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Fixed-point theorems of $F^{\star}-(\psi,\phi)$ integral-type contractive conditions on 1_E-complete multiplicative partial cone metric spaces over Banach algebras and applications

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Abstract

In this paper, we introduce some user-friendly versions of integral-type fixed-point results and give some modifications of the classical Banach contraction principle by constructing a special type of contractive restrictions of integral forms for weak contraction mappings defined on 1_E-complete multiplicative partial cone metric spaces over Banach algebras and formulate some existence and uniqueness results regarding the fixed-point theorems using some integrative conditions. Moreover, we validate the significance our results and exploit them to find the unique solution of a fractional nonlinear differential equation of Caputo type, which complements some previously well-known generalizations found in the literature.

Mathematics Subject Classification: Primary 47H09; secondary 47H10

Keywords: Multiplicative algebra cones; Multiplicative segments; Multiplicative partial cone metric spaces over Banach algebras; Integral-type inequalities; Fixed points

1 Introduction

One of the most frequently cited results of functional analysis is the Banach contraction mapping principle. It is widely regarded as the source of the metric fixed-point theory, and its importance lies in its broad applicability in a number of branches of mathematics. We are interested in the generalization given by Branchiari [1].

In 2002, Branchiari [1] introduced the integral version of the Banach contraction principle and studied the existence of fixed points for a single-valued mapping satisfying interesting integral-type contractive conditions in complete metric spaces. Later, many authors in [2–5] extended the result of Branciari and proved some fixed-point theorems on different spaces involving more general contractive conditions.

In 2010, Khojasteh et al. [6] defined a new notion of integral with respect to a cone and proved some fixed-point theorems for integral-type contractions in the setting of cone metric spaces.

In 2018, Langpoklakpam Premila Devi and Laishram Shambhu Singh [7] proved generalized results to C-class functions using contractive conditions of integral type on complete S_b-metric spaces. In 2018, Ansari et al. [8] generalized the results via C-class functions under new contractive conditions of integral type on complete S-metric spaces.

In 2020, Amjad Ali et al. [9] developed some common fixed-point theorems for generalized mappings and used their results to solve a system of functional equations and Voltera-type integral equations, and at the same year, Sahar Mohamed Ali Abou Bakr [10] proved the existence of a common fixed point of $(A;B)$ generalized cyclic $\phi - abc$ weak nonexpansive mappings.

In 2021, Sahar Mohamed Ali Abou Bakr [11] gave more generalizations of [7] and [8]. Moreover, as a particular case, the author proved the existence of a fixed point for both cyclic $\Omega_{S,(abef)}$-weak contractions and cyclic $\Omega_{S,(abef)}$-weak nonexpansive mappings. On the other side, the author extended the studies in the settings of cone metric spaces, generalized cone metric spaces, and cone b metric spaces in [12–14]. See [10] and references therein.

In 2022, Amjad Ali et al. [15] generalized the notion of θ-contractions in the framework of b-metric to the case of nonlinear θ_b-contraction mappings and compiled their work by an application of nonlinear θ_b-contractions to Liouville–Caputo fractional differential equations. Also, Amjad Ali et al. [16] took a different approach in the case of b-metric-like and orthogonal b-metric-like spaces via a hybrid pair of operators to present some results in fixed-point theory and included some applications in the field of nonlinear analysis to highlight the usability and validity of the theoretical results. Then Amjad Ali et al. [17] developed some fixed-point theorems and the notion of F-contractions to the case of nonlinear (F, FH)-dynamic-iterative scheme for Branciari Ćirić-type contractions in controlled metric spaces; they also provided an application to the Liouville–Caputo fractional derivatives and fractional differential equations.

In 2023, Sahar Mohamed Ali Abou Bakr [18] proved the existence of coupled coincidence points of generalized contraction mappings where parametric contractions are vectors.

The present paper is organized into three main sections. Firstly, we introduce the notion of integration with respect to the multiplicative algebra cone. Secondly, we manifest some fixed-point results under general $F^\star - (\psi, \phi)$ integral-type contractive inequalities. Ultimately, we present an application and numerical example of our core result. In fact, we introduce some user-friendly versions of integral-type fixed-point results. Besides, the new results are supported by illustrative examples. As an application, we prove the existence and uniqueness of a solution for fractional-order differential equation to validate the significance of the results that complement a number of previously well-known generalizations found in the literature.

2 Preliminaries

Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R} = (-\infty, \infty)$, and $\mathbb{R}^+ = (0, \infty)$.

Consistently with [19], we will need the following definitions.

Definition 2.1 *A multiplicative normed space is a vector space E (with either the real or complex numbers as its field of scalars) with zero vector θ, in which to each vector x, there corresponds a real number, denoted $\rho(x)$ and called the multiplicative norm of x, satisfying the following conditions:*

(i) $\rho(x) \geq 1$ *for all* $x \in E$;

(ii) $\rho(x) = 1$ *if and only if* $x = \theta$;

(iii) $\rho(\alpha x) = \rho(x)^{|\alpha|}$ *for all* $x \in E$ *and all scalars* α;

(iv) $\rho(x \pm y) \leq \rho(x) \times \rho(y)$ *for all* $x, y \in E$.

The pair $(E, \rho(\cdot))$ is called a multiplicative normed space.

To fix notations, let $C_{\mathbb{R}}[a,b]$ be the set of all continuous real-valued functions defined on the closed bounded interval of real numbers $[a,b]$; that is,

$$C_{\mathbb{R}}[a,b] := \{f : [a,b] \to \mathbb{R}, f \text{ is a continuous function on } [a,b]\},$$

and let $\|f\| := \max_{t\in[a,b]} |f(t)|$. The multiplicative norm on $C_{\mathbb{R}}[a,b]$ is defined by $\rho_\infty(f) = r^{\|f\|}$, where $r > 1$ is any fixed real number (see [19]).

Definition 2.2 *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a multiplicative normed space $(E, \rho(\cdot))$ is said to be multiplicative convergent to $x_0 \in E$ if for any $\varepsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $\rho(x_n - x_0) < \varepsilon$ for all $n \geq n_0$. Mathematically, $x_n \xrightarrow{\rho} x_0$ as $n \to \infty$ if $\lim_{n\to\infty} \rho(x_n - x_0) = 1$.*

3 Multiplicative algebra cones in Banach algebras

The aim of this section is to stress the aspects of multiplicative algebra cones in ordered Banach algebras. We suppose the reader is acquainted with the definition of a Banach algebra; if not, see [20] for a review. Let E be a Banach algebra with multiplicative element denoted by 1_E, and let x be an element in E. The main tools for dealing with operations on exponential terms on E are defined subject to the following three key formulas:

In the case of positive integer powers, x^n $(n \in \mathbb{N})$ is the n-fold product of x by itself, that is, $x^n := \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$, and $x^0 := 1_E$. By a root we mean a fractional power x^r, where $r = \frac{1}{m}$ for $m \in \mathbb{N}$, which is an element $y \in E$ such that $x := y^m$. We now turn to the fractional exponent case: in general, $x^{\frac{n}{m}}$ $(n, m \in \mathbb{N})$ is defined as $x^{\frac{n}{m}} := \underbrace{x^{\frac{1}{m}} \cdot \ldots \cdot x^{\frac{1}{m}}}_{n \text{ times}}$ or, equivalently, $x^{\frac{n}{m}} := (\underbrace{x \cdot \ldots \cdot x}_{n \text{ times}})^{\frac{1}{m}}$.

Definition 3.1 *Let E be a real Banach algebra with multiplicative element 1_E. A subset M $(\neq \emptyset)$ of E is called a multiplicative algebra cone if M satisfies the following conditions:*

(i) *M is closed, $1_E \in M$, and $M \neq \{1_E\}$;*

(ii) $M \cdot M \subset M$;

(iii) *$M^a \subset M$ $(a \geq 0)$, that is, $x^a \in M$ for all $x \in M$ and $a \geq 0$, where $x^a = \lim_{n\to\infty} x^{a_n}$ for a sequence a_n of rational numbers converging to a;*

(iv) *$M \bigcap \frac{1}{M} = \{1_E\}$, that is, if $x, \frac{1_E}{x} \in M$, then $x = 1_E$.*

We proceed by giving the following example.

Example 3.2 Let $E := M_n(\mathbb{R})$ be the algebra of all real n-by-n matrices with the standard matrix multiplication and with Frobenius multiplicative norm defined by $\rho_F(A) := r^{(\sum_{i,j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}}$ for $A \in M_n(\mathbb{R})$, where $r > 1$ is a fixed real number. Clearly, $(M_n(\mathbb{R}), \rho_F(\cdot))$ is a Banach algebra, and the identity matrix $I_n = \{a_{ij} : a_{ii} = 1 \text{ and } a_{ij} = 0 \text{ whenever } i \neq j, 1 \leq i,j \leq n\}$ serves as the multiplicative element in $M_n(\mathbb{R})$. Let M be the family of all diagonal matrices in $M_n(\mathbb{R})$ with diagonal elements $a_{ii} \geq 1$ for $1 \leq i \leq n$, that is,

$$M := \{A = [a_{ij}]_{n\times n} \in E : a_{ij} = 0 \text{ when } i \neq j, \text{ and } a_{ii} \geq 1 \text{ for } 1 \leq i,j \leq n\}.$$

Thus $M \subset E$ is a multiplicative algebra cone in $M_n(\mathbb{R})$.

Definition 3.3 *Let x and y be elements of the Banach algebra E ordered by a multiplicative algebra cone M. The preference ordering $\preceq$ with respect to M is defined by $x \preceq y$ if and only if $\frac{y}{x} \in M$. We say that $x \prec y$ if and only if $\frac{y}{x} \in M$, but $x \neq y$. Further, $x \ll y$ if and only if $\frac{y}{x} \in \operatorname{Int}(M)$, where $\operatorname{Int}(M)$ denotes the interior of M, if there are any. If $\operatorname{Int}(M) \neq \emptyset$, then M is called solid. Note that $\frac{y}{x} \in \operatorname{Int}(M)$ implies $\frac{y}{x} \in M$, but the converse is not always the case.*

A significant property that the multiplicative algebra cone M may have is normality.

Definition 3.4 *Let M be a multiplicative algebra cone. We say that M is normal if there is $K > 0$ such that for all $x, y \in E$, $\rho(x) \leq \rho(y)^K$ whenever $1_E \preceq x \preceq y$. The least positive number satisfying the above inequality is called the multiplicative normality constant of K.*

Lemma 3.5 *If E is a real Banach algebra with multiplicative algebra cone M and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in M such that $x_n \overset{\rho}{\to} 1_E$ (or, equivalently, $\rho(x_n - 1_E) \underset{n\to\infty}{\longrightarrow} 1$), then $\{x_n\}_{n\in\mathbb{N}}$ is a c-sequence, that is, for each $1_E \ll c$, there exists $n_c \in \mathbb{N}$ such that $x_n \ll c$ for all $n \geq n_c$.*

Remark 3.6 For the moment, it is worth remarking that the converse of Lemma 3.5 holds whenever the underlying multiplicative algebra cone M is necessarily normal. Indeed, for each $\varepsilon > 1$, there exists $1_E \ll c$ such that $\rho(c) < (\frac{\varepsilon}{\rho(1_E)})^{\frac{1}{K}}$, where K is the multiplicative normality constant of M. For this c, there is some $n_c \in \mathbb{N}$ such that $x_n \ll c$ for all $n \geq n_c$ and for all $n \geq n_c$, we have

$$\rho(x_n - 1_E) \leq \rho(x_n) \times \rho(1_E) \leq \rho(c)^K \times \rho(1_E) < \frac{\varepsilon}{\rho(1_E)} \times \rho(1_E) = \varepsilon.$$

This is equivalent to $x_n \overset{\rho}{\to} 1_E$ as $n \to \infty$.

