^{\rho\beta}\|u_\varepsilon(s)\|_{L^{\rho q}(\Omega)}^\rho\,ds \\
&\quad\le Ce^{rT}T^\beta\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \frac{C}{r}T^\beta\big(e^{rT}-1\big)\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))} \\
&\qquad + \frac{C}{r}T^\beta M\big(e^{rT}-1\big) + \frac{C}{1-\rho\beta}R^\rho Me^{rT}T^{1-\rho\beta+\beta} \le R.
\end{aligned}$$

If $u_\varepsilon, v_\varepsilon \in \mathbf{B}_R$, we have

$$\begin{aligned}
&\|\Lambda u_\varepsilon - \Lambda v_\varepsilon\|_{X_\rho} \\
&\quad\le \sup_{t\in(0,T]} t^\beta \int_0^t \big\|\mathrm{T}_\alpha^\varepsilon(t-s)\big(Nu_\varepsilon(s) - Nv_\varepsilon(s)\big)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\le MC \sup_{t\in(0,T]} t^\beta \int_0^t e^{r(t-s)}\big(1 + \|u_\varepsilon(s)\|_{L^{\rho q}(\Omega)}^{\rho-1} + \|v_\varepsilon(s)\|_{L^{\rho q}(\Omega)}^{\rho-1}\big) \\
&\qquad \times \big\|u_\varepsilon(s) - v_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le MC\|u_\varepsilon - v_\varepsilon\|_{X_\rho} \sup_{t\in(0,T]} e^{rt}t^\beta \int_0^t e^{-rs}s^{-\beta}\,ds \\
&\quad + MC\|u_\varepsilon - v_\varepsilon\|_{X_\rho} \sup_{t\in(0,T]} e^{rt}t^\beta \int_0^t e^{-rs}s^{-\beta}s^{-\beta(\rho-1)}\|u_\varepsilon\|_{X_\rho}^{\rho-1}\,ds \\
&\quad + MC\|u_\varepsilon - v_\varepsilon\|_{X_\rho} \sup_{t\in(0,T]} e^{rt}t^\beta \int_0^t e^{-rs}s^{-\beta}s^{-\beta(\rho-1)}\|v_\varepsilon\|_{X_\rho}^{\rho-1}\,ds,
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\|\Lambda u_\varepsilon - \Lambda v_\varepsilon\|_{X_\rho} \\
&\quad\le CMe^{rT}T^\beta\Gamma(1-\beta)r^{\beta-1}\|u_\varepsilon - v_\varepsilon\|_{X_\rho} \\
&\qquad + CMe^{rT}T^\beta\Gamma(1-\rho\beta)r^{\rho\beta-1}\|u_\varepsilon - v_\varepsilon\|_{X_\rho}\|u_\varepsilon\|_{X_\rho}^{\rho-1} \\
&\qquad + CMe^{rT}T^\beta\Gamma(1-\rho\beta)r^{\rho\beta-1}\|u_\varepsilon - v_\varepsilon\|_{X_\rho}\|v_\varepsilon\|_{X_\rho}^{\rho-1} \\
&\quad\le CMe^{rT}T^\beta\Gamma(1-\beta)r^{\beta-1}\|u_\varepsilon - v_\varepsilon\|_{X_\rho} \\
&\qquad + CMR^{\rho-1}e^{rT}T^\beta\Gamma(1-\rho\beta)r^{\rho\beta-1}\|u_\varepsilon - v_\varepsilon\|_{X_\rho} \\
&\qquad + CMR^{\rho-1}e^{rT}T^\beta\Gamma(1-\rho\beta)r^{\rho\beta-1}\|u_\varepsilon - v_\varepsilon\|_{X_\rho} \\
&\quad\le CMe^{rT}T^\beta\|u_\varepsilon - v_\varepsilon\|_{X_\rho}\big(r^{\beta-1}\Gamma(1-\beta) + 2R^{\rho-1}r^{\rho\beta-1}\Gamma(1-\rho\beta)\big) \\
&\quad\le \frac{1}{2}\|u_\varepsilon - v_\varepsilon\|_{X_\rho}.
\end{aligned}
$$

This yields that Λ is a contraction operator on X_ρ. We shall prove that u_ε is an L^q-mild solution and is unique in X_ρ. Indeed, we have

$$
\begin{aligned}
&\big\|u_\varepsilon(t) - u_{0,\varepsilon}\big\|_{L^q(\Omega)} \\
&\quad\le \big\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\big\|_{L^q(\Omega)} \\
&\qquad + \left\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds\right\|_{L^q(\Omega)} + \left\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)\,ds\right\|_{L^q(\Omega)} \\
&\quad\le C\big\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\big\|_{L^{\rho q}(\Omega)} + C\int_0^t e^{r(t-s)}\big\|f_\varepsilon(s)\big\|_{L^q(\Omega)}\,ds \\
&\qquad + MC\int_0^t e^{r(t-s)}\big(1 + \big\|u_\varepsilon(s)\big\|_{L^q(\Omega)}^\rho\big)\,ds \\
&\quad\le C\big\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\big\|_{L^{\rho q}(\Omega)} + C\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))}e^{rt}\int_0^t e^{-rs}\,ds \\
&\qquad + MC\int_0^t e^{r(t-s)}\,ds + MC\|u_\varepsilon\|_{X_\rho}^\rho e^{rt}\int_0^t e^{-rs}s^{-\rho\beta}\,ds,
\end{aligned}
$$

thus, as $t\to 0^+$, we obtain

$$
\begin{aligned}
\big\|u_\varepsilon(t) - u_{0,\varepsilon}\big\|_{L^q(\Omega)} &\le C\big\|S_\alpha^\varepsilon(t)u_{0,\varepsilon} - u_{0,\varepsilon}\big\|_{L^{\rho q}(\Omega)} + \frac{C}{r}\big(e^{rt}-1\big)\|f_\varepsilon\|_{C([0,T],L^{\rho q}(\Omega))} \\
&\quad + \frac{MC}{r}\big(e^{rt}-1\big) + \frac{MC}{1-\rho\beta}e^{rt}t^{1-\rho\beta}\|u_\varepsilon\|_{X_\rho} \to 0.
\end{aligned}
$$

Therefore, $u_\varepsilon \in C([0,T],L^q(\Omega))\cap C((0,T],L^{\rho q}(\Omega))$ is an L^q-mild solution for the nonlinear approximating problem (1.12).

