Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

A new type of Szász–Mirakjan operators based on q-integers

Pembe Sabancigil[1*], Nazim Mahmudov[1,2] and Gizem Dagbasi[1]

*Correspondence:
pembe.sabancigil@emu.edu.tr
[1] Department of Mathematics, Eastern Mediterranean University, Famagusta, T. R. Northern Cyprus Mersin 10, Turkey
Full list of author information is available at the end of the article

Abstract

In this article, by using the notion of quantum calculus, we define a new type Szász–Mirakjan operators based on the q-integers. We derive a recurrence formula and calculate the moments $\Phi_{n,q}(t^m;x)$ for $m = 0, 1, 2$ and the central moments $\Phi_{n,q}((t-x)^m;x)$ for $m = 1, 2$. We give estimation for the first and second-order central moments. We present a Korovkin type approximation theorem and give a local approximation theorem by using modulus of continuity. We obtain a local direct estimate for the new Szász–Mirakjan operators in terms of Lipschitz-type maximal function of order α. Finally, we prove a Korovkin type weighted approximation theorem.

Keywords: q-calculus; q-Bernstein-polynomials; q-Szász-operators; Moments; Modulus of continuity

1 Introduction

Approximation theory is one of the most important research areas in mathematics, which appeared in the nineteenth century. Since then, it has been studied by many mathematicians all over the world. The main goal of this theory is to produce a representation of any given function using other functions that have a simpler structure and more elementary properties such as differentiability and integrability. Positive linear operators have an important place in approximation theory and the theory of these operators has been an important area of research in the last three decades. Bernstein polynomials are the most popular and have been used to approximate functions in many areas of mathematics and also in some other fields. The first generalization of Bernstein polynomials using the concept of q-integers is introduced by A. Lupaş [13] in 1987. Later, in 1996, a different generalization of Bernstein polynomials using q-integers, is introduced by G.M. Phillips [16]. Until today, there are many generalizations of some positive linear operators based on q-integers. It is proved by A. Lupas [13] and G.M. Phillips [16] that the rate of convergence of q-generalizations of these operators are better than the classical ones.

Szász–Mirakjan operator [18] defined by O. Szász in 1950 is as follows:

For $f \in C[0,\infty)$

$$S_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty), n = 1, 2, \dots,$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$.

The moments of the Szász–Mirakjan operator can be found in [11].

The operators $S_n(f;x)$ defined by O. Szász generalized the Bernstein polynomials to the infinite interval $[0,\infty)$ and they have an important place among all the operators that can be used to approximate functions on the unbounded intervals. Szász–Mirakjan operators have a simple structure and they have been widely examined in the recent years. Many authors from this area introduced and discussed different modifications of classical Szász–Mirakjan operators and also Szász–Mirakjan operators based on the q-integers (see [2, 3, 5, 7, 9, 14, 15, 17]).

For $0 < q < 1$, the q-Szász–Mirakjan operators defined by A. Aral are as follows (see [2]):

$$S_n^q(f;x) = E_q\left(-[n]_q\frac{x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right)\frac{([n]_q x)^k}{[k]_q!(b_n)^k},$$

where $0 \leq x < \alpha_q(n)$, $\alpha_q(n) = \frac{b_n}{(1-q)[n]_q}$, $f \in C(\mathbb{R}_0)$ and b_n is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$. The operators S_n^q are positive and linear and reduce to the classical Szász–Mirakjan operators in the case $q = 1$.

On the other hand, q-parametric Szász–Mirakjan operator defined by N.I. Mahmudov is as follows (see [14]):

For $n \in \mathbb{N}$, $0 < q < 1$ and $f : [0,\infty) \longrightarrow \mathbb{R}$

$$S_{n,q}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-2}[n]_q}\right)s_{n,k}(q,x),$$

where

$$s_{n,k}(q,x) = q^{\frac{k(k-1)}{2}}\frac{[n]_q^k x^k}{[k]_q!}\frac{1}{E_q([n]_q x)}.$$

Like the classical Szász–Mirakjan operator S_n, Mahmudov's operator $S_{n,q}$ is also positive and linear.

In this paper, motivated by the studies mentioned above, we define a new generalization of the Szász–Mirakjan operators based on the q-integers.

The paper is organized as follows. In Sect. 2, we define new type Szász–Mirakjan operators based on the q-integers, $\Phi_{n,q}(f;x)$. We derive a recurrence formula and use this recurrence formula to calculate the moments $\Phi_{n,q}(t^m;x)$ for $m = 0,1,2$ and the central moments $\Phi_{n,q}((t-x)^m;x)$ for $m = 1,2$. We also present an estimation for the first and the second order central moments. In Sect. 3, we give a Korovkin-type approximation theorem and an estimation of the rate of convergence by using modulus of continuity. In Sect. 4, we present a local approximation theorem by using first and second order modulus of continuity and obtain a local direct estimate for the new Szász–Mirakjan operators in terms of Lipschitz-type maximal function of order α. In Sect. 5, we prove a Korovkin-type weighted approximation theorem.

2 Operators and estimation of their moments

Basic concepts and notations of the q-calculus and applications of q-calculus in operator theory can be found in [12] and [4].

Let $B_m[0,\infty) = \{f : |f(x)| \leq M_f(1 + x^m),\ x \in [0,\infty),\ m > 0$ and M_f is a constant depending on $f\}$,

$$C_m[0,\infty) = \left\{f \in B_m[0,\infty) \cap C[0,\infty) : \|f\|_m := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^m} < \infty\right\},$$
$$C_m^*[0,\infty) = \left\{f \in C_m[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty\right\}.$$

The spaces mentioned above are equipped with the norm

$$\|f\|_m = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^m}.$$

We introduce new type Szász–Mirakjan operators based on the q-integers as follows:

Definition 1 Let $0 < q < 1$ and $n \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$, a new type of the Szász–Mirakjan operators based on the q-integers is defined as follows:

$$\Phi_{n,q}(f;x) = \frac{1}{\varepsilon^{[n]_q x}} \sum_{k=0}^{\infty} (1+1)_q^k \left(\frac{[n]_q x}{2}\right)^k \frac{1}{[k]_q!} f\left(\frac{[k]_q}{[n]_q}\right), \tag{1}$$

where $(1+1)_q^k = \prod_{j=0}^{k-1}(1+q^j)$ and $\varepsilon^{[n]_q x} = \sum_{k=0}^{\infty}(1+1)_q^k (\frac{[n]_q x}{2})^k \frac{1}{[k]_q!}$.