Definition 3.7 [6] *Suppose that C is a normal cone in E and $a, b \in E$ with $a < b$. Then define $[a,b] := \{x \in E : x = tb + (1-t)a \text{ for some } t \in [0,1]\}$.*

Definition 3.8 [6] *The set $\{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$ is called a partition for $[a,b]$ if and only if the sets $\{[x_{i-1}, x_i)\}_{i=1}^{n}$ are pairwise disjoint and*

$$[a,b] = \left\{\bigcup_{i=1}^{n} [x_{i-1}, x_i)\right\} \cup \{b\}.$$

Definition 3.9 *Suppose that the multiplicative algebra cone M in E is normal, and let $a, b \in M$ be any elements such that $a \preceq b$. Then*

(i) *The multiplicative segment between a and b denoted by $[a,b]_m$ is defined as*

$$[a,b]_m := \left\{c : c = a^t \cdot b^{(1-t)} \text{ for some } t \in [0,1]\right\} \subset M.$$

(ii) *The partition of the multiplicative segment $[a,b]_m$ is any finite ordered subset $\mathcal{Q} = \{a = x_0, x_1, x_2, \dots, x_{n-1}, x_n = b\}$ satisfying the following conditions: $\{x_i : 1 \le i \le n-1\} \subseteq (a,b)_m$, $x_i \prec x_{i+1}$ for all $i \in \{0, 1, \dots, n-1\}$, and $[a,b]_m = \{\bigcup_{i=1}^{n}[x_{i-1}, x_i)_m\} \bigcup \{b\}$.*

The collection of all partitions of a multiplicative segment $[a,b]_m$ is denoted $\mathcal{Q}[a,b]_m$.

Example 3.10 Consider the Banach algebra $E := (C_{\mathbb{R}}[0,1], \rho_\infty(\cdot))$ in the order of the multiplicative algebra cone $M = \{f \in E : f(x) \ge 1 \text{ on } x \in [0,1]\}$ with multiplicative normality constant $K = 1$. Define multiplication in $C_{\mathbb{R}}[0,1]$ in the natural way. Let $f(x) = e^x$ be the exponential function with the natural base e, and let $g(x) = e - \cos(1) + \cos(x)$. Since $e^x \le (e - \cos(1) + \cos(x))$ for all $x \in [0,1]$, we have $f \preceq g$. The multiplicative segment between the mappings f and g is indicated by

$$[f,g]_m = \left\{h_t : \left(h_t(x) = e^{tx}\left(e - \cos(1) + \cos(x)\right)^{(1-t)} \text{ for all } x \in [0,1]\right) \text{ for some } t \in [0,1]\right\}.$$

Specifically, we will stick to the following mappings defined over the values of $x \in [0,1]$ as follows:

$$h_1(x) := f(x) = e^x;$$

$$h_{0.9}(x) := e^{0.9x}\left(e - \cos(1) + \cos(x)\right)^{0.1};$$

$$h_{\frac{4}{5}}(x) := e^{\frac{4}{5}x}\left(e - \cos(1) + \cos(x)\right)^{\frac{1}{5}};$$

$$h_{0.75}(x) := e^{0.75x}\left(e - \cos(1) + \cos(x)\right)^{0.25};$$

$$h_{\frac{2}{3}}(x) := e^{\frac{2}{3}x}\left(e - \cos(1) + \cos(x)\right)^{\frac{1}{3}};$$

$$h_{0.5}(x) := \sqrt{e^x\left(e - \cos(1) + \cos(x)\right)};$$

$$h_{0.22}(x) := e^{0.22x}\left(e - \cos(1) + \cos(x)\right)^{0.78};$$

$$h_0(x) : g(x) = e - \cos(1) + \cos(x).$$

Clearly, $\mathcal{Q}_1 = \{h_1, h_{\frac{4}{5}}, h_{0.75}, h_{0.5}, h_0\}$, $\mathcal{Q}_2 = \{h_1, h_{0.9}, h_{\frac{2}{3}}, h_{0.22}, h_0\}$, and $\mathcal{Q}_3 := (\mathcal{Q}_1 \cup \mathcal{Q}2) = \{h_1, h_{0.9}, h_{\frac{4}{5}}, h_{0.75}, h_{\frac{2}{3}}, h_{0.5}, h_{0.22}, h_0\}$ are partitions of the multiplicative segment $[f,g]_m$. We present the graphs for the mappings in partition $\mathcal{Q}_3$ in Fig. 1.

Observe that $x = 1 \in [0,1]$ is a coincidence point of the mappings f, g, and h_t for any $h_t \in \mathcal{Q}_3$. Indeed, $x = 1$ is a coincidence point of all mappings belonging to the multiplicative segment $[e^x, (e - \cos(1) + \cos(x))]_m$.

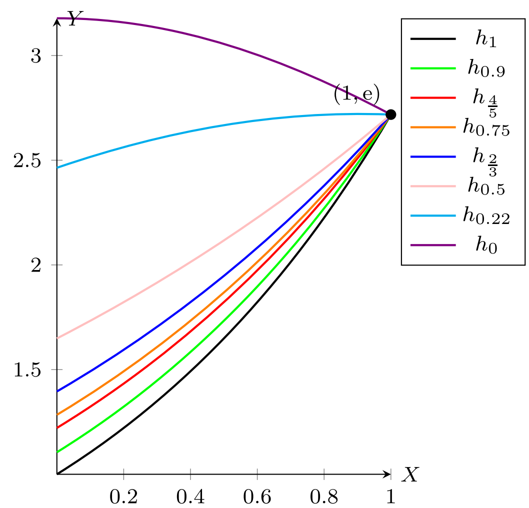

Figure 1 Graph of the mappings in partition $\mathcal{Q}_3 := (\mathcal{Q}_1 \cup \mathcal{Q}_2)$ of the multiplicative segment $[e^x, (e - \cos(1) + \cos(x))]_m$ on the closed interval $[0, 1]$ in Example 3.10

Example 3.11 Let $E := C_{\mathbb{R}}[1,2]$ be equipped with the multiplicative norm $\rho_\infty(\cdot)$ and usual multiplication. Then E is a Banach algebra. Let

$$M = \{f \in E : f(x) \geq 1 \text{ on } x \in [1,2]\}.$$

Thus $M \subset E$ is a multiplicative algebra cone with multiplicative normality constant $K = 1$.

Note that $(1 + \ln(x)) \geq 1$ for $x \in [1,2]$, and thus $1_E \preceq (1 + \ln(x))$, where 1_E in $C_{\mathbb{R}}[1,2]$ is the constant mapping $f(x) = 1$ for all $x \in [1,2]$. The multiplicative segment between 1_E and $(1 + \ln(x))$ is presented by

$$\bigl[1_E, \bigl(1 + \ln(x)\bigr)\bigr]_m = \bigl\{h_t : \bigl(h_t(x) = \bigl(1 + \ln(x)\bigr)^{(1-t)} \text{ for all } x \in [1,2]\bigr) \text{ for some } t \in [0,1]\bigr\}.$$

Naturally, $\mathcal{Q} = \{1_E, (1 + \ln(x))^{0.7}, \sqrt{1 + \ln(x)}, (1 + \ln(x))^{\frac{1}{3}}, (1 + \ln(x))\}$ is a partition of $[1_E, (1 + \ln(x))]_m$. The set of mappings in the partition $\mathcal{Q}$ is depicted in Fig. 2.

We can see that $x = 1 \in [1,2]$ is a common fixed point of the mappings

$$\bigl\{f(x) = 1, \bigl(1 + \ln(x)\bigr)^{0.7}, \sqrt{1 + \ln(x)}, \bigl(1 + \ln(x)\bigr)^{\frac{1}{3}}, \bigl(1 + \ln(x)\bigr)\bigr\}.$$

The common fixed point of these mappings can be seen in Fig. 2. Moreover, $x = 1$ is a common fixed point of any mapping $h_t \in [1_E, (1 + \ln(x))]_m$.

Example 3.12 Let $E := (C_{\mathbb{R}}[0, 2\pi], \rho_\infty(.))$, and define the multiplication in E pointwise. Then E is a real Banach algebra. Consider a multiplicative algebra cone $M = \{h \in E : h(x) \geq 1 \text{ on } x \in [0, 2\pi]\}$ in E with multiplicative normality constant $K = 1$. Consider the following mappings:

$$f(x) = \begin{cases} 2 + \sin x & \text{if } x \in [0, \pi], \\ 2 - \sin x & \text{if } x \in [\pi, 2\pi], \end{cases} \qquad g(x) = \begin{cases} 4 - \sin x & \text{if } x \in [0, \pi], \\ 4 + \sin x & \text{if } x \in [\pi, 2\pi]. \end{cases}$$

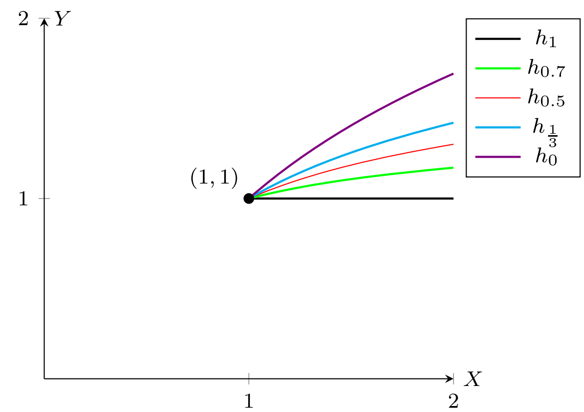

Figure 2 The mappings in the partition $\mathcal{Q}$ of the multiplicative segment between $f(x) = 1$ and $(1 + \ln(x))$ on the closed interval $[1, 2]$ in Example 3.11

Evidently, we have $(2 + \sin x) \le (4 - \sin x)$ for all $x \in I = [0, \pi]$ and $(2 - \sin x) \le (4 + \sin x)$ for all $x \in J = [\pi, 2\pi]$.

Mathematically, the multiplicative segment between f and g can be expressed as

$$[f, g]_m = \begin{cases} \{h_t : (h_t(x) = (2 + \sin x)^t (4 - \sin x)^{(1-t)} \text{ for all } x \in I) \text{ for some } t \in [0, 1]\}, \\ \{k_t : (k_t(x) = (2 - \sin x)^t (4 + \sin x)^{(1-t)} \text{ for all } x \in J) \text{ for some } t \in [0, 1]\}. \end{cases}$$

We remark that

$$\mathcal{Q} = \begin{cases} \{h_1(x), h_{\frac{1}{2}}(x), h_{0.35}(x), h_{\frac{1}{4}}(x), h_0(x)\} & \text{if } x \in I, \\ \{k_1(x), k_{\frac{1}{2}}(x), k_{0.35}(x), k_{\frac{1}{4}}(x), k_0(x)\} & \text{if } x \in J, \end{cases}$$

is a partition of $[f, g]_m$. The sets of mappings in the partition $\mathcal{Q}$ can be seen graphically in Fig. 3.