If $v_\varepsilon \in X_\rho$ is a solution of problem (1.12), taking $0 < T' \le T$ such that $\|v_\varepsilon\|_{X_\rho}^{T'} \le R$, the uniqueness in $\mathbf{B}_R$ implies that $u_\varepsilon(t,x) = v_\varepsilon(t,x)$ for all $t \in [0,T']$. Now, we set

$$R' = \max\Big\{\sup_{t\in(0,T]} t^\beta \big\|u_\varepsilon(t)\big\|_{L^{\rho q}(\Omega)}, \sup_{t\in(0,T]} t^\beta \big\|v_\varepsilon(t)\big\|_{L^{\rho q}(\Omega)}\Big\}.$$

For $t \in [T',T]$, we have

$$\begin{aligned}
&t^\beta \big\|u_\varepsilon(t) - v_\varepsilon(t)\big\|_{L^{\rho q}(\Omega)} \\
&\quad\le MCt^\beta \int_0^t e^{r(t-s)}\big(1 + \big\|u_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1} + \big\|v_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1}\big)\big\|u_\varepsilon(s) - v_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\le MCt^\beta e^{rt} \int_0^t e^{-rs}s^{-\beta}\big(1 + s^{-(\rho-1)\beta}\|u_\varepsilon\|_{X_\rho}^{\rho-1} + s^{-(\rho-1)\beta}\|v_\varepsilon\|_{X_\rho}^{\rho-1}\big) \\
&\qquad\times s^\beta \big\|u_\varepsilon(s) - v_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\le MC\big(1 + 2T'^{-(\rho-1)\beta}R'^{\rho-1}\big)e^{rt}\int_{T'}^t e^{-rs}s^{-\beta}s^\beta \big\|u_\varepsilon(s) - v_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds.
\end{aligned}$$

Defining a function $\eta : [0,T] \to [0,+\infty)$ by $\eta_\varepsilon(t) = t^\beta\|u_\varepsilon(t) - v_\varepsilon(t)\|_{L^{\rho q}(\Omega)}$, we have

$$\eta_\varepsilon(t) \le MC\big(1 + 2T'^{-(\rho-1)\beta}R'^{\rho-1}\big)e^{rt}\int_0^t e^{-rs}s^{-\beta}\eta_\varepsilon(s)\,ds,$$

so, due to the singular Gronwall's lemma [4, Theorem 4], we obtain the uniqueness.

In order to obtain the continuous dependence on the initial data of the solutions, for $T > 0$, we assume that u_ε and v_ε are solutions of equation (1.12) starting from $u_{0,\varepsilon}, v_{0,\varepsilon} \in L^{\rho q}(\Omega)$, respectively. From the choice of T, it follows that

$$\begin{aligned}
&t^\beta \big\|u_\varepsilon(t) - v_\varepsilon(t)\big\|_{L^{\rho q}(\Omega)} \\
&\quad\le Ce^{rt}t^\beta\|u_{0,\varepsilon} - v_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} \\
&\qquad+ MC\int_0^t t^\beta e^{r(t-s)}\big(1 + \big\|u_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1} + \big\|v_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1}\big)\big\|u_\varepsilon(s) - v_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\le Ce^{rT}T^\beta\|u_{0,\varepsilon} - v_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} \\
&\qquad+ CMe^{rT}T^\beta\big\|u_\varepsilon(s) - v_\varepsilon(s)\big\|_{X_\rho}\left(\frac{1}{r^{1-\beta}}\Gamma(1-\beta) + \frac{2}{r^{1-\rho\beta}}R^{\rho-1}\Gamma(1-\rho\beta)\right) \\
&\quad\le C\|u_{0,\varepsilon} - v_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \frac{1}{2}\|u_\varepsilon - v_\varepsilon\|_{X_\rho}.
\end{aligned}$$

For the nonlinear limit problem (1.13), let $\mathbf{B}_R$ be a closed ball in X_ρ, centered at the origin and with radius R, and define a map $\Gamma : \mathbf{B}_R \to \mathbf{B}_R$ by

$$\Gamma u(t) = \mathrm{S}_\alpha(t)u_0 + \int_0^t \mathrm{T}_\alpha(t-s)f(s)\,ds + \int_0^t \mathrm{T}_\alpha(t-s)Nu(s)\,ds. \tag{4.2}$$

For $u \in X_\rho$, we have

$$\begin{aligned}
\|\Gamma u\|_{X_\rho} &\le \left\|\mathrm{S}_\alpha(t)u_0\right\|_{X_\rho} + \left\|\int_0^t \mathrm{T}_\alpha(t-s)f(s)\,ds\right\|_{X_\rho} \\
&\quad + \left\|\int_0^t \mathrm{T}_\alpha(t-s)Nu(s)\,ds\right\|_{X_\rho} \\
&\le \sup_{t\in(0,T]} t^\beta \left\|\mathrm{S}_\alpha(t)u_0\right\|_{L^{\rho q}(\Omega)} + \sup_{t\in(0,T]} t^\beta \int_0^t \left\|\mathrm{T}_\alpha(t-s)f(s)\right\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad + \sup_{t\in(0,T]} t^\beta \int_0^t \left\|\mathrm{T}_\alpha(t-s)Nu(s)\right\|_{L^{\rho q}(\Omega)}\,ds.
\end{aligned}$$