Note that if we take $q = 1$, the operators $\Phi_{n,q}(f;x)$ reduce to the classical Szasz–Mirakjan operators $S_n(f;x)$.

Moments and central moments play an important role in approximation theory. In the following lemma we derive a recurrence formula for $\Phi_{n,q}(t^{m+1};x)$ which will be used to calculate moments $\Phi_{n,q}(t^m;x)$ for $m = 0,1,2$ and the central moments $\Phi_{n,q}((t-x)^m;x)$ for $m = 1,2$.

Lemma 2 *Let $0 < q < 1$, $m \in \mathbb{Z}^+ \cup \{0\}$ and $n \in \mathbb{N}$. For the operators $\Phi_{n,q}(f;x)$, we have*

$$\Phi_{n,q}\left(t^{m+1};x\right) = \frac{x}{2} \sum_{j=0}^{m} \binom{m}{j} \frac{q^j}{[n]_q^{m-j}} \left\{\Phi_{n,q}\left(t^j;x\right) + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} \Phi_{n,q}\left(t^j;qx\right)\right\}. \tag{2}$$

Proof By using the definition of the operators $\Phi_{n,q}(f;x)$, we have

$$\begin{aligned}
\Phi_{n,q}\left(t^{m+1};x\right) &= \frac{1}{\varepsilon^{[n]_q x}} \sum_{k=0}^{\infty} (1+1)_q^k \left(\frac{[n]_q x}{2}\right)^k \frac{1}{[k]_q!} \frac{[k]_q^{m+1}}{[n]_q^{m+1}} \\
&= \frac{1}{\varepsilon^{[n]_q x}} \sum_{k=0}^{\infty} (1+1)_q^{k+1} \left(\frac{[n]_q x}{2}\right)^{k+1} \frac{1}{[k+1]_q!} \frac{[k+1]_q^{m+1}}{[n]_q^{m+1}} \\
&= \frac{1}{\varepsilon^{[n]_q x}} \sum_{k=0}^{\infty} (1+1)_q^{k+1} \left(\frac{[n]_q x}{2}\right)^{k+1} \frac{1}{[k]_q!} \frac{[k+1]_q^{m}}{[n]_q^{m+1}}.
\end{aligned}$$

With the help of the binomial formula, we can write $[k+1]_q^m = (1+q[k]_q)^m = \sum_{j=0}^{m}\binom{m}{j} \times (q[k]_q)^j$. Thus

$$\begin{aligned}\Phi_{n,q}\left(t^{m+1};x\right) &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{k=0}^{\infty}(1+1)_q^{k+1}\left(\frac{[n]_q x}{2}\right)^{k+1}\frac{1}{[k]_q!}\frac{1}{[n]_q^{m+1}}\sum_{j=0}^{m}\binom{m}{j}\left(q[k]_q\right)^j \\ &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{\infty}(1+1)_q^{k+1}\left(\frac{[n]_q x}{2}\right)^{k+1}\frac{1}{[k]_q!}\frac{1}{[n]_q^{m+1}}\left(q[k]_q\right)^j,\end{aligned}$$

now by using the identity $(1+1)_q^{k+1} = (1+1)_q^k(1+q^k)$, we get

$$\begin{aligned}\Phi_{n,q}\left(t^{m+1};x\right) &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{\infty}(1+1)_q^k\left(1+q^k\right)\left(\frac{[n]_q x}{2}\right)^{k+1}\frac{1}{[k]_q!}\frac{1}{[n]_q^{m+1}}\left(q[k]_q\right)^j \\ &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{\infty}(1+1)_q^k\left(\frac{[n]_q x}{2}\right)^{k+1}\frac{1}{[k]_q!}\frac{1}{[n]_q^{m+1}}\left(q[k]_q\right)^j \\ &\quad + \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{\infty}q^k(1+1)_q^k\left(\frac{[n]_q x}{2}\right)^{k+1}\frac{1}{[k]_q!}\frac{1}{[n]_q^{m+1}}\left(q[k]_q\right)^j \\ &= S_1 + S_2,\end{aligned}$$

where

$$\begin{aligned}S_1 &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{\infty}(1+1)_q^k\left(\frac{[n]_q x}{2}\right)^{k+1}\frac{1}{[k]_q!}\frac{1}{[n]_q^{m+1}}\left(q[k]_q\right)^j \\ &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}q^j\frac{x}{2}\frac{1}{[n]_q^{m-j}}\binom{m}{j}\sum_{k=0}^{\infty}(1+1)_q^k\left(\frac{[n]_q x}{2}\right)^{k}\frac{1}{[k]_q!}\frac{[k]_q^j}{[n]_q^j} \\ &= \sum_{j=0}^{m}q^j\frac{x}{2}\frac{1}{[n]_q^{m-j}}\binom{m}{j}\Phi_{n,q}\left(t^j;x\right)\end{aligned}$$

and

$$\begin{aligned}S_2 &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{\infty}q^k(1+1)_q^k\left(\frac{[n]_q x}{2}\right)^{k+1}\frac{1}{[k]_q!}\frac{1}{[n]_q^{m+1}}\left(q[k]_q\right)^j \\ &= \frac{1}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}q^j\frac{x}{2}\frac{1}{[n]_q^{m-j}}\binom{m}{j}\sum_{k=0}^{\infty}(1+1)_q^k\left(\frac{[n]_q qx}{2}\right)^{k}\frac{1}{[k]_q!}\frac{[k]_q^j}{[n]_q^j} \\ &= \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}q^j\frac{x}{2}\frac{1}{[n]_q^{m-j}}\binom{m}{j}\frac{1}{\varepsilon^{[n]_q qx}}\sum_{k=0}^{\infty}(1+1)_q^k\left(\frac{[n]_q qx}{2}\right)^{k}\frac{1}{[k]_q!}\frac{[k]_q^j}{[n]_q^j} \\ &= \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\sum_{j=0}^{m}q^j\frac{x}{2}\frac{1}{[n]_q^{m-j}}\binom{m}{j}\Phi_{n,q}\left(t^j;qx\right).\end{aligned}$$

Now combining S_1 and S_2, we obtain

$$\Phi_{n,q}\left(t^{m+1};x\right)=\frac{x}{2}\sum_{j=0}^{m}q^j\frac{1}{[n]_q^{m-j}}\binom{m}{j}\left\{\Phi_{n,q}\left(t^j;x\right)+\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}\Phi_{n,q}\left(t^j;qx\right)\right\}.$$ □

From Lemma 2, by using the recurrence formula (2), we obtain explicit formulas for the moments $\Phi_{n,q}(t^j;x)$ for $j=0,1,2$.

