Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some variants of the hybrid extragradient algorithm in Hilbert spaces

Yasir Arfat[1], Poom Kumam[1,2,3*], Muhammad Aqeel Ahmad Khan[4], Thidaporn Seangwattana[5] and Zaffar Iqbal[6]

*Correspondence:
poom.kum@kmutt.ac.th
[1]KMUTT Fixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
[2]Center of Excellence in Theoretical and Computational Science (TaCS-CoE), SCL 802 Fixed Point Laboratory, Science Laboratory Building, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
Full list of author information is available at the end of the article

Abstract

This paper provides convergence analysis of some variants of the hybrid extragradient algorithm (HEA) in Hilbert spaces. We employ the HEA to compute the common solution of the equilibrium problem and split fixed-point problem associated with the finite families of $\Bbbk$-demicontractive mappings. We also incorporate appropriate numerical results concerning the viability of the proposed variants with respect to various real-world applications.

Mathematics Subject Classification: 47H05; 47H10; 47J25; 49M30; 54H25

Keywords: Hybrid extragradient algorithm; Equilibrium problem; Fixed-point problem; Strong convergence

1 Introduction

The class of split inverse problems (SIP) plays a prominent role in medical image reconstruction and in signal processing. One of the important generalizations of the SIP is the split common fixed-point problem (SCFPP). The class of SCFPP associated with a variety of nonlinear mappings has been analyzed in the framework of Hilbert as well as Banach spaces. In this paper, we are interested in solving the SCFPP for finite families of $\Bbbk$-demicontractive mappings in Hilbert spaces.

In 1994, Blum and Oettli [14] proposed the (monotone-) equilibrium problem (EP) theory in Hilbert spaces. Since then, several iterative algorithms have been employed to compute the optimal solution of the (monotone-) EP as well as EP together with the fixed-point problem (FPP). In 2006, Tada and Takahashi [32] suggested a hybrid framework for the analysis of monotone EP and FPP in Hilbert spaces. However, the iterative algorithm proposed in [32] fails for the case of pseudomonotone EP. To overcome this drawback, Anh [2] employed the hybrid extragradient method, based on the seminal work of Korpelevich [27], to compute the optimal common solution of the pseudomonotone EP and the FPP.

Inspired and motivated by the ongoing research, it is natural to study the pseudomonotone EP together with the SCFPP associated with the class of $\Bbbk$-demicontractive mappings in Hilbert spaces. We propose some accelerated variants, based on the inertial extrapolation technique [29] (see also [1, 3–12, 15–17, 19, 21–24]), of the hybrid extragradient algorithm in Hilbert spaces.

Springer

The rest of the paper is organized as follows. We present some relevant preliminary concepts and useful results regarding the pseudomonotone EP and SCFPP in Sect. 2. Section 3 comprises strong convergence results of the proposed variants of the hybrid inertial extragradient algorithm under a suitable set of constraints. In Sect. 4, we provide numerical results for the demonstration of the main results in Sect. 3 as well as the viability of the proposed variants with respect to various real-world applications.

2 Preliminaries

Let $\mathcal{K}$ be a nonempty, closed, convex subset of a real Hilbert space $\mathcal{H}_1$. The metric projector $\mathcal{P}_{\mathcal{K}}$ from $\mathcal{H}_1$ onto $\mathcal{K}$ is defined for each $\mu \in \mathcal{H}_1$ there exists a unique nearest point $\mathcal{P}_{\mathcal{K}}\mu$ in $\mathcal{K}$ such that $\|\mu - \mathcal{P}_{\mathcal{K}}\mu\| \leq \|\mu - \nu\|$, $\forall \nu \in \mathcal{K}$. A subset $\mathcal{X}$ of $\mathcal{K}$ is said to be proximal if for each $\mu \in \mathcal{K}$, there exists $\nu \in \mathcal{X}$ such that $d(\mu, \mathcal{X}) = \|\mu - \nu\|$. Throughout the rest of the paper, we denote by $\mathcal{CB}(\mathcal{H}_1)$, $\mathcal{CC}(\mathcal{H}_1)$, and $\mathcal{PB}(\mathcal{H}_1)$ the families of nonempty, closed, bounded subsets, nonempty, closed, convex subsets, and nonempty, proximal, bounded subsets of $\mathcal{H}_1$. The Hausdorff metric on $\mathcal{CB}(\mathcal{H}_1)$ is defined as:

$$\mathcal{D}(A, B) := \max\left\{\sup_{\mu \in A} d(\mu, B), \sup_{\nu \in B} d(\nu, A)\right\}, \quad \forall A, B \in \mathcal{CB}(\mathcal{H}_1),$$

where $d(\mu, B) = \inf_{b \in B} \|\mu - b\|$.

Let $\mathcal{T} : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a multivalued mapping with a nonempty, closed, and convex fixed-point set denoted by $\mathrm{Fix}(\mathcal{T}) := \{\nu \in \mathcal{H}_1; \nu \in \mathcal{T}\nu\}$. Recall that the multivalued mapping $\mathcal{T}$ is said to be a (i) contraction if there exists $\Bbbk \in (0,1)$ such that $\mathcal{D}(\mathcal{T}\mu, \mathcal{T}\nu) \leq \Bbbk\|\mu - \nu\|$ for all $\mu, \nu \in \mathcal{H}_1$; (ii) nonexpansive if $\mathcal{D}(\mathcal{T}\mu, \mathcal{T}\nu) \leq \|\mu - \nu\|$ for all $\mu, \nu \in \mathcal{H}_1$; (iii) quasinonexpansive if $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$ and $\mathcal{D}(\mathcal{T}\mu, \nu) \leq \|\mu - \nu\|$ for all $\mu \in \mathcal{H}_1$, $\nu \in \mathrm{Fix}(\mathcal{T})$; (iv) demicontractive [20] if $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$, and there exists $\Bbbk \in [0,1)$ such that $\mathcal{D}(\mathcal{T}\mu, \mathcal{T}\nu)^2 \leq \|\mu - \nu\|^2 + \Bbbk d(\mu, \mathcal{T}\nu)^2$ for all $\mu \in \mathcal{H}_1$, $\nu \in \mathrm{Fix}(\mathcal{T})$. It is worth mentioning that every multivalued quasinonexpansive mapping $\mathcal{T}$ with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$ is demicontractive, but not all multivalued demicontractive mappings are quasinonexpansive (see Example 2.2 in Ref. [25] for the proper inclusion).

Recall that the best approximation operator $\mathcal{P}_{\mathcal{T}}$ of a multivalued mapping $\mathcal{T} : \mathcal{H}_1 \to \mathcal{PB}(\mathcal{H}_1)$ is defined as $\mathcal{P}_{\mathcal{T}}(\mu) := \{\nu \in \mathcal{T}\mu : d(\mu, \mathcal{T}\mu) = \|\mu - \nu\|\}$. Observe that $\mathrm{Fix}(\mathcal{T}) = \mathrm{Fix}(\mathcal{P}_{\mathcal{T}})$ and $\mathcal{P}_{\mathcal{T}}$ satisfies the endpoint condition, i.e., $\mathcal{T}\mu = \{\mu\}$, for all $\mu \in \mathrm{Fix}(\mathcal{T})$. Meanwhile, there is an example for the best approximation operator $\mathcal{P}_{\mathcal{T}}$ that is nonexpansive, but $\mathcal{T}$ is not necessarily nonexpansive [31]. Recall also that the multivalued mapping $\mathcal{T} : \mathcal{H}_1 \to \mathcal{CB}(\mathcal{H}_1)$ satisfies the demiclosedness principle at 0 if for any sequence (x_k) in $\mathcal{H}_1$ that converges weakly to $\mu \in \mathcal{H}_1$ and the sequence $(\|x_k - y_k\|)$ converges strongly to 0, where $y_k \in \mathcal{T}x_k$, then $\mu \in \mathrm{Fix}(\mathcal{T})$. The Hilbert space $\mathcal{H}_1$ satisfies Opial's condition if for a sequence $(\nu_k) \subset \mathcal{H}_1$ with $\nu_k \rightharpoonup \nu$ then the inequality $\liminf_{k\to\infty} \|\nu_k - \nu\| < \liminf_{k\to\infty} \|\nu_k - \mu\|$ holds for all $\mu \in \mathcal{H}_1$ with $\nu \neq \mu$. Moreover, $\mathcal{H}_1$ satisfies the Kadec–Klee property, i.e., if $\nu_k \rightharpoonup \nu$ and $\|\nu_k\| \to \|\nu\|$ as $k \to \infty$, then $\|\nu_k - \nu\| \to 0$ as $k \to \infty$.

Let $g : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ be a monotone bifunction, i.e., $g(\mu, \nu) + g(\nu, \mu) \leq 0$, for all $\mu, \nu \in \mathcal{H}_1$, then the equilibrium problem associated with the bifunction g is to find $\mu \in \mathcal{H}_1$ such that $g(\mu, \nu) \geq 0$ for all $\nu \in \mathcal{H}_1$. The set of solutions of the equilibrium problem is denoted by $\mathcal{EP}(g)$.