Using Lemmas 2.3–2.4, one gets

$$\begin{aligned}
\|\Gamma u\|_{X_\rho} &\le C\|u_0\|_{L^q(\Omega)} + C\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1-\beta}\left\|f(s)\right\|_{L^q(\Omega)}\,ds \\
&\quad + CM\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}\left(1+\left\|u(s)\right\|_{L^{\rho q}(\Omega)}^\rho\right)ds \\
&\le C\|u_0\|_{L^q(\Omega)} + \frac{C}{\alpha-\beta}\left\|f(s)\right\|_{C([0,T];L^q(\Omega))}T^\alpha + \frac{CM}{\alpha}T^{\alpha+\beta} \\
&\quad + CM\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}\left\|u(s)\right\|_{L^{\rho q}(\Omega)}^\rho\,ds \\
&\le C\|u_0\|_{L^q(\Omega)} + \frac{C}{\alpha-\beta}\left\|f(s)\right\|_{C([0,T];L^q(\Omega))}T^\alpha + \frac{CM}{\alpha}T^{\alpha+\beta} \\
&\quad + CMR^\rho\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}s^{-\rho\beta}\,ds \\
&\le C\|u_0\|_{L^q(\Omega)} + \frac{C}{\alpha-\beta}\left\|f(s)\right\|_{C([0,T],L^q(\Omega))}T^\alpha + \frac{CM}{\alpha}T^{\alpha+\beta} \\
&\quad + CMR^\rho T^{\alpha-\rho\beta+\beta}\mathrm{B}(\alpha,1-\rho\beta) \le R.
\end{aligned}$$

For $u, v \in \mathbf{B}_R$, we also get

$$\begin{aligned}
&\|\Gamma u - \Gamma v\|_{X_\rho} \\
&\quad\le \sup_{t\in(0,T]} t^\beta \int_0^t \left\|\mathrm{T}_\alpha(t-s)\big(Nu(s)-Nv(s)\big)\right\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\le MC\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}\left(1+\left\|u(s)\right\|_{L^{\rho q}(\Omega)}^{\rho-1}\right. \\
&\qquad \left.+\left\|v(s)\right\|_{L^{\rho q}(\Omega)}^{\rho-1}\right)\left\|u(s)-v(s)\right\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\le MC\|u-v\|_{X_\rho}\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}s^{-\beta}\,ds \\
&\qquad + MC\|u-v\|_{X_\rho}\sup_{t\in[0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}s^{-\beta}s^{-(\rho-1)\beta}\left(\|u\|_{X_\rho}^{\rho-1} + \|v)\|_{X_\rho}^{\rho-1}\right)ds \\
&\quad\le MC\|u-v\|_{X_\rho}T^\alpha\mathrm{B}(\alpha,1-\beta)
\end{aligned}$$

$$
\begin{aligned}
&+ 2MC\|u-v\|_{X_\rho} R^{\rho-1} T^{\alpha+\beta-\rho\beta} \mathrm{B}(\alpha, 1-\rho\beta) \\
&\leq MC\|u-v\|_{X_\rho} \bigl(T^\alpha \mathrm{B}(\alpha, 1-\beta) + 2R^{\rho-1} T^{\alpha+\beta-\rho\beta} \mathrm{B}(\alpha, 1-\rho\beta)\bigr) \\
&\leq \frac{1}{2}\|u-v\|_{X_\rho}.
\end{aligned}
$$

The contraction principle implies the existence of a mild solution u in $\mathbf{B}_R$. We will prove that u is an L^q-mild solution and is unique in X_ρ. Indeed, we have

$$
\begin{aligned}
&\bigl\|u(t)-u_0\bigr\|_{L^q(\Omega)} \\
&\quad\leq \bigl\|S_\alpha(t)u_0 - u_0\bigr\|_{L^q(\Omega)} \\
&\qquad + \int_0^t \bigl\|\mathrm{T}_\alpha(t-s)f(s)\bigr\|_{L^q(\Omega)}\,ds + \int_0^t \bigl\|\mathrm{T}_\alpha(t-s)Nu(s)\bigr\|_{L^q(\Omega)}\,ds \\
&\quad\leq \bigl\|S_\alpha(t)u_0 - u_0\bigr\|_{L^q(\Omega)} + C\int_0^t (t-s)^{\alpha-1}\bigl\|f(s)\bigr\|_{L^q(\Omega)}\,ds \\
&\qquad + CM\int_0^t (t-s)^{\alpha-1}\bigl(1+\bigl\|u(s)\bigr\|_{L^q(\Omega)}^\rho\bigr)\,ds \\
&\quad\leq \bigl\|S_\alpha(t)u_0 - u_0\bigr\|_{L^q(\Omega)} + C\|f\|_{C([0,T],L^q(\Omega))}\int_0^t (t-s)^{\alpha-1}\,ds \\
&\qquad + CM\int_0^t (t-s)^{\alpha-1}\,ds + CM\int_0^t (t-s)^{\alpha-1}\bigl\|u(s)\bigr\|_{L^q(\Omega)}^\rho\,ds,
\end{aligned}
$$

thus, as $t\to 0^+$, we obtain

$$
\begin{aligned}
\bigl\|u(t)-u_0\bigr\|_{L^q(\Omega)} \leq{}& \bigl\|S_\alpha(t)u_0 - u_0\bigr\|_{L^q(\Omega)} + C\alpha^{-1}t^\alpha\|f\|_{C([0,T],L^q(\Omega))} + CM\alpha^{-1}t^\alpha \\
&+ CMt^{\alpha-\rho\beta}\mathrm{B}(\alpha, 1-\rho\beta)\|u\|_{X_\rho}^\rho \to 0.
\end{aligned}
$$

Therefore, $u\in \mathrm{C}([0,T],L^q(\Omega))\cap \mathrm{C}((0,T],L^{\rho q}(\Omega))$ is an L^q-mild solution for the limit problem (1.13). If $v\in X_\rho$ is a solution of problem (1.13), we may take $0<T'\leq T$ such that $\|v\|_{X_\rho}^{T'}\leq R$. Then, by the uniqueness in $\mathbf{B}_R$, we have $u(t,x)=v(t,x)$ for all $t\in[0,T']$. Set

$$
R' = \max\Bigl\{\sup_{t\in(0,T]} t^\beta\bigl\|u(t)\bigr\|_{L^{\rho q}(\Omega)}, \sup_{t\in(0,T]} t^\beta\bigl\|v(t)\bigr\|_{L^{\rho q}(\Omega)}\Bigr\}.
$$