Remark 3 Let $0<q<1$ and $n\in\mathbb{N}$. We have

$$\Phi_{n,q}(1;x)=1,$$

$$\Phi_{n,q}(t;x)=\frac{x}{2}\left\{1+\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}\right\},$$

$$\Phi_{n,q}\left(t^2;x\right)=\frac{x^2}{4}\left(q+q\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}+q^2\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}+q^2\frac{\varepsilon^{[n]_qq^2x}}{\varepsilon^{[n]_qx}}\right)+\frac{x}{2[n]_q}\left(1+\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}\right).$$

Now by using the linearity property of the operators $\Phi_{n,q}(f;x)$ and Remark 3, we obtain central moments $\Phi_{n,q}((t-x)^j;x)$ for $j=1,2$.

Lemma 4 *Let $0<q<1$ and $n\in\mathbb{N}$. For every $x\in[0,\infty)$, we have the following equalities*:

$$\Phi_{n,q}\left((t-x);x\right)=\frac{x}{2}\left(\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}-1\right)\quad \textit{and} \tag{3}$$

$$\begin{aligned}&\Phi_{n,q}\left((t-x)^2;x\right)\\&\quad=\frac{x^2}{4}\left(q+q\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}+q^2\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}+q^2\frac{\varepsilon^{[n]_qq^2x}}{\varepsilon^{[n]_qx}}-4\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}\right)+\frac{x}{2[n]_q}\left(1+\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}\right).\end{aligned} \tag{4}$$

In the following lemma, we give estimations for the first- and second-order central moments.

Lemma 5

$$(i)\quad \left|\Phi_{n,q}\left((t-x);x\right)\right|\le\frac{x^2}{2}\left(1-q^n\right),$$

$$(ii)\quad \Phi_{n,q}\left((t-x)^2;x\right)\le\frac{x^2}{4}\left(2(1-q)(2+q)+2x\left(1-q^n\right)\right)+\frac{x}{[n]_q}. \tag{5}$$

Proof (i) From the previous lemma, we know that $|\Phi_{n,q}((t-x);x)|=|\frac{x}{2}(\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}-1)|$. We start by finding an estimation for $|\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}-1|$.

$$\begin{aligned}\frac{\varepsilon^{[n]_qqx}}{\varepsilon^{[n]_qx}}-1&=\frac{1}{\varepsilon^{[n]_qx}}\left(\varepsilon^{[n]_qqx}-\varepsilon^{[n]_qx}\right)\\&=\frac{1}{\varepsilon^{[n]_qx}}\sum_{k=0}^{\infty}(1+1)_q^k\left(\frac{[n]_qx}{2}\right)^k\frac{1}{[k]_q!}\left(q^k-1\right)\\&=(q-1)\frac{1}{\varepsilon^{[n]_qx}}\sum_{k=0}^{\infty}(1+1)_q^{k+1}\left(\frac{[n]_qx}{2}\right)^{k+1}\frac{1}{[k]_q!}\end{aligned}$$

$$
\begin{aligned}
&= \frac{[n]_q x}{2}(q-1)\frac{1}{\varepsilon^{[n]_q x}}\left(\varepsilon^{[n]_q x} + \varepsilon^{[n]_q qx}\right) = \frac{x}{2}\left(q^n - 1\right)\frac{1}{\varepsilon^{[n]_q x}}\left(\varepsilon^{[n]_q x} + \varepsilon^{[n]_q qx}\right) \\
&= \frac{x}{2}\left(q^n - 1\right)\left(1 + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right). \qquad (6)
\end{aligned}
$$

Now we get

$$
\left|\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} - 1\right| = \left|\frac{x}{2}\left(q^n - 1\right)\left(1 + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right)\right|,
$$

which implies

$$
\begin{aligned}
1 - \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} &= \frac{x}{2}\left(1 - q^n\right)\left(1 + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right). \qquad (7) \\
&\leq x\left(1 - q^n\right). \qquad (8)
\end{aligned}
$$

Thus

$$
\left|\Phi_{n,q}\left((t-x);x\right)\right| \leq \frac{x^2}{2}\left(1 - q^n\right).
$$

(ii) By using Lemma 4, we write

$$
\begin{aligned}
&\Phi_{n,q}\left((t-x)^2;x\right) \\
&\quad = \left|\Phi_{n,q}\left((t-x)^2;x\right)\right| \\
&\quad = \left|\frac{x^2}{4}\left(q + q\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} + q^2\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} + q^2\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} - 4\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right) + \frac{x}{2[n]_q}\left(1 + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right)\right| \\
&\quad \leq \frac{x^2}{4}\left|q + q\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} + q^2\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} + q^2\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} - 4\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right| + \frac{x}{2[n]_q}\left(1 + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right).
\end{aligned}
$$

We start with the estimation of the term in modulus.

$$
\begin{aligned}
&\left|q + q\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} + q^2\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} + q^2\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} - 4\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right| \\
&\quad = \left|(q-1) + \left(1 - \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right) + \left(q + q^2 - 2\right)\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right. \qquad (9) \\
&\qquad \left. + \left(q^2 - 1\right)\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} + \left(\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} - \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right)\right| \\
&\quad = |I_1 + I_2 + I_3 + I_4 + I_5| \\
&\quad \leq |I_1| + |I_2| + |I_3| + |I_4| + |I_5|,
\end{aligned}
$$

where

$$
\begin{aligned}
&|I_1| = |q-1| = 1 - q, \\
&|I_2| = \left|1 - \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right| = 1 - \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} \leq x\left(1 - q^n\right) \quad \text{from part } (i). \qquad (10)
\end{aligned}
$$

For $|I_3|$ and $|I_4|$, since $\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} \leq 1$ and $\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} \leq 1$, we can write

$$|I_3| = \left|(q + q^2 - 2)\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right| = (2 - q - q^2)\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} \leq 2 - q - q^2,$$
$$|I_4| = \left|(q^2 - 1)\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}}\right| = (1 - q^2)\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} \leq 1 - q^2.$$

For $|I_5|$, we write

$$\begin{aligned}|I_5| &= \left|\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q x}} - \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right| \\ &= \left|\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\left(\frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q qx}} - 1\right)\right| \leq 1 - \frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q qx}}.\end{aligned}$$