Assumption 2.1 ([13, 14]) Let $g : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ be the bifunction satisfying the following assumptions:

(A1): $g(\mu,\nu) \geq 0 \Rightarrow g(\mu,\nu) \leq 0$, for all $\mu,\nu \in \mathcal{H}_1$ (pseudomonotonicity);
(A2): There exist constants $d_1 > 0$ and $d_2 > 0$ for Lipschitz-type continuity such that

$$g(\mu,\nu) + g(\nu,\xi) \geq g(\mu,\xi) - d_1\|\mu-\nu\|^2 - d_2\|\nu-\xi\|^2, \quad \text{for all } \mu,\nu,\xi \in \mathcal{H}_1;$$

(A3): If $\mu,\nu \in \mathcal{H}_1$ and two sequences (μ_k), (ν_k) such that $\mu_k \rightharpoonup \mu$ and $\nu_k \rightharpoonup \nu$, respectively, then $f(\mu_k,\nu_k) \to f(\mu,\nu)$ (weakly continuity property of g);
(A4): For a fixed $\mu \in \mathcal{H}_1$, the function $g(\mu,\cdot)$ is convex and subdifferentiable on $\mathcal{H}_1$.

The set $\mathcal{EP}(g)$ is weakly closed and convex provided that the bifunction g satisfies Assumption 2.1. For a finite family of bifunctions g_i satisfying Assumption 2.1, we can compute the same Lipschitz coefficients (d_1,d_2) for the family of bifunctions g_i by employing the condition (A2) as

$$g_i(\mu,\xi) - g_i(\mu,\nu) - g_i(\nu,\xi) \leq d_{1,i}\|\mu-\nu\|^2 + d_{2,i}\|\nu-\xi\|^2 \leq d_1\|\mu-\nu\|^2 + d_2\|\nu-\xi\|^2,$$

where $(d_1,d_2) = \max_{1\leq i\leq M}(d_{1,i},d_{2,i})$. Therefore, $g_i(\mu,\nu) + g_i(\nu,\xi) \geq g_i(\mu,\xi) - d_1\|\mu-\nu\|^2 - d_2\|\nu-\xi\|^2$. In addition, for all $j = 1,2,\ldots N$, let $\mathcal{T}_j : \mathcal{H}_1 \to \mathcal{CB}(\mathcal{H}_1)$ and $\mathcal{S}_j : \mathcal{H}_2 \to \mathcal{CB}(\mathcal{H}_2)$ be finite families of multivalued demicontractive mappings with constants $\Bbbk_j$ and $\tilde{\Bbbk}_j$, respectively, such that $\mathcal{T}_j - \mathrm{Id}$ and $\mathcal{S}_j - \mathrm{Id}$ are demiclosed at zero. If we assume $\Theta : \mathcal{H}_1 \to \mathcal{H}_2$ to be a bounded linear operator then the solution set of the SCFPP for two finite families of multivalued mappings $(\mathcal{T}_j)_{j=1}^N$ and $(\mathcal{S}_j)_{j=1}^N$, is denoted as

$$\Omega =: \left\{ \nu \in \bigcap_{j=1}^{N} \mathrm{Fix}(\mathcal{T}_j) : \Theta\nu \in \bigcap_{j=1}^{N} \mathrm{Fix}(\mathcal{S}_j) \right\}.$$

In [34], it was shown that the fixed-point set of a multivalued demicontractive mapping is closed and convex provided it satisfies the endpoint condition. In a similar fashion, we can choose $(\Bbbk,\tilde{\Bbbk}) = \sup_{1\leq j\leq N}(\Bbbk_j,\tilde{\Bbbk}_j)$. Suppose that $\Gamma := (\bigcap_{i=1}^M \mathcal{EP}(g_i)) \cap \Omega \neq \emptyset$. Then, we are interested in the following problem:

$$\nu^* \in \Gamma. \tag{2.1}$$

Lemma 2.2 *Let $\mu,\nu \in \mathcal{H}_1$ and $\theta \in \mathbb{R}$, then*

$$\|\mu+\nu\|^2 \leq \|\mu\|^2 + 2\langle \nu, \mu+\nu\rangle.$$

Lemma 2.3 ([26]) *Let $\mathcal{T} : \mathcal{H}_1 \to \mathcal{CB}(\mathcal{H}_1)$ be a $\Bbbk$-demicontractive multivalued mapping. If $\mu \in \mathrm{Fix}(\mathcal{T})$ such that $\mathcal{T}\mu = (\mu)$, then the following inequalities hold: for all $\tilde{\mu} \in \mathcal{H}_1$, $\tilde{\nu} \in \mathcal{T}\tilde{\mu}$,*
(I) $\langle \tilde{\mu}-\tilde{\nu}, \mu-\tilde{\nu}\rangle \leq \frac{1+\Bbbk}{2}\|\tilde{\mu}-\tilde{\nu}\|^2$;
(II) $\langle \tilde{\mu}-\tilde{\nu}, \tilde{\mu}-\mu\rangle \geq \frac{1-\Bbbk}{2}\|\tilde{\mu}-\tilde{\nu}\|^2$.

Lemma 2.4 ([18]) *Let $\mathcal{H}_1$ be a Hilbert space and (ν_k) be a sequence in $\mathcal{H}_1$. Then, for any given $(\alpha_k)_{k=1}^\infty \subset (0,1)$ with $\sum_{k=1}^\infty \alpha_k = 1$ and for any positive integer i,j with $i \leq j$,*

$$\left\| \sum_{k=1}^{\infty} \alpha_k \nu_k \right\|^2 \leq \sum_{k=1}^{\infty} \alpha_k \|\nu_k\|^2 - \alpha_i\alpha_j\|\nu_i - \nu_j\|^2.$$

Lemma 2.5 ([33]) *Assume a convex and subdifferentiable function $h : \mathcal{K} \to \mathbb{R}$ defined on a nonempty, closed, convex subset $\mathcal{K}$ of a real Hilbert space $\mathcal{H}_1$. A point ν_* solves the convex problem* $\min\{h(\nu) : \nu \in \mathcal{K}\}$ *if and only if* $0 \in \partial h(\nu_*) + N_{\mathcal{K}}(\nu_*)$, *where* $\partial h(\cdot)$ *indicates the subdifferential of h and $N_{\mathcal{K}}(\bar{\nu})$ is the normal cone of $\mathcal{K}$ at $\bar{\nu}$.*

Lemma 2.6 ([28]) *Let $\mathcal{K}$ be a nonempty, closed, convex subset of a real Hilbert space $\mathcal{H}_1$. For every $p, q, r \in \mathcal{H}_1$ and $\gamma \in \mathbb{R}$, the following set is closed and convex*:

$$D = \left\{\nu \in \mathcal{K} : \|q - \nu\|^2 \le \|p - \nu\|^2 + \langle r, \nu\rangle + \gamma\right\}.$$

3 Algorithm and convergence analysis

Our main iterative algorithm of this section has the following architecture (Algorithm 1).

Theorem 3.1 *Let the following conditions*:

(C1) $\sum_{k=1}^{\infty} \xi_k \|\nu_k - \nu_{k-1}\| < \infty$;
(C2) $\sum_{j=1}^{N} \tilde{\alpha}_{k,j} = 1$ *and* $\liminf_{k\to\infty} \tilde{\alpha}_{k,j} > 0$, for all $j = 1, 2, \dots N$;
(C3) $\sum_{j=0}^{N} \tilde{\beta}_{k,j} = 1$ and $\liminf_{k\to\infty} \tilde{\beta}_{k,j} > 0$, for all $j = 1, 2, \dots N$;

hold. Then, the sequence (ν_k) generated by Algorithm 1 converges strongly to an element in Γ.