For $t\in[T',T]$, we have

$$
\begin{aligned}
&t^\beta\bigl\|u(t)-v(t)\bigr\|_{L^{\rho q}(\Omega)} \\
&\quad\leq MCt^\beta\int_0^t (t-s)^{\alpha-1}\bigl(1+\bigl\|u(s)\bigr\|_{L^{\rho q}(\Omega)}^{\rho-1} + \bigl\|v(s)\bigr\|_{L^{\rho q}(\Omega)}^{\rho-1}\bigr)\bigl\|u(s)-v(s)\bigr\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\leq MCt^\beta\int_0^t (t-s)^{\alpha-1}\bigl(1+s^{-(\rho-1)\beta}\|u\|_{X_\rho}^{\rho-1} + s^{-(\rho-1)\beta}\|v\|_{X_\rho}^{\rho-1}\bigr) \\
&\qquad\times \bigl\|u(s)-v(s)\bigr\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad\leq MC\bigl(1+2\bigl(T'\bigr)^{-(\rho-1)\beta}R'^{\rho-1}\bigr)\bigl(T'\bigr)^{-\beta}t^\beta\int_0^t (t-s)^{\alpha-1}s^\beta\bigl\|u(s)-v(s)\bigr\|_{L^{\rho q}(\Omega)}\,ds.
\end{aligned}
$$

Now, let $\xi : [0,T] \to [0,+\infty)$ be defined by $\xi(t) = t^{\beta}\|u(t)-v(t)\|_{L^{\rho q}(\Omega)}$, thus we have

$$\xi(t) \le MC\big(1+2\big(T'\big)^{-(\rho-1)\beta}R'^{\rho-1}\big)\big(T'\big)^{-\beta}t^{\beta}\int_0^t (t-s)^{\alpha-1}\xi(s)\,ds.$$

Apply the singular Gronwall's Lemma [4, Theorem 4], we obtain the uniqueness of the solution.

In order to obtain the continuous dependence on the initial data of the solution, we assume that u and v are solutions of problem (1.13) starting from $u_0, v_0 \in L^{\rho q}(\Omega)$. From the choice of T, it follows that

$$\begin{aligned} t^{\beta}\big\|u(t)-v(t)\big\|_{L^{\rho q}(\Omega)} &\le C\|u_0-v_0\|_{L^q(\Omega)} \\ &\quad + MC\sup_{t\in(0,T]} t^{\beta}\int_0^t (t-s)^{\alpha-1}\big(1+\big\|u(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1}+\big\|v(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1}\big) \\ &\quad \times \big\|u(s)-v(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\ &\le C\|u_0-v_0\|_{L^q(\Omega)} \\ &\quad + MC\|u-v\|_{X_\rho}\big(T^{\alpha}\mathrm{B}(\alpha,1-\beta)+2R^{\rho-1}T^{\alpha+\beta-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta)\big) \\ &\le C\|u_0-v_0\|_{L^q(\Omega)}+\frac{1}{2}\|u-v\|_{X_\rho}, \end{aligned}$$

which finishes the proof of Theorem 3.2. □

To give the proof of Theorems 3.3 and 3.4, we need the fact that the solutions $u, u_\varepsilon \in L^{\infty}_{\rho}((0,T), L^{\rho q}(\Omega))$, where the space

$$\begin{aligned} &L^{\infty}_{\rho}\big((0,T),L^{\rho q}(\Omega)\big) \\ &\quad = \Big\{u\in L^{\infty}\big((0,T),L^{\rho q}(\Omega)\big): \sup_{t\in(0,T)} t^{\frac{\alpha n(\rho-1)}{2\rho q}}\big\|u(t)\big\|_{L^{\rho q}(\Omega)}<\infty\Big\} \end{aligned} \tag{4.3}$$

is endowed with the norm

$$\|u\|_{L^{\infty}_{\rho}((0,T),L^{\rho q}(\Omega))} = \sup_{t\in(0,T)} t^{\frac{\alpha n(\rho-1)}{2\rho q}}\big\|u(t)\big\|_{L^{\rho q}(\Omega)}. \tag{4.4}$$

Lemma 4.1 *If u_ε, u are defined by equations (2.6) and (2.7), respectively, then we have*

$$\|u_\varepsilon\|_{L^{\infty}_{\rho}((0,T),L^{\rho q}(\Omega))} \le C\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)}+\|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))}\big), \tag{4.5}$$

$$\|u\|_{L^{\infty}_{\rho}((0,T),L^{\rho q}(\Omega))} \le C\|u_0\|_{L^q(\Omega)}+\frac{C}{\alpha-\beta}T^{\alpha}\|f\|_{C([0,T],L^q(\Omega))}. \tag{4.6}$$

Proof From equations (2.6) and (2.7), and estimates (2.14) and (2.15), we get

$$\begin{aligned} \big\|u_\varepsilon(t)\big\|_{L^{\rho q}(\Omega)} &\le \big\|\mathrm{S}^{\varepsilon}_{\alpha}(t)\big\|_{\mathcal{L}(L^{\rho q}(\Omega))}\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)}+\big\|\mathrm{T}^{\varepsilon}_{\alpha}(t)\big\|_{\mathcal{L}(L^{\rho q}(\Omega))} * \big\|f_\varepsilon(t)\big\|_{L^{\rho q}(\Omega)} \\ &\le C\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)}+C\|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))} \\ &\le C\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)}+\|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))}\big), \end{aligned}$$

and

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty_\rho((0,T),L^{\rho q})} &\le CT^\beta\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))}\big)\\ &\le C\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))}\big),\\ \|u(t)\|_{L^{\rho q}(\Omega)} &\le \|\mathrm{S}_\alpha(t)u_0\|_{L^{\rho q}(\Omega)} + \|\mathrm{T}_\alpha(t)*f(t)\|_{L^{\rho q}(\Omega)}\\ &\le Ct^{-\beta}\|u_0\|_{L^q(\Omega)} + Ct^{-\beta+\alpha-1}*\|f(t)\|_{L^q(\Omega)},\end{aligned}$$

therefore

$$\begin{aligned}\|u\|_{L^\infty_\rho((0,T);L^{\rho q})} &\le C\sup_{t\in(0,T)} t^{-\beta}t^\beta\|u_0\|_{L^q(\Omega)} + C\sup_{t\in(0,T)} t^\beta\big(t^{-\beta+\alpha-1}*\|f(t)\|_{L^q(\Omega)}\big)\\ &\le C\|u_0\|_{L^q(\Omega)} + \frac{C}{\alpha-\beta}T^\alpha\|f\|_{C([0,T],L^q(\Omega))},\end{aligned}$$

