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On wavelets Kantorovich (p,q)-Baskakov operators and approximation properties

Alexander E. Moreka[1], Santosh Kumar[2,3*] and M. Mursaleen[4,5]

*Correspondence: drsengar2002@gmail.com
[2]Department of Mathematics, College of Natural and Applied Sciences, University of Dar es Salaam, Dar es Salaam, 35062, Tanzania
[3]Department of Mathematics, North Eastern Hill University, Shillong, Meghalaya 793022, India
Full list of author information is available at the end of the article

Abstract

In this paper, we generalize and extend the Baskakov-Kantorovich operators by constructing the (p,q)-Baskakov Kantorovich operators

$$(\Upsilon_{n,b,p,q}h)(x) = [n]_{p,q} \sum_{b=0}^{\infty} q^{b-1} \upsilon_{b,n}^{p,q}(x) \int_{\mathbb{R}} h(y)\Psi\left([n]_{p,q} \frac{q^{b-1}}{p^{n-1}} y - [b]_{p,q}\right) d_{p,q}y.$$

The modified Kantorovich (p,q)-Baskakov operators do not generalize the Kantorovich q-Baskakov operators. Thus, we introduce a new form of this operator. We also introduce the following useful conditions, that is, for any $0 \leq b \leq \omega$, such that $\omega \in \mathbb{N}$, Ψ_ω is a continuous derivative function, and $0 < q < p \leq 1$, we have $\int_{\mathbb{R}} x^b \Psi_\omega(x)\, d_{p,q}x = 0$. Also, for every $\Psi \in L_\infty$,

(a) there exists a finite constant γ such that $\gamma > 0$ with the property $\Psi \subset [0,\gamma]$,
(b) its first ω moment vanishes, that is, for $1 \leq b \leq \omega$, we have that $\int_{\mathbb{R}} y^b \Psi(y)\, d_{p,q}y = 0$,
(c) and $\int_{\mathbb{R}} \Psi(y)\, d_{p,q}y = 1$.

Furthermore, we estimate the moments and norm of the new operators. And finally, we give an upper bound for the operator's norm.

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1 Introduction and preliminaries

It has been three decades since Alexandru Lupas [18] for the first time introduced the notion of quantum calculus in the field of approximation theory. Since there, the area has become more active in research due to its application in different fields of science, engineering, and mathematics. Many researchers work on the extension of operators (see in Aral *et al.* [5]), the extended operators were known as exponential-type operators, which include Baskakov operators, Szász-Mirakyan operators, Meyer-König-Zeller operators, Picard operators, Weierstrass operators, and Bleiman, Butzer and Hahn operators. Moreover, q-analogue of standard integral operators of Kantorovich- and Durrmeyer-type was introduced.

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The classical Baskakov operators for functions that are continuous on $[0,\infty)$ were established by Baskakov [6], and the Baskakov-Kantorovich operators using integration were constructed by Ditzian and Totik [10]. In 2009, Zhang and Zhu [27] investigated some preservation properties, including monotonicity, smoothness, and convexity of Baskakov-Kantorovich operators. Aral and Gupta [3] and Radu [26] generalized the Baskakov operators using a q-integer. With the use of q-integration Gupta and Radu [12] proposed the following Kantorovich variant of the q-Baskakov operators:

$$(\xi_{n,b,q}h)(x) = [n]_q \sum_{b=0}^{\infty} q^{b-1} C_{b,n}^q(x) \int_{\frac{q[b]_q}{[n]_q}}^{\frac{[b+1]_q}{[n]_q}} h\big(q^{1-b}y\big)\, d_q y, \tag{1.1}$$

where

$$C_{b,n}^q = \binom{n+b-1}{b} \frac{x^b}{(1+x)_q^{n+b}} q^{\frac{b(b-1)}{2}}.$$

Recently, many researchers have established and investigated the approximation properties of positive linear operators by employing the techniques of post-quantum calculus. Several operators have been defined, and their approximation properties are discussed in [14–17, 20–24].

(p,q)-calculus play a vital role in differential equations, physical sciences, hypergeometric series, and oscillator algebra. For example, Burban [8] uses the concept of (p,q)-calculus to present the (p,q)-analogue of two-dimensional conformal field theory based on the (p,q)-deformation of the $su(1,1)$ subalgebra of the Virasoro algebra.

Based on (p,q)-calculus, Aral and Gupta [4] constructed the (p,q)-analogue of classical Baskakov operators for $x \in [0,\infty)$ and $0 < q < p \leq 1$, which is given as

$$(\Gamma_{n,p,q}h)(x) = \sum_{b=0}^{\infty} \upsilon_{n,b}^{p,q}(x) h\left(\frac{p^{n-1}[b]_{p,q}}{q^{b-1}[n]_{p,q}}\right), \tag{1.2}$$

where

$$\upsilon_{n,b}^{p,q}(x) = \binom{n+b-1}{b}_{p,q} p^{\frac{b+n(n-1)}{2}} q^{\frac{b(b-1)}{2}} \frac{x^b}{(1 \oplus x)_{p,q}^{n+b}}.$$

Furthermore, Aral and Gupta [4] give the moments of operators (1.2). That is, for $e_j(x) = x^j$, such that $j = 0, 1, 2$ and $n \in \mathbb{N}$, the following holds:

$$(\Gamma_{n,p,q}e_0)(x) = 1, (\Gamma_{n,p,q}e_1)(x) = x, (\Gamma_{n,p,q}e_2)(x) = x^2 + x\frac{p^{n-1}}{[n]_{p,q}}\left(1 + \frac{p}{q}x\right) \tag{1.3}$$

for $x \in [0,1]$ and $0 < q < p \leq 1$.

Acar *et al.* [1] defined a (p,q)-analogue of modified Kantrovich-Baskakov operators, such that for $x \in [0,\infty)$ and $0 < q < p \leq 1$, the Kantrovich variant of modified (p,q)-Baskakov operators is presented as follows:

$$(B_{n,p,q}h)(x) = [n]_{p,q} \sum_{b=0}^{\infty} \upsilon_{n,b}^{p,q}(x) q^{-b} \int_{\frac{p[b]_{p,q}}{[n]_{p,q}}}^{\frac{[b+1]_{p,q}}{[n]_{p,q}}} h\left(\frac{p^{n-1}y}{q^{b-1}}\right) d_{p,q}y. \tag{1.4}$$

Definition 1.1 Suppose that $0 < q < p \leq 1$, then for any nonnegative integer n, we have the (p,q)-integer denoted by $[n]_{p,q}$ and defined as:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} \quad \text{and} \quad [0]_{p,q} = 0.$$