From Fig. 3 we can point out that $\frac{\pi}{2} \in I = [0, \pi]$ is the coincidence point of the mappings $\{2 + \sin x, 4 - \sin x, h_t(x)\}$ for any mapping $h_t(x) \in [f, g]_m$. Further, $\frac{3\pi}{2} \in J = [\pi, 2\pi]$ is the coincidence point of the mappings $\{2 - \sin x, 4 + \sin x, k_t(x)\}$ for any mapping $k_t(x) \in [f, g]_m$.

Definition 3.13 [6] *For each partition Q of $[a, b] = \{\bigcup_{i=1}^{n} [x_{i-1}, x_i)\} \bigcup \{b\}$ and each increasing function $\varphi : [a, b] \to C$, the cone lower summation and cone upper summation are defined as*

$$L_n^{Con}(\varphi, Q) = \sum_{0 \le i \le n-1} \varphi(x_i) \|x_{i+1} - x_i\|,$$

$$U_n^{Con}(\varphi, Q) = \sum_{0 \le i \le n-1} \varphi(x_{i+1}) \|x_{i+1} - x_i\|,$$

respectively.

Lemma 3.14 *Let E be the real multiplicative Banach algebra of real-valued functions defined on an arbitrary closed bounded interval $J \subseteq \mathbb{R}$, and let the underlying multiplicative*

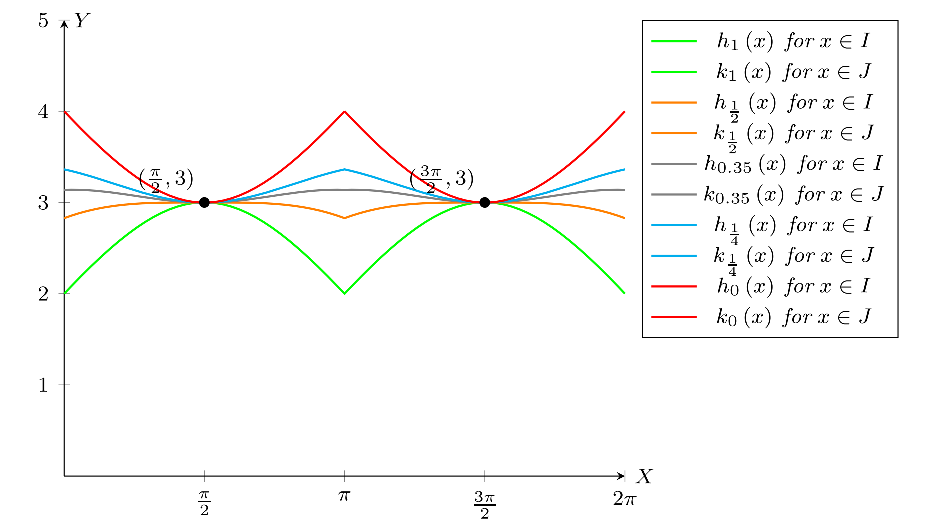

Figure 3 Graphical representation of partition $\mathcal{Q}$ in Example 3.12

algebra cone M be normal. For any $f, g \in E$, we define

$$f \preceq g \quad \text{if and only if} \quad \left(\frac{g}{f}\right)(x) \geq 1 \quad \text{for all } x \in J.$$

Let $f, g \in M$ with $f \preceq g$. If $\bar{x} \in J$ is a coincidence point of f and g, then $\bar{x}$ is a coincidence point of f, g, and h for any $h \in [f, g]_m$.

Proof Let $h \in [f, g]_m$. Then $h = f^t \times g^{(1-t)}$ for some $t \in [0, 1]$. Assume that $\bar{x} \in J$ is a coincidence point of f and g. Then there must be some $y \in \mathbb{R}$ such that $f(\bar{x}) = g(\bar{x}) = y$. For this $\bar{x}$, we find that

$$h(\bar{x}) = \left(f^t \times g^{(1-t)}\right)(\bar{x}) = \left(f^t(\bar{x})\right) \times \left(g^{(1-t)}(\bar{x})\right) = y^t \times y^{(1-t)} = y.$$

Hence $h(\bar{x}) = f(\bar{x}) = g(\bar{x}) = y$, and so $\bar{x}$ is a coincidence point of f, g, and h for any $h \in [f, g]_m$. □

Definition 3.15 *Suppose that $\varphi : [a, b]_m \to M$ is any increasing mapping with respect to $\preceq$. Let $\mathcal{Q} = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$ be any partition of $[a, b]_m$. Then we define the multiplicative lower product $L_n^M(\varphi, \mathcal{Q})$ and the multiplicative upper product $U_n^M(\varphi, \mathcal{Q})$ of φ, respectively, as follows:*

$$L_n^M(\varphi, \mathcal{Q}) = \prod_{0 \leq i \leq n-1} \varphi(x_i)^{\rho(x_{i+1} - x_i)},$$
$$U_n^M(\varphi, \mathcal{Q}) = \prod_{0 \leq i \leq n-1} \varphi(x_{i+1})^{\rho(x_{i+1} - x_i)}.$$

Definition 3.16 *The mapping $\varphi : [a, b]_m \to M$ is called a multiplicative integrable mapping on $[a, b]_m$ with respect to the multiplicative algebra cone M or, for simplicity, a multi-*

plicative integrable mapping if and only if for any partition $\mathcal{Q}$ of $[a,b]_m$, we have

$$\lim_{n\to\infty} L_n^M(\varphi,\mathcal{Q}) = \lim_{n\to\infty} U_n^M(\varphi,\mathcal{Q}) = P^M.$$

The limit P^M is called the multiplicative integral of φ on $[a,b]_m$, and we denote it by $\int_a^b \varphi(t)^{d_M(t)}$.

We denote the set of all multiplicative integrable mappings $\varphi : [a,b]_m \to M$ by $\mathcal{L}^1([a,b]_m, M)$.

Lemma 3.17 *Let $[a,b]_m \subset [a,c]_m$ and $\varphi \in \{\mathcal{L}^1([a,b]_m,M), \mathcal{L}^1([a,c]_m,M)\}$. Then*

$$\int_a^b \varphi(t)^{d_M(t)} \preceq \int_a^c \varphi(t)^{d_M(t)}.$$

Proof Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the resulting points of the partitions of $[a,b]_m$ and $[b,c]_m$, respectively:

$$\mathcal{Q}_1 = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\},$$
$$\mathcal{Q}_2 = \{b = x_n, x_{n+1}, \dots, x_{m-1}, x_m = c\}_{m>n}.$$

The set $\bar{\mathcal{Q}} = \{a = x_0, x_1, \dots, x_{n-1}, b = x_n, x_{n+1}, \dots, x_{m-1}, x_m = c\}_{m>n}$ is a partition of $[a,c]_m$ because $x_i \prec x_{i+1}$ and $x_j \prec x_{j+1}$ for every $i \in \{0,1,\dots,n-1\}$ and every $j \in \{n, n+1, \dots, m\}$. We have

$$\begin{aligned} L_n^M(\varphi,\mathcal{Q}_1) &= \prod_{0\le i\le n-1} \varphi(x_i)^{\rho(x_{i+1}-x_i)} \\ &\preceq \left\{ \prod_{0\le i\le n-1} \varphi(x_i)^{\rho(x_{i+1}-x_i)} \right\} \cdot \left\{ \prod_{n\le j\le m-1} \varphi(x_j)^{\rho(x_{j+1}-x_j)} \right\} \\ &= \prod_{k\in\{0,1,\dots,n-1,n,n+1,\dots,m-1\}} \varphi(x_k)^{\rho(x_{k+1}-x_k)} \\ &= L_m^M(\varphi,\bar{\mathcal{Q}}). \end{aligned}$$

Consequently, $\int_a^b \varphi(t)^{d_M(t)} \preceq \int_a^c \varphi(t)^{d_M(t)}$. □

Definition 3.18 *The mapping $\varphi : M \to M$ is called a submultiplicative integrable if and only if for all $a, b \succ 1_E$, we have*

$$\int_{1_E}^{a\cdot b} \varphi(t)^{d_M(t)} \preceq \int_{1_E}^{a} \varphi(t)^{d_M(t)} \cdot \int_{1_E}^{b} \varphi(t)^{d_M(t)}.$$

Definition 3.19 *The multiplicative integrable mapping $\varphi \in \mathcal{L}^1([1_E,c]_m, M)$ is called completely multiplicative integrable on M if the following conditions hold*:

(i) *φ is a nonvanishing mapping*;
(ii) *φ is submultiplicative integrable on each multiplicative segment $[1_E,c]_m$ for any $c \in M$*;
(iii) *$\int_{1_E}^c \varphi(t)^{d_M(t)} \succ 1_E$ for each $c \succ 1_E$.*

4 Topological structure on multiplicative partial cone metric spaces over Banach algebras

For completeness, we begin with the following preliminary notes.

Let $(E, \|\cdot.\|)$ be a real Banach space with the zero vector θ. A proper nonempty closed subset C of E is called a cone if $C + C \subset C$, $\lambda C \subset C$ for $\lambda \geq 0$, and $C \cap (-C) = \{\theta\}$. If the cone C has a nonempty interior $\mathrm{Int}(C)$, then it is called solid. Each cone induces a partial ordering on E, denoted by $\preceq$ and defined by $x \preceq y$ if and only if $y - x \in C$. The notation $x \prec y$ indicates that $x \preceq y$ and $x \neq y$, and the notation $x \ll y$ indicates that $y - x \in \mathrm{Int}(C)$. The cone C is called normal if there is a constant number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of C (see [21]).

Let X be a nonempty set, and let C be a cone of a real Banach space E. A partial cone metric on X is a mapping $p : X^2 \to C$ such that for each $x, y, z \in X$,

$(\mathrm{PCM}_1) : \theta \preceq p(x,x) \preceq p(x,y)$;

$(\mathrm{PCM}_2) :$ if $p(x,x) = p(x,y) = p(y,y)$, then $x = y$;

$(\mathrm{PCM}_3) : p(x,y) = p(y,x)$;

$(\mathrm{PCM}_4) : p(x,y) \preceq p(x,z) + p(z,y) - p(z,z)$.

The quadruple (X, E, C, p) is called a partial cone metric space (see [22, 23]).

Each partial cone metric p on X over a solid cone generates a topology $\Im_p$ on X, which has a base of the family of open balls $\{B_p(x;c) : x \in X, \theta \ll c\}$, where $B_p(x;c) = \{y \in X : p(x,y) \ll p(x,x) + c\}$ for each $x \in X$ and each $c \in \mathrm{Int}(C)$.

Let (X, E, C, p) be a partial cone metric space over a solid cone C of a topological vector space E.

(*i*) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges [22] to $x \in X$ (denoted by $x_n \stackrel{\Im_p}{\to} x$) if for each $c \in \mathrm{Int}(C)$, there exists a positive integer n_0 such that $p(x_n, x) \ll p(x,x) + c$ for each $n \geq n_0$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X strongly converges [23] to $x \in X$ (denoted $x_n \stackrel{s-\Im_p}{\to} x$) if $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x,x)$.

(*ii*) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is θ-Cauchy if for each $c \in \mathrm{Int}(C)$, there exists a positive integer n_0 such that $p(x_n, x_m) \ll c$ for $m, n \geq n_0$. The partial cone metric space (X, E, C, p) is θ-complete if each θ-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of X converges to a point $x \in X$ such that $p(x,x) = \theta$.

Let (X, E, C, p) be a partial cone metric space over a solid cone C of a normed space $(E, \|\cdot\|)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is Cauchy [22, 23] if there exists an element $u \in C$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = u$. The partial cone metric space (X, E, C, p) is complete [22, 23] if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of X strongly converges to some point $x \in X$ with $p(x,x) = u$.