Algorithm 1 Hybrid Inertial Extragradient Algorithm (Alg.1)

Initialization: Choose arbitrarily $\nu_0, \nu_1 \in \mathcal{C}_0 = \mathcal{H}_1$. Set $k \ge 1$ and nonincreasing sequences $(\tilde{\alpha}_{k,j}), (\tilde{\beta}_{k,j}) \subset (0,1)$, $0 < \vartheta < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$ and $\lambda \in (0, \frac{1-\tilde{\Bbbk}}{\|\Theta\|^2})$. Choose the inertial parameter

$$\xi_k = \begin{cases} \min\{\frac{p_k}{\|\nu_k - \nu_{k-1}\|}, \xi\} & \text{if } \nu_k \ne \nu_{k-1}; \\ \xi & \text{otherwise,} \end{cases}$$

where $\{p_k\}$ is a positive sequence such that $\sum_{k=1}^{\infty} p_k < \infty$ and $\xi \in [0,1)$.
Iterative Steps: Given $\nu_k \in \mathcal{H}_1$, calculate:
Step 1. Compute

$$\begin{cases} b_k = \nu_k + \xi_k(\nu_k - \nu_{k-1}); \\ u_k = \operatorname{argmin}\{\vartheta g_i(b_k, \tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2 : \tilde{y} \in \mathcal{K}\}, & i = 1, 2, \dots, M; \\ w_k = \operatorname{argmin}\{\vartheta g_i(u_k, \tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2 : \tilde{y} \in \mathcal{K}\}, & i = 1, 2, \dots, M; \\ y_k = w_k + \sum_{j=1}^{N} \tilde{\alpha}_{k,j} \lambda \Theta^*(\mathcal{S}_j(\Theta w_k) - \Theta w_k), & j = 1, 2, \dots, N; \\ z_k = \tilde{\beta}_{k,0} y_k + \sum_{j=1}^{N} \tilde{\beta}_{k,j} \mathcal{T}_j y_k, & j = 1, 2, \dots, N. \end{cases}$$

If $z_k = y_k = w_k = u_k = b_k = \nu_k$ then terminate and ν_k is the required solution. Otherwise,
Step 2. Construct

$$\mathcal{C}_{k+1} = \left\{z^* \in \mathcal{C}_k : \left\|z_k - z^*\right\|^2 \le \left\|\nu_k - z^*\right\|^2 + \xi_k^2 \|\nu_k - \nu_{k-1}\|^2 + 2\xi_k \left\langle \nu_k - z^*, \nu_k - \nu_{k-1}\right\rangle\right\},$$
$$\nu_{k+1} = \mathcal{P}_{\mathcal{C}_{k+1}}^{\mathcal{H}_1} \nu_1, \forall k \ge 1.$$

Put $k =: k + 1$ and execute **Step 1** again.

The following result is crucial for the strong convergence result of Algorithm 1.

Lemma 3.2 ([2, 30]) *Suppose that $\nu^* \in \mathcal{EP}(g_i)$, then*

$$\left\|w_k - \nu^*\right\|^2 \leq \left\|b_k - \nu^*\right\|^2 - (1 - 2\vartheta d_1)\|u_k - b_k\|^2 - (1 - 2\vartheta d_2)\|u_k - w_k\|^2,$$

where ν_k, b_k, u_k, and ν_k are defined in Algorithm 1.

Proof of Theorem 3.1 *Step* 1. Algorithm 1 is well defined.

It is obvious by recalling Lemma 2.6 that the set $\mathcal{C}_k$ is closed and convex. Moreover, the set Ω is closed and convex. Therefore, Γ is nonempty, closed, and convex. For any $\nu^* \in \Gamma$, it follows from Algorithm 1 that

$$\begin{aligned}\left\|b_k - \nu^*\right\|^2 &= \left\|\nu_k - \nu^* + \xi_k(\nu_k - \nu_{k-1})\right\|^2 \\ &\leq \left\|\nu_k - \nu^*\right\|^2 + \xi_k^2\|\nu_k - \nu_{k-1}\|^2 + 2\xi_k\left\langle \nu_k - \nu^*, \nu_k - \nu_{k-1}\right\rangle. \end{aligned} \tag{3.1}$$

Recalling the estimate (3.1), Lemmas 2.3 and 2.4, and Lemma 3.2, we obtain

$$\begin{aligned}\left\|y_k - \nu^*\right\|^2 &= \left\|w_k + \sum_{j=1}^{N}\tilde{\alpha}_{k,j}\lambda\Theta^*\big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big) - \nu^*\right\|^2 \\ &= \left\|\sum_{j=1}^{N}\tilde{\alpha}_{k,j}\big(w_k - \nu^* + \lambda\Theta^*\big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big)\big)\right\|^2 \\ &\leq \sum_{j=1}^{N}\tilde{\alpha}_{k,j}\left\|w_k - \nu^* + \lambda\Theta^*\big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big)\right\|^2 \\ &= \sum_{j=1}^{N}\tilde{\alpha}_{k,j}\Big(\left\|w_k - \nu^*\right\|^2 + \lambda^2\left\|\Theta^*\big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big)\right\|^2 \\ &\quad + 2\lambda\left\langle w_k - \nu^*, \Theta^*\big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big)\right\rangle\Big) \\ &\leq \sum_{j=1}^{N}\tilde{\alpha}_{k,j}\Big(\left\|w_k - \nu^*\right\|^2 + \lambda^2\|\Theta\|^2\left\|\mathcal{S}_j(\Theta w_k) - \Theta w_k\right\|^2 \\ &\quad + 2\lambda\left\langle w_k - \nu^*, \Theta^*\big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big)\right\rangle\Big). \end{aligned} \tag{3.2}$$

Putting $M_k = 2\lambda\langle w_k - \nu^*, \Theta^*(\mathcal{S}_j(\Theta w_k) - \Theta w_k)\rangle$, since $\mathcal{S}_j$ is $\tilde{\Bbbk}_j$-demicontractive, then by Lemma 2.3, we have

$$\begin{aligned} M_k &= 2\lambda\left\langle \Theta\big(w_k - \nu^*\big), \mathcal{S}_j(\Theta w_k) - \Theta w_k\right\rangle \\ &= 2\lambda\left\langle \Theta\big(w_k - \nu^*\big) + \big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big) - \big(\mathcal{S}_j(\Theta w_k) - \Theta w_k\big), \mathcal{S}_j(\Theta w_k) - \Theta w_k\right\rangle \\ &= 2\lambda\left\langle \mathcal{S}_j(\Theta w_k) - \Theta\nu^*, \mathcal{S}_j(\Theta w_k) - \Theta w_k\right\rangle - \left\|\mathcal{S}_j(\Theta w_k) - \Theta w_k\right\|^2 \\ &\leq 2\lambda\left(\frac{1 + \tilde{\Bbbk}_j}{2}\right)\left\|\mathcal{S}_j(\Theta w_k) - \Theta w_k\right\|^2 - \left\|\mathcal{S}_j(\Theta w_k) - \Theta w_k\right\|^2 \\ &= -(1 - \tilde{\Bbbk}_j)\lambda\left\|\mathcal{S}_j(\Theta w_k) - \Theta w_k\right\|^2 \end{aligned}$$

$$\leq -(1-\tilde{\Bbbk})\lambda\left\|\mathcal{S}_j(\Theta w_k)-\Theta w_k\right\|^2. \tag{3.3}$$

Utilizing (3.2) and (3.3), we obtain

$$\left\|y_k-\nu^*\right\|^2 \leq \left\|w_k-\nu^*\right\|^2-\sum_{j=1}^N \tilde{\alpha}_{k,j}\lambda\big(1-\tilde{\Bbbk}-\lambda\|\Theta\|^2\big)\left\|\mathcal{S}_j(\Theta w_k)-\Theta w_k\right\|^2. \tag{3.4}$$

Since $\mathcal{T}_j$ is $\Bbbk_j$-demicontractive and by using Lemma 2.4, we have

$$\begin{aligned}
\left\|z_k-\nu^*\right\|^2 &= \left\|\tilde{\beta}_{k,0}y_k+\sum_{j=1}^N \tilde{\beta}_{k,j}\mathcal{T}_j y_k-\nu^*\right\|^2 \\
&\leq \tilde{\beta}_{k,0}\left\|y_k-\nu^*\right\|^2+\sum_{j=1}^N \tilde{\beta}_{k,j}\left\|\mathcal{T}_j y_k-\nu^*\right\|^2-\tilde{\beta}_{k,0}\tilde{\beta}_{k,j}\|y_k-\mathcal{T}_j y_k\|^2 \\
&= \tilde{\beta}_{k,0}\left\|y_k-\nu^*\right\|^2+\sum_{j=1}^N \tilde{\beta}_{k,j}d\big(\mathcal{T}_j y_k-\mathcal{T}_j\nu^*\big)-\tilde{\beta}_{k,0}\tilde{\beta}_{k,j}\|y_k-\mathcal{T}_j y_k\|^2 \\
&\leq \tilde{\beta}_{k,0}\left\|y_k-\nu^*\right\|^2+\sum_{j=1}^N \tilde{\beta}_{k,j}\mathcal{H}\big(\mathcal{T}_j y_k-\mathcal{T}_j\nu^*\big)-\tilde{\beta}_{k,0}\tilde{\beta}_{k,j}\|y_k-\mathcal{T}_j y_k\|^2 \\
&\leq \tilde{\beta}_{k,0}\left\|y_k-\nu^*\right\|^2+\sum_{j=1}^N \tilde{\beta}_{k,j}\big(\left\|y_k-\nu^*\right\|^2+\Bbbk_j d(y_k,\mathcal{T}_j y_k)\big) \\
&\quad -\tilde{\beta}_{k,0}\tilde{\beta}_{k,j}\|y_k-\mathcal{T}_j y_k\|^2 \\
&\leq \tilde{\beta}_{k,0}\left\|y_k-\nu^*\right\|^2+\sum_{j=1}^N \tilde{\beta}_{k,j}\left\|y_k-\nu^*\right\|^2\|^2+\sum_{j=1}^N \tilde{\beta}_{k,j}\Bbbk\|y_k-\mathcal{T}_j y_k)\right\|^2 \\
&\quad -\tilde{\beta}_{k,0}\tilde{\beta}_{k,j}\|y_k-\mathcal{T}_j y_k\|^2 \\
&= \left\|y_k-\nu^*\right\|^2\|^2-(\tilde{\beta}_{k,0}-\Bbbk)\tilde{\beta}_{k,j}\|y_k-\mathcal{T}_j y_k\|^2 \\
&\leq \left\|w_k-\nu^*\right\|^2-\sum_{j=1}^N \tilde{\alpha}_{k,j}\lambda\big(1-\tilde{\Bbbk}-\lambda\|\Theta\|^2\big)\left\|\mathcal{S}_j(\Theta w_k)-\Theta w_k\right\|^2 \\
&\quad -(\tilde{\beta}_{k,0}-\Bbbk)\tilde{\beta}_{k,j}\|y_k-\mathcal{T}_j y_k\|^2.
\end{aligned} \tag{3.5}$$

This shows that Γ is contained in $\mathcal{C}_k$, for all $k\geq 0$. Recalling the definition of the set $\mathcal{C}_k$ the above estimate infers that Algorithm 1 is well defined.