Definition 1.2 The (p,q)-power basis is also known as (p,q)-binomial expansion, that is, for $n \in \mathbb{N}$, we have:

$$(ax+by)^n_{p,q} = \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \binom{n}{k}_{p,q} a^{n-k} b^k x^{n-k} y^k.$$

Definition 1.3 The (p,q)-derivative of the function $f : \mathbb{R} \to \mathbb{R}$ is denoted as $D_{p,q}f$ and is defined as:

$$(D_{p,q}f)(y) = \frac{f(py) - f(qy)}{(p-q)y}, \quad \text{for } y \neq 0.$$

If f is differentiable at $y = 0$, then $(D_{p,q}f)(0) = f'(0)$ holds. The assertions below hold true

$$D_{p,q}(y \oplus b)^n_{p,q} = [n]_{p,q}(py \oplus b)^{n-1}_{p,q} \quad \text{for } n \geq 1,$$
$$D_{p,q}(b \oplus y)^n_{p,q} = -[n]_{p,q}(b \oplus qy)^{n-1}_{p,q} \quad \text{for } n \geq 1,$$

and $D_{p,q}(b \oplus y)^0_{p,q} = 0$.

The formula for product and quotient (p,q)-derivative is:

$$D_{p,q}\big(f(x)h(x)\big) = h(qx)D_{p,q}\big(f(x)\big) + f(qx)D_{p,q}\big(h(x)\big)$$
$$\text{and} \quad D_{p,q}\left(\frac{f(x)}{h(x)}\right) = \frac{h(qx)D_{p,q}(f(x)) - f(qx)D_{p,q}(h(x))}{h(qx)h(px)} \quad \text{respectively.}$$

Definition 1.4 Let $f : C[a,b] \to \mathbb{R}$ for $b > a$, then the (p,q)-integration of f is:

$$\int_0^a h(y)\, d_{p,q}y = (p-q)a \sum_{i=0}^{\infty} h\left(\frac{q^i}{p^{i+1}} a\right) \frac{q^i}{p^{i+1}} \quad \text{when } \left|\frac{p}{q}\right| < 1,$$
$$\int_0^a h(y)\, d_{p,q}y = (q-p)a \sum_{i=0}^{\infty} h\left(\frac{p^i}{q^{i+1}}\right) \frac{p^i}{q^{i+1}} \quad \text{when } \left|\frac{q}{p}\right| < 1. \tag{1.5}$$

The integral in Equation (1.5) is not always positive unless is assumed that h is nondecreasing function. Therefore, Acar *et al.* [1] introduced the following (p,q)-integration to avoid some technical error during the construction of the Kantorovich modification of various operators:

$$\int_a^b h(y)\, d_{p,q}y = (p-q)(b-a) \sum_{n=0}^{\infty} h\left(a + (b-a)\frac{q^n}{p^{n+1}}\right) \frac{q^n}{p^{n+1}} \quad \text{when } \left|\frac{q}{p}\right| < 1,$$
$$\int_a^b h(y)\, d_{p,q}y = (q-p)(b-a) \sum_{n=0}^{\infty} h\left(a + (b-a)\frac{p^n}{q^{n+1}}\right) \frac{p^n}{q^{n+1}} \quad \text{when } \left|\frac{p}{q}\right| < 1. \tag{1.6}$$

2 Construction of operators

Refer to some critical facts on wavelets as defined by Meyer [19] and Graps [11]. The wavelets formed by dilation and translation of a single function Ψ (known as basic wavelets or mother wavelets) are the set of functions that take the form

$$\Psi_{\eta,\lambda}(x) = \eta^{-\frac{1}{2}}\Psi\left(\frac{x-\lambda}{\eta}\right), \quad \text{for } \eta > 0 \text{ and } \lambda \in \mathbb{R}.$$

If a and b are integers, then in the Franklin-Stromberg theory, we replace the constant η by 2^a, and $2^a b$ replaces λ. Given an arbitrary function $h \in L_2$ in analysis of this function, the wavelets will take the significant part of orthonormal basis, and the function h is defined as:

$$h(x) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\beta(a,b)\Psi_{a,b}(x),$$

where,

$$\beta(a,b) = 2^{\frac{a}{2}}\int_{\mathbb{R}} h(x)\Psi\left(2^a x - b\right)dx.$$

An orthonormal basis for $L_2(\mathbb{R})$ defined in the form $2^{\frac{a}{2}}\Psi_\omega(2^a x - b)$, where a and b are integers, and ω is the positive integer, was constructed by Daubechies [9] with $[0, 2\omega+1]$ as Ψ_ω support. If $\alpha\omega$ is the order of continuous derivatives of Ψ_ω and α is a positive constant, then for any $0 \leq b \leq \omega$, where ω is a natural number, we have:

$$\int_{\mathbb{R}} x^b \Psi_\omega(x)\,dx = 0. \tag{2.1}$$

Now, if we put $\omega = 0$, the system reduces to a Haar system. Using wavelets in the construction of Baskakov-type operators, Agratini [2] established the following condition for $\Psi \in L_\infty$,

(a) there exists a finite constant γ such that $\gamma > 0$ with the property $\Psi \subset [0,\gamma]$,
(b) its first ω moment vanishes; that is, for $1 \leq b \leq \omega$, we have that $\int_{\mathbb{R}} y^b \Psi(y)\,dy = 0$,
(c) $\int_{\mathbb{R}} \Psi(y)\,dy = 1$.

With the use of the Haar basis, Agratini [2] constructed wavelet Baskakov operators and defined them as:

$$(\mu_{n,b}h)(x) = n\sum_{b=0}^{\infty}\binom{n+b-1}{b}\frac{x^b}{(1+x)^{n+b}}\int_{\mathbb{R}} h(y)\Psi(ny-b)\,dy. \tag{2.2}$$

The operators defined by Equation (2.2) extend the Baskakov-Kantorovich operators defined by Ditzian and Totik [10]. Furthermore, for $\Psi \subset [0,\gamma]$, Agratini [2] expressed the operators $\mu_{n,b}h$ in the form of:

$$(\mu_{n,b}h)(x) = \sum_{b=0}^{\infty}\binom{n+b-1}{b}\frac{x^b}{(1+x)^{n+b}}\int_0^{\gamma} h\left(\frac{y+b}{n}\right)\Psi(y)\,dy. \tag{2.3}$$

In construction of the q-Baskakov-type operators, Nasiruzzaman *et al.* [25] introduced other conditions. Let a positive constant be γ, and let $\Psi_\omega(x)$ be any continuous derivatives of order $\gamma\omega$. Also, for $0 \le b \le \omega$, such that $\omega \in \mathbb{N}$ and $q > 0$, we have:

$$\int_{\mathbb{R}} x^b \Psi_\omega(x)\, d_q x = 0. \tag{2.4}$$

Note that if we put $q = 1$, Equation (2.4) reduces to Equation (2.1), and if we take $\omega = 0$ and $q = 1$, the system goes to the Haar basis. So, Nasiruzzaman *et al.* [25] provide the following conditions $\forall \Psi \in L_\infty$

(a) there exists a finite constant γ such that $\gamma > 0$ with the property $\Psi \subset [0,\gamma]$,
(b) its first ω moment vanishes; that is, for $1 \le b \le \omega$, we have that $\int_{\mathbb{R}} y^b \Psi(y)\, d_q y = 0$,
(c) $\int_{\mathbb{R}} \Psi(y)\, d_q y = 1$.