Definition 4.1 *Let M be a multiplicative algebra cone in the real Banach algebra E. Suppose that X is a nonempty set. We call the mapping $\mathfrak{p} : X^2 \to M$ a multiplicative partial cone metric on X if for all $x, y, z \in X$, the following axioms are satisfied*:

$(\mathrm{M-PCM}_1) : 1_E \preceq \mathfrak{p}(x,x) \preceq \mathfrak{p}(x,y)$;

$(\mathrm{M-PCM}_2) :$ *if* $\mathfrak{p}(x,x) = \mathfrak{p}(y,y) = \mathfrak{p}(x,y)$, *then* $x = y$;

$(\mathrm{M-PCM}_3) : \mathfrak{p}(x,y) = \mathfrak{p}(y,x)$;

$(\mathrm{M-PCM}_4) : \mathfrak{p}(x,y) \preceq \frac{\mathfrak{p}(x,z)\cdot\mathfrak{p}(z,y)}{\mathfrak{p}(z,z)}$.

The quadruple $(X, E, M, \mathfrak{p})$ is called a multiplicative partial cone metric space over a Banach algebra E.

Observe that for any $x, y \in X$, if $\mathfrak{p}(x,y) = 1_E$, then $x = y$.Equivalently, if $x \neq y$, then $1_E \prec \mathfrak{p}(x,y)$.

Example 4.2 Let $E := \mathbb{R}^2$ be with the coordinatewise multiplication

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 y_1, x_2 y_2).$$

Endow E with the multiplicative norm $\rho((x,y)) = r^{\max\{|x|,|y|\}}$, where $r > 1$. Thus $(\mathbb{R}^2, \rho(\cdot))$ is a Banach algebra. Let $M = \{(x,y) \in E : x, y \geq 1\}$, and let $X = \mathbb{R}_0^+$ be the set of all nonnegative real numbers. Also, let $\beta \geq 0$ and $a > 1$ be given fixed real numbers, and let the mapping $\mathfrak{p} : X^2 \to M$ be defined by

$$\mathfrak{p}(x,y) = \left(a^{\max\{x,y\}}, a^{\beta \max\{x,y\}}\right).$$

Then $(\mathbb{R}_0^+, \mathbb{R}^2, M, \mathfrak{p})$ is a multiplicative partial cone metric space over Banach algebra $\mathbb{R}^2$.

It is worth mentioning that it is possible to characterize the multiplicative partial cone metric $\mathfrak{p}$ in the following way: for a nonempty set X, let $\mathfrak{p}(x,y) = a^{p(x,y)}$ for all $x, y \in X$, where $a > 1$. Then $\mathfrak{p}$ defines a multiplicative partial cone metric whenever p defines a partial cone metric. Now solving for $p(x,y)$, we have $p(x,y) = \log_a(\mathfrak{p}(x,y))$, unless $a = e$, and then p is not a partial cone metric.

The following example of a multiplicative partial cone metric space shows the reason why these spaces are worth considering. We will show this by answering the following question: if (X, E, M, p^*) is a complete multiplicative partial cone metric space and we define the function p by $p(x,y) = \ln(p^*(x,y))$ for $x, y \in X$, then is p a partial cone metric?

Let $E := (C_{\mathbb{R}}[0,1], \rho_\infty(\cdot))$ and $X = M := \{f \in E : f(x) \geq 1 \text{ on } x \in [0,1]\}$, and let $p^* : C_{\mathbb{R}}[0,1] \times C_{\mathbb{R}}[0,1] \longrightarrow M$ be defined by

$$p^*(f,g) = \begin{cases} f & \text{if } f = g, \\ fg & \text{otherwise,} \end{cases}$$

where fg is the pointwise multiplication of two real mappings f and g, that is, $(fg)(x) = f(x)g(x)$ for $f, g \in X$. For a multiplicative algebra cone M, define the partial ordering $\preceq$ with respect to M by

$$f \preceq g \quad \text{if and only if} \quad \left(\frac{g}{f}\right)(x) \geq 1 \quad \text{for all } x \in [0,1],$$

where $\geq$ is the usual order on the elements of $\mathbb{R}$.

We state that $(M, C_{\mathbb{R}}[0,1], M, p^*)$ is multiplicative partial cone metric space over a Banach algebra $C_{\mathbb{R}}[0,1]$.

$(\mathrm{M}-\mathrm{PCM}_1)$ For any $f, g \in X$, if $f = g$, then $p^*(f,f) = p^*(f,g) = f$, whereas if $f \neq g$, then $p^*(f,f) = f \preceq fg = p^*(f,g)$.

$(\mathrm{M}-\mathrm{PCM}_2)$ For any $f, g \in X$, $p^*(f,f) = p^*(f,g) = p^*(g,g)$ whenever $f = g$.

$(\mathrm{M}-\mathrm{PCM}_3)$ For any $f, g \in X$, if $f = g$, then $p^*(f,g) = f = p^*(g,f)$. If $f \neq g$, then $p^*(f,g) = fg = gf = p^*(g,f)$.

($\mathrm{M}-\mathrm{PCM}_4$) The multiplicative triangle inequality is easy to verify by looking shortly at the cases. For arbitrary $f,g,h \in X$, we include the following five cases:

$$p^*(f,g) = f = \frac{p^*(f,h)\cdot p^*(h,g)}{p^*(h,h)} \quad \text{when } f = g = h;$$
$$p^*(f,g) = f \preceq fgh = \frac{p^*(f,h)\cdot p^*(h,g)}{p^*(h,h)} \quad \text{when } f = g \neq h;$$
$$p^*(f,g) = fg = \frac{p^*(f,h)\cdot p^*(h,g)}{p^*(h,h)} \quad \text{when } f \neq g = h;$$
$$p^*(f,g) = fg = \frac{p^*(f,h)\cdot p^*(h,g)}{p^*(h,h)} \quad \text{when } f = h \neq g;$$
$$p^*(f,g) = fg \preceq fgh = \frac{p^*(f,h)\cdot p^*(h,g)}{p^*(h,h)} \quad \text{when } f \neq g \neq h.$$

The above discussion confirms that the multiplicative triangle inequality holds for all $f,g,h \in X$. This completes the verification.

Now define the function p by $p(f,g) = \ln(p^*(f,g))$ for $f,g \in X$. Then

$$p(f,g) = \begin{cases} \ln(f) & \text{if } f = g, \\ \ln(f) + \ln(g) & \text{if } f \neq g. \end{cases}$$

The mapping p is not a partial cone metric, since axiom (PCM_1) is not satisfied for all $f,g \in X$ with $f \neq g$. Indeed, if $g : [0,1] \to \mathbb{R}$ is a continuous mapping such that $0 < g(x) < 1$ for $x \in [0,1]$, then $\ln(f(x)) \npreceq \ln(f(x)) + \ln(g(x))$ for $f \in C_{\mathbb{R}}[0,1]$. For example, take $g(x) = \sin(x)$ for $x \in (0, \frac{\pi}{2})$.

It is mildly interesting that we can redefine $p^* : C_{\mathbb{R}}[0,1] \times C_{\mathbb{R}}[0,1] \longrightarrow M$ by

$$p^*(f,g) = \begin{cases} 1_E & \text{if } f = g, \\ fg & \text{otherwise.} \end{cases}$$

Then $(M, C_{\mathbb{R}}[0,1], M, p^*)$ is a complete multiplicative partial cone metric space over a Banach algebra $C_{\mathbb{R}}[0,1]$. Again, if we define the function p by $p(f,g) = \ln(p^*(f,g))$ for all $f,g \in X$, then

$$p(f,g) = \begin{cases} 0 & \text{if } f = g, \\ \ln(f) + \ln(g) & \text{if } f \neq g. \end{cases}$$

Clearly, p is not a partial cone metric.

In what follows, we suppose that E is a Banach algebra over the real field $\mathbb{R}$ with multiplicative identity 1_E, $M \subset E$ is a multiplicative algebra cone with $\mathrm{Int}(M) \neq \emptyset$, and $\preceq$ is a partial ordering with respect to M.

Proposition 4.3 *Every multiplicative partial cone metric* $\mathfrak{p}$ *generates some topology* $\Im_{\mathfrak{p}}$ *on* X. *More precisely,*

$$\Im_{\mathfrak{p}} = \left\{ U \subseteq X : \textit{for every } x \in U, \textit{ there exists } c \in \mathrm{Int}(M) \textit{ such that } \mathfrak{B}_{\mathfrak{p}}(x;c) \subseteq U \right\} \cup \{\phi\}.$$

The basis of $\Im_{\mathfrak{p}}$ is the family of multiplicative open balls given as

$$\mathfrak{B}_{\mathfrak{p}}(x;c) := \left\{ y \in X : 1_E \ll \frac{c \cdot \mathfrak{p}(x,x)}{\mathfrak{p}(x,y)} \right\},$$

where (x,c) is any element in $X \times \operatorname{Int}(M)$.

Definition 4.4 *Let $(X,E,M,\mathfrak{p})$ be a multiplicative partial cone metric space over a Banach algebra E, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then*:

(i) *$\{x_n\}_{n\in\mathbb{N}}$ is said to be multiplicative convergent, with respect to $\Im_{\mathfrak{p}}$, to some $x \in X$ if for each $1_E \ll c$, there exists $n_c \in \mathbb{N}$ such that $\mathfrak{p}(x_n,x) \ll c \cdot \mathfrak{p}(x,x)$ for all $n \geq n_c$. This type of convergence is denoted by $x_n \xrightarrow{\Im_{\mathfrak{p}}} x$.*

(ii) *$\{x_n\}_{n\in\mathbb{N}}$ is said to be strongly multiplicative convergent to some $x \in X$ if $\lim_{n\to\infty} \mathfrak{p}(x_n,x) = \lim_{n\to\infty} \mathfrak{p}(x_n,x_n) = \mathfrak{p}(x,x)$. This type of convergence is denoted by $x_n \xrightarrow{s-\Im_{\mathfrak{p}}} x$.*

(iii) *$\{x_n\}_{n\in\mathbb{N}}$ is said to be multiplicative Cauchy if there exists $u \in M$ such that $\lim_{n,m\to\infty} \mathfrak{p}(x_n,x_m) = u$.*

(iv) *$\{x_n\}_{n\in\mathbb{N}}$ is said to be 1_E-Cauchy if for each $1_E \ll c$, there exists $n_c \in \mathbb{N}$ such that $\mathfrak{p}(x_n,x_{n+p}) \ll c$ for all $n \geq n_c$ and $p \in \mathbb{N}$ or, equivalently, $\mathfrak{p}(x_n,x_{n+p}) \xrightarrow{\rho} 1_E$ for all $p \in \mathbb{N}$.*

(v) *$(X,E,M,\mathfrak{p})$ is called a multiplicative complete if each multiplicative Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is strongly multiplicative convergent to some point $x \in X$ such that $\mathfrak{p}(x,x) = u$.*

(vi) *$(X,E,M,\mathfrak{p})$ is called 1_E-complete if each 1_E-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of X is multiplicative convergent with respect to $\Im_{\mathfrak{p}}$ to some point $x \in X$ such that $\mathfrak{p}(x,x) = 1_E$.*

Remark 4.5 It follows from Lemma (3.5) and Remark (3.6) that if $x_n \xrightarrow{s-\Im_{\mathfrak{p}}} x$ for some $x \in X$, then $x_n \xrightarrow{\Im_{\mathfrak{p}}} x$. The converse is still valid whenever the underlying multiplicative algebra cone M of $(E,\rho(\cdot))$ is normal.