Step 2. The limit $\lim_{k\to\infty}\|\nu_k-\nu_1\|$ exists.

From $\nu_{k+1}=\mathcal{P}^{\mathcal{H}_1}_{\mathcal{C}_{k+1}}\nu_1$, we have $\langle \nu_{k+1}-\nu_1,\nu_{k+1}-p\rangle\leq 0$ for each $p\in\mathcal{C}_{k+1}$. In particular, we have $\langle \nu_{k+1}-\nu_1,\nu_{k+1}-\nu^*\rangle\leq 0$ for each $\nu^*\in\Gamma$. This establishes that the sequence $(\|\nu_k-\nu_1\|)$ is bounded. Nevertheless, from $\nu_k=\mathcal{P}^{\mathcal{H}_1}_{\mathcal{C}_k}\nu_1$ and $\nu_{k+1}=\mathcal{P}^{\mathcal{H}_1}_{\mathcal{C}_{k+1}}\nu_1\in\mathcal{C}_{k+1}$, we have that

$$\|\nu_k-\nu_1\|\leq\|\nu_{k+1}-\nu_1\|.$$

This infers that $(\|\nu_k-\nu_1\|)$ is nondecreasing and consequently

$$\lim_{k\to\infty}\|\nu_k-\nu_1\| \quad \text{exists.} \tag{3.6}$$

Step 3. $\tilde{\nu} \in \Gamma$.

We first compute

$$\begin{aligned} \|\nu_{k+1} - \nu_k\|^2 &= \|\nu_{k+1} - \nu_1 + \nu_1 - \nu_k\|^2 \\ &= \|\nu_{k+1} - \nu_1\|^2 + \|\nu_k - \nu_1\|^2 - 2\langle \nu_k - \nu_1, \nu_{k+1} - \nu_1\rangle \\ &= \|\nu_{k+1} - \nu_1\|^2 + \|\nu_k - \nu_1\|^2 - 2\langle \nu_k - \nu_1, \nu_{k+1} - \nu_k + \nu_k - \nu_1\rangle \\ &= \|\nu_{k+1} - \nu_1\|^2 - \|\nu_k - \nu_1\|^2 - 2\langle \nu_k - \nu_1, \nu_{k+1} - \nu_k\rangle \\ &\leq \|\nu_{k+1} - \nu_1\|^2 - \|\nu_k - \nu_1\|^2. \end{aligned}$$

Employing the lim sup and recalling (3.6), we have

$$\lim_{k\to\infty} \|\nu_{k+1} - \nu_k\| = 0. \tag{3.7}$$

By recalling (b_k) from Algorithm 1 and the condition (C1), we have

$$\lim_{k\to\infty} \|b_k - \nu_k\| = 0. \tag{3.8}$$

By recalling the estimates (3.7), (3.8), and the following triangle inequality:

$$\|b_k - \nu_{k+1}\| \leq \|b_k - \nu_k\| + \|\nu_k - \nu_{k+1}\|,$$

we have

$$\lim_{k\to\infty} \|b_k - \nu_{k+1}\| = 0. \tag{3.9}$$

Recall that $\nu_{k+1} \in \mathcal{C}_{k+1}$, therefore we have

$$\|z_k - \nu_{k+1}\| \leq \|\nu_k - \nu_{k+1}\| + 2\xi_k\|\nu_k - \nu_{k-1}\| + 2\xi_k\langle \nu_k - \nu_{k+1}, \nu_k - \nu_{k-1}\rangle.$$

Recalling the estimate (3.7) and the condition (C1), the above estimate infers that

$$\lim_{k\to\infty} \|z_k - \nu_{k+1}\| = 0. \tag{3.10}$$

By employing the estimates (3.7), (3.10), and the following triangular inequality:

$$\|z_k - \nu_k\| \leq \|z_k - \nu_{k+1}\| + \|\nu_{k+1} - \nu_k\|,$$

we obtain

$$\lim_{k\to\infty} \|z_k - \nu_k\| = 0. \tag{3.11}$$

In view of Lemma 3.2, it is easy to obtain the following variant of the estimate (3.5):

$$(1 - 2\vartheta d_1)\|u_k - b_k\|^2 - (1 - 2\vartheta d_2)\|u_k - w_k\|^2$$

$$\leq \left(\left\|\nu_k - \nu^*\right\| + \left\|z_k - \nu^*\right\|\right)\|\nu_k - z_k\| + \xi_k^2\|\nu_k - \nu_{k-1}\|^2 + 2\xi_k\left\|\nu_k - \nu^*\right\|\|\nu_k - \nu_{k-1}\|.$$

Recalling the estimate (3.11) and the condition (C1), we have

$$(1 - 2\vartheta d_1)\lim_{k\to\infty}\|u_k - b_k\|^2 - (1 - 2\vartheta d_2)\lim_{k\to\infty}\|u_k - w_k\|^2 = 0. \tag{3.12}$$

The estimate (3.12) implies that

$$\lim_{k\to\infty}\|u_k - b_k\|^2 = \lim_{k\to\infty}\|u_k - w_k\|^2 = 0. \tag{3.13}$$

We can also extract the following two inequalities from the estimate (3.5):

$$\begin{aligned}
&\sum_{j=1}^{N}\tilde{\alpha}_{k,j}\lambda\left(1 - \tilde{\mathbb{k}} - \lambda\|\Theta\|^2\right)\left\|\mathcal{S}_j(\Theta w_k) - \Theta w_k\right\|^2 \\
&\quad\leq \left\|w_k - \nu^*\right\|^2 - \left\|z_k - \nu^*\right\|^2 \\
&\quad\leq \left(\left\|\nu_k - \nu^*\right\| + \left\|z_k - \nu^*\right\|\right)\|\nu_k - z_k\| + \xi_k^2\|\nu_k - \nu_{k-1}\|^2 \\
&\qquad + 2\xi_k\left\langle \nu_k - \nu^*, \nu_k - \nu_{k-1}\right\rangle
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&(\tilde{\beta}_{k,0} - \mathbb{k})\tilde{\beta}_{k,j}\|y_k - \mathcal{T}_j y_k\|^2 \\
&\quad\leq \left\|w_k - \nu^*\right\|^2 - \left\|z_k - \nu^*\right\|^2 \\
&\quad\leq \left(\left\|\nu_k - \nu^*\right\| + \left\|z_k - \nu^*\right\|\right)\|\nu_k - z_k\| + \xi_k^2\|\nu_k - \nu_{k-1}\|^2 \\
&\qquad + 2\xi_k\left\langle \nu_k - \nu^*, \nu_k - \nu_{k-1}\right\rangle.
\end{aligned} \tag{3.15}$$

Utilizing (3.14) and the conditions (C1) and (C2), we have

$$\lim_{k\to\infty}\left\|\mathcal{S}_j(\Theta w_k) - \Theta w_k\right\| = 0, \quad \text{for all } j = 1, 2, \dots N. \tag{3.16}$$

Since $\liminf_{k\to\infty}(\tilde{\beta}_{k,0} - \mathbb{k})\tilde{\beta}_{k,j} > 0$, we obtain from (3.15) that

$$\lim_{k\to\infty}\left\|(\mathrm{Id} - \mathcal{T}_j)y_k\right\| = 0 \quad \text{for all } j = 1, 2, \dots N. \tag{3.17}$$

In order to establish the claim of this section, we first show that $\nu^* \in \bigcap_{i=1}^{M}\mathcal{EP}(g_i)$.