The following is the q-analogy for Baskakov-Kantorovich-type wavelet operators introduced by Nasiruzzaman *et al.* [25]

$$(\delta_{n,b,q} h)(x) = [n]_q \sum_{b=0}^{\infty} q^{b-1} C_{b,n}^{q} \int_{\mathbb{R}} h(y)\Psi\big(q^{b-1}[n]_q y - [b]_q\big)\, d_q y. \tag{2.5}$$

The operators defined by Equation (2.5) extended the Kantorovich q-Baskakov operators defined by Radu [26]. That is, if we choose $\omega = 0$ and Ψ Haar basis, Equation (2.5) reduces to operators defined by Equation (1.1). Additionally, by choosing $\omega = 0$, $q = 1$, and Ψ Haar basis, we get the Kantorovich modification of Baskakov operators defined by Ditzian and Totik [10]. For $\Psi \subset [0,\gamma]$, operators defined by Equation (2.5) can be rewritten as:

$$(\delta_{n,b,q} h)(x) = \sum_{b=0}^{\infty} C_{b,n}^{q} \int_0^{\gamma} h\left(\frac{y + [b]_q}{q^{b-1}[n]_q}\right)\Psi(y)\, d_q y. \tag{2.6}$$

Taking $q = 1$, Equation (2.6) reduces to classical Baskakov-Kantorovich wavelet operators defined by Equation (2.3).

In this section, the Kantorovich (p,q)-Baskakov operators is constructed with the help of Daubechies compactly-supported wavelets. Since the modified Kantorovich (p,q)-Baskakov operators defined by Equation (1.4) do not generalize the Kantorovich q-Baskakov operators expressed by Equation (1.1), that is, for $q = 1$, Equation (1.4) is not equal to Equation (1.1). Therefore, Equation (1.4) must be rewritten. Now, to achieve our objective of constructing the Kantorovich (p,q)-Baskakov operators, we define the new operators, generalizing the operators defined by Equation (1.1), as follows:

$$\big(B^*_{n,p,q} h\big)(x) = [n]_{p,q} \sum_{b=0}^{\infty} \upsilon_{n,b}^{p,q}(x) q^{b-1} \int_{\frac{q[b]_{p,q}}{[n]_{p,q}}}^{\frac{[b+1]_{p,q}}{[n]_{p,q}}} h\left(\frac{q^{b-1} y}{p^{n-1}}\right) d_{p,q} y. \tag{2.7}$$

We see that by choosing $p = 1$, Equation (2.7) reduces to Equation (1.1). Hence, Equation (1.1) is generalized by Equation (2.7).

Lemma 2.1 *For $n \in \mathbb{N}$ and $0 < q < p \leq 1$. The following holds*

$$\int_{\frac{q[b]_{p,q}}{[n]_{p,q}}}^{\frac{[b+1]_{p,q}}{[n]_{p,q}}} d_{p,q}y = \frac{p^b}{[n]_{p,q}},$$

$$\int_{\frac{q[b]_{p,q}}{[n]_{p,q}}}^{\frac{[b+1]_{p,q}}{[n]_{p,q}}} y\, d_{p,q}y = \frac{qp^b[b]_{p,q}}{[n]^2_{p,q}} + \frac{p^{2b}}{(p+q)[n]^2_{p,q}},$$

$$\int_{\frac{q[b]_{p,q}}{[n]_{p,q}}}^{\frac{[b+1]_{p,q}}{[n]_{p,q}}} y^2\, d_{p,q}y = \frac{q^2p^b[b]^2_{p,q}}{[n]^3_{p,q}} + 2\frac{p^{2b}q[n]_{p,q}}{(p+q)[n]^3_{p,q}} + \frac{p^{3b}}{[n]^3_{p,q}(p^2+pq+q^2)}.$$

Proof Using Equation (1.6), we have that

$$\begin{aligned}
\int_{\frac{q[b]_{p,q}}{[n]_{p,q}}}^{\frac{[b+1]_{p,q}}{[n]_{p,q}}} d_{p,q}y &= (p-q)\left(\frac{[b+1]_{p,q}}{[n]_{p,q}} - \frac{q[b]_{p,q}}{[n]_{p,q}}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} \\
&= \frac{(p-q)}{[n]_{p,q}p}\left(p^b + q[b]_{p,q} - q[b]_{p,q}\right)\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^n \\
&= \frac{p^b(p-q)}{[n]_{p,q}p}\frac{p}{(p-q)} \\
&= \frac{p^b}{[n]_{p,q}}.
\end{aligned}$$

The two remaining parts can be proved similarly as the first part. Note that for some simple calculation, we have $[b+1]_{p,q} = p^b + q[b]_{p,q}$ and $\sum_{n=0}^{\infty}(\frac{q}{p})^n = \frac{p}{p-q}$. □

Though we have some conditions that are considered in the construction of the operators, we have to introduce other conditions to make the wavelets useful in our study. Let a positive constant be γ and Ψ_ω be a continuous derivative of order $\gamma\omega$; on top of that, suppose that for any $0 \leq b \leq \omega$ such that $\omega \in \mathbb{N}$ and $q > 0$, we have:

$$\int_{\mathbb{R}} x^b \Psi_\omega(x)\, d_{p,q}x = 0. \tag{2.8}$$

Choosing $q = 1$, the system reduces to Equation (2.4), and for $p = q = 1$, the system reduces to Equation (2.1). Equation (2.8) becomes a Haar system by choosing $\omega = 0$ an $p = q = 1$. Now, we present the following conditions. For every $\Psi \in L_\infty$,

(a) there exists a finite constant γ such that $\gamma > 0$ with the property $\Psi \subset [0, \gamma]$,
(b) its first ω moment vanishes; that is, for $1 \leq b \leq \omega$, we have that $\int_{\mathbb{R}} y^b \Psi(y)\, d_{p,q}y = 0$,
(c) $\int_{\mathbb{R}} \Psi(y)\, d_{p,q}y = 1$.

