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A new class of fractional inequalities through the convexity concept and enlarged Riemann–Liouville integrals

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Abstract

Fractional inequalities play a crucial role in building mathematical mechanisms and their related solution functions across the majority of practical science domains. A variety of mathematical disciplines are significantly impacted by convexity as well. In this article, we describe and verify many new fractional inequalities using a thorough kind of Riemann–Liouville integral and the convexity criterion of the functions. Our approach for dealing with fractional integral inequalities is clear and easy to use, and the current study is a new addition to the literature. Additionally, it is simple to observe that all the inequalities produced are extensive and may be broken down into several and different inequalities that were previously in the literature.

1 Introduction

Researchers in geometry and analysis find convex analysis to be one of their most interesting fields. Convex functions differ from other topics due to their geometric, differential, and other supportive characteristics. Additionally, there are numerous applications for the convex set and convex function in mathematical physics, optimization theory, and others [1–6]. As fractional calculus has advanced so quickly in recent years, the connection to convexity has grown stronger.

Actually, fractional calculus provides several potentially useful techniques for resolving differential and integral equations, and it also addresses a number of other issues involving unique mathematical physics functions in addition to their generalizations and extensions in one or more variables.

In recognition of the significance of fractional calculus, scientists have used it to develop a number of fractional integral inequalities that have proven to be extremely valuable in approximation theory. By utilizing inequalities like the Hermite–Hadamard, Simpson's, midpoint, Ostrowski, and trapezoidal inequalities, for instance, we can ascertain the boundaries of formulas employed in numerical integration. The Hermite–Hadamard-type and trapezoidal-type inequalities were first established using the Riemann–Liouville fractional integrals by Sarikaya et al. in [7]. Following that, Set demonstrated in [8] how the Riemann–Liouville fractional operator produces the Ostrowski inequality for differentiable functions. Using harmonic convexity, Iscan and Wu established Hermite–Hadamard-type inequalities in [9]. A fractional integral operator and p-convex function are used to create a new version of Hermite–Hadamard inequalities in [10]. Additionally, Park in [11] studied the $\mathcal{MT}$-convex functions using Riemann–Liouville fractional integrals and the Hermite–Hadamard inequality for $\mathcal{MT}$-convex functions was also proven by him. Generalized fractional integrals were recently described by Sarikaya and Ertugral [12], who used these integrals to demonstrate a generalization of Hermite–Hadamard-type inequalities for convex functions. We direct readers to these sources [13–18] for additional information.

This article uses a comprehensive kind of Riemann–Liouville integral and the convexity condition of the functions to explain and prove some novel fractional integral inequalities. Our technique for dealing with fractional integral inequalities is simple and straightforward, and the current study is a fresh contribution to the body of literature. Additionally, it is easy to see that all of the inequalities generated are significant and may be divided into a number of other, distinct inequalities that have already been published in the literature. We set up our results, which are divided into the following sections, using generalized fractional integrals. The first section provides an introduction to some fundamental concepts, relevant terminology, and the outcomes that pertain to our key goals. Examining the Hermite–Hadamard inequality's error estimates with the expanded (μ,θ)-order Riemann–Liouville integrals is covered in the second section that also studies related inequalities. Finally, the conclusion is provided at the end of our study.

2 Preliminaries

Before moving on to the essential outcomes of the research, it is necessary to have a discussion about some of the pertinent terminology and results. A convex function $\mathfrak{M} : [a,b] \to \mathbf{R}$ is one that satisfies the following inequality:

$$\mathfrak{M}\big(\lambda x_1 + (1-\lambda)x_2\big) \le \lambda \mathfrak{M}(x_1) + (1-\lambda)\mathfrak{M}(x_2), \quad \lambda \in [0,1], \text{ and } x_1, x_2 \in [a,b]. \tag{1}$$

If the additive inverse of the function $\mathfrak{M}$ is convex, then the function $\mathfrak{M}$ is said to have a concave form. The following Hermite–Hadamard inequality gives us a tangible and geometric description of a convex function.

Theorem 2.1 [19] *Given that $\mathfrak{M} : [a,b] \to \mathbf{R}$ is a convex function, we acquire the subsequent inequality*:

$$\mathfrak{M}\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \mathfrak{M}(t)\,dt \le \frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}. \tag{2}$$

If $\mathfrak{M}$ is concave, the converse direction of (2) *is valid.*

The following are some basics and concepts of fractional calculus that will be utilized during the current research.

Definition 2.1 [20] Let $\mathfrak{M} \in L^1[a,b]$, $a, b \in \mathbf{R}$ with $a < b$. The fractional r-order integrals of Riemann–Liouville are suggested as:

$$\mathcal{N}_{a+}^{r}\mathfrak{M}(x) = \frac{1}{\Gamma(r)}\int_a^x (x-t)^{r-1}\mathfrak{M}(t)\,dt, \quad x > a, \tag{3}$$

$$\mathcal{N}^{r}_{b-}\mathfrak{M}(x) = \frac{1}{\Gamma(r)}\int_{x}^{b}(t-x)^{r-1}\mathfrak{M}(t)\,dt, \quad x < b, \tag{4}$$

respectively. Here, $\Gamma(r) = \int_0^\infty t^{r-1}\exp(t)\,dt$, and $\mathcal{N}^{0}_{a+}\mathfrak{M}(x) = \mathcal{N}^{0}_{b-}\mathfrak{M}(x) = \mathfrak{M}(x)$.