Observe that

$$u_k = \arg\min\left\{\vartheta g_i(b_k, \tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2 : \tilde{y} \in \mathcal{K}\right\}.$$

Recalling Lemma 2.5, we obtain

$$0 \in \partial_2\left\{\vartheta g_i(b_k, \tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2\right\}(u_k) + N_{\mathcal{K}}(u_k).$$

This implies the existence of $p \in \partial_2 g_i(b_k, u_k)$ and $\bar{p} \in N_{\mathcal{K}}(u_k)$ such that

$$\vartheta p + b_k - u_k + \bar{p}. \tag{3.18}$$

Since $\bar{p} \in N_{\mathcal{K}}(u_k)$ and $\langle \bar{p}, \tilde{y} - u_k \rangle \leq 0$ for all $\tilde{y} \in \mathcal{K}$. Hence, recalling the estimate (3.18), we have

$$\vartheta \langle p, \tilde{y} - u_k \rangle \geq \langle u_k - b_k, \tilde{y} - u_k \rangle, \quad \forall \tilde{y} \in \mathcal{K}. \tag{3.19}$$

Since $p \in \partial_2 g_i(b_k, u_k)$,

$$g_i(b_k, \tilde{y}) - g_i(b_k, u_k) \geq \langle p, \tilde{y} - u_k \rangle, \quad \forall \tilde{y} \in \mathcal{K}. \tag{3.20}$$

Therefore, recalling the estimates (3.19) and (3.20), we obtain

$$\vartheta \big(g_i(b_k, \tilde{y}) - g_i(b_k, u_k) \big) \geq \langle u_k - b_k, \tilde{y} - u_k \rangle, \quad \forall \tilde{y} \in \mathcal{K}. \tag{3.21}$$

Since (ν_k) is bounded, there exists a subsequence (ν_{k_t}) of (ν_k) such that $\nu_{k_t} \rightharpoonup \tilde{\nu} \in \mathcal{H}_1$ as $t \to \infty$. This also implies that $z_{k_t} \rightharpoonup \tilde{\nu}$ and $w_{k_t} \rightharpoonup \tilde{\nu}$ as $t \to \infty$. Since $u_k \rightharpoonup \tilde{\nu}$, therefore, recalling the assumption (A3) and the estimate (3.21), we deduce that $0 \leq g_i(\tilde{\nu}, y)$ for all $y \in \mathcal{K}$ and $i \in \{1, 2, 3, \ldots, M\}$. This infers that $\tilde{\nu} \in \bigcap_{i=1}^{M} \mathcal{EP}(g_i)$.

Finally, we show that $\tilde{\nu} \in \Omega$. Since $x_{k_t} \rightharpoonup \tilde{\nu}$ as $t \to \infty$, recalling the demiclosed principal along with the estimate (3.16) and (3.17), we have $\tilde{\nu} \in \Omega$. Hence, $\tilde{\nu} \in \Gamma$.

Step 4. $\nu_k \to \nu^* = \mathcal{P}_{\Gamma}^{\mathcal{H}_1} \nu_1$.

Since $\nu^* = \mathcal{P}_{\Gamma}^{\mathcal{H}_1} \nu_1$ and $\tilde{\nu} \in \Gamma$, we have

$$\big\| \nu_1 - \nu^* \big\| \leq \| \nu_1 - \tilde{\nu} \| \leq \liminf_{t \to \infty} \| \nu_1 - \nu_{k_t} \| \leq \limsup_{t \to \infty} \| \nu_1 - \nu_{k_t} \| \leq \| \nu_1 - \tilde{\nu} \|.$$

Recalling the uniqueness of the metric projection operator yields that $\tilde{\nu} = \nu^* = \mathcal{P}_{\Gamma}^{\mathcal{H}_1} \nu_1$. This completes the proof. □

If for each $j = 1, 2, \ldots, N$, let $S_j = \mathrm{Id}$, then we have the following result:

Corollary 3.3 *Let $\mathcal{K} \subseteq \mathcal{H}_1$ be a nonempty, closed, convex subset of a real Hilbert space $\mathcal{H}_1$ and let $g_i : \mathcal{K} \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a finite family of bifunctions satisfying Assumptions 2.1 for all $i = 1, 2, \ldots, M$. For all $j = 1, 2, \ldots, N$, let $\mathcal{T}_j : \mathcal{H}_1 \to \mathcal{CB}(\mathcal{H}_1)$ be a finite family of multivalued $\Bbbk_j$-demicontractive mappings such that $\mathcal{T}_j - \mathrm{Id}$ are demiclosed at zero. Assume that $\Gamma := \bigcap_{i=1}^{M} \mathcal{EP}(g_i) \cap \bigcap_{j=1}^{N} \mathrm{Fix}(\mathcal{T}_j) \neq \emptyset$ and calculate*

$$\begin{cases} b_k = \nu_k + \xi_k(\nu_k - \nu_{k-1}); & \\ u_k = \operatorname{argmin}\{\vartheta g_i(b_k, \tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2 : \tilde{y} \in \mathcal{K}\}, & i = 1, 2, \ldots, M; \\ w_k = \operatorname{argmin}\{\vartheta g_i(u_k, \tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2 : \tilde{y} \in \mathcal{K}\}, & i = 1, 2, \ldots, M; \\ z_k = \tilde{\beta}_{k,0} w_k + \sum_{j=1}^{N} \tilde{\beta}_{k,j} \mathcal{T}_j w_k, & j = 1, 2, \ldots, N; \\ \mathcal{C}_{k+1} = \{z^* \in \mathcal{C}_k : \|z_k - z^*\|^2 & \\ \qquad \leq \|\nu_k - z^*\|^2 + \xi_k^2 \|\nu_k - \nu_{k-1}\|^2 + 2\xi_k \langle \nu_k - z^*, \nu_k - \nu_{k-1} \rangle\}; & \\ \nu_{k+1} = \mathcal{P}_{\mathcal{C}_{k+1}}^{\mathcal{H}_1} \nu_1, & \forall k \geq 1. \end{cases} \tag{3.22}$$

Assume that the conditions (C1) *and* (C3) *hold, then the sequence* (ν_k) *generated by* (3.22) *converges strongly to an element in* Γ.

As a direct application of Theorem 3.1, we have the following result for the variational inequality problem (i.e., find $\bar{\nu} \in \mathcal{K}$ for which $\langle A\bar{\nu}, \bar{\mu} - \bar{\nu}\rangle \geq 0\ \forall \bar{\mu} \in \mathcal{K}$, where $A : \mathcal{K} \to \mathcal{H}_1$ is a nonlinear, monotone mapping defined on a nonempty, closed, convex subset $\mathcal{K} \subseteq \mathcal{H}_1$):

Theorem 3.4 *Let* $\mathcal{K} \subseteq \mathcal{H}_1$ *be a nonempty, closed, convex subset of a real Hilbert space* $\mathcal{H}_1$ *and for each* $i = 1, 2, \ldots, M$ *let* $A_i : \mathcal{K} \to \mathcal{H}_1$ *be a finite family of pseudomonotone and L-Lipschitz continuous mappings. For all* $j = 1, 2, \ldots, N$, *let* $\mathcal{T}_j : \mathcal{H}_1 \to \mathcal{CB}(\mathcal{H}_1)$ *and* $\mathcal{S}_j : \mathcal{H}_2 \to \mathcal{CB}(\mathcal{H}_2)$ *be two finite families of multivalued, demicontractive mappings with constants* $\Bbbk_j$ *and* $\tilde{\Bbbk}_j$, *respectively, such that* $\mathcal{T}_j - \mathrm{Id}$ *and* $\mathcal{S}_j - \mathrm{Id}$ *are demiclosed at zero. Let* $\Gamma = \bigcap_{i=1}^{M} VI(\mathcal{K}, A_i) \cap \Omega \neq \emptyset$. *Let* ξ_k *be a bounded real sequence and* $0 < \lambda < \frac{1-\tilde{\Bbbk}}{\|\Theta\|^2}$. *Given* $\nu_k \in \mathcal{H}_1$, *calculate*

$$\begin{cases} b_k = \nu_k + \xi_k(\nu_k - \nu_{k-1}); & \\ u_k = P_{\mathcal{C}}(b_k - \vartheta A_i(b_k)), & i = 1, 2, \ldots, M; \\ w_k = P_{\mathcal{C}}(b_k - \vartheta A_i(u_k)), & i = 1, 2, \ldots, M; \\ y_k = w_k + \sum_{j=1}^{N} \tilde{\alpha}_{k,j} \lambda \Theta^*(\mathcal{S}_j(\Theta w_k) - \Theta w_k), & j = 1, 2, \ldots, N; \\ z_k = \tilde{\beta}_{k,0} y_k + \sum_{j=1}^{N} \tilde{\beta}_{k,j} \mathcal{T}_j y_k, & j = 1, 2, \ldots, N; \\ \mathcal{C}_{k+1} = \{z^* \in \mathcal{C}_k : \|z_k - z^*\|^2 & \\ \qquad \leq \|\nu_k - z^*\|^2 + \xi_k^2 \|\nu_k - \nu_{k-1}\|^2 + 2\xi_k \langle \nu_k - z^*, \nu_k - \nu_{k-1}\rangle\}, & \\ \nu_{k+1} = \mathcal{P}_{\mathcal{C}_{k+1}}^{\mathcal{H}_1} \nu_1, & \forall k \geq 1. \end{cases} \tag{3.23}$$

Assume that the conditions (C1)–(C3) *hold, then the sequence* (ν_k) *generated by* (3.23) *converges strongly to an element in* Γ.