For $x \in [0, \infty)$ and $0 < q < p \leq 1$, below is the (p,q)-analogue of the Baskakov-Kantorovich-type wavelet operators:

$$(\Upsilon_{n,b,p,q}h)(x) = [n]_{p,q}\sum_{b=0}^{\infty} q^{b-1}\upsilon^{p,q}_{b,n}(x)\int_{\mathbb{R}} h(y)\Psi\left([n]_{p,q}\frac{q^{b-1}}{p^{n-1}}y - [b]_{p,q}\right) d_{p,q}y. \tag{2.9}$$

The operators $\Upsilon_{n,b,p,q}$ extend Kantorovich (p,q)-Baskakov operators defined by Equation (2.7), that is, for the choice of $\omega = 0$ and Ψ Haar basis, Equation (2.9) reduces to Equation (2.7), in addition to that, for $p = 1$, Equation (2.9) reduces to Kantorovich q-Baskakov operators defined by Equation (1.1). Furthermore, for the choice of $\omega = 0$, $p = q = 1$ and Ψ Haar basis, Equation (2.9) reduces to classical Kantorovich-Baskakov operators defined by Ditzian and Totik [10]. Now, for $\Psi \subset [0,\gamma]$, Equation (2.9) is rewritten in the form:

$$(\Upsilon_{n,b,p,q}h)(x) = \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \int_0^{\gamma} h\left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}\right)\Psi(y)\,d_{p,q}y. \tag{2.10}$$

Note that clearly for the choice of $p = 1$, Equations (2.9) and (2.10) reduce to q-Baskakov Kantorovich wavelet operators defined by Equations (2.5) and (2.6), respectively. Similarly, for the choice of $p = q = 1$, the Equations (2.9) and (2.10) reduce to classical Baskakov-Kantorovich wavelet operators defined by the Equations (2.3) and (2.4), respectively.

3 Main results

3.1 Moments of $\Upsilon_{n,b,p,q}$

In this section, we present the moments of wavelet Kantorovich (p,q)-Baskakov operators. That is for $0 < q < p \leq 1$, we have the following theorem:

Theorem 3.1 *Let $e_j = y^j$, $\forall 0 \leq j \leq \omega$ and $\omega \in \mathbb{N}$. Then, for $x \in [0,\infty]$, we have* $(\Upsilon_{n,b,p,q}e_j)(x) = (\Gamma_{n,b,p,q}e_j)(x)$.

Proof To prove this theorem, refer to Inequality (2.10). Now, Equation (2.10) can be written as:

$$\begin{aligned}
(\Upsilon_{n,b,p,q}e_j)(x) &= \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \int_{\mathbb{R}} \left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}\right)^j \Psi(y)\,d_{p,q}y \\
&= \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \left(\frac{p^{n-1}}{[n]_{p,q}q^{b-1}}\right)^j \int_{\mathbb{R}} (y+[b]_{p,q})^j \Psi(y)\,d_{p,q}y \\
&= \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \left(\frac{p^{n-1}}{[n]_{p,q}q^{b-1}}\right)^j \int_{\mathbb{R}} \left(\sum_{a=0}^{\infty} \binom{j}{a} y^a [b]_{p,q}^{j-a}\right) \Psi(y)\,d_{p,q}y \\
&= \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \left(\frac{p^{n-1}}{[n]_{p,q}q^{b-1}}\right)^j \\
&\quad \times \left[\int_{\mathbb{R}} [b]_{p,q}^j \Psi(y)\,d_{p,q}y + \int_{\mathbb{R}} \sum_{a=1}^{\infty} \binom{j}{a} y^a [b]_{p,q}^{j-a} \Psi(y)\,d_{p,q}y\right].
\end{aligned}$$

From condition (b) above, given that $\int_{\mathbb{R}} \sum_{a=1}^{\infty} \binom{j}{a} y^a [b]_{p,q}^{j-a} \Psi(y)\,d_{p,q}y = 0$, we have

$$\begin{aligned}
(\Upsilon_{n,b,p,q}e_j)(x) &= \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \left(\frac{p^{n-1}}{[n]_{p,q}q^{b-1}}\right)^j \int_{\mathbb{R}} [b]_{p,q}^j \Psi(y)\,d_{p,q}y \\
&= \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \left(\frac{[b]_{p,q}p^{n-1}}{[n]_{p,q}q^{b-1}}\right)^j \int_{\mathbb{R}} \Psi(y)\,d_{p,q}y.
\end{aligned}$$

And from condition (c) above, given that $\int_{\mathbb{R}} \Psi(y)\, d_{p,q}y = 1$, we have:

$$\begin{aligned}(\Upsilon_{n,b,p,q}e_j)(x) &= \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x)\left(\frac{[b]_{p,q}p^{n-1}}{[n]_{p,q}q^{b-1}}\right)^j \\ &= (\Gamma_{n,b,p,q,}, e_j)(x).\end{aligned}$$

□

Remark 3.1 Theorem (3.1) implies that the moments of the operators $\Upsilon_{n,b,p,q,}$ defined by Equation (2.9) is the same as that of the operators $\Gamma_{n,b,p,q,}$ defined by Aral and Gupta [4]. So, we have the following Lemma proved by Aral and Gupta [4].

Lemma 3.1 [4] *For* $x \in [0,1]$ *and* $0 < q < p \le 1$, *we have*

i. $(\Upsilon_{n,b,p,q}e_0)(x) = 1$,
ii. $(\Upsilon_{n,b,p,q}e_1)(x) = x$,
iii. $(\Upsilon_{n,b,p,q}e_2)(x) = x^2 + x\frac{p^{n-1}}{[n]_{p,q}}(1 + \frac{p}{q}x)$.

3.2 Characterization of second-order Lipschitz functions

In this section, we shall present some Bernstein-Markov types of inequality of Kantorovich (p,q)-Baskakov operators, which will be used as our preliminary result to state our main result of this section. The following facts are also needed:

Peetre's K-functional defined as:

$$K_2(h,y) = \inf_{g\in C''[0,\infty)\cap C_B[0,\infty)}\left\{\|h-g\|_\infty + y\left\|g''\right\|_\infty\right\} \quad \text{for } y>0. \tag{3.1}$$

For $g \notin L_\infty[0,\infty)$, $\|h-g\|_\infty = \infty$, the K-functional defined by equality (3.1) is equivalent to the modulus of smoothness. Johnen [13] gives the following relation: for some constant $\varrho > 0$ and any $\nu > 0$, we have:

$$\varrho^{-1}\omega_2(h,y) \le K_2\left(h,y^2\right) \le \varrho\omega_2(h,y) \quad \text{such that} \quad h\in C_B \text{ and } 0<y\le\nu, \tag{3.2}$$

where

$$\omega_2(h,t) = \sup_{0<g\le t}\left|\triangle_g^2 h\right|_\infty,$$

and

$$\triangle_g^2 h(x) = \begin{cases} h(x+g) - 2h(x) + h(x-g) & \text{for } g\le x,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 3.2 *For all* $h \in C[0,\infty)\cap L_\infty[0,\infty)$, *then the following inequalities hold for* $0 < q < p \le 1$;

i. $\|\Upsilon_{n,b,p,q}h\|_\infty \le \alpha\|h\|_\infty\|\Psi\|_\infty$,
ii. $\|\Upsilon'_{n,b,p,q}h\|_\infty \le 2\alpha[n]_{p,q}\|h\|_\infty\|\Psi\|_\infty$,
iii. $\|\Upsilon''_{n,b,p,q}h\|_\infty \le 4\alpha[n]_{p,q}[n+1]_{p,q}\|h\|_\infty\|\Psi\|_\infty$.