Now from the Equation (7), we know that

$$1 - \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} = \frac{x}{2}(1 - q^n)\left(1 + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right),$$

which gives us the following explicit formulas

$$\begin{aligned}\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} &= \frac{2 + x(q^n - 1)}{2 - x(q^n - 1)}, \\ \frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q qx}} &= \frac{2 + qx(q^n - 1)}{2 - qx(q^n - 1)},\end{aligned} \tag{11}$$

thus

$$\begin{aligned}|I_5| &\leq 1 - \frac{\varepsilon^{[n]_q q^2 x}}{\varepsilon^{[n]_q qx}} \\ &= 1 - \frac{2 + qx(q^n - 1)}{2 - qx(q^n - 1)} = \frac{2qx(1 - q^n)}{2 + qx(1 - q^n)} \leq x(1 - q^n).\end{aligned} \tag{12}$$

For the estimation of the second term, we use again the fact that $\frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}} \leq 1$, so

$$\frac{x}{2[n]_q}\left(1 + \frac{\varepsilon^{[n]_q qx}}{\varepsilon^{[n]_q x}}\right) \leq \frac{x}{[n]_q},$$

and we conclude that

$$\begin{aligned}&\Phi_{n,q}\left((t - x)^2; x\right) \\ &\quad\leq \frac{x^2}{4}\left((1 - q) + x(1 - q^n) + (2 - q - q^2) + (1 - q^2) + x(1 - q^n)\right) + \frac{x}{[n]_q} \\ &\quad= \frac{x^2}{4}\left(2(1 - q)(2 + q) + 2x(1 - q^n)\right) + \frac{x}{[n]_q}.\end{aligned}$$

□

To show that the first- and the second-order central moments approach zero under some conditions, we need to prove the following limits.

Lemma 6 *Assume that* $q = q_n \in (0,1)$, $q_n \to 1$ *and* $q_n^n \to b$ *as* $n \to \infty$. *Then we have*

$$(a)\quad \lim_{n\to\infty} \frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} = \frac{2+x(b-1)}{2-x(b-1)},$$

$$(b)\quad \lim_{n\to\infty}\left(\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}x}} - \frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\right) = \frac{(2+x(b-1))2x(b-1)}{(2-x(b-1))^2},$$

$$(c)\quad \lim_{n\to\infty}\left(q_n + q_n\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} + q_n^2\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} + q_n^2\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}x}} - 4\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\right) = \frac{4x^2(b-1)^2}{(2-x(b-1))^2}.$$

Proof (a) From Equation (11), we know that

$$\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} = \frac{2+x(q_n^n-1)}{2-x(q_n^n-1)}, \tag{13}$$

since $q_n \to 1$ and $q_n^n \to b$ as $n \to \infty$, we get

$$\lim_{n\to\infty}\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} = \lim_{n\to\infty}\frac{2+x(q_n^n-1)}{2-x(q_n^n-1)} = \frac{2+x(b-1)}{2-x(b-1)}.$$

(b) For the proof of this part, we write

$$\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}x}} - \frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} = \frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\left(\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}q_n x}} - 1\right),$$

and from part (a), we can easily see that

$$\lim_{n\to\infty}\left(\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}q_n x}} - 1\right) = \frac{2x(b-1)}{2-x(b-1)}.$$

Thus we get

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}x}} - \frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\right)\\ &\quad= \lim_{n\to\infty}\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\left(\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}q_n x}} - 1\right) = \frac{(2+x(b-1))2x(b-1)}{(2-x(b-1))^2}.\end{aligned}$$

(c) From the Equation (9), we know that

$$\begin{aligned}&q_n + q_n\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} + q_n^2\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}} + q_n^2\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}x}} - 4\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\\ &\quad= (q_n-1) + \left(1 - \frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\right) + \left(q_n + q_n^2 - 2\right)\frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\\ &\qquad+ \left(q_n^2-1\right)\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}x}} + \left(\frac{\varepsilon^{[n]_{q_n}q_n^2 x}}{\varepsilon^{[n]_{q_n}x}} - \frac{\varepsilon^{[n]_{q_n}q_n x}}{\varepsilon^{[n]_{q_n}x}}\right)\\ &\quad= I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n} + I_{5,n},\end{aligned}$$

thus

$$\begin{aligned}&\lim_{n\to\infty}\left(q_n+q_n\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_n^2x}}{\varepsilon^{[n]_{q_n}x}}-4\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right)\\&\quad=\lim_{n\to\infty}(I_{1,n}+I_{2,n}+I_{3,n}+I_{4,n}+I_{5,n}).\end{aligned}$$

Since $q_n\to 1$ as $n\to\infty$, we have

$$\lim_{n\to\infty}I_{1,n}=\lim_{n\to\infty}I_{3,n}=\lim_{n\to\infty}I_{4,n}=0.$$

Now from part (a)

$$\lim_{n\to\infty}I_{2,n}=1-\frac{2+x(b-1)}{2-x(b-1)}=-\frac{2x(b-1)}{2-x(b-1)}$$

and from part (b)

$$\lim_{n\to\infty}I_{5,n}=\lim_{n\to\infty}\left(\frac{\varepsilon^{[n]_{q_n}q_n^2x}}{\varepsilon^{[n]_{q_n}x}}-\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right)=\frac{(2+x(b-1))2x(b-1)}{(2-x(b-1))^2}.$$

Thus we get

$$\lim_{n\to\infty}\left(q_n+q_n\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_n^2x}}{\varepsilon^{[n]_{q_n}x}}-4\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right)=\frac{4x^2(b-1)^2}{(2-x(b-1))^2}.\qquad\square$$

Corollary 7 *Assume that* $q=q_n\in(0,1)$, $q_n\to 1$ *and* $q_n^n\to 1$ *as* $n\to\infty$. *Then we have*

$$(i)\qquad \lim_{n\to\infty}\Phi_{n,q_n}\big((t-x);x\big)=0$$

and

$$(ii)\qquad \lim_{n\to\infty}\Phi_{n,q_n}\big((t-x)^2;x\big)=0.$$

Proof (i) From Equation (3) and Lemma 6, it is clear that, if $q_n\to 1$ and $q_n^n\to 1$ as $n\to\infty$, then

$$\lim_{n\to\infty}\Phi_{n,q_n}\big((t-x);x\big)=\lim_{n\to\infty}\frac{x}{2}\left(\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}-1\right)=0.$$