Proof Let $g_i(\bar{\nu}, \bar{\mu}) = \langle A_i(\bar{\nu}), \bar{\mu} - \bar{\nu}\rangle$ for all $\nu, y \in \mathcal{K}$ and $i = 1, 2, \ldots, M$. Since A_i is L-Lipschitz continuous, we observe that for all $\bar{\nu}, \bar{\mu}, \bar{\xi} \in \mathcal{C}$

$$\begin{aligned} g_i(\bar{\nu}, \bar{\mu}) + g_i(\bar{\mu}, \bar{\xi}) - g_i(\bar{\nu}, \bar{\xi}) &= \langle A_i(\bar{\nu}), \bar{\mu} - \bar{\nu}\rangle + \langle A_i(\bar{\mu}), \bar{\xi} - \bar{\mu}\rangle - \langle A_i(\bar{\nu}), \bar{\xi} - \bar{\nu}\rangle \\ &= -\langle A_i(\bar{\mu}) - A_i(\bar{\nu}), \bar{\mu} - \bar{\xi}\rangle \\ &\geq -\|A_i(\bar{\mu}) - A_i(\bar{\nu})\| \|\bar{\mu} - \bar{\xi}\| \\ &\geq -L\|\bar{\mu} - \bar{\nu}\| \|\bar{\mu} - \bar{\xi}\| \\ &\geq -\frac{L}{2}\|\bar{\mu} - \bar{\nu}\|^2 - \frac{L}{2}\|\bar{\mu} - \bar{\xi}\|^2. \end{aligned}$$

This infers that g_i is Lipschitz-type continuous with $d_1 = d_2 = \frac{L}{2}$. Moreover, the pseudomonotonicity of A_i ensures the pseudomonotonicity of g_i. From Algorithm 1, we have

$$\begin{aligned} u_k &= \arg\min\left\{\vartheta\langle A_i(\nu_k), \mu - \nu_k\rangle + \frac{1}{2}\|\nu_k - \mu\|^2 : \mu \in \mathcal{K}\right\}, \\ w_k &= \arg\min\left\{\vartheta\langle A_i(u_k), \mu - u_k\rangle + \frac{1}{2}\|\nu_k - \mu\|^2 : \mu \in \mathcal{K}\right\}. \end{aligned}$$

Equivalently, we have

$$u_k = \arg\min\left\{\frac{1}{2}\|\tilde{y} - (\nu_k - \vartheta A_i(\nu_k)\|^2 : \tilde{y} \in \mathcal{K}\right\} = P_{\mathcal{K}}\big(\nu_k - \vartheta A_i(\nu_k)\big),$$
$$w_k = \arg\min\left\{\frac{1}{2}\|\tilde{y} - (\nu_k - \vartheta A_i(u_k)\|^2 : \tilde{y} \in \mathcal{K}\right\} = P_{\mathcal{K}}\big(\nu_k - \vartheta A_i(u_k)\big).$$

Recalling the proof of Theorem 3.1 with the above-mentioned $g_i(\nu,\mu)$ for all $i \in \{1,2,\dots,M\}$ leads to the desired result. □

Setting a terminating criterion by fixing $k > k_{\max}$ for an appropriately chosen large number $k_{\max}$, we now propose a Halpern-type variant of Algorithm 1.

Theorem 3.5 *If* $\Gamma \neq \emptyset$ *such that the conditions* (C1)–(C3) *with* $C^* = \lim_{k\to\infty} \tilde{\gamma}_k = 0$ *hold. Then, the sequence* (ν_k) *generated by Algorithm 2 converges strongly to an element in* Γ.

Proof Observe that the set $\mathcal{C}_k$ can be expressed as:

$$\mathcal{C}_k = \Big\{\nu^* \in \mathcal{C}_k : \big\|h_k - \nu^*\big\|^2 \leq \tilde{\gamma}_k\big\|q - \nu^*\big\|^2 + (1-\tilde{\gamma}_k)\big(\|\nu_k - \nu_{k+1}\|^2 + \xi_k^2\|\nu_k - \nu_{k-1}\|^2 + 2\xi_k\langle \nu_k - \nu_{k+1}, \nu_k - \nu_{k-1}\rangle\big)\Big\}.$$

Algorithm 2 Hybrid Inertial Halpern-Extragradient Algorithm (Alg.2)

Initialization: Choose arbitrarily $q \in \mathcal{H}_1$ and $\nu_0, \nu_1 \in \mathcal{C}_0 = \mathcal{H}_1$, set $k \geq 1$ and nonincreasing sequence $(\tilde{\alpha}_{k,j}), (\tilde{\beta}_{k,j}) \subset (0,1)$, $0 < \vartheta < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$, $\xi_k \subset [0,1)$.

Iterative Steps: Given $\nu_k \in \mathcal{H}_1$, calculate:

Step 1. Compute

$$\begin{cases} b_k = \nu_k + \xi_k(\nu_k - \nu_{k-1}); & \\ u_k = \operatorname{argmin}\{\vartheta g_i(b_k,\tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2 : \tilde{y} \in \mathcal{K}\}, & i = 1,2,\dots,M; \\ w_k = \operatorname{argmin}\{\vartheta g_i(u_k,\tilde{y}) + \frac{1}{2}\|b_k - \tilde{y}\|^2 : \tilde{y} \in \mathcal{K}\}, & i = 1,2,\dots,M; \\ y_k = w_k + \sum_{j=1}^{N} \tilde{\alpha}_{k,j}\lambda\Theta^*(\mathcal{S}_j(\Theta w_k) - \Theta w_k), & j = 1,2,\dots,N, \\ z_k = \tilde{\beta}_{k,0} y_k + \sum_{j=1}^{N} \tilde{\beta}_{k,j}\mathcal{T}_j y_k, & j = 1,2,\dots,N, \\ h_k = \tilde{\gamma}_k q + (1-\tilde{\gamma}_k) z_k. & \end{cases}$$

If $h_k = z_k = y_k = w_k$ then stop and ν_k is the solution of problem Γ. Otherwise,

Step 2. Compute

$$\mathcal{C}_{k+1} = \Big\{z^* \in \mathcal{C}_k : \big\|h_k - z^*\big\|^2 \leq \big\|\nu_k - z^*\big\|^2 + \xi_k^2\|\nu_k - \nu_{k-1}\|^2 + 2\xi_k\big\langle \nu_k - z^*, \nu_k - \nu_{k-1}\big\rangle\Big\};$$
$$\nu_{k+1} = \mathcal{P}^{\mathcal{H}_1}_{\mathcal{C}_{k+1}}\nu_1, \quad \forall k \geq 1.$$

Put $k =: k+1$ and go back to **Step 1.**

Recalling the proof of Theorem 3.1, it infers that (i) the sets $\Gamma$and$\mathcal{C}_k$ are closed and convex, satisfying $\Gamma \subset \mathcal{C}_{k+1}$ for all $k \geq 0$; (ii) (ν_k) is bounded such that

$$\lim_{k\to\infty} \|\nu_{k+1} - \nu_k\| = 0. \tag{3.24}$$

Since $\nu_{k+1} = \mathcal{P}^{\mathcal{H}_1}_{\mathcal{C}_k}(q) \in \mathcal{C}_k$ and by the definition of $\mathcal{C}_k$, we have

$$\begin{aligned}\|h_k - \nu_{k+1}\|^2 \leq \tilde{\gamma}_k\|q - \nu_{k+1}\|^2 + (1 - \tilde{\gamma}_k) \\ &\times \left(\|\nu_k - \nu_{k+1}\|^2 + \xi_k^2\|\nu_k - \nu_{k-1}\|^2 \right. \\ &\left. + 2\xi_k\langle \nu_k - \nu_{k+1}, \nu_k - \nu_{k-1}\rangle\right).\end{aligned}$$

Recalling the estimate (3.24), the conditions (C1)–(C3) and the boundedness of (ν_k), we obtain

$$\lim_{k\to\infty} \|h_k - \nu_{k+1}\| = 0,$$

implying that

$$\lim_{k\to\infty} \|h_k - \nu_k\| = 0. \tag{3.25}$$

Also, observe that

$$\begin{aligned}&(1 - \tilde{\gamma}_k)(1 - 2\vartheta d_1)\|u_k - \nu_k\|^2 + (1 - 2\vartheta d_2)\|u_k - w_k\|^2 \\ &\quad\leq \tilde{\gamma}_k\left(\|q - \nu^*\|^2 - \|\nu_k - \nu^*\|^2\right) + \|\nu_k - z_k\|\left(\|\nu_k - \nu^*\| + \|z_k - \nu^*\|\right),\end{aligned}$$

for each $\nu^* \in \Gamma$. Recalling the estimate (3.24), the conditions (C1)–(C3), and the boundedness of (ν_k), we obtain

$$\lim_{k\to\infty} \|u_k - \nu_k\| = 0 = \lim_{k\to\infty} \|w_k - \nu_k\|, \quad i \in \{1, 2, \dots, M\}.$$