ending the proof of Lemma 4.1. □

Lemma 4.2 *If u_ε and u are defined by equations (2.8) and (2.9), respectively, then*

$$\|u_\varepsilon\|_{L^\infty_\rho((0,T),L^{\rho q}(\Omega))} \le MC\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))}\big), \tag{4.7}$$

$$\begin{aligned}\|u\|_{L^\infty_\rho((0,T),L^{\rho q})} &\le C\big(\|u_0\|_{L^q(\Omega)} + \|u_0\|^\rho_{L^q(\Omega)}\\ &\quad + \|f\|_{C([0,T],L^q(\Omega))} + \|f\|^\rho_{C([0,T],L^q(\Omega))}\big),\end{aligned} \tag{4.8}$$

where $C = C(\alpha, r, \theta, T, \rho) > 0$ is a constant.

Proof From equations (2.8), (2.9) and Lemmas 2.3–2.5, we get

$$\begin{aligned}&\|u_\varepsilon(t)\|_{L^{\rho q}(\Omega)}\\ &\quad\le \|\mathrm{S}^\varepsilon_\alpha(t)\|_{\mathcal{L}(L^{\rho q}(\Omega))}\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)}\\ &\qquad + \|\mathrm{T}^\varepsilon_\alpha(t)\|_{\mathcal{L}(L^{\rho q}(\Omega))}*\|f_\varepsilon(t)\|_{L^{\rho q}(\Omega)} + \|\mathrm{T}^\varepsilon_\alpha(t)\|_{\mathcal{L}(L^{\rho q}(\Omega))}*\|Nu_\varepsilon(t)\|_{L^{\rho q}(\Omega)}\\ &\quad\le C\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + C\|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))} + MC\int_0^t\big(1+\|u_\varepsilon\|^\rho_{L^{\rho q}(\Omega)}\big)\,ds\\ &\quad\le C\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))} + MT\big) + MC\int_0^t\|u_\varepsilon(s)\|^\rho_{L^{\rho q}(\Omega)}\,ds.\end{aligned}$$

Applying a generalization of Gronwall's inequality proved in [3], we get

$$\begin{aligned}&\|u_\varepsilon(t)\|_{L^{\rho q}(\Omega)}\\ &\quad\le \big[C\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))} + MT\big)^{1-\rho} + (\rho-1)tMC\big]^{\rho-1}\\ &\quad\le C\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))} + MT\big) + C\big[(\rho-1)TMC\big]^{\rho-1},\end{aligned}$$

thus

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty_\rho((0,T),L^{\rho q})} &\le CT^\beta\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))} + MT\big)\\ &\quad + CT^\beta\big[(\rho-1)TMC\big]^{\rho-1}\\ &\le CM\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))}\big)\end{aligned}$$

and

$$\begin{aligned}\big\|t^\beta u(t)\big\|_{L^{\rho q}(\Omega)} &\le t^\beta\big\|\mathrm{S}_\alpha(t)u_0\big\|_{L^{\rho q}(\Omega)} + t^\beta\big\|\mathrm{T}_\alpha(t)*f(t)\big\|_{L^{\rho q}(\Omega)}\\ &\quad + t^\beta\big\|\mathrm{T}_\alpha(t)*Nu(t)\big\|_{L^{\rho q}(\Omega)}\\ &\le Ct^\beta t^{-\beta}\|u_0\|_{L^q(\Omega)} + t^\beta Ct^{-\beta+\alpha-1}*\big\|f(t)\big\|_{L^q(\Omega)}\\ &\quad + t^\beta MCt^{\alpha-1}*\big(1+\|u\|^\rho_{L^{\rho q}(\Omega)}\big)\\ &\le C\bigg(\|u_0\|_{L^q(\Omega)} + \frac{t^\alpha}{\alpha-\beta}\|f\|_{C([0,T],L^q(\Omega))} + \frac{t^{\alpha+\beta}}{\alpha}\bigg)\\ &\quad + Ct^\beta\int_0^t(t-s)^{\alpha-1}s^{-\rho\beta}s^{\rho\beta}\big\|u(s)\big\|^\rho_{L^{\rho q}(\Omega)}\,ds\\ &\le C\bigg(\|u_0\|_{L^q(\Omega)} + \frac{T^\alpha}{\alpha-\beta}\|f\|_{C([0,T],L^q(\Omega))} + \frac{T^{\alpha+\beta}}{\alpha}\bigg)\\ &\quad + CT^\beta T^{-\rho\beta}\int_0^t(t-s)^{\alpha-1}s^{\rho\beta}\big\|u(s)\big\|^\rho_{L^{\rho q}(\Omega)}\,ds.\end{aligned}$$

Applying the singular Gronwall's inequality [4, Theorem 4], we get

$$\begin{aligned}&\big\|t^\beta u(t)\big\|_{L^{\rho q}(\Omega)}\\ &\quad\le C\bigg(\|u_0\|_{L^q(\Omega)} + \frac{T^\alpha}{\alpha-\beta}\|f\|_{C([0,T],L^q(\Omega))} + \frac{T^{\alpha+\beta}}{\alpha}\bigg)\\ &\qquad + C\alpha^{-1}t^\alpha T^{\beta-\rho\beta}\bigg(\|u_0\|_{L^q(\Omega)} + \frac{T^\alpha}{\alpha-\beta}\|f\|_{C([0,T],L^q(\Omega))} + \frac{T^{\alpha+\beta}}{\alpha}\bigg)^\rho,\end{aligned}$$

therefore,

$$\begin{aligned}&\|u\|_{L^\infty_\rho((0,T),L^{\rho q})}\\ &\quad\le C\bigg(\|u_0\|_{L^q(\Omega)} + \frac{T^\alpha}{\alpha-\beta}\|f\|_{C([0,T],L^q(\Omega))} + \frac{T^{\alpha+\beta}}{\alpha}\bigg)\\ &\qquad + C\frac{T^{\beta-\rho\beta+\alpha}}{\alpha}\bigg(\|u_0\|_{L^q(\Omega)} + \frac{T^\alpha}{\alpha-\beta}\|f\|_{C([0,T],L^q(\Omega))} + \frac{T^{\alpha+\beta}}{\alpha}\bigg)^\rho\\ &\quad\le C\big(\|u_0\|_{L^q(\Omega)} + \|u_0\|^\rho_{L^q(\Omega)} + \|f\|_{C([0,T],L^q(\Omega))} + \big\|f(t)\big\|^\rho_{C([0,T],L^q(\Omega))}\big),\end{aligned}$$