Proof Take into consideration that

$$\int_0^\gamma h\left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}\right)\Psi(y)\,d_{p,q}y = \mathcal{J}_h(n,b,p,q).$$

Thus, for every $h \in C[0,\infty) \cap L_\infty[0,\infty)$, the inequality

$$\left|\mathcal{J}_h(n,b,p,q)\right| \le \alpha\|h\|_\infty\|\Psi\|_\infty \quad \text{holds.}$$

From Equation (2.10), we have

$$\begin{aligned}
\|\Upsilon_{n,b,p,q}h\| &= \left|\sum_{b=0}^{\infty}\upsilon_{b,n}^{p,q}(x)\int_0^\gamma h\left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}\right)\Psi(y)\,d_{p,q}y\right| \\
&= \left|\sum_{b=0}^{\infty}\upsilon_{b,n}^{p,q}(x)\mathcal{J}_h(n,b,p,q)\right| \\
&\le \sum_{b=0}^{\infty}\upsilon_{b,n}^{p,q}(x)\left|\mathcal{J}_h(n,b,p,q)\right| \\
&\le \alpha\|h\|_\infty\|\Psi\|_\infty.
\end{aligned}$$

Employing the definition of (p,q)-derivative to prove inequalities (ii) and (iii). That is, in $\upsilon_{b,n}^{p,q}(x)$, we have that

$$\begin{aligned}
D_{p,q}\left(\frac{x^b}{(1\oplus x)_{p,q}^{n+b}}\right) &= \frac{(1+qx)_{p,q}^{n+b}[b]_{p,q}x^{b-1} + (qx)^b xpq[n+b]_{p,q}(1+qx)_{p,q}^{n+b-1}}{(1+qx)_{p,q}^{n+b}(1+px)_{p,q}^{n+b}} \\
&= \frac{1}{x(1+qx)}\left(\frac{(1+qx)[b]_{p,q} + xq^{b+1}p[n+b]_{p,q}}{(1+px)_{p,q}^{n+b}}\right)x^b.
\end{aligned}$$

Here, $D_{p,q}(x^b) = [b]_{p,q}x^{b-1}$, $D_{p,q}(b\oplus x)_{p,q}^{n+b} = -[n+b]_{p,q}pq(1\oplus qx)_{p,q}^{n+b-1}$. By making some simple computations, we have $[n+b]_{p,q} = p^n[b]_{p,q} + q^b[n]_{p,q}$.

Therefore, using the facts above, we have that

$$\begin{aligned}
D_{p,q}\left(\upsilon_{b,n}^{p,q}(x)\right) &= \frac{1}{x(1+qx)}\left((1+qx)[b]_{p,q} + xq^{b+1}p\left(p^n[b]_{p,q} + q^b[n]_{p,q}\right)\right) \\
&\quad\times \binom{n+b-1}{b}_{p,q} p^{\frac{b+n(n-1)}{2}} q^{\frac{b(b-1)}{2}} \frac{x^b}{(1+px)_{p,q}^{n+b}} \\
&\simeq \frac{[n]_{p,q}}{x(1+qx)}\left(\left(1+qx+q^{b+1}p^{n+1}x\right)\frac{[b]_{p,q}}{[n]_{p,q}} + q^{2b+1}px\right)\upsilon_{b,n}^{p,q}(x), \quad \text{for } x>0.
\end{aligned}$$

For every $b \in \mathbb{N}$, we have the following equality:

$$\frac{[b]_{p,q}}{[n]_{p,q}}\frac{q^{1-b}}{p^{n+1}}\upsilon_{b,n}^{p,q}(x) = x\upsilon_{b-1,n+2}^{p,q}(x).$$

That is,

$$\frac{[b]_{p,q}}{[n]_{p,q}}\upsilon_{b,n}^{p,q}(x) = xp^{n+1}q^{b-1}\upsilon_{b-1,n+2}^{p,q}(x).$$

Hence, we get that

$$\begin{aligned}
D_{p,q}\upsilon_{b,n}^{p,q}(x) &\simeq \frac{[n]_{p,q}}{(1+qx)}\big(\big(1+qx+q^{b+1}p^{n+1}x\big)p^{n+1}q^{b-1}\upsilon_{b-1,n+2}^{p,q}(x)+q^{2b+1}p\upsilon_{b,n}^{p,q}(x)\big)\\
&= \frac{[n]_{p,q}}{(1+qx)}\sum_{b=0}^{\infty}\mathcal{J}_h(n,b,p,q)\big(\big(1+qx+q^{b+1}p^{n+1}x\big)p^{n+1}q^{b-1}\upsilon_{b-1,n+2}^{p,q}(x)\\
&\quad+q^{2b+1}p\upsilon_{b,n}^{p,q}(x)\big)\\
&= \frac{[n]_{p,q}}{(1+qx)}\Bigg[\sum_{b=0}^{\infty}\big(1+qx+q^{b+1}p^{n+1}x\big)p^{n+1}q^{b-1}\upsilon_{b-1,n+2}^{p,q}(x)\mathcal{J}_h(n,b,p,q)\\
&\quad+\sum_{b=0}^{\infty}q^{2b+1}p\upsilon_{b,n}^{p,q}(x)\mathcal{J}_h(n,b,p,q)\Bigg]\\
&= \frac{[n]_{p,q}}{(1+qx)}\Bigg[\sum_{b=0}^{\infty}\big(1+qx+q^{b+1}p^{n+1}x\big)p^{n}q^{b}\upsilon_{b,n+1}^{p,q}(x)\mathcal{J}_h(n,b+1,p,q)\\
&\quad+\sum_{b=0}^{\infty}q^{2b+1}p\upsilon_{b,n}^{p,q}(x)\mathcal{J}_h(n,b,p,q)\Bigg].
\end{aligned}$$