(ii) From Equation (4), we write

$$\begin{aligned}&\lim_{n\to\infty}\Phi_{n,q_n}\big((t-x)^2;x\big)\\&\quad=\lim_{n\to\infty}\frac{x^2}{4}\left(q_n+q_n\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_n^2x}}{\varepsilon^{[n]_{q_n}x}}-4\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right)\\&\qquad+\lim_{n\to\infty}\frac{x}{2[n]_{q_n}}\left(1+\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right).\end{aligned}$$

Now again from Lemma 6, it is clear that, if $q_n \to 1$ and $q_n^n \to 1$ as $n \to \infty$, then

$$\lim_{n\to\infty}\left(q_n + q_n \frac{\varepsilon^{[n]_{q_n} q_n x}}{\varepsilon^{[n]_{q_n} x}} + q_n^2 \frac{\varepsilon^{[n]_{q_n} q_n x}}{\varepsilon^{[n]_{q_n} x}} + q_n^2 \frac{\varepsilon^{[n]_{q_n} q_n^2 x}}{\varepsilon^{[n]_{q_n} x}} - 4\frac{\varepsilon^{[n]_{q_n} q_n x}}{\varepsilon^{[n]_{q_n} x}}\right) = 0.$$

Let us show that also the second term approaches zero. If $q_n \to 1$, then for any fixed positive integer m, we have $[n]_{q_n} \geq [m]_{q_n}$ when $n \geq m$. Therefore, $\liminf_{n\to\infty}[n]_{q_n} \geq \lim_{n\to\infty}[m]_{q_n} = m$. Since m has been chosen arbitrarily, it follows that $[n]_{q_n} \to \infty$. Hence, $\frac{1}{[n]_{q_n}} \to 0$. Thus we get $\lim_{n\to\infty} \Phi_{n,q_n}((t-x)^2;x) = 0$. □

3 Direct approximation results

In this section, we prove a Korovkin-type approximation theorem and give a rate of convergence for the operators $\Phi_{n,q}(f;x)$.

Theorem 8 *Let q_n be a sequence such that $q_n \in (0,1)$. For each $f \in C_2^*[0,\infty)$, $\Phi_{n,q_n}(f;x)$ converges to f uniformly on $[0,D]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof Suppose that $\lim_{n\to\infty} q_n = 1$ and $D > 0$ is fixed. Consider the lattice homomorphism $T_D : C[0,\infty) \to C[0,D]$ defined by

$$T_D(f) := f_{|[0,D]}.$$

We can obviously see that

$$T_D\big(\Phi_{n,q_n}(1)\big) = T_D(1), \qquad T_D\big(\Phi_{n,q_n}(t)\big) \to T_D(t) \quad \text{and} \quad T_D\big(\Phi_{n,q_n}(t^2)\big) \to T_D(t^2)$$

uniformly on $[0,D]$. From the proposition 4.2.5, (6) of [1], we can say that $C_2^*[0,\infty)$ is isomorphic to $C[0,1]$ and the set $\{1,t,t^2\}$ is a Korovkin set in $C_2^*[0,\infty)$. So the universal Korovkin-type property (property (vi) of Thm. 4.1.4 in [1]) implies that

$$\Phi_{n,q_n}(f;x) \to f(x) \quad \text{uniformly on } [0,D] \text{ as } n \to \infty$$

provided $f \in C_2^*[0,\infty)$ and $D > 0$.

For the converse result, we use contradiction method. Assume that $\lim_{n\to\infty} q_n \neq 1$. Then it must have a subsequence $q_{n_k} \in (0,1)$ such that $q_{n_k} \to \beta \in [0,1)$ as $k \to \infty$.

Thus from Equation (13) and the fact that $\lim_{k\to\infty}(q_{n_k})^{n_k} = 0$,

$$\frac{\varepsilon^{[n_k] q_{n_k} x}}{\varepsilon^{[n_k] x}} \to \frac{2-x}{2+x} \quad \text{as } k \to \infty \text{ and } \frac{2-x}{2+x} \neq 1 \text{ for } x \in (0,\infty)$$

and we get

$$\begin{aligned}\Phi_{n_k,q_{n_k}}(t;x) - x &= \frac{x}{2}\left\{1 + \frac{\varepsilon^{[n_k] q_{n_k} x}}{\varepsilon^{[n_k] x}}\right\} - x \\ &\nrightarrow 0.\end{aligned}$$

This leads to a contradiction. Thus $\lim_{n\to\infty} q_n = 1$ as $n \to \infty$. □

Theorem 9 *Let* $0 < q < 1, f \in C_2[0,\infty)$ *and* $\omega_{A+1}(f,\delta) = \sup\{|f(t)-f(x)| : |t-x| \leq \delta, x, t \in [0, A+1]\}$ *be the modulus of continuity of* f *on the closed interval* $[0, A+1]$, *where* $A > 0$. *Then we have*

$$\left\| \Phi_{n,q}(f;x) - f(x) \right\|_{C[0,A]} \leq 4M_f\left(1+A^2\right)\alpha_n(A) + 2\omega_{A+1}\left(f;\sqrt{\alpha_n(A)}\right), \tag{14}$$

where $\alpha_n(A) = \frac{A^2}{4}(2(1-q)(2+q) + 2A(1-q^n)) + \frac{A}{[n]_q}$.

Proof For $x \in [0, A]$ and $t \geq 0$, we have

$$\left|f(t) - f(x)\right| \leq 4M_f\left(1+A^2\right)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{A+1}(f;\delta)$$

(see Equation 3.3 in [10]).

By using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\Phi_{n,q}(f;x) - f(x)\right| \\
&\quad\leq \Phi_{n,q}\left(\left|f(t)-f(x)\right|;x\right) \\
&\quad\leq 4M_f\left(1+A^2\right)\Phi_{n,q}\left((t-x)^2;x\right) + \left(1 + \Phi_{n,q}\left(\frac{|t-x|}{\delta};x\right)\right)\omega_{A+1}(f;\delta) \\
&\quad\leq 4M_f\left(1+A^2\right)\Phi_{n,q}\left((t-x)^2;x\right) + \omega_{A+1}(f;\delta)\left(1 + \frac{1}{\delta}\left(\Phi_{n,q}\left((t-x)^2;x\right)\right)^{\frac{1}{2}}\right).
\end{aligned}$$

For $x \in [0, A]$, using Lemma 5,

$$\begin{aligned}
\Phi_{n,q}\left((t-x)^2;x\right) &\leq \frac{x^2}{4}\left(2(1-q)(2+q) + 2x\left(1-q^n\right)\right) + \frac{x}{[n]_q} \\
&\leq \frac{A^2}{4}\left(2(1-q)(2+q) + 2A\left(1-q^n\right)\right) + \frac{A}{[n]_q} \\
&= \alpha_n(A).
\end{aligned}$$