ending the proof of Lemma 4.2. □

Proof of Theorem 3.3. The space $Y_\rho = L^p((0,T),L^{\rho q}(\Omega))$ is defined in (3.3). Using equations (2.8) and (2.9), as well as Lemmas 2.4–2.7, we have

$$\begin{aligned}&\|u_\varepsilon - u\|_{Y_\rho}\\ &\quad\le \big\|\mathrm{S}^\varepsilon_\alpha(t)(u_{0,\varepsilon}-u_0)\big\|_{Y_\rho} + \big\|\big(\mathrm{S}^\varepsilon_\alpha(t)-\mathrm{S}_\alpha(t)\big)u_0\big\|_{Y_\rho}\end{aligned}$$

$$
\begin{aligned}
&+\left\| \int_0^t \big(\mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s) - \mathrm{T}_\alpha(t-s)f(s)\big)\,ds \right\|_{Y_\rho} \\
&\le C\Big\{\|u_{0,\varepsilon}-u_0\|_{L^{\rho q}(\Omega)} + \big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon)-\chi_0\big|\big)\|u_0\|_{L^{\rho q}(\Omega)}\Big\} \\
&\quad + C\Big\{\|f_\varepsilon - f\|_{L^p((0,T),L^1(\Omega))} + \big[\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} \\
&\quad + \big|\chi_0 - \chi(\varepsilon)\big|\big]\|f\|_{L^p((0,T),L^1(\Omega))}\Big\} \\
&\le C\Big\{\|u_{0,\varepsilon}-u_0\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon - f\|_{C([0,T],L^{\rho q}(\Omega))} \\
&\quad + \big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon)-\chi_0\big|\big)\|u_0\|_{L^{\rho q}(\Omega)} \\
&\quad + \big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \big|\chi_0-\chi(\varepsilon)\big|\big)\|f\|_{C([0,T],L^{\rho q}(\Omega))}\Big\},
\end{aligned}
$$

ending the proof of Theorem 3.3. □

Proof of Theorem 3.4. From Lemma 4.2, we have

$$
\begin{aligned}
&\|u_\varepsilon\|_{L_\rho^\infty((0,T),L^{\rho q})} \le CM\big(\|u_{0,\varepsilon}\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\Omega))}\big) \le C, \\
&\|u\|_{L_\rho^\infty((0,T),L^{\rho q})} \le C\big(\|u_0\|_{L^q(\Omega)} + \|u_0\|_{L^q(\Omega)}^\rho \\
&\qquad\qquad + \|f\|_{C([0,T],L^q(\Omega))} + \|f\|_{C([0,T],L^q(\Omega))}^\rho\big) \le C.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&t^\beta \big\|u_\varepsilon(t) - u(t)\big\|_{L^{\rho q}(\Omega)} \\
&\quad = t^\beta \bigg\| \mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon} + \int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\,ds + \int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)\,ds \\
&\qquad - \mathrm{S}_\alpha(t)u_0 - \int_0^t \mathrm{T}_\alpha(t-s)f(s) - \int_0^t \mathrm{T}_\alpha(t-s)Nu(s)\,ds \bigg\|_{L^{\rho q}(\Omega)} \\
&\quad \le t^\beta \big\|\mathrm{S}_\alpha^\varepsilon(t)(u_{0,\varepsilon}-u_0)\big\|_{L^{\rho q}(\Omega)} + t^\beta \big\|\big(\mathrm{S}_\alpha^\varepsilon(t) - \mathrm{S}_\alpha(t)\big)u_0\big\|_{L^{\rho q}(\Omega)} \\
&\qquad + t^\beta \int_0^t \big\|\mathrm{T}_\alpha^\varepsilon(t-s)\big(f_\varepsilon(s) - f(s)\big)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\qquad + t^\beta \int_0^t \big\|\big(\mathrm{T}_\alpha^\varepsilon(t-s) - \mathrm{T}_\alpha(t-s)\big)f(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\qquad + t^\beta \int_0^t \big\|\mathrm{T}_\alpha^\varepsilon(t-s)\big(Nu_\varepsilon(s) - Nu(s)\big)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\qquad + t^\beta \int_0^t \big\|\big(\mathrm{T}_\alpha^\varepsilon(t-s) - \mathrm{T}_\alpha(t-s)\big)Nu(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad \le D_\varepsilon^1(t) + Mt^\beta \big\|\mathrm{T}_\alpha^\varepsilon(t) - \mathrm{T}_\alpha(t)\big\|_{\mathcal{L}(L^{\rho q}(\Omega))} * \big(1 + \big\|u(t)\big\|_{L^{\rho q}(\Omega)}^\rho\big) \\
&\qquad + Mt^\beta e^{rt} \int_0^t e^{-rs}\big(1 + \big\|u_\varepsilon(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1} \\
&\qquad + \big\|u(s)\big\|_{L^{\rho q}(\Omega)}^{\rho-1}\big)\big\|u_\varepsilon(s) - u(s)\big\|_{L^{\rho q}(\Omega)}\,ds \\
&\quad \le D_\varepsilon^1(t) + D_\varepsilon^2(t)
\end{aligned}
$$

$$+ Me^{rT} T^{\beta} \int_0^t \bigl(1 + 2Cs^{\beta-\rho\beta}\bigr) s^{-\beta} s^{\beta} \bigl\| u_\varepsilon(s) - u(s) \bigr\|_{L^{\rho q}(\Omega)} \, ds,$$