Recalling $h_k = \tilde{\gamma}_k q + (1 - \tilde{\gamma}_k)z_k$ and the conditions (C2) and (C3) with C^*, we obtain

$$\|z_k - \nu_k\| \leq \frac{1}{(1 - \tilde{\gamma}_k)}\|h_k - \nu_k\| + \frac{\tilde{\gamma}_k}{(1 - \tilde{\gamma}_k)}\|q - \nu_k\|.$$

Recalling the estimate (3.25) again, the above estimate implies that

$$\lim_{k\to\infty} \|z_k - \nu_k\| = 0.$$

The rest of the proof of Theorem 3.5 is similar to the proof of Theorem 3.1 and is therefore omitted here. □

Remark 3.6 From the numerical standpoint, the condition (C1) can easily be aligned in an algorithm as $\|\nu_k - \nu_{k-1}\|$ is a priorly known before selecting ξ_k satisfying $0 \leq \xi_k \leq \widehat{\xi_k}$, where

$$\widehat{\xi_k} = \begin{cases} \min\{\frac{\sigma_k}{\|\nu_k-\nu_{k-1}\|}, \xi\} & \text{if } \nu_k \neq \nu_{k-1}; \\ \xi & \textit{otherwise}, \end{cases}$$

where $\{\sigma_k\}$ is a positive sequence such that $\sum_{k=1}^{\infty} \sigma_k < \infty$ and $\xi_k \in [0,1)$.

4 Numerical experiment and results

This section provides the effective viability of our algorithm supported by a suitable example.

Example 4.1 Let $\mathcal{H}_1 = \mathbb{R} = \mathcal{H}_2$ be the set of all real numbers with the inner product defined by $\langle \mu, \nu \rangle = \mu\nu$, for all $\mu, \nu \in \mathbb{R}$ and the induced usual norm $|\cdot|$. For each $i = \{1,2,3,\dots,M\}$ and $\mathcal{K} = [0,1] \subset \mathcal{H}_1$, let the bifunctions $g_i(\mu,\nu) : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ be defined by $g_i(\mu,\nu) = h_i(\mu)(\nu - \mu)$ with

$$h_i(\mu) = \begin{cases} 0, & \text{if } 0 \le \mu \le \varrho_i, \\ \sin(\mu - \varrho_i) + \exp(\mu - \varrho_i) - 1, & \text{if } \varrho_i \le \mu \le 1, \end{cases}$$

where $0 < \varrho_1 < \varrho_2 < \cdots < \varrho_m < 1$. It is easy to prove that $g_i(\mu,\nu)$ is pseudomonotone satisfying the Assumptions 2.1 with $h_i(\mu)$ being 4-Lipschitz continuous. Observe that $\mathcal{EP}(g_i) = [0, \varrho_i]$ if and only if $0 \le \mu \le \varrho_i$ for all $\nu \in [0,1]$. Hence, $\bigcap_{i=1}^{M} \mathcal{EP}(g_i) = [0, \varrho_1]$. For each $j = 1,2,\dots,N$, let $\mathcal{T}_j$ and $\mathcal{S}_j$ be defined as:

$$\mathcal{T}_j(\mu) = \begin{cases} 0, & \text{if } \mu < 0; \\ [\frac{\mu}{j+1}, \mu], & \text{if } \mu \ge 0 \end{cases}$$

and

$$\mathcal{S}_j(\mu) = \begin{cases} [0, \frac{|\mu|}{j+2}], & \text{if } \mu < j+2; \\ [1, j+1], & \text{if } \mu \ge j+2. \end{cases}$$

It is not difficult to show that $\mathcal{T}_j$ and $\mathcal{S}_j$ are 0-demicontractive, and $\mathrm{Id} - \mathcal{T}_j$ and $\mathrm{Id} - \mathcal{S}_j$ are demiclosed at zero for all $j = 1,2,\dots,N$. We also define a bounded linear operator $\Theta : \mathbb{R} \to \mathbb{R}$ by $\Theta\mu = 3\mu$. Thus, $\Theta^*\mu = 3\mu$ and $\Theta = 3$. It is clear that $0 \in \Omega$, where $\Omega = \{\mu \in \bigcap_{j=1}^{N} \mathrm{Fix}(\mathcal{T}_j) : \Theta\mu \in \bigcap_{j=1}^{N} \mathrm{Fix}(\mathcal{S}_j)\}$. Hence, $\Gamma = \bigcap_{i=1}^{M} \mathcal{EP}(g_i) \cap \Omega = 0$.

Set $\xi_k = \xi = 0.5$, $\tilde{\alpha}_{k,j} = \tilde{\alpha}_k = \frac{1}{2^{k+1}}$, $\tilde{\beta}_{k,j} = \tilde{\beta}_k = \frac{1}{2^k}$, $\vartheta = \frac{1}{7}$, $\varrho_i = \frac{i}{(M+1)}$, $M = 2 \times 10^5$ and $N = 3 \times 10^5$. Since

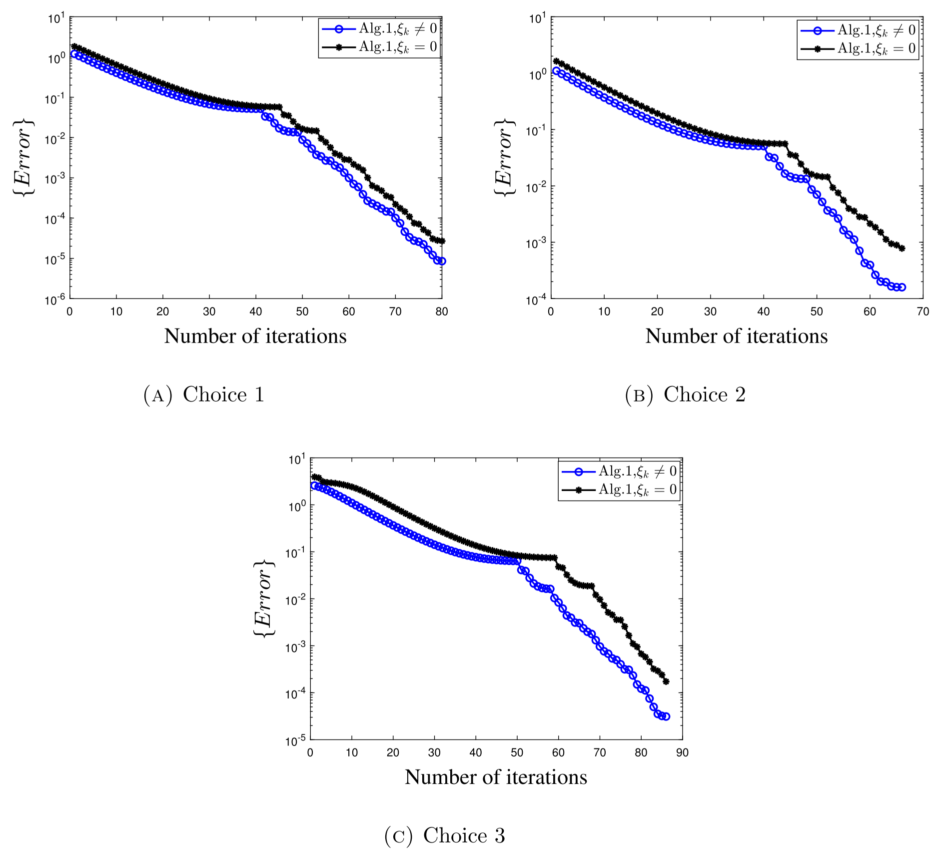
$$\begin{cases} \min\{\frac{1}{k^2\|\nu_k-\nu_{k-1}\|}, 0.5\} & \text{if } \nu_k \neq \nu_{k-1}; \\ 0.5 & \text{otherwise}. \end{cases}$$

The terminating criteria of Algorithm 1 is set as Error $= E_k = \|\nu_k - \nu_{k-1}\| < 10^{-6}$. Table 1 summarizes the computation of Algorithm 1 and its variant.

The terminating criterion E_k and (ν_k) summarized in Table 1 for Algorithm 1 are depicted in Fig. 1.