Then,

$$\begin{aligned}
\Upsilon'_{n,b,p,q,}h(x) &\simeq \Bigg|\frac{[n]_{p,q}}{(1+qx)}\Bigg[\sum_{b=0}^{\infty}\big(1+qx+q^{b+1}p^{n+1}x\big)p^{n}q^{b}\upsilon_{b,n+1}^{p,q}(x)\mathcal{J}_h(n,b+1,p,q)\\
&\quad+\sum_{b=0}^{\infty}q^{2b+1}p\upsilon_{b,n}^{p,q}(x)\mathcal{J}_h(n,b,p,q)\Bigg]\Bigg|\\
&\leq \frac{[n]_{p,q}}{(1+qx)}\Bigg[\sum_{b=0}^{\infty}\big|\big(1+qx+q^{b+1}p^{n+1}x\big)p^{n}q^{b}\upsilon_{b,n+1}^{p,q}(x)\mathcal{J}_h(n,b+1,p,q)\big|\\
&\quad+\sum_{b=0}^{\infty}\big|q^{2b+1}p\upsilon_{b,n}^{p,q}(x)\mathcal{J}_h(n,b,p,q)\big|\Bigg]\\
&\leq \frac{[n]_{p,q}}{(1+qx)}\Bigg[\sum_{b=0}^{\infty}\upsilon_{b,n+1}^{p,q}(x)\big|\mathcal{J}_h(n,b+1,p,q)\big|+\big|\Upsilon_{n,b,p,q,h}(x)\big|\Bigg]\\
&\leq [n]_{p,q}\big[\alpha\|h\|_\infty\|\Psi\|_\infty+\alpha\|h\|_\infty\|\Psi\|_\infty\big]\\
&= 2\alpha[n]_{p,q}\|h\|_\infty\|\Psi\|_\infty.
\end{aligned}$$

In a similar way, we have that

$$\begin{aligned}
\big(\Upsilon''_{n,b,p,q,}h\big)(x) &\leq \frac{[n+1]_{p,q}}{(1+qx)}\big(\Upsilon'_{n,b,p,q,}h\big)(x)+\Bigg[\frac{[n]_{p,q}[n+1]_{p,q}}{(1+qx)}\big(\Upsilon'_{n,b,p,q,}h\big)(x)\\
&\quad\times\Bigg(\sum_{b=0}^{\infty}\upsilon_{b,n+2}^{p,q}(x)\mathcal{J}_h(n,b+2,p,q)+\sum_{b=0}^{\infty}\upsilon_{b,n+1}^{p,q}(x)\mathcal{J}_h(n,b+1,p,q)\Bigg)\Bigg].
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\|\Upsilon''_{n,b,p,q,}h\| &\leq \left| \frac{[n+1]_{p,q}}{(1+qx)} \big(\Upsilon'_{n,b,p,q,}h\big)(x) + \left[\frac{[n]_{p,q}[n+1]_{p,q}}{(1+qx)} \big(\Upsilon'_{n,b,p,q,}h\big)(x) \right.\right. \\
&\quad \left.\left. \times \left(\sum_{b=0}^{\infty} \upsilon_{b,n+2}^{p,q}(x)\mathcal{J}_h(n,b+2,p,q) + \sum_{b=0}^{\infty} \upsilon_{b,n+1}^{p,q}(x)\mathcal{J}_h(n,b+1,p,q) \right) \right] \right| \\
&\leq \frac{[n+1]_{p,q}}{(1+qx)} \|\Upsilon'_{n,b,p,q,}h\| + \left[\frac{[n]_{p,q}[n+1]_{p,q}}{(1+qx)} \|\Upsilon'_{n,b,p,q,}h\| \right. \\
&\quad \left. \times \left(\sum_{b=0}^{\infty} \upsilon_{b,n+2}^{p,q}(x)\big|\mathcal{J}_h(n,b+2,p,q)\big| + \sum_{b=0}^{\infty} \upsilon_{b,n+1}^{p,q}(x)\big|\mathcal{J}_h(n,b+1,p,q)\big| \right) \right] \\
&\leq [n+1]_{p,q}\|\Upsilon'_{n,b,p,q,}h\| + \big[[n]_{p,q}[n+1]_{p,q}\big(\alpha\|h\|_\infty\|\Psi\|_\infty + \alpha\|h\|_\infty\|\Psi\|_\infty\big)\big] \\
&\leq 2[n]_{p,q}[n+1]_{p,q}\alpha\|h\|_\infty\|\Psi\|_\infty + 2[n]_{p,q}[n+1]_{p,q}\alpha\|h\|_\infty\|\Psi\|_\infty \\
&= 4[n]_{p,q}[n+1]_{p,q}\alpha\|h\|_\infty\|\Psi\|_\infty.
\end{aligned}$$

□

Theorem 3.3 *Given that $0 < q < p \leq 1$ and K_2 is the Peetre's K-functional, then for all $h \in C''[0,\infty) \cap C_B[0,\infty)$, we have the following*

$$\big|(\Upsilon_{n,b,p,q}h)(x) - h(x)\big| \leq \big(\gamma\|\Psi\|_\infty + 1\big)K_2\left(h, \frac{\gamma^2}{3[n]_{p,q}^2} + \frac{x}{[n]_{p,q}}\left(1 + \frac{p}{q}x\right)\right).$$

Proof Let $h \in C''[0,\infty) \cap C_B[0,\infty)$, the Taylor series expansion of a function h is given as

$$g(y) = g(x) + (y-x)g'(x) + \int_x^y (y-v)g''(v)\,d_{p,q}v.$$

Therefore, using Equation (1.3) and Theorem 3.1, we have

$$\begin{aligned}
\big|(\Upsilon_{n,b,p,q}g)(x) - g(x)\big| &= \left| \Upsilon_{n,b,p,q}\left(\left(\int_x^y (y-v)g''(v)d_{p,q(v)} \right), x \right) \right| \\
&= \left| \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \int_0^{\gamma} \left(\int_x^{\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}} \left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}} - v \right) \right.\right. \\
&\quad \left.\left. \times \big|g''(x)\big|\,d_{p,q}v \right) \Psi(y)\,d_{p,q}y \right| \\
&\leq \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \int_0^{\gamma} \left(\left| \int_x^{\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}} \frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}} - v \right| \right. \\
&\quad \left. \times \big|g''(x)\big|\,d_{p,q}v \right) \|\Psi\|_\infty\,d_{p,q}y \\
&\leq \|\Psi\|_\infty\|g''\|_\infty \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \int_0^{\gamma} \left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}} - x \right)^2 d_{p,q}y \\
&= \gamma\|\Psi\|_\infty\|g''\|_\infty \left(\frac{\gamma^2}{3[n]_{p,q}^2} + \frac{\gamma}{[n]_{p,q}}\mu_{n,1,p,q}(x) + \mu_{n,2,p,q}(x) \right) \\
&= \gamma\|\Psi\|_\infty\|g''\|_\infty \left(\frac{\gamma^2}{3[n]_{p,q}^2} + \frac{x}{[n]_{p,q}}\left(1 + \frac{p}{q}x\right) \right).
\end{aligned}$$