where

$$\begin{aligned}
D_\varepsilon^1(t) &= t^{\beta} \bigl\| \mathrm{S}_\alpha^\varepsilon(t)(u_{0,\varepsilon} - u_0) \bigr\|_{L^{\rho q}(\Omega)} + t^{\beta} \bigl\| \bigl(\mathrm{S}_\alpha^\varepsilon(t) - \mathrm{S}_\alpha(t)\bigr) u_0 \bigr\|_{L^{\rho q}(\Omega)} \\
&\quad + t^{\beta} \int_0^t \bigl\| \mathrm{T}_\alpha^\varepsilon(t-s)\bigl(f_\varepsilon(s) - f(s)\bigr) \bigr\|_{L^{\rho q}(\Omega)} \, ds \\
&\quad + t^{\beta} \int_0^t \bigl\| \bigl(\mathrm{T}_\alpha^\varepsilon(t-s) - \mathrm{T}_\alpha(t-s)\bigr) f(s) \bigr\|_{L^{\rho q}(\Omega)} \, ds, \\
D_\varepsilon^2(t) &= Mt^{\beta} \bigl\| \mathrm{T}_\alpha^\varepsilon(t) - \mathrm{T}_\alpha(t) \bigr\|_{\mathcal{L}(L^{\rho q}(\Omega))} * \bigl(1 + \bigl\| u(t) \bigr\|_{L^{\rho q}(\Omega)}^{\rho}\bigr).
\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}
& t^{\beta} \bigl\| u_\varepsilon(t) - u(t) \bigr\|_{L^{\rho q}(\Omega)} \\
&\quad \le D_\varepsilon^1(t) + D_\varepsilon^2(t) + Me^{rT} T^{\beta} \int_0^t \exp\biggl\{ \frac{(1-\rho\beta)(s^{1-\beta}) + 2C(1-\beta)(s^{1-\rho\beta})}{(1-\beta)(1-\rho\beta)} \biggr\} \\
&\qquad \times \bigl[D_\varepsilon^1(s) + D_\varepsilon^2(s)\bigr]\bigl(s^{-\beta} + 2Cs^{-\rho\beta}\bigr) \, ds.
\end{aligned}$$

Taking the Y_ρ-norm and using Young's inequality, we get

$$\begin{aligned}
\|u_\varepsilon - u\|_{Y_\rho} &\le \bigl(\bigl\| D_\varepsilon^1(t) \bigr\|_{L^p(0,T)} + \bigl\| D_\varepsilon^2(t) \bigr\|_{L^p(0,T)} \bigr) \\
&\qquad + Me^{rT} T^{\beta+1} \exp\biggl(\frac{T^{1-\beta}}{1-\beta} + \frac{2CT^{1-\rho\beta}}{1-\rho\beta} \biggr) \\
&\qquad + \bigl(\bigl\| D_\varepsilon^1(t) \bigr\|_{L^p(0,T)} + \bigl\| D_\varepsilon^2(t) \bigr\|_{L^p(0,T)} \bigr) \biggl(\frac{T^{1-\beta}}{1-\beta} + \frac{2CT^{1-\rho\beta}}{1-\rho\beta} \biggr) \\
&\quad \le C\bigl(\bigl\| D_\varepsilon^1(t) \bigr\|_{L^p(0,T)} + \bigl\| D_\varepsilon^2(t) \bigr\|_{L^p(0,T)} \bigr).
\end{aligned}$$

Applying Young's inequality again, from Lemmas 2.4–2.7, we deduce that

$$\begin{aligned}
\bigl\| D_\varepsilon^1(t) \bigr\|_{L^p(0,T)} &\le C\bigl\{ \|u_{0,\varepsilon} - u_0\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon - f\|_{C([0,T],L^{\rho q}(\Omega))} \\
&\qquad + \bigl(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \bigl|\chi(\varepsilon) - \chi_0\bigr|\bigr) \|u_0\|_{L^{\rho q}(\Omega)} \\
&\qquad + \bigl(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \bigl|\chi(\varepsilon) - \chi_0\bigr|\bigr) \|f\|_{C([0,T],L^{\rho q}(\Omega))} \bigr\}, \\
\bigl\| D_\varepsilon^2(t) \bigr\|_{L^p(0,T)} &\le C \bigl\| t^{\beta}\bigl(\mathrm{T}_\alpha^\varepsilon(t) - \mathrm{T}_\alpha(t)\bigr) \bigr\|_{L^p((0,T),\mathcal{L}(L^{\rho q}(\Omega)))} \\
&\qquad \times \bigl(T + \bigl\| t^{-\beta} t^{\beta} u(t) \bigr\|_{L^1((0,T),L^{\rho q}(\Omega))}^{\rho}\bigr) \\
&\quad \le C \bigl\| t^{\beta}\bigl(\mathrm{T}_\alpha^\varepsilon(t) - \mathrm{T}_\alpha(t)\bigr) \bigr\|_{L^p((0,T),\mathcal{L}(L^{\rho q}(\Omega)))} \\
&\qquad \times \biggl(T + \frac{T^{1-\rho\beta}}{1-\rho\beta} \bigl\| t^{\beta} u(t) \bigr\|_{L^\infty((0,T),L^{\rho q}(\Omega))}^{\rho}\biggr) \\
&\quad \le C \bigl\| t^{\beta}\bigl(\mathrm{T}_\alpha^\varepsilon(t) - \mathrm{T}_\alpha(t)\bigr) \bigr\|_{L^p((0,T),\mathcal{L}(L^{\rho q}(\Omega)))} \\
&\qquad \times \biggl\{T + \frac{T^{1-\rho\beta}}{1-\rho\beta} C\bigl[\|u_0\|_{L^{\rho q}(\Omega)} + \|u_0\|_{L^{\rho q}(\Omega)}^{\rho}
\end{aligned}$$

$$+ \|f\|_{C([0,T]),L^{\rho q}(\Omega))} + \|f\|^{\rho}_{C([0,T],L^{\rho q}(\Omega))}\Big]\Big\},$$

therefore,

$$\begin{aligned}\|u_\varepsilon - u\|_{Y_\rho} \le C\big\{&\|u_{0,\varepsilon} - u_0\|_{L^{\rho q}(\Omega)} + \|f_\varepsilon - f\|_{C([0,T],L^{\rho q}(\Omega))}\\ &+ \big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\big)\|u_0\|_{L^{\rho q}(\Omega)}\\ &+ \big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + \big|\chi(\varepsilon) - \chi_0\big|\big)\big(\|u_0\|_{L^{\rho q}(\Omega)} + \|u_0\|^{\rho}_{L^{\rho q}(\Omega)}\\ &+ \|f\|_{C([0,T],L^{\rho q}(\Omega))} + \|f\|^{\rho}_{C([0,T],L^{\rho q}(\Omega))}\big)\big\},\end{aligned}$$