By the following example we show, from the point of topology, that the real novelty of multiplicative partial cone metric spaces is in the convergence structure. In particular, it also shows that the multiplicative partial cone metric $\mathfrak{p}$ and the partial cone metric p are topologically different.

Let $E := \mathbb{R}$, $X = C := [0,\infty)$, and let $p : [0,\infty)^2 \to [0,\infty)$ be defined by

$$p(x,y) = \begin{cases} (\frac{1}{2})^{\max\{x,y\}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then p is a partial cone metric. Let $\{x_n\}_{n\in\mathbb{N}} = \{\frac{1}{n}\}_{n\in\mathbb{N}} \in X$. Clearly, $p(x_n,0) = (\frac{1}{2})^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, and $p(0,0) = 0$. We will now show that $x_n \overset{\Im_p}{\nrightarrow} 0 \in X$. If we take $c = \frac{1}{2} \in \operatorname{Int}(C) = (0,\infty)$, then at that point, we will never find $n_c \in \mathbb{N}$ such that

$$\frac{1}{2} - \left(\frac{1}{2}\right)^{\frac{1}{n}} > 0 \quad \text{for all } n \geq n_c.$$

If we now consider $E := \mathbb{R}$, $M := [1,\infty)$, $X = [0,\infty)$, and

$$\begin{cases} \mathfrak{p} : [0,\infty)^2 \to [1,\infty), \\ (x,y) \mapsto e^{\max\{x,y\}}, \end{cases}$$

then $\mathfrak{p}$ is a multiplicative partial cone metric. Choosing the sequence $\{x_n\}_{n\in\mathbb{N}} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in X, it is clear that $x_n \xrightarrow{\Im_\mathfrak{p}} 0$. Indeed, for every $c \in \operatorname{Int}(M) = (1,\infty)$, we have

$$\lim_{n\to\infty} \frac{c}{(e)^{\frac{1}{n}}} = c > 1.$$

Thus the sequence $\{x_n\}_{n\in\mathbb{N}} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ is multiplicative convergent in the multiplicative partial cone metric space $([0,\infty),\mathbb{R},[1,\infty),\mathfrak{p})$, but it is not convergent in the partial cone metric space $([0,\infty),\mathbb{R},[0,\infty),p)$. Hence topologies induced on $X = [0,\infty)$ by p and $\mathfrak{p}$ are different.

Lemma 4.6 *Suppose that $\varphi : M \to M$ is completely multiplicative integrable mapping on M. Let $\mathfrak{p} : X^2 \to M$ be a multiplicative partial cone metric defined on a nonempty set X. Then for all $x,y,z \in X$, we have*:

(i) $1_E \preceq \int_{1_E}^{\mathfrak{p}(x,x)} \varphi(t)^{d_M(t)} \preceq \int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}$ *and* $1_E \preceq \int_{1_E}^{\mathfrak{p}(y,y)} \varphi(t)^{d_M(t)} \preceq \int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}$;

(ii) *if* $\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)} = \int_{1_E}^{\mathfrak{p}(x,x)} \varphi(t)^{d_M(t)} = \int_{1_E}^{\mathfrak{p}(y,y)} \varphi(t)^{d_M(t)}$, *then* $x = y$;

(iii) $\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)} = \int_{1_E}^{\mathfrak{p}(y,x)} \varphi(t)^{d_M(t)}$;

(iv) $\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}) \preceq \int_{1_E}^{\frac{\mathfrak{p}(x,z)\cdot\mathfrak{p}(z,y)}{\mathfrak{p}(z,z)}} \varphi(t)^{d_M(t)}$.

Remark 4.7 If $\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)} = 1_E$, then $\mathfrak{p}(x,y) = 1_E$, so that $x = y$.

Remark 4.8 If we take $\varphi := 1_E$, then $\int_{1_E}^{\mathfrak{p}(x,y)} (1_E)^{d_M(t)} = \mathfrak{p}(x,y)$, and we recover Definition 4.1.

5 Main fixed-point results

We begin the section by first introducing the following classes of mappings defined on multiplicative algebra cone M.

Definition 5.1 *Let Ψ be the class of all continuous self-mappings defined on a multiplicative algebra cone M that satisfies the following conditions*:

(i) *every $\psi \in \Psi$ is sequentially continuous*;

(ii) *every $\psi \in \Psi$ is strongly monotonic increasing, that is, $u \preceq v$ if and only if $\psi(u) \preceq \psi(v)$ for all $u,v \in M$*;

(iii) *$\psi(u) = 1_E$ if and only if $u = 1_E$.*

Definition 5.2 *By Φ we denote the class of all self-mappings ϕ defined on a multiplicative algebra cone M and satisfying the following properties*:

(i) *$\phi(u) \succ 1_E$ for all $u \succ 1_E$*;

(ii) *$\phi(1_E) \succeq 1_E$.*

Definition 5.3 *By $\mathcal{C}^*$ we denote the set of all continuous mappings $F^\star : M \times M \to M$ satisfying the following axioms*:

(i) $F^{\star}(u,v) \preceq u$;
(ii) $F^{\star}(u,v) = u$ *implies that either* $u = 1_E$ *or* $v = 1_E$ *for all* $u,v \in M$.

Example 5.4 The following functions $F^{\star} : M \times M \to M$ are elements of $\mathcal{C}^*$ for all $u,v \in M$:
(i) $F^{\star}(u,v) = \frac{u}{v}$;
(ii) $F^{\star}(u,v) = u^k$, $0 < k < 1$.

We further assume that E is a Banach algebra over the real field $\mathbb{R}$ with multiplicative identity 1_E, $M \subseteq E$ is a normal multiplicative algebra cone with $\mathrm{Int}(M) \neq \emptyset$, and $\preceq$ is a partial ordering with respect to the multiplicative algebra cone M.

The following theorem is the main result in this paper.

Theorem 5.5 *Let* $(X,E,M,\mathfrak{p})$ *be a* 1_E*-complete multiplicative partial cone metric space over Banach algebra* E*. In a nonempty set* X*, let* $T : X \to X$ *be a self-mapping. Suppose that there exists a constant* $k \in (0,1)$ *such that the integral inequality*

$$\psi\left(\int_{1_E}^{\mathfrak{p}(Tx,Ty)} \varphi(t)^{d_M(t)}\right) \preceq F^{\star}\left(\psi\left(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}\right)^k, \phi\left(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}\right)^k\right)$$

is satisfied for all $x,y \in X$*, where* $(\psi,\phi,F^{\star}) \in \Psi \times \Phi \times \mathcal{C}^*$*, and* $\varphi : M \to M$ *is a completely multiplicative integrable mapping on* M*. Then the mapping* T *admits a unique fixed-point in* X*.*

Proof For a generic point x_0 in X, we define the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by the recurrence relation $x_{n+1} = Tx_n := T^{n+1}x_0$ for all $n \in \mathbb{N}_0$.

A direct computation show that

$$\begin{aligned}
\psi\left(\int_{1_E}^{\mathfrak{p}(x_n,x_{n+1})} \varphi(t)^{d_M(t)}\right) &= \psi\left(\int_{1_E}^{\mathfrak{p}(Tx_{n-1},Tx_n)} \varphi(t)^{d_M(t)}\right) \\
&\preceq F^{\star}\left(\psi\left(\int_{1_E}^{\mathfrak{p}(x_{n-1},x_n)} \varphi(t)^{d_M(t)}\right)^k, \phi\left(\int_{1_E}^{\mathfrak{p}(x_{n-1},x_n)} \varphi(t)^{d_M(t)}\right)^k\right) \\
&\preceq \psi\left(\int_{1_E}^{\mathfrak{p}(x_{n-1},x_n)} \varphi(t)^{d_M(t)}\right)^k
\end{aligned}$$

for certain $k \in (0,1)$ and for all $n \in \mathbb{N}$. Since ψ is strongly increasing, we deduce that

$$\int_{1_E}^{\mathfrak{p}(x_n,x_{n+1})} \varphi(t)^{d_M(t)} \preceq \left(\int_{1_E}^{\mathfrak{p}(x_{n-1},x_n)} \varphi(t)^{d_M(t)}\right)^k \quad \text{for all } n \in \mathbb{N}.$$

As before, we have

$$\int_{1_E}^{\mathfrak{p}(x_{n-1},x_n)} \varphi(t)^{d_M(t)} \preceq \left(\int_{1_E}^{\mathfrak{p}(x_{n-2},x_{n-1})} \varphi(t)^{d_M(t)}\right)^k.$$

Repeating the previous process over and over again, we speculate that

$$\int_{1_E}^{\mathfrak{p}(x_n,x_{n+1})} \varphi(t)^{d_M(t)} \preceq \left(\int_{1_E}^{\mathfrak{p}(x_{n-1},x_n)} \varphi(t)^{d_M(t)}\right)^k \preceq \left(\int_{1_E}^{\mathfrak{p}(x_{n-2},x_{n-1})} \varphi(t)^{d_M}(t)\right)^{k^2}$$

$$\preceq \cdots \preceq \left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)} \varphi(t)^{d_M(t)}\right)^{k^n}.$$

To give more details, let us distinguish the following two cases.

(i) If $x_0 = x_n$ for all $n \geq 2$, the we see that

$$\begin{aligned}\int_{1_E}^{\mathfrak{p}(x_0,x_1)} \varphi(t)^{d_M(t)} &= \int_{1_E}^{\mathfrak{p}(x_0,Tx_0)} \varphi(t)^{d_M(t)} \\ &= \int_{1_E}^{\mathfrak{p}(x_n,Tx_n)} \varphi(t)^{d_M(t)} \\ &= \int_{1_E}^{\mathfrak{p}(x_n,x_{n+1})} \varphi(t)^{d_M(t)} \\ &\preceq \left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)} \varphi(t)^{d_M(t)}\right)^{k^n}.\end{aligned}$$

Henceforth, we possess

$$\left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)} \varphi(t)^{d_M(t)}\right)^{k^n-1} \in M.$$

On the other hand, since we are treating $k \in (0,1)$, and thus $1 - k^n > 0$, there must be

$$\left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)} \varphi(t)^{d_M(t)}\right)^{1-k^n} \in M.$$

Reviewing all the above discussions, we have only one possibility that $\int_{1_E}^{\mathfrak{p}(x_0,x_1)} \varphi(t)^{d_M(t)} = 1_E$, and so $\mathfrak{p}(x_0,x_1) = 1_E$, which means $x_0 = Tx_0$, and so x_0 is a fixed point of T.

Therefore we presume that $x_1 \neq x_0$, i.e., $\mathfrak{p}(x_0,x_1) \succ 1_E$. In view of Definition 3.19, since for each $c \succ 1_E$, $\int_{1_E}^{c} \varphi(t)^{d_M(t)} \succ 1_E$, we have $\int_{1_E}^{\mathfrak{p}(x_0,x_1)} \varphi(t)^{d_M(t)} \succ 1_E$.

(ii) If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T, and there is nothing to prove. By the previous arguments we suppose that no two consecutive elements are equal in the sequence $\{x_n\}_{n\in\mathbb{N}}$, i.e., $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Consequently, $\mathfrak{p}(x_n,x_{n+1}) \succ 1_E$ for all $n \in \mathbb{N}_0$, which in turn implies that $\int_{1_E}^{\mathfrak{p}(x_n,x_{n+1})} \varphi(t)^{d_M(t)} \succ 1_E$.