Using Theorem 3.2 (i) and taking infimum over all $g \in C''[0,\infty) \cap C_B[0,\infty)$, we have the following:

$$\begin{aligned}\left|(\Upsilon_{n,b,p,q}h)(x) - h(x)\right| \le \inf_{g\in C''[0,\infty)\cap C_B[0,\infty)} &\left\{ \left\| \Upsilon_{n,b,p,q}(h-g)\right\|_\infty + \|h-g\|_\infty \right.\\ &\left. + \left|(\Upsilon_{n,b,p,q}g)(x) - g(x)\right| \right\}.\end{aligned}$$

That is,

$$\begin{aligned}\left|(\Upsilon_{n,b,p,q}h)(x) - h(x)\right| &\le \inf_{g\in C''[0,\infty)\cap C_B[0,\infty)} \left\{ \left(\gamma\|\Psi\|_\infty + 1\right)\|h-g\|_\infty + \gamma\|\Psi\|_\infty \left\|g''\right\|_\infty \right.\\ &\quad \left. + \gamma\|\Psi\|_\infty\left\|g''\right\|_\infty \left(\frac{\gamma^2}{3[n]_{p,q}^2} + \frac{x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right)\right)\right\}\\ &\le \left(\gamma\|\Psi\|_\infty + 1\right) \inf_{g\in C''[0,\infty)\cap C_B[0,\infty)} \left\{ \|h-g\|_\infty \right.\\ &\quad \left. + \left\|g''\right\|_\infty \left(\frac{\gamma^2}{3[n]_{p,q}^2} + \frac{x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right)\right)\right\}\\ &= \left(\gamma\|\Psi\|_\infty + 1\right)K_2\left(\frac{\gamma^2}{3[n]_{p,q}^2} + \frac{x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right)\right). \qquad \square\end{aligned}$$

From Theorem 3.3, let $N = \gamma\|\Psi\|_\infty$, $\xi = \frac{\gamma^2}{3}$ and $\varphi_{n,b,p,q} = x(1+\frac{p}{q}x)$. The following corollary holds for the operators $(\Upsilon_{n,b,p,q}h)$.

Corollary 3.1 *For any* $0 < q < p \le 1$ *and* $h \in C_B[0,\infty)$, *then*

$$\left|(\Upsilon_{n,b,p,q}h)(x) - h(x)\right| \le (N+1)K_2\left(h, \frac{\xi}{[n]_{p,q}^2} + \frac{\varphi_{n,b,p,q}}{[n]_{p,q}}\right),$$

where K_2 *defined by Equality* (3.1).

Furthermore, if Theorem 3.3 and Inequality (3.2) are well known, we easily obtain the following results explained by corollary below.

Corollary 3.2 *For any* $0 < \rho < 2$ *and* $h \in C_B[0,\infty)$, *if* $\omega_2(h,y) = O(y^\rho)$ *then*

$$\left|(\Upsilon_{n,b,p,q}h)(x) - h(x)\right| \le k\left(\frac{\xi}{[n]_{p,q}^2} + \frac{\varphi_{n,b,p,q}}{[n]_{p,q}}\right)^{\frac{\rho}{2}}$$

holds for the operators $(\Upsilon_{n,b,p,q}h)$. *Such that* $k > 0$ *is a constant,* $\xi = \frac{\gamma^2}{3}$ *and* $\varphi_{n,b,p,q} = x(1+\frac{p}{q}x)$.

Remark 3.2 Corollary 3.2 gives the main result of this section. The function φ control the rate of convergence of the operators $\Upsilon_{n,b,p,q}$.

3.3 The norm of the operators $\Upsilon_{n,b,p,q}$ in L_p

In this section, we present the norm of $\Upsilon_{n,b,p,q}$ in L_p space. We shall use r to avoid confusion of using p as its already used in different definition and facts of (p,q)-calculus. Therefore, we shall write L_r instead of L_p. In this section, we shall use Theorem 3.2 and the Riesz-Thorin theorem given below (also see in the book [7]).

Theorem 3.4 (*Riesz-Thorin*) *Suppose that* $0 < r < \infty$ *and* $1 < s < \infty$, *also let* $T : S_0(d\mu;\mathbb{C}) \to L^{\mathrm{loc}}(d\nu;\mathbb{C})$ *be a linear operator. Now, if in the domain of* T *the following inequalities are valid*:

$$\|T(h)\|_{sk} \le N_k\|h\|_{rk}, \quad k = 0, 1,$$

then T *extends by continuity to an operator acting from* $L_{r(\vartheta)}(d\mu;\mathbb{C})$ *into* $L_{s(\vartheta)}(d\mu;\mathbb{C})$. *The norm of this operator does not exceed* $N_0^{1-\vartheta}N_1^{\vartheta}$.

Now, we state the main theorem of this section.

Theorem 3.5 *Let* $n > 1$, $1 \le r \le \infty$, $h \in L_r[0,\infty)$. *Then, we have*

$$\|\Upsilon_{n,b,p,q}h\|_r \le C_r\|h\|_r,$$

where C_r *is a constant.*

Remark 3.3 We shall not directly prove Theorem 3.5 because by the Riesz-Thorin Theorem 3.4 and Theorem 3.2, we shall consider only one case for $r = 1$. Therefore, we prove Theorem 3.6 below, and then we generalize it to Theorem 3.4 by finding the value of C_r.

Theorem 3.6 *Suppose that* $\gamma \le k$ *for* $k \in \mathbb{Z}^+$, *then for any* $h \in L_1[0,\infty)$, *we have the following*

$$\|\Upsilon_{n,b,p,q}h\|_1 \le N_1\|h\|_1,$$

where $N_1 = \frac{k[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty$.