Thus we get

$$\left|\Phi_{n,q}(f;x) - f(x)\right| \leq 4M_f\left(1+A^2\right)\alpha_n(A) + \omega_{A+1}(f;\delta)\left(1 + \frac{1}{\delta}\left(\alpha_n(A)\right)^{\frac{1}{2}}\right).$$

Now, choosing $\delta = \sqrt{\alpha_n(A)}$, we obtain the desired result. □

4 Local approximation

In this section, we examine local approximation properties of the operators $\Phi_{n,q}(f;x)$ and we give a local direct estimate in terms of Lipschitz-type maximal function of order α. Let $C_B[0,\infty)$ denote the space of all bounded, real valued continuous functions on $[0,\infty)$. This space is equipped with the norm

$$\|f\| = \sup_{x\in[0,\infty)} \left|f(x)\right|.$$

On the other hand, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \{ \|f-g\| + \delta \|g''\| \}, \quad \delta \geq 0,$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By Theorem 2.4 in [6], there exists an absolute constant $L > 0$ such that

$$K_2(f;\delta) \leq L\omega_2(f;\sqrt{\delta}), \tag{15}$$

where $\omega_2(f;\delta)$ is the second-order modulus of smoothness defined as

$$\omega_2(f;\delta) = \sup_{0<\omega\leq\delta} \sup_{x\in[0,\infty)} \left| f(x+2\omega) - 2f(x+\omega) + f(x) \right|.$$

In the following theorem we give a local approximation for the operators $\Phi_{n,q}(f;x)$ in terms of the first modulus of continuity and the second modulus of smoothness.

Theorem 10 *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, there exists a constant $L > 0$ such that*

$$\left| \Phi_{n,q}(f;x) - f(x) \right| \leq L\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\beta_n(x)\right),$$

where

$$\begin{aligned}
\delta_n(x) &= \Phi_{n,q}\left((t-x)^2;x\right) + \left(\Phi_{n,q}\left((t-x);x\right)\right)^2 \\
&= \frac{x^2}{4}\left(q + q\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}} + q^2\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}} + q^2\frac{\varepsilon^{[n]q^2x}}{\varepsilon^{[n]x}} - 4\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}}\right) + \frac{x}{2[n]}\left(1 + \frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}}\right) \\
&\quad + \frac{x^2}{4}\left(\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}} - 1\right)^2
\end{aligned}$$

and

$$\beta_n(x) = \left| \Phi_{n,q}\left((t-x);x\right) \right| = \frac{x}{2}\left(1 - \frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}}\right).$$

Proof Let

$${}^*\Phi_{n,q}(f;x) = \Phi_{n,q}(f;x) + f(x) - f\left(\rho_n(x)\right),$$

where $f \in C_B[0,\infty]$, $\rho_n(x) = \Phi_{n,q}((t-x);x) + x = \frac{x}{2}(\frac{\varepsilon^{[n]qqx}}{\varepsilon^{[n]qx}} + 1)$. Note that ${}^*\Phi_{n,q}((t-x);x) = 0$. Using the Taylor's formula, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)\,ds, \quad g \in C_B^2[0,\infty).$$

Applying ${}^*\Phi_{n,q}$ to both sides of the above equation, we have

$${}^*\Phi_{n,q}(g;x) - g(x)$$

$$
\begin{aligned}
&= {}^{*}\Phi_{n,q}\big((t-x)g'(x);x\big) + {}^{*}\Phi_{n,q}\left(\int_x^t (t-s)g''(s)\,ds;x\right)\\
&= g'(x)^{*}\Phi_{n,q}\big((t-x);x\big) + \Phi_{n,q}\left(\int_x^t (t-s)g''(s)\,ds;x\right) - \int_x^{\rho_n(x)} \big(\rho_n(x)-s\big)g''(s)\,ds\\
&= \Phi_{n,q}\left(\int_x^t (t-s)g''(s)\,ds;x\right) - \int_x^{\rho_n(x)} \big(\rho_n(x)-s\big)g''(s)\,ds.
\end{aligned}
$$

On the other hand,

$$
\left|\int_x^t (t-s)g''(s)\,ds\right| \le \int_x^t (t-s)\big|g''(s)\big|\,ds \le \|g''\|\int_x^t (t-s)\,ds \le \|g''\|(t-s)^2
$$

and

$$
\left|\int_x^{\rho_n(x)} \big(\rho_n(x)-s\big)g''(s)\,ds\right| \le \|g''\|\big(\rho_n(x)-x\big)^2 = \|g''\|\big(\Phi_{n,q}(t-x;x)\big)^2,
$$

which implies

$$
\begin{aligned}
|{}^{*}\Phi_{n,q}(g;x)-g(x)| &\le \left|\Phi_{n,q}\left(\int_x^t (t-s)g''(s)\,ds;x\right)\right| + \left|\int_x^{\rho_n(x)} \big(\rho_n(x)-s\big)g''(s)\,ds\right|\\
&\le \|g''\|\big\{\Phi_{n,q}\big((t-x)^2;x\big) + \big(\Phi_{n,q}(t-x);x\big)^2\big\}\\
&= \|g''\|\delta_n(x).
\end{aligned}
\tag{16}
$$

We also have

$$
|{}^{*}\Phi_{n,q}(f;x)| \le \big|\Phi_{n,q}(f;x)\big| + \big|f(x)\big| + \big|f\big(\rho_n(x)\big)\big| \le \Phi_{n,q}\big(|f|;x\big) + 2\|f\| \le 3\|f\|.
$$

Using (16) and the uniform boundedness of ${}^{*}\Phi_{n,q}$, we get

$$
\begin{aligned}
&\big|\Phi_{n,q}(f;x)-f(x)\big|\\
&\quad\le \big|{}^{*}\Phi_{n,q}(f-g;x)\big| + \big|{}^{*}\Phi_{n,q}(g;x)-g(x)\big| + \big|f(x)-g(x)\big| + \big|f\big(\rho_n(x)\big)-f(x)\big|\\
&\quad\le 4\|f-g\| + \|g''\|\delta_n(x) + \omega\big(f,\beta_n(x)\big).
\end{aligned}
$$