In conclusion, we presume that $x_n \neq x_m$ for all idiosyncratic $m,n \in \mathbb{N}$. We complete the proof in three steps as follows.

Step 1. We will prove that $\{x_n\}_{n\in\mathbb{N}}$ is 1_E-Cauchy. For any $n,p \in \mathbb{N}$, we have the following estimates:

$$\begin{aligned}\int_{1_E}^{\mathfrak{p}(x_n,x_{n+p})} \varphi(t)^{d_M(t)} &\preceq \int_{1_E}^{\frac{\prod_{i=n}^{n+p-1} \mathfrak{p}(x_i,x_{i+1})}{\prod_{i=n+1}^{n+p-1} \mathfrak{p}(x_i,x_i)}} \varphi(t)^{d_M(t)} \\ &\preceq \int_{1_E}^{\prod_{i=n}^{n+p-1} \mathfrak{p}(x_i,x_{i+1})} \varphi(t)^{d_M(t)} \\ &\preceq \prod_{i=n}^{n+p-1} \left(\int_{1_E}^{\mathfrak{p}(x_i,x_{i+1})} \varphi(t)^{d_M(t)}\right)\end{aligned}$$

$$\begin{aligned}
&\preceq \prod_{i=n}^{n+p-1}\left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)}\varphi(t)^{d_M(t)}\right)^{k^i}\\
&=\left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)}\varphi(t)^{d_M(t)}\right)^{\sum_{i=n}^{n+p-1}k^i}\\
&\prec\left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)}\varphi(t)^{d_M(t)}\right)^{k^n\sum_{j=0}^{\infty}k^j}\\
&=\left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)}\varphi(t)^{d_M(t)}\right)^{\frac{k^n}{1-k}}.
\end{aligned}$$

Since ψ is strongly increasing, we clearly have

$$\psi\left(\int_{1_E}^{\mathfrak{p}(x_n,x_{n+p})}\varphi(t)^{d_M(t)}\right)\preceq\psi\left(\int_{1_E}^{\mathfrak{p}(x_0,x_1)}\varphi(t)^{d_M(t)}\right)^{\frac{k^n}{1-k}}.$$

Since $k<1$, we can readily infer that $(\int_{1_E}^{\mathfrak{p}(x_0,x_1)}\varphi(t)^{d_M(t)})^{\frac{k^n}{1-k}}\underset{n\to\infty}{\longrightarrow}1_E$.

This, together with the premise that ψ is continuous, implies

$$\psi\left(\int_{1_E}^{\mathfrak{p}(x_n,x_{n+p})}\varphi(t)^{d_M(t)}\right)\underset{n\to\infty}{\longrightarrow}1_E.$$

Taking into account that ψ is sequentially continuous leads us to $\int_{1_E}^{\mathfrak{p}(x_n,x_{n+p})}\varphi(t)^{d_M(t)}\underset{n\to\infty}{\longrightarrow}1_E$, and thus $\mathfrak{p}(x_n,x_{n+p})\underset{n\to\infty}{\longrightarrow}1_E$ for all $p\in\mathbb{N}$. This shows that $\{x_n\}_{n\in\mathbb{N}}$ is 1_E-Cauchy. By the 1_E-completeness nature of $(X,E,M,\mathfrak{p})$ there must be an element (say) $\bar{x}\in X$ for which $x_n\overset{\Im_{\mathfrak{p}}}{\longrightarrow}\bar{x}$ with $\mathfrak{p}(\bar{x},\bar{x})=1_E$. Recall that the multiplicative algebra cone M is normal and thus $x_n\overset{s-\Im_{\mathfrak{p}}}{\longrightarrow}\bar{x}$, i.e., $\lim_{n\to\infty}\mathfrak{p}(x_n,\bar{x})=\mathfrak{p}(\bar{x},\bar{x})=1_E$.

Step 2. We are required to show that this limit $\bar{x}$ is a fixed point of the mapping T. To see this, we derive

$$\begin{aligned}
\psi\left(\int_{1_E}^{\mathfrak{p}(x_{n+1},T\bar{x})}\varphi(t)^{d_M(t)}\right)&=\psi\left(\int_{1_E}^{\mathfrak{p}(Tx_n,T\bar{x})}\varphi(t)^{d_M(t)}\right)\\
&\preceq F^{\star}\left(\psi\left(\int_{1_E}^{\mathfrak{p}(x_n,\bar{x})}\varphi(t)^{d_M(t)}\right)^k,\phi\left(\int_{1_E}^{\mathfrak{p}((x_n,\bar{x})}\varphi(t)^{d_M(t)}\right)^k\right)\\
&\preceq\psi\left(\int_{1_E}^{\mathfrak{p}(x_n,\bar{x})}\varphi(t)^{d_M(t)}\right)^k.
\end{aligned}$$

This, coupled with the assumption that ψ is a strongly increasing mapping, gives

$$\int_{1_E}^{\mathfrak{p}(x_{n+1},T\bar{x})}\varphi(t)^{d_M(t)}\preceq\left(\int_{1_E}^{\mathfrak{p}(x_n,\bar{x})}\varphi(t)^{d_M(t)}\right)^k.$$

Thus $\int_{1_E}^{\mathfrak{p}(x_{n+1},T\bar{x})}\varphi(t)^{d_M(t)}\underset{n\to\infty}{\longrightarrow}1_E$, and so $\mathfrak{p}(x_{n+1},T\bar{x})\underset{n\to\infty}{\longrightarrow}1_E$. At the moment,

$$\mathfrak{p}(\bar{x},T\bar{x})\overset{(\text{M}-\text{PCM}_4)}{\preceq}\mathfrak{p}(\bar{x},x_{n+1})\cdot\mathfrak{p}(x_{n+1},T\bar{x}).$$

Letting n tend to ∞ on both sides, we get $\mathfrak{p}(\bar{x},T\bar{x})=1_E$, which says that $\bar{x}=T\bar{x}$.

Step 3. To prove the theorem, we just need to prove the uniqueness of the fixed point of T. Let, on the contrary, there exist $\hat{x} \in X$ such that $T\bar{x} = \bar{x} \neq \hat{x} = T\hat{x}$. We state that

$$\begin{aligned} \psi\left(\int_{1_E}^{\mathfrak{p}(\bar{x},\hat{x})} \varphi(t)^{d_M(t)}\right) &= \psi\left(\int_{1_E}^{\mathfrak{p}(T\bar{x},T\hat{x})} \varphi(t)^{d_M(t)}\right) \\ &\preceq F^\star\left(\psi\left(\int_{1_E}^{\mathfrak{p}(\bar{x},\hat{x})} \varphi(t)^{d_M(t)}\right)^k, \phi\left(\int_{1_E}^{\mathfrak{p}(\bar{x},\hat{x})} \varphi(t)^{d_M(t)}\right)^k\right) \\ &\preceq \psi\left(\int_{1_E}^{\mathfrak{p}(\bar{x},\hat{x})} \varphi(t)^{d_M(t)}\right)^k. \end{aligned}$$

Therefore we must have that

$$\begin{aligned} \int_{1_E}^{\mathfrak{p}(\bar{x},\hat{x})} \varphi(t)^{d_M(t)} &\preceq \left(\int_{1_E}^{\mathfrak{p}(\bar{x},\hat{x})} \varphi(t)^{d_M(t)}\right)^k \\ &\prec \int_{1_E}^{\mathfrak{p}(\bar{x},\hat{x})} \varphi(t)^{d_M(t)}, \end{aligned}$$

which is a contradiction to our assumption that T has another fixed point $\hat{x}$. Thus T possesses a unique fixed point $\bar{x} \in X$ such that $\mathfrak{p}(\bar{x},\bar{x}) = 1_E$. This completes the proof. □

Corollary 5.6 *Let* $(X,E,M,\mathfrak{p})$ *be a* 1_E*-complete multiplicative partial cone metric space over a Banach algebra* E. *In a nonempty set* X, *let* $T : X \to X$ *be a self-mapping. Suppose that there exist* $N \in \mathbb{N}$ *and* $k \in (0,1)$ *such that the integral inequality*

$$\psi\left(\int_{1_E}^{\mathfrak{p}(T^N x,T^N y)} \varphi(t)^{d_M(t)}\right) \preceq F^\star\left(\psi\left(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}\right)^k, \phi\left(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}\right)^k\right)$$

is satisfied for all $x,y \in X$, *where* $(\psi,\phi,F^\star) \in \Psi \times \Phi \times \mathcal{C}^*$, *and* $\varphi : M \to M$ *is a completely multiplicative integrable mapping on* M. *Then the mapping* T *admits a unique fixed point in* X.

Corollary 5.7 *Let* $(X,E,M,\mathfrak{p})$ *be a* 1_E*-complete multiplicative partial cone metric space over a Banach algebra* E. *In a nonempty set* X, *let* $T : X \to X$ *be a self-mapping. Suppose that there exists a constant* $k \in (0,1)$ *such that the integral inequality*

$$\psi\left(\int_{1_E}^{\mathfrak{p}(Tx,Ty)} \varphi(t)^{d_M(t)}\right) \preceq \frac{\psi(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)})^k}{\phi(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)})^k}$$

is satisfied for all $x,y \in X$, *where* $(\psi,\phi) \in \Psi \times \Phi$, *and* $\varphi : M \to M$ *is a completely multiplicative algebra integrable mapping on* M. *Then the mapping* T *admits a unique fixed point in* X.

Proof Along the hypotheses stated in Theorem 5.5, define $F^\star : M \times M \to M$ by $F^\star(u,v) = \frac{u}{v}$ for $u,v \in M$. □

Corollary 5.8 *Let* $(X,E,M,\mathfrak{p})$ *be a* 1_E*-complete multiplicative partial cone metric space over a Banach algebra* E. *In a nonempty set* X, *let* $T : X \to X$ *be a self-mapping. Suppose* *that there exists a constant $k \in (0,1)$ such that the integral inequality*

$$\psi\left(\int_{1_E}^{\mathfrak{p}(Tx,Ty)} \varphi(t)^{d_M(t)}\right) \preceq \psi\left(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}\right)^k$$

is satisfied for all $x, y \in X$, where $\psi \in \Psi$, and $\varphi : M \to M$ is a completely multiplicative algebra integrable mapping on M. Then the mapping T admits a unique fixed point in X.

Proof In Theorem 5.5, take $F^\star : M \times M \to M$ defined by $F^\star(u, v) = u$. □

Corollary 5.9 *Let $(X, E, M, \mathfrak{p})$ be a 1_E-complete multiplicative partial cone metric space over a Banach algebra E. In a nonempty set X, let $T : X \to X$ be a self-mapping. Suppose that there exists a constant $k \in (0,1)$ such that the integral inequality*

$$\int_{1_E}^{\mathfrak{p}(Tx,Ty)} \varphi(t)^{d_M(t)} \preceq \left(\int_{1_E}^{\mathfrak{p}(x,y)} \varphi(t)^{d_M(t)}\right)^k$$

is satisfied for all $x, y \in X$, where $\varphi : M \to M$ is a completely multiplicative integrable mapping on M. Then the mapping T admits a unique fixed point in X.