Table 1 Summary of the Numerical Computation for Algorithm 1

	No. of Iter. $\xi_k = 0$	Alg.1, $\xi_k \neq 0$	CPU(s) $\xi_k = 0$	Alg.1, $\xi_k \neq 0$
Choice 1. $\nu_0 = (5)$, $\nu_1 = (2)$	90	82	0.076019	0.068766
Choice 2. $\nu_0 = (4.4)$, $\nu_1 = (1.8)$	85	68	0.074187	0.067455
Choice 3. $\nu_0 = (-9)$, $\nu_1 = (4)$	102	88	0.075195	0.063578



(A) Choice 1

(B) Choice 2

(C) Choice 3

Figure 1 Comparison of Alg.1, $\xi_k \neq 0$, $\xi_k = 0$

We can see from Table 1 and Fig. 1 that Alg.1 with $\xi_k \neq 0$ outperforms Alg.1 with $\xi_k = 0$ with respect to the reduction in the error, time consumption, and the number of iterations required for the convergence towards the common solution.

5 Conclusions

In this paper, we have investigated an inertial-based, parallel, hybrid, extragradient algorithm for constructing iteratively a common solution of the pseudomonotone EP and the SCFPP associated with the finite families demicontractive mappings in Hilbert spaces. The abstract formalism of the problem has been strengthened with the computer-assisted simulation for the algorithm via an appropriate numerical example. We emphasize that our proposed abstract formalism together with the computer-assisted iterative algorithm arise naturally in various forms of real-world applications and would be an important topic of future research.

Acknowledgements
The author Yasir Arfat was supported by the Petchra Pra Jom Klao Ph.D Research Scholarship from King Mongkut's University of Technology Thonburi, Thailand (Grant No.16/2562).

Funding
The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research was funded by the National Science, Research and Innovation Fund (NSRF), King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-66-36. This research has received funding support from the NSRF via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation [grant number B39G660025].

Availability of data and materials
Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
YA and TS wrote the initial draft of the manuscript while MAAK gave the idea. YA and ZI performed the computational work. MAAK and PK supervised the final draft of the manuscript.

Author details
[1]KMUTT Fixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand. [2]Center of Excellence in Theoretical and Computational Science (TaCS-CoE), SCL 802 Fixed Point Laboratory, Science Laboratory Building, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand. [3]Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan. [4]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan. [5]Faculty of Science Energy and Environment, King Mongkut's University of Technology North Bangkok, Rayong Campus (KMUTNB), 21120, Rayong, Thailand. [6]Department of Mathematics, University of Gujrat, Gujrat, 50700, Pakistan.

Received: 8 April 2022 Accepted: 16 October 2023 Published online: 12 January 2024

References

1. Alvarez, F., Attouch, H.: An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Anal. **9**, 3–11 (2001)
2. Anh, P.N.: A hybrid extragradient method for pseudomonotne equilibrium problems and fixed point problems. Bull. Malays. Math. Sci. Soc. **36**(1), 107–116 (2013)
3. Arfat, Y., Iyiola, O.S., Khan, M.A.A., Kumam, P., Kumam, W., Sitthithakerngkiet, K.: Convergence analysis of the shrinking approximants for fixed point problem and generalized split common null point problem. J. Inequal. Appl. **67**, 1–21 (2022)
4. Arfat, Y., Khan, M.A.A., Kumam, P., Kumam, W., Sitthithakerngkiet, K.: Iterative solutions via some variants of extragradient approximants in Hilbert spaces. AIMS Math. **7**(8), 13910–13926 (2022)
5. Arfat, Y., Kumam, P., Khan, M.A.A., Cho, Y.J.: A hybrid steepest-descent algorithm for convex minimization over the fixed point set of multivalued mappings. Carpath. J. Math. **39**(1), 303–314 (2023)
6. Arfat, Y., Kumam, P., Khan, M.A.A., Iyiola, O.S.: Multi-inertial parallel hybrid projection algorithm for generalized split null point problems. J. Appl. Math. Comput. **68**, 3179–3198 (2022)
7. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiamsunthorn, P.S.: Parallel shrinking inertial extragradient approximants for pseudomonotone equilibrium, fixed point and generalized split null point problem. Ric. Mat. (2021). https://doi.org/10.1007/s11587-021-00647-4
8. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiamsunthorn, P.S.: Shrinking approximants for fixed point problem and generalized split null point problem in Hilbert spaces. Optim. Lett. **16**, 1895–1913 (2022). https://doi.org/10.1007/s11590-021-01810-4
9. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiamsunthorn, P.S.: An accelerated Visco-Cesaro means Tseng type splitting method for fixed point and monotone inclusion problems. Carpath. J. Math. **38**(2), 281–297 (2022)
10. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiamsunthorn, P.S.: An inertial extragradient algorithm for equilibrium and generalized split null point problems. Adv. Comput. Math. **48**, 53 (2022)
11. Arfat, Y., Kumam, P., Khan, M.A.A., Ngiamsunthorn, P.S., Kaewkhao, A.: A parallel hybrid accelerated extragradient algorithm for pseudomonotone equilibrium, fixed point, and split null point problems. Adv. Differ. Equ. **2021**, 364 (2021)
12. Arfat, Y., Kumam, P., Ngiamsunthorn, P.S., Khan, M.A.A.: An accelerated projection based parallel hybrid algorithm for fixed point and split null point problems in Hilbert spaces. Math. Methods Appl. Sci. **1**(19) (2021). https://doi.org/10.1002/mma.7405
13. Bianchi, M., Schaible, S.: Generalized monotone bifunctions and equilibrium problems. J. Optim. Theory Appl. **90**, 31–43 (1996)
14. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)

15. Ceng, L.C., Coroian, I., Qin, X., Yao, J.C.: A general viscosity implicit iterative algorithm for split variational inclusions with hierarchical variational inequality constraints. Fixed Point Theory **20**, 469–482 (2019)
16. Ceng, L.C., Petrusel, A., Qin, X., Yao, J.C.: A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems. Fixed Point Theory **21**, 93–108 (2020)
17. Ceng, L.C., Shang, M.J.: Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings. Optimization **70**, 715–740 (2021)
18. Chang, S.S., Kim, J.K., Wang, X.R.: Modified block iterative algorithm for solving convex feasibility problems in Banach spaces. J. Inequal. Appl. **14** (2010)
19. Chen, J.Z., Ceng, L.C., Qiu, Y.Q., Kong, Z.R.: Extra-gradient methods for solving split feasibility and fixed point problems. Fixed Point Theory Appl. **2015**, 192 (2015)
20. Chidume, C.E., Bello, A.U., Ndambomve, P.: Strong and Δ-convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in CAT(0) spaces. Abstr. Appl. Anal. **2014**, Article ID 805168 (2014)
21. Cui, H.H., Ceng, L.C.: Iterative solutions of the split common fixed point problem for strictly pseudo-contractive mappings. J. Fixed Point Theory Appl. **20**, 1–12 (2018)
22. Cui, H.H., Zhang, H.X., Ceng, L.C.: An inertial Censor-Segal algorithm for split common fixed-point problems. Fixed Point Theory **22**, 93–103 (2021)
23. Guan, J.L., Ceng, L.C., Hu, B.: Strong convergence theorem for split monotone variational inclusion with constraints of variational inequalities and fixed point problems. J. Inequal. Appl. **2018**, 311 (2018)
24. He, L., Cui, Y.L., Ceng, L.C., Zhao, T.Y., Wang, D.Q., Hu, H.Y.: Strong convergence for monotone bilevel equilibria with constraints of variational inequalities and fixed points using subgradient extragradient implicit rule. J. Inequal. Appl. **2021**, 146 (2021)
25. Jailoka, P., Suantai, S.: Split null point problems and fixed point problems for demicontractive multivalued mappings. Mediterr. J. Math. **15**, 1–19 (2018)
26. Jailoka, P., Suantai, S.: The split common fixed point problem for multivalued demicontractive mappings and its applications. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**, 689–706 (2019)
27. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Èkon. Mat. Metody **12**, 747–756 (1976)
28. Martinez-Yanes, C., Xu, H.K.: Strong convergence of CQ method for fixed point iteration processes. Nonlinear Anal. **64**, 2400–2411 (2006)
29. Polyak, B.T.: Some methods of speeding up the convergence of iteration methods. USSR Comput. Math. Math. Phys. **4**(5), 1–17 (1964)
30. Quoc, T.D., Muu, L.D., Hien, N.V.: Extragradient algorithms extended to equilibrium problems. Optimization **57**, 749–776 (2008)
31. Song, Y., Cho, Y.J.: Some note on Ishikawa iteration for multi-valued mappings. Bull. Korean Math. Soc. **48**, 575–584 (2011)
32. Tada, A., Takahashi, W.: Strong Convergence Theorem for an Equilibrium Problem and a Nonexpansive Mapping. Nonlinear Anal. Conv. Anal. Yokohama Pub., Yokohama (2006)
33. Tiel, J.: Convex Analysis. An Introductory Text. Wiley, Chichester (1984)
34. Tufa, A.R., Zegeye, H., Thuto, M.: Convergence theorems for non-self mappings in CAT(0) spaces. Numer. Funct. Anal. Optim. **38**, 705–722 (2017)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.