If we take the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$, we obtain

$$
\big|\Phi_{n,q}(f;x)-f(x)\big| \le 4K_2\big(f;\delta_n(x)\big) + \omega\big(f,\beta_n(x)\big),
$$

which together with (15) gives the proof of the theorem □

Theorem 11 *Let* $\alpha \in (0,1]$ *and* A *be any subset of the interval* $[0,\infty)$. *Then, if* $f \in C_B[0,\infty)$ *is locally* $Lip(\alpha)$; *i.e., the condition*

$$
\big|f(y)-f(x)\big| \le L|y-x|^{\alpha}, \quad y \in A \text{ and } x \in [0,\infty)
\tag{17}
$$

holds, then, for each $x \in [0,\infty)$, *we have*

$$
\big|\Phi_{n,q}(f;x)-f(x)\big| \le L\big\{\lambda_{n,q}^{\frac{\alpha}{2}}(x) + 2\big(d(x,A)\big)^{\alpha}\big\},
$$

where

$$\lambda_{n,q}(x) = \frac{x^2}{4}\left(q + q\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}} + q^2\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}} + q^2\frac{\varepsilon^{[n]q^2x}}{\varepsilon^{[n]x}} - 4\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}}\right) + \frac{x}{2[n]}\left(1 + \frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}}\right),$$

L is a constant depending on α *and* f, $d(x,A)$ *is the distance between* x, *and* A *defined as*

$$d(x,A) = \inf\{|t-x| : t \in A\}.$$

Proof Assume that $\overline{A}$ is the closure of A in $[0,\infty)$. Then, there exists a point $x_0 \in \overline{A}$ such that $|x - x_0| = d(x,A)$. By using the triangle inequality

$$\left|f(t) - f(x)\right| \le \left|f(t) - f(x_0)\right| + \left|f(x) - f(x_0)\right|$$

and (17), we get

$$\begin{aligned}\left|\Phi_{n,q}(f;x) - f(x)\right| &\le \Phi_{n,q}\left(\left|f(t) - f(x_0)\right|;x\right) + \Phi_{n,q}\left(\left|f(x) - f(x_0)\right|;x\right)\\ &\le L\left\{\Phi_{n,q}\left(|t-x_0|^\alpha;x\right) + |x-x_0|^\alpha\right\}\\ &\le L\left\{\Phi_{n,q}\left(|t-x|^\alpha + |x-x_0|^\alpha;x\right) + |x-x_0|^\alpha\right\}\\ &\le L\left\{\Phi_{n,q}\left(|t-x|^\alpha;x\right) + 2|x-x_0|^\alpha\right\}.\end{aligned}$$

Now, taking $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$ in the Hölder inequality, we get

$$\begin{aligned}&\left|\Phi_{n,q}(f;x) - f(x)\right|\\ &\quad\le L\left\{\left[\Phi_{n,q}\left(|t-x|^{\alpha p};x\right)\right]^{\frac{1}{p}}\left[\Phi_{n,q}\left(1^q;x\right)\right]^{\frac{1}{q}} + 2\left(d(x,A)\right)^\alpha\right\}\\ &\quad= L\left\{\left[\Phi_{n,q}\left(|t-x|^2;x\right)\right]^{\frac{\alpha}{2}} + 2\left(d(x,A)\right)^\alpha\right\}\\ &\quad= L\left\{\left[\frac{x^2}{4}\left(q + q\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}} + q^2\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}} + q^2\frac{\varepsilon^{[n]q^2x}}{\varepsilon^{[n]x}} - 4\frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}}\right) + \frac{x}{2[n]}\left(1 + \frac{\varepsilon^{[n]qx}}{\varepsilon^{[n]x}}\right)\right]^{\frac{\alpha}{2}}\right.\\ &\qquad\left. + 2\left(d(x,A)\right)^\alpha\right\}\\ &\quad= L\left\{\lambda_{n,q}(x)^{\frac{\alpha}{2}} + 2\left(d(x,A)\right)^\alpha\right\}\end{aligned}$$

and the proof is completed. □

5 Weighted approximation

In this section, we study weighted approximation theorem for the operators $\Phi_{n,q}(f;x)$.

Theorem 12 *Let* $q = q_n \in (0,1)$, $q_n \to 1$ *and* $q_n^n \to 1$ *as* $n \to \infty$. *Then for each* $f \in C_3^*[0,\infty)$, *one has*

$$\lim_{n\to\infty}\left\|\Phi_{n,q_n}(f;x) - f(x)\right\|_3 = 0.$$

Proof For the proof of this theorem, we will use Korovkin-type theorem on weighted approximation ([8]) and Remark 3. Thus, it will be sufficient to verify the following condition for $m = 0, 1, 2$:

$$\lim_{n\to\infty}\left\|\Phi_{n,q_n}\left(t^m;x\right)-x^m\right\|_3=0.$$

Since $\Phi_{n,q_n}(1;x)=1$, it is obvious for $m=0$.

For $m=1$, we have

$$\begin{aligned}\lim_{n\to\infty}\left\|\Phi_{n,q_n}(t;x)-x\right\|_3&=\lim_{n\to\infty}\sup_{x\geq 0}\frac{|\Phi_{n,q_n}(t;x)-x|}{1+x^3}\\&=\lim_{n\to\infty}\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x}{2}+\frac{x}{2}\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}-x\right|\\&=\lim_{n\to\infty}\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x}{2}\left(\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}-1\right)\right|\\&=\lim_{n\to\infty}\sup_{x\geq 0}\frac{x}{2(1+x^3)}\left|\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}-1\right|\end{aligned}$$

now from Inequality (10), since $|\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}-1|\leq x(1-q_n^n)$, we get

$$\lim_{n\to\infty}\left\|\Phi_{n,q_n}(t;x)-x\right\|_3\leq\lim_{n\to\infty}\sup_{x\geq 0}\frac{x^2}{2(1+x^3)}\left(1-q_n^n\right)\leq\lim_{n\to\infty}\left(1-q_n^n\right)=0.$$