Proof The result follows trivially by using the definition of ψ in Corollary 5.8. □

Corollary 5.10 *Let $(X, E, M, \mathfrak{p})$ be a 1_E-complete multiplicative partial cone metric space over a Banach algebra E. In a nonempty set X, let $T : X \to X$ be a self-mapping. Suppose that there exists a constant $k \in (0,1)$ such that the inequality*

$$\psi\left(\mathfrak{p}(Tx, Ty)\right) \preceq F^\star\left(\psi\left(\mathfrak{p}(x,y)\right)^k, \phi\left(\mathfrak{p}(x,y)\right)^k\right)$$

is satisfied for all $x, y \in X$, where $(\psi, \phi, F^\star) \in \Psi \times \Phi \times \mathcal{C}^*$. Then the mapping T admits a unique fixed point in X.

Proof Take $\varphi := 1_E$ in Corollary 5.6. □

Corollary 5.11 *Let $(X, E, M, \mathfrak{p})$ be a 1_E-complete multiplicative partial cone metric space over a Banach algebra E. In a nonempty set X, let $T : X \to X$ be a self-mapping. Suppose that there exists a constant $k \in (0,1)$ such that the inequality*

$$\mathfrak{p}(Tx, Ty) \preceq \mathfrak{p}(x,y)^k$$

is satisfied for all $x, y \in X$. Then the mapping T admits a unique fixed point in X.

Proof Take $\varphi := 1_E$ in Corollary 5.9. □

Let us consider a nontrivial counterexample for Corollary 5.11.

Example 5.12 Consider the Banach algebra $E := (C_{\mathbb{R}}[1,\infty), \rho_\infty(\cdot))$ in the order of the multiplicative algebra cone $M := \{f \in E : f(x) \geq 1 \text{ on } x \in [1,\infty)\}$, and define the multiplication in the natural way. For $X = M$, define the mapping $\mathfrak{p} : X^2 \to M$ for $f, g \in X$ by

$$\mathfrak{p}(f,g) = \begin{cases} 1_E & \text{if } f = g, \\ fg & \text{otherwise,} \end{cases}$$

where fg is the pointwise multiplication of two real mappings f and g, that is,

$(fg)(x) = f(x)g(x)$ for all $x \in [1,\infty)$, and 1_E (the multiplicative element in $C_{\mathbb{R}}[1,\infty)$) is understood as the constant mapping $f(x) = 1$ for all $x \in [1,\infty)$. Then $(M, C_{\mathbb{R}}[1,\infty), M, \mathfrak{p})$ defines a multiplicative partial cone metric space over the Banach algebra $C_{\mathbb{R}}[1,\infty)$.

Step 1. We will show that $(M, C_{\mathbb{R}}[1,\infty), M, \mathfrak{p})$ is 1_E-complete. If $\{f_n\}_{n\in\mathbb{N}}$ is any 1_E-Cauchy sequence in X, then for each $1_E \ll c$, there exists $n_c \in \mathbb{N}$ such that $\mathfrak{p}(f_n, f_m) \ll c$ for all $m > n \geq n_c$. It is clear that $f_n(x) \leq f_n(x)f_m(x) = (f_nf_m)(x)$ for all $x \in [1,\infty)$, that is, $f_n \preceq f_nf_m$ for all $m, n \in \mathbb{N}$ with $m > n$. Thus, for each $1_E \ll c$, there exists $n_c \in \mathbb{N}$ such that

$$f_n \preceq f_nf_m = \mathfrak{p}(f_n,f_m) \ll c \quad \text{for all } n \geq n_c. \tag{5.1}$$

This implies that $\{f_n\}_{n\in\mathbb{N}}$ is a c-sequence in $X = M$ for all $1_E \ll c$ (for the purpose at hand, exceptionally in this space, we proved that any 1_E-Cauchy sequence is a c-sequence in $X = M$ for all $1_E \ll c$). We need here to consider two possible situations.

Case 1: If $f_n = 1_E$, that is, $f_n(x) = 1$ for all $n \in \mathbb{N}$ and $x \in [1,\infty)$, then $\mathfrak{p}(f_n, 1_E) = \mathfrak{p}(1_E, 1_E) = 1_E$, and thus $\mathfrak{p}(f_n, 1_E) \ll \mathfrak{p}(1_E, 1_E) \cdot c$ makes sense for all $1_E \ll c$ and $n \in \mathbb{N}$. This says that $\{f_n\}_{n\in\mathbb{N}}$ is multiplicative convergent to 1_E in X.

Case 2: If $f_n \neq 1_E$, then $\mathfrak{p}(f_n, 1_E) = f_n \overset{(5.1)}{\ll} c = c \cdot \mathfrak{p}(1_E, 1_E)$ for all $1_E \ll c$ and $n \geq n_c$. This means that $\{f_n\}_{n\in\mathbb{N}}$ is multiplicative convergent to 1_E in X.

Both cases ensure that every 1_E-Cauchy sequence in X is multiplicative convergent to $1_E \in X$ with $\mathfrak{p}(1_E, 1_E) = 1_E$. Therefore $(M, C_{\mathbb{R}}[1,\infty), M, \mathfrak{p})$ is 1_E-complete multiplicative partial cone metric space over Banach algebra $C_{\mathbb{R}}[1,\infty)$.

Step 2. Define the self-mapping $T : X \to X$ by $f(x) \mapsto e^{f(x)}$ for all $f \in X$ and $x \in [1,\infty)$. Let $f(x) = x$ and $g(x) = \ln(x)$ for $x \in [1,\infty)$. For this case, in general, the contractive inequality condition on the mapping T is not satisfied for all $k \in (0,1)$. Indeed, we have

$$\begin{aligned} \mathfrak{p}(Tf, Tg) &= \mathfrak{p}\left(e^x, x\right) \\ &= xe^x \\ &\npreceq \left(x\ln(x)\right)^k \\ &= \left(\mathfrak{p}(f,g)\right)^k. \end{aligned}$$

For $x = 1$, it is clear that $e \npreceq (\ln(1))^k$ for all $k \in (0,1)$.

We eventually have that the mapping T is a fixed-point free, since there is no real-valued function $f \in X$ such that $f(x) - e^{f(x)} = 0$ for $x \in [1,\infty)$.

6 Application to fractional calculus: involvement of fixed-point result for solving Caputo fractional boundary value problem

As an application of the main proved fixed-point results, we investigate the existence and uniqueness of analytical solution for nonlinear fractional differential equations involving the Caputo fractional-order derivative. First, we forward the theoretical background materials that will be utilized throughout this section (see [24, 25]).

If $\delta > 0$ is a given positive real number and $\Gamma(\delta)$ denotes the gamma function, the extended form of the factorial function to complex (real) numbers, then

$$\Gamma(\delta) := \int_0^\infty y^{\delta-1}e^{-y}\,dy.$$

In particular, if n is a natural number, then we have

$$\Gamma(1) = \int_0^\infty e^{-y}\,dy = 1 \quad \text{and} \quad \Gamma(n) = \int_0^\infty y^{n-1}e^{-y}\,dy = n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

If α is a number calculated to be within $n-1 \leq \alpha < n$, then $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α, and for a function h on the interval $[a,b]$, the αth Caputo fractional-order derivative of h is defined by

$${}^c\mathrm{D}_a^\alpha\big(h(t)\big) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)\,ds.$$

In this section, we illustrate the main theorem of this paper. In fact, we investigate a solution of the Caputo fractional derivative boundary value problem in Banach algebras stated as follows. Given a continuous function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and a finite set of n real constants $x_0, x_1, \ldots, x_{n-2}, x_{n-1} := x_b$, the problem is to find an n times differentiable function $x \in C_{\mathbb{R}}^{n-1}[a,b]$ satisfying the following conditions for some α within $n-1 \leq \alpha < n$:

$$\begin{cases} {}^c\mathrm{D}_a^\alpha(x(t)) = f(t, x(t)) & \text{for } t \in [a,b], \\ x^{(j)}(a) = x_j, \quad j = 0, 1, \ldots, n-2, & x^{(n-1)}(b) = x_b, \end{cases} \tag{6.1}$$

where ${}^c\mathrm{D}_a^\alpha$ is the Caputo fractional derivative of order α.

Lemma 6.1 [26] *Let $h : [a,b] \to \mathbb{R}$ be a continuous function. A function $x(t)$ is a solution of the fractional boundary value problem*

$$\begin{cases} {}^c\mathrm{D}_a^\alpha(x(t)) = h(t), \quad t \in [a,b], \\ x^{(j)}(a) = x_j, \quad j = 0, 1, \ldots, n-2, \qquad x^{(n-1)}(b) = x_b, \end{cases}$$

if and only if $x(t)$ is a solution of the fractional integral equation

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\,ds + \left(\frac{x_b}{(n-1)!} + \frac{h(a)(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)(t-a)^{n-1} \\ & - \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}h(s)\,ds + \sum_{j=0}^{n-2}\frac{x_j}{j!}(t-a)^j. \end{aligned}$$

Note that we usually can select a positive constant $L > 0$ such that

$$0 < L(b-a)^\alpha\left(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right) < 1. \tag{6.2}$$

We have the following:

Theorem 6.2 *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous mapping satisfying the uniform Lipschitz condition with respect to the second coordinate, that is,*

$$\begin{aligned} &\big|f\big(t,x(t)\big) - f\big(t,y(t)\big)\big| \le L\big|x(t) - y(t)\big| \\ &\text{for all } \big(t,x(t)\big),\big(t,y(t)\big) \in [a,b] \times \mathbb{R} \end{aligned} \tag{6.3}$$

with some constant $L > 0$ satisfying inequality (6.2). Then equation (6.1) has a unique solution on $[a,b]$.

Proof Problem (6.1) can be transformed into a fixed-point problem as follows. Let us define the self-mapping $T : C^{n-1}_{\mathbb{R}}[a,b] \to C^{n-1}_{\mathbb{R}}[a,b]$ by

$$\begin{aligned} T\big(x(t)\big) = {}& \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f\big(s,x(s)\big)\,ds \\ &+ \left(\frac{x_b}{(n-1)!} + \frac{f(a,x(a))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)(t-a)^{n-1} \\ &- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} f\big(s,x(s)\big)\,ds + \sum_{j=0}^{n-2} \frac{x_j}{j!}(t-a)^j. \end{aligned}$$

Obviously, the fixed points of the mapping T are solutions of equation (6.1). We prove that T has a fixed point. For that reason, set $E := (C^{n-1}_{\mathbb{R}}[a,b], \rho_\infty(\cdot))$; for the Banach algebra E, the associated multiplicative algebra cone $M = \{x \in E : x(t) \ge 1 \text{ on } t \in [a,b]\}$ is solid, $\operatorname{Int}(M) \neq \emptyset$, and normal with multiplicative normality constant $K = 1$. Let $X = C^{n-1}_{\mathbb{R}}[a,b]$, and let $\mathfrak{p} : X^2 \to M$ be defined for all $x, y \in X$ by $\mathfrak{p}(x,y)(t) = e^{(\|x-y\|_\infty)t}$ for all $t \in [a,b]$. The space $(C^{n-1}_{\mathbb{R}}[a,b], C^{n-1}_{\mathbb{R}}[a,b], M, \mathfrak{p})$ is a 1_E-complete multiplicative partial cone metric space over a Banach algebra $C^{n-1}_{\mathbb{R}}[a,b]$.