Proof From Equation (2.10) for any $h \in L_1[0,\infty)$, we have

$$\begin{aligned}
\|\Upsilon_{n,b,p,q}h\|_1 &= \left\| \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \int_0^{\gamma} h\left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}\right)\Psi(y)\,d_{p,q}y \right\|_1 \\
&\le \int_0^{\infty/A} \sum_{b=0}^{\infty} \upsilon_{b,n}^{p,q}(x) \left(\int_0^{\gamma} \left| h\left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}\right)\right| \|\Psi\|_\infty\,d_{p,q}y \right) d_{p,q}x \\
&\le \|\Psi\|_\infty \sum_{b=0}^{\infty} \left(\int_0^{\gamma} \left| h\left(\frac{p^{n-1}(y+[b]_{p,q})}{[n]_{p,q}q^{b-1}}\right)\right| d_{p,q}y \right) \int_0^{\infty/A} \upsilon_{b,n}^{p,q}(x)\,d_{p,q}x.
\end{aligned}$$

However, by doing simple calculations in (p,q)-calculus, we have the following facts;

$$\int_0^{\infty/A} \frac{x^b}{(1\oplus x)_{p,q}^{n+b}}\,d_{p,q}x = \frac{1}{\mathcal{K}(A,b+1)}\beta_{p,q}(b+1,n-1).$$

But

$$\beta_{p,q}(b+1,n-1)=\frac{[b]_{p,q}}{[n-1]_{p,q}}\beta_{p,q}(b,n),$$

and

$$\mathcal{K}(x,y)=\frac{1}{x+1}x^{y}\left(1+\frac{1}{x}\right)_{p,q}^{y}(1+x)_{p,q}^{1-y}.$$

This implies that

$$\int_{0}^{\infty/A}\upsilon_{b,n}^{p,q}(x)\,d_{p,q}x=\frac{1}{[n-1]_{p,q}}\frac{q^{b}}{p^{n}}.$$

Hence we have that

$$\begin{aligned}\|\Upsilon_{n,b,p,q}h\|_1&=\frac{[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty\sum_{b=0}^{\infty}\int_{\frac{p^{n-1}[b]_{p,q}}{[n]_{p,q}q^{b-1}}}^{\frac{p^{n-1}(\gamma+[b]_{p,q})}{[n]_{p,q}q^{b-1}}}\left|h(x)\right|d_{p,q}x\\&\le\frac{[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty\sum_{b=0}^{\infty}\int_{\frac{p^{n-1}[b]_{p,q}}{[n]_{p,q}q^{b-1}}}^{\frac{p^{n-1}(k+[b]_{p,q})}{[n]_{p,q}q^{b-1}}}\left|h(x)\right|d_{p,q}x\\&=\frac{[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty\sum_{b=0}^{\infty}\sum_{i=0}^{k-1}\int_{\frac{p^{n-1}[b+i]_{p,q}}{[n]_{p,q}q^{b-1}}}^{\frac{p^{n-1}[b+i+1]_{p,q}}{[n]_{p,q}q^{b-1}}}\left|h(x)\right|d_{p,q}x.\end{aligned}$$

Is the same as writing

$$\begin{aligned}\|\Upsilon_{n,b,p,q}h\|_1&=\frac{[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty\sum_{i=0}^{k-1}\left(\sum_{b=0}^{\infty}\int_{\frac{p^{n-1}[b+i]_{p,q}}{[n]_{p,q}q^{b-1}}}^{\frac{p^{n-1}[b+i+1]_{p,q}}{[n]_{p,q}q^{b-1}}}\left|h(x)\right|d_{p,q}x\right)\\&\le\frac{[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty\sum_{i=0}^{k-1}\|h\|_1\\&=\frac{[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty\|h\|_1\sum_{i=0}^{k-1}1\\&=N_1\|h\|_1.\end{aligned}$$

□

Now, let us calculate the value of C_r, which is the upper bound of the operator's norm. According to the Riesz-Thorin Theorem 3.4, the norm of the operator $\Upsilon_{n,b,p,q}$ does not exceed $N_1^{1-\vartheta}N_0^{\vartheta}$. But, in our case, $N_0^{\vartheta}=N^{\vartheta}$ and $1-\vartheta=\frac{1}{r}$ for $\vartheta\in(0,1)$. Then, from Corollary 3.1, we have $N=\gamma\|\Psi\|_\infty$, and from Theorem 3.6, we have $N_1=\frac{k[n]_{p,q}}{[n-1]_{p,q}}\|\Psi\|_\infty$. So, we have

$$\begin{aligned}
C_r &= N_1^{1-\vartheta} N^{\vartheta} \\
&= \left(\frac{k[n]_{p,q}}{[n-1]_{p,q}} \|\Psi\|_{\infty} \right)^{\frac{1}{r}} \left(\gamma \|\Psi\|_{\infty} \right)^{1-\frac{1}{r}} \\
&= \gamma^{1-\frac{1}{r}} k^{\frac{1}{r}} \left(\frac{[n]_{p,q}}{[n-1]_{p,q}} \right)^{\frac{1}{r}} \|\Psi\|_{\infty}.
\end{aligned}$$

By choosing $\Psi = \chi_{[0,1)}$, we get $\|\Psi\|_{\infty} = 1$, $\gamma = 1$, and k becomes γ. Then, we have the following estimation of operators $\Upsilon_{n,b,p,q}$:

$$\|\Upsilon_{n,b,p,q} h\|_r \leq \left(\frac{[n]_{p,q}}{[n-1]_{p,q}} \right)^{\frac{1}{r}} \|h\|_r.$$

4 Conclusion

The constructed operators $\Upsilon_{n,b,p,q}$ generalize the Kantorovich wavelets q-Baskakov operators and Kantorovich-Baskakov wavelets operators; that is, for the choice of $p = 1$, we get operators defined by Equations (2.5) and (2.6), and also by choosing $p = q = 1$, operators defined by the Equations (2.9) and (2.10) reduce to Equations (2.2) and (2.3), respectively. Furthermore, operators $\Upsilon_{n,b,p,q}$ extend and generalize the Kantorovich q-Baskakov operators defined by Equation (1.1) and the classical Kantorovich-Baskakov defined by Baskakov [6] because, by choosing $\omega = 0$, $p = 1$ and Ψ a Haar basis, we get $\xi_{n,b,q}$. In addition to that, for $q = 1$, we get precisely the classical Kantorovich-Baskakov operators defined by Baskakov [6]. Using these facts, we can conclude that the constructed operators are more general as they generalize classical, q, and wavelets Kantorovich-Baskakov operators. We also observed that the moments of wavelets Kantorovich (p,q)-Baskakov operators are the same as that of clasical Kantorovich (p,q)-Baskakov operators.

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Author details
[1]Department of Computer Systems and Mathematics, Ardhi University, Dar es Salaam, Tanzania. [2]Department of Mathematics, College of Natural and Applied Sciences, University of Dar es Salaam, Dar es Salaam, 35062, Tanzania. [3]Department of Mathematics, North Eastern Hill University, Shillong, Meghalaya 793022, India. [4]Department of Mathematics, Aligarh Muslim University, Aligarh, India. [5]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan.

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