For $m=2$, we have

$$\begin{aligned}&\lim_{n\to\infty}\left\|\Phi_{n,q_n}\left(t^2;x\right)-x^2\right\|_3\\&\quad=\lim_{n\to\infty}\sup_{x\geq 0}\frac{|\Phi_{n,q_n}(t^2;x)-x^2|}{1+x^3}\\&\quad=\lim_{n\to\infty}\left\{\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}\left(q_n+q_n\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}+q_n^2\frac{\varepsilon^{[n]_{q_n}q_n^2x}}{\varepsilon^{[n]_{q_n}x}}\right)\right.\right.\\&\qquad\left.\left.+\frac{x}{2[n]_{q_n}}\left(1+\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right)-x^2\right|\right\}\\&\quad\leq\lim_{n\to\infty}\left\{\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}(q_n-1)\right|+\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}(q_n-1)\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right|\right.\\&\qquad+\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}\left(\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}-1\right)\right|\\&\qquad+\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}\left(q_n^2-1\right)\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right|+\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}\left(\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}-1\right)\right|\\&\qquad+\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}\left(q_n^2-1\right)\frac{\varepsilon^{[n]_{q_n}q_n^2x}}{\varepsilon^{[n]_{q_n}x}}\right|+\sup_{x\geq 0}\frac{1}{1+x^3}\left|\frac{x^2}{4}\left(\frac{\varepsilon^{[n]_{q_n}q_n^2x}}{\varepsilon^{[n]_{q_n}x}}-1\right)\right|\\&\qquad\left.+\sup_{x\geq 0}\frac{1}{1+x^3}\frac{x}{2[n]_{q_n}}\left(1+\frac{\varepsilon^{[n]_{q_n}q_nx}}{\varepsilon^{[n]_{q_n}x}}\right)\right\}.\end{aligned}$$

The terms with $(q_n - 1)$ and $(q_n^2 - 1)$ go to zero since $q_n \to 1$. Now from Inequalities (10) and (12), we have

$$\sup_{x\geq 0} \frac{1}{1+x^3} \left| \frac{x^2}{4} \left(\frac{\varepsilon^{[n]_{q_n} q_n x}}{\varepsilon^{[n]_{q_n} x}} - 1 \right) \right| \leq \sup_{x\geq 0} \frac{x^3}{4(1+x^3)} \left(1 - q_n^n\right) \leq 1 - q_n^n$$

and

$$\sup_{x\geq 0} \frac{1}{1+x^3} \left| \frac{x^2}{4} \left(\frac{\varepsilon^{[n]_{q_n} q_n^2 x}}{\varepsilon^{[n]_{q_n} x}} - 1 \right) \right| \leq \sup_{x\geq 0} \frac{x^3}{4(1+x^3)} \left(1 - q_n^n\right) \leq 1 - q_n^n,$$

thus

$$\lim_{n\to\infty} \left\| \Phi_{n,q_n}\left(t^2; x\right) - x^2 \right\|_3 = 0$$

and we conclude that

$$\lim_{n\to\infty} \left\| \Phi_{n,q_n}\left(t^m; x\right) - x^m \right\|_3 = 0, \quad m = 0, 1, 2. \qquad \square$$

Acknowledgements
The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their valuable comments and suggestions.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
PS and GD prepared the original draft. NM reviewed and edited the manuscript. All authors read and approved the original manuscript.

Author details
[1]Department of Mathematics, Eastern Mediterranean University, Famagusta, T. R. Northern Cyprus Mersin 10, Turkey. [2]Research Center of Econophysics, Azerbaijan State University of Economics (UNEC), Istiqlaliyyat Str. 6, Baku, 1001, Azerbaijan.

Received: 12 June 2023 Accepted: 20 October 2023 Published online: 31 October 2023

References

1. Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and Its Applications. De Gruyter Studies in Mathematics, vol. 17. Walter de Gruyter&Co., Berlin (1994)
2. Aral, A.: A generalization of Szász–Mirakyan operators based on q-integers. Math. Comput. Model. **47**(9–10), 1052–1062 (2008). https://doi.org/10.1016/j.mcm.2007.06.018
3. Aral, A., Gupta, V.: The q-derivative and applications to q-Szasz Mirakyan operators. Calcolo **43**(3), 151–170 (2006). https://doi.org/10.1007/s10092-006-0119-3
4. Aral, A., Gupta, V., Agarwal, R.: Applications of q-Calculus in Operator Theory. Springer, Berlin (2013)
5. Aral, A., Limmam, M.L., Özsarac, F.: Approximation properties of Szász–Mirakyan–Kantorovich type operators. Math. Methods Appl. Sci. **42**(16), 5233–5240 (2019). https://doi.org/10.1002/mma.5280
6. DeVore, R.A., Lorentz, G.G.: Constructive Approximation. Springer, Berlin (1993)
7. Doğru, O., Gadjieva, E.: Ağırlıklıuzaylarda Szász tipinde operatörler dizisinin sürekli fonksiyonlara yaklaşımı. II.Kızılırmak UluslararasıFen Bilimleri Kongresi Bildiri Kitabı, Kırıkkale pp. 29–37 (1998)
8. Gadzhiev, A.D.: A problem on the convergence of a sequence of positive linear operators on unbounded sets, and theorems that are analogous to P.P. Korovkin's theorem. Dokl. Akad. Nauk **218**(5), 1001–1004 (1974)

9. Gal, S., Mahmudov, N., Kara, M.: Approximation by complex q-Szász–Kantorovich operators in compact disks, $q > 1$. Complex Anal. Oper. Theory **7**, 1853–1867 (2013). https://doi.org/10.1007/s11785-012-0257-3
10. Gupta, V., Aral, A.: Convergence of the q-analogue of Szász-beta operators. Apply. Math. Comput. **216**(2), 374–380 (2010). https://doi.org/10.1016/j.amc.2010.01.018
11. Gupta, V., Rassias, T.M.: Moments of Linear Positive Operators and Approximation. Series: SpringerBriefs in Mathematics. Springer Nature, Switzerland (2019)
12. Kac, V., Cheung, P.: Quantum Calculus. Universitext, New York (2002)
13. Lupas, A.: A q-analogue of the Bernstein operator. Univ. Cluj-Napoca Semin. Numer. Stat. Calc. **9**, 85–92 (1987)
14. Mahmudov, N.I.: On q-parametric Szász–Mirakjan operators. Mediterr. J. Math. **7**(3), 297–311 (2010). https://doi.org/10.1007/s00009-010-0037-0
15. Mahmudov, N.I., Gupta, V.: On certain q-analogue of Szász Kantorovich operators. J. Appl. Math. Comput. **37**, 407–419 (2011). https://doi.org/10.1007/s12190-010-0441-4
16. Phillips, G.M.: Bernstein polynomials based on the q-integers. The heritage of P.L. Chebyshev. Ann. Numer. Math. **4**, 511–518 (1997)
17. Sabancigil, P., Kara, M., Mahmudov, N.: Higher order Kantorovich-type Szasz–Mirakjan operators. J. Inequal. Appl. **2022**, Article ID 91 (2022)
18. Szász, O.: Generalization of S. Bernstein's polynomials to the infinite interval. J. Res. Natl. Bur. Stand. **45**, 239–245 (1950)

Publisher's Note