Now, with the help of this frame, we are going to show that all the hypotheses of Corollary 5.11 are satisfied. For all $x, y \in C^{n-1}_{\mathbb{R}}[a,b]$ and all $t \in [a,b]$, observe the following fractional integral inequalities:

$$\begin{aligned} \big|Tx(t) - Ty(t)\big| \le {}& \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\ &+ \left(\frac{(t-a)^{n-1}(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)\big|f\big(a,x(a)\big) - f\big(a,y(a)\big)\big| \\ &+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds. \end{aligned}$$

Thus we must have

$$\begin{aligned} \big|Tx(t) - Ty(t)\big| \le {}& \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\ &+ \left(\frac{(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\right)\big|f\big(a,x(a)\big) - f\big(a,y(a)\big)\big| \\ &+ \frac{(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds. \end{aligned}$$

Since $x, y \in C_{\mathbb{R}}^{n-1}[a,b]$ are a continuous functions on a compact set, we get

$$\begin{aligned}
\left|Tx(t) - Ty(t)\right| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \big(L\left|x(s) - y(s)\right|\big)\,ds \\
&\quad + \left(\frac{(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\right)\big(L\left|x(a) - y(a)\right|\big) \\
&\quad + \frac{(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \big(L\left|x(s) - y(s)\right|\big)\,ds \\
&\leq \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sup_{s\in[a,b]} \left|x(s) - y(s)\right|\,ds \\
&\quad + L\left(\frac{(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\right) \sup_{s\in[a,b]} \left|x(s) - y(s)\right| \\
&\quad + \frac{L(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \sup_{s\in[a,b]} \left|x(s) - y(s)\right|\,ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|Tx(t) - Ty(t)\right| &\leq L\|x-y\|_{\infty}\left(\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\,ds + \left(\frac{(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\right)\right. \\
&\quad \left. + \frac{(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n}\,ds\right) \\
&\leq L\|x-y\|_{\infty}\left(\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{(n-1)(b-a)^{\alpha}}{(n-1)!\Gamma(\alpha-n+2)}\right) \\
&\quad + L\|x-y\|_{\infty}\left(\frac{(b-a)^{\alpha}}{(n-1)!(\alpha-n+1)\Gamma(\alpha-n+1)}\right) \\
&= L(b-a)^{\alpha}\|x-y\|_{\infty}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right).
\end{aligned}$$

For the least upper bound, we have

$$\|Tx - Ty\|_{\infty} \leq L(b-a)^{\alpha}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right)\|x-y\|_{\infty}.$$

If $k < 1$ is as $k := L(b-a)^{\alpha}(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)})$, then we have the following result:

$$e^{(\|Tx-Ty\|_{\infty})t} \leq e^{k(\|x-y\|_{\infty})t}, \quad x, y \in C_{\mathbb{R}}^{n-1}[a,b], t \in [a,b].$$

Consequently,

$$\mathfrak{p}(Tx, Ty) \preceq \mathfrak{p}(x,y)^k$$

with certain $k \in (0,1)$. This shows that all the requirements of Corollary 5.11 are satisfied for the defined mapping T. Then T has a fixed point, which is a solution for equation (6.1) in X. Since the fixed point of T is unique, the solution of (6.1) is also unique in X. □

7 Pertinent example

We provide a numerical example to support our application of the previous section, Corollary 5.11.

Consider the nonlinear Caputo fractional differential equation

$$^{c}\mathrm{D}_0^{\alpha}\big(x(t)\big) = \frac{t^2}{10} + \frac{\sin|x(t)|}{(50+t^2)} \quad \text{for } t \in \mathbb{I} = [0,1], n-1 \le \alpha < n,$$

subject to the initial conditions

$$x^{(j)}(0) = 0, \quad j = 0,1,\dots,n-2, \qquad x^{(n-1)}(1) = 1.$$

In this case,

$$f\big(t,x(t)\big) = \frac{t^2}{10} + \frac{\sin|x(t)|}{(50+t^2)}$$

is continuous for all $t \in \mathbb{I}$. Further, let $x(t), y(t) \in \mathbb{R}_0^+$ and $t \in \mathbb{I}$. Thus we have

$$\big|f\big(t,x(t)\big) - f\big(t,y(t)\big)\big| \le \frac{1}{50}\big|x(t) - y(t)\big|.$$

Hence (6.3) is satisfied with $L = \frac{1}{50}$.

We also need to check that

$$k := L\left(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right) < 1. \tag{7.1}$$

For instance, if we take $\alpha = \frac{5}{2}$, then $n = [\alpha] + 1 = 3$, and we have the following boundary value problem:

$$\begin{cases} ^{c}\mathrm{D}_0^{\frac{5}{2}}(x(t)) = \frac{t^2}{10} + \frac{\sin|x(t)|}{(50+t^2)} & \text{for } t \in \mathbb{I}, \\ x^{(j)}(0) = 0, \quad j = 0,1, & x''(1) = 1. \end{cases} \tag{7.2}$$

Correspondingly, we have

$$k = \frac{1}{50}\left(\frac{1}{\Gamma(\frac{7}{2})} + \frac{3}{(2!)\Gamma(\frac{3}{2})}\right) = \frac{1}{50}\left(\frac{4}{15\sqrt{\pi}} + \frac{3}{\sqrt{\pi}}\right) = \frac{49}{750\sqrt{\pi}} = 0.036860 < 1.$$

Hence (7.1) is also satisfied. Then Theorem 6.2 guarantees that the considered system (7.2) has a unique solution on $\mathbb{I}$ for $\alpha \in [2,3)$. 3

8 Conclusion

A metric sort distance, convergence structures, and some more topological inquiries in multiplicative partial cone metric spaces over Banach algebras were contemplated. We set up the idea for defining these generalizations of metric locations by supplanting the arrangement of a Banach space by an arranged multiplicative Banach algebra. Additionally, the multiplicative partial cone metric is valued in an ordering multiplicative algebra cone in the Banach algebra having a nonempty interior. We stretched out Banach's contraction mapping theory for fixed points of contractions to such spaces and built up synthesis of fixed-point results for contractive sort mappings fulfilling a contractivity kind condition including a new integration structure in terms of these spaces by embedding the presumption of normality of the multiplicative algebra cone. A supporting nontrivial counterexample to the main theorem and some illustrative examples are additionally given. One of our results is an exceptionally significant instrument in solving a certain problem in fractional calculus pursued by a numerical example.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
NF found initial conceptualizations, originated the idea of this research, supervised the results and investigated applications and examples. SM found initial conceptualizations, originated the idea of this research, supervised the result, investigated applications and examples and wrote reviews and editing HA analyzed the ideas,supervised the results, investigated applications and examples and wrote reviews and editing. SS analyzed the ideas, registered and wrote the results, provide references of previous studies, investigated applications and examples to support the main results and wrote reviews and editing. All authors read and approved the final manuscript.

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References

1. Branciari, A.: A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Sci. **29**, 531–536 (2002). https://doi.org/10.1155/S0161171202007524
2. Rhoades, B.E.: Two fixed-point theorems for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. **63**, 4007–4013 (2003). https://doi.org/10.1155/S0161171203208024
3. Aliouche, A.: A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type. J. Math. Anal. Appl. **322**, 796–802 (2006)
4. Djoudi, A., Aliouche, A.: Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type. J. Math. Anal. Appl. **329**, 31–45 (2007)
5. Vijayaraju, P., Rhoades, B.E., Mohanraj, R.: A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type. Int. J. Math. Sci. (2005)
6. Khojasteh, F., Goodarzi, Z., Razani, A.: Some fixed point theorems of integral type contraction in cone metric spaces. J. Fixed Point Theory Appl. (2010). https://doi.org/10.1155/2010/189684
7. Premila Devi, L., Singh, L.S.: Fixed point results of integral type on complete S_b-metric space using C-class functions. Int. J. Adv. Math. **2018**, 1–15 (2018)
8. Ansari, A.H., Dhamodharan, D., Yumnam, R., Krishnakumar, R.: C-class function on new contractive conditions of integral type on complete S-metric spaces. J. Glob. Res. Math. Arch. **5**, 46–63 (2018)
9. Ali, A., Uddin, F., Arshad, M., Rashid, M.: Hybrid fixed point results via generalized dynamic process for $F-HRS$ type contractions with application. Phys. A, Stat. Mech. Appl. **538**, Article ID 122669 (2020). https://doi.org/10.1016/j.physa.2019.122669
10. Mohamed Ali, S., Bakr, A.: A study on common fixed point of joint $(A;B)$ generalized cyclic $\phi-abc$ weak nonexpansive mappings and generalized cyclic $abc;r$ contractions in quasi metric spaces. Abstr. Appl. Anal. (2020). https://doi.org/10.1155/2020/9427267
11. Mohamed Ali, S., Bakr, A.: Cyclic $G-\Omega$-weak contraction-weak nonexpansive mappings and some fixed point theorems in metric spaces. Abstr. Appl. Anal. (2021). https://doi.org/10.1155/2021/6642564
12. Mohamed Ali, S., Bakr, A.: Theta cone metric spaces and some fixed point theorems. J. Math. (2020). https://doi.org/10.1155/2020/8895568
13. Mohamed Ali, S., Bakr, A.: Coupled fixed-point theorems in theta-cone-metric spaces. J. Math. (2021). https://doi.org/10.1155/2021/6617738
14. Mohamed Ali, S., Bakr, A.: Coupled fixed point theorems for some type of contraction mappings in b-cone and b-theta cone metric spaces. J. Math. (2021). https://doi.org/10.1155/2021/5569674

15. Ali, A., Arshad, M., Hussain, A., Hussain, N., Alsulami, S.M.: On new generalized θ_b contractions and related fixed point theorems. J. Inequal. Appl. **2022**, 37 (2022). https://doi.org/10.1186/s13660-022-02770-8
16. Ali, A., Hussain, A., Arshad, M., Sulami, H.A., Tariq, M.: Certain new development to the orthogonal binary relations. Symmetry **14**, Article ID 1954 (2022). https://doi.org/10.3390/sym14101954
17. Ali, A., Arshad, M., Emeer, E., Aydi, H., Mukheimer, A., Odayeh, A.: Certain dynamic iterative scheme families and multi-valued fixed point results. AIMS Math. **7**(7), 12177–12202 (2022)
18. Mohamed Ali Abou Bakr, S., Hussein, H.K.: Some results on coupled coincidence points in vector quasi cone metric spaces over Banach algebras with satisfactory cones. J. Math. Comput. Sci. **31**(3), 305–317 (2023). https://doi.org/10.22436/jmcs.031.03.06
19. Chugh, R., Nandal, A., Kumar, N.: Multiplicative normed linear space and its topological properties. Indian J. Sci. Technol. **10**, 1–9 (2017). https://doi.org/10.17485/ijst/2017/v10i10/74294
20. Rudin, W.: Functional Analysis, 2nd edn. McGraw-Hill, New York (1991)
21. Huang, L.G., Zhang, X.: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**, 1468–1476 (2007)
22. Sonmez, A.: Fixed point theorems in partial cone metric spaces (2011). arXiv:1101.2741
23. Sonmez, A.: On partial cone metric space (2012). arXiv:1207.6766
24. Benchohra, M., Hamani, S., Ntouyas, S.K.: Boundary value problems for differential equations with fractional order. Surv. Math. Appl. **3**, 1–12 (2008)
25. Agarwal, R.P., Benchohra, M., Hamani, S.: Boundary value problems for fractional differential equations. Georgian Math. J. **16**, 401–411 (2009). https://doi.org/10.1515/GMJ.2009.401
26. Yufeng, S., Zheng, Z., Jie, S.: Existence and uniqueness for the boundary value problems of nonlinear fractional differential equation. Appl. Math. **8**, 312–323 (2017). https://doi.org/10.4236/am.2017.83026

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