completing the proof of Theorem 3.4. □

5 Discussion

This paper mainly discusses two problems. One is the well-posedness of mild solutions of integro-differential time-fractional order ordinary differential equations (1.1), (1.7), (1.12), and (1.13). The other is the convergence of mild solutions $u_\varepsilon(t,x)$ to the initial boundary value problems (1.1) and (1.12) which contain the small positive scale parameter ε, and the asymptotic behavior of the approximating mild solution $u_\varepsilon(t,x)$ near the limit mild solution $u(t,x)$ as $\varepsilon \to 0^+$, where $u(t,x)$ is the mild solution of the limit problems (1.7) and (1.13), respectively.

An interesting problem is that the second-order differential operator A in equation (1.1) and equation (1.12) has the following divergence form:

$$A := A(x,t,D_x) = \sum_{i,j=1}^{n} D_{x_i}\big(a_{ij}(x,t)D_{x_j}\big) + a_0(x). \tag{5.1}$$

In contrast to the differential operator $A = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}) + a_0(x)$ defined in equation (1.5), the difference is that $a_{ij}(t,x)$ depends on the time variable t. For the special case $a_{ij}(t,x) = K(t)\tilde{a}_{ij}(x)$, one can see Remark 2.1. For the problems (1.1) and (1.12) with A defined in equation (5.1), our method is invalid. So the question is, for example, how to characterize the analytic semigroup of the operator A? And how to define a mild solution by the Laplace transform?

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Availability of data and materials
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions

Y. Z. carried out the fractional partial differential equations and Y. T. carried out the reaction diffusion equations. All authors carried out the proofs and conceived the study. All authors read and approved the final manuscript.

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References

1. Andrade, B., Siracusa, G., Viana, A.: A nonlinear fractional diffusion equation: well-posedness, comparison results, and blow-up. J. Math. Anal. Appl. **505**(2), 125524 (2022)
2. Bazhlekova, E.: Fractional evolution equations in banach spaces. PhD Thesis, Eindhoven University of Technology, Eindhoven (2001)
3. Bihari, I.: A generalisation of a lemma of Bellman and its application to uniqueness problems of differential equations. Acta Math. Acad. Sci. Hung. **7**, 81–94 (1956)
4. Boukerrioua, K., Diabi, D., Kilani, B.: Some new Gronwall–Bihari type inequalities and its application in the analysis for solutions to fractional differential equations. Int. J. Math. Comput. Meth. **5**, 60–68 (2020)
5. Chen, J., Tang, Y.: Homogenization of nonlocal nonlinear p-Laplacian equation with variable index and periodic structure. J. Math. Phys. **64**(6), 061502 (2023)
6. Chen, J., Tang, Y.: Homogenization of nonlinear nonlocal diffusion equation with periodic and stationary structure. Netw. Heterog. Media **18**(3), 1118–1177 (2023)
7. Dos Santos, J.P.C.: Fractional resolvent operator with $\alpha \in (0,1)$ and applications. Fract. Differ. Calc. **9**(2), 187–208 (2019)
8. Dos Santos, J.P.C., Guzzo, S.M., Rabelo, M.N.: Asymptotically almost periodic solutions for abstract partial neutral integro-differential equation. Adv. Differ. Equ. **2010**, 310951 (2010)
9. Dos Santos, J.P.C., Henríquez, H.: Existence of s-asymptotically ω-periodic solutions to abstract integro-differential equations. Appl. Math. Comput. **256**, 109–118 (2015)
10. Dos Santos, J.P.C., Henríquez, H., Henández, E.: Existence results for neutral integro-differential equations with unbounded delay. J. Integral Equ. Appl. **23**(2), 289–330 (2011)
11. Goldstein, J.A.: Semigroup of Linear Operators and Applications, Oxford, New York (1985)
12. Grimmer, R., Prichard, A.: Analytic resolvent operators for integral equations in Banach space. J. Differ. Equ. **50**(2), 234–259 (1983)
13. Gu, C., Tang, Y.: Chaotic characterization of one dimensional stochastic fractional heat equation. Chaos Solitons Fractals **145**, 110780 (2021)
14. Gu, C., Tang, Y.: Global solution to the Cauchy problem of fractional drift diffusion system with power-law nonlinearity. Netw. Heterog. Media **18**(1), 109–139 (2023)
15. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
16. Li, Y.: Regularity of mild solutions for fractional abstract Cauchy problem with order $\alpha \in (1,2)$. Z. Angew. Math. Phys. **66**, 3283–3298 (2015)
17. Lorenzi, A., Messina, F.: Approximation of solutions to linear integro-differential parabolic equations in L^p-spaces. J. Math. Anal. Appl. **333**, 642–656 (2007)
18. Lorenzi, A., Messina, F.: Approximation of solutions to non-linear integro-differential parabolic equations in L^p-spaces. Differ. Integral Equ. **20**(6), 693–720 (2007)
19. Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhaüser, Basel (1995)
20. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity. Imperial College Press, London (2010)
21. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
22. Prüss, J.: Evolutionary Integral Equations and Applications. Monographs in Mathematics, vol. 87. Birkhäuser, Basel (1993)
23. Wang, R.N., Chen, D.H., Xiao, T.J.: Abstract fractional Cauchy problems with almost sectorial operators. J. Differ. Equ. **252**, 202–235 (2012)
24. Yang, X., Tang, Y.: Decay estimates of nonlocal diffusion equations in some particle systems. J. Math. Phys. **60**(4), 043302 (2019)
25. Zhang, Q., Li, Y.: Global well-posedness and blow-up solutions of the Cauchy problem for a time-fractional superdiffusion equation. J. Evol. Equ. **19**, 271–303 (2019)
26. Zhao, Y., Tang, Y.: Approximation of solutions to integro-differential time fractional wave equations in L^p-space. Netw. Heterog. Media **18**(3), 1024–1058 (2023)

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