In their paper [21], Jarad et al. offered further exhaustive fractional integrals. In addition, they contributed a few features as well as links that included a variety of fractional integrals.

Definition 2.2 [21] Let $\mathfrak{M} \in L^1[a,b]$. If $\mu > 0$ and $\theta \in (0,1]$, the expanded (μ,θ)-order Riemann–Liouville integrals are defined by:

$$^{\mu}\mathcal{N}^{\theta}_{a+}\mathfrak{M}(x) = \frac{1}{\Gamma(\mu)}\int_{a}^{x}\left(\frac{(x-a)^{\theta}-(t-a)^{\theta}}{\theta}\right)^{\mu-1}\frac{\mathfrak{M}(t)}{(t-a)^{1-\theta}}\,dt, \quad x > a \tag{5}$$

and

$$^{\mu}\mathcal{N}^{\theta}_{b-}\mathfrak{M}(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{b}\left(\frac{(b-x)^{\theta}-(b-t)^{\theta}}{\theta}\right)^{\mu-1}\frac{\mathfrak{M}(t)}{(b-t)^{1-\theta}}\,dt, \quad x < b, \tag{6}$$

respectively.

It is interesting that Sarikaya et al. [7] established the subsequent inequality of Hermite–Hadamard type by employing Riemann–Liouville integrals $\mathcal{N}^{\mu}_{a+}$ and $\mathcal{N}^{\mu}_{b-}$.

Theorem 2.2 *Let $\mathfrak{M} \in L_1[a,b]$. If $\mathfrak{M}$ is convex and $\mathfrak{M} > 0$, then we have*:

$$\mathfrak{M}\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[\mathcal{N}^{\mu}_{a^+}\mathfrak{M}(b) + \mathcal{N}^{\mu}_{b^-}\mathfrak{M}(a)\right] \le \frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}. \tag{7}$$

In addition, Sarikaya and Yldrm [15] gave another inequality of Hermite–Hadamard type about the operators $\mathcal{N}^{\mu}_{a+}$ and $\mathcal{N}^{\mu}_{b-}$.

Theorem 2.3 *Suppose $\mathfrak{M} : [a,b] \to \mathbf{R}$ is convex, $\mathfrak{M} \in L_1[a,b]$, and $\mathfrak{M} > 0$. Then, the next inequalities are correct*:

$$\mathfrak{M}\left(\frac{a+b}{2}\right) \le \frac{2^{\mu-1}\Gamma(\mu+1)}{(b-a)^{\mu}}\left[\mathcal{N}^{\mu}_{(\frac{a+b}{2})^+}\mathfrak{M}(b) + \mathcal{N}^{\mu}_{(\frac{a+b}{2})^-}\mathfrak{M}(a)\right] \le \frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}. \tag{8}$$

Set et al. [16] proposed an important Hermite–Hadamard-type inequality by making use of the fractional integrals in equations (5) and (6).

Theorem 2.4 *Given that $\mathfrak{M}$ is a convex and positive function that goes from $[a,b]$ to $\mathbf{R}$, with $\mathfrak{M}$ being in $L_1[a,b]$, the operators $^{\mu}\mathcal{N}^{\theta}_{a+}$ and $^{\mu}\mathcal{N}^{\theta}_{b-}$ fulfill the following inequality*:

$$\mathfrak{M}\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\mu+1)\theta^{\mu}}{2(b-a)^{\theta\mu}}\left[{}^{\mu}\mathcal{N}^{\theta}_{a+}\mathfrak{M}(b) + {}^{\mu}\mathcal{N}^{\theta}_{b-}\mathfrak{M}(a)\right] \le \frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}, \tag{9}$$

where $Re(\mu) > 0$ and $\theta \in [0,1]$.

Based on Hermite–Hadamard form, Gözpınar [17] created an inequality for convex and positive functions that included the fractional operators (5) and (6) in the following manner:

Theorem 2.5 *Assume that $\mathfrak{M}$ is a convex and positive function that goes from $[a,b]$ to $\mathbf{R}$, with $\mathfrak{M}$ being in $L_1[a,b]$. Then, the operators ${}^{\mu}\mathcal{N}^{\theta}_{a+}$ and ${}^{\mu}\mathcal{N}^{\theta}_{b-}$ attain the following inequality*:

$$\begin{aligned}\mathfrak{M}\left(\frac{a+b}{2}\right) &\le \frac{2^{\theta\mu-1}\Gamma(\mu+1)\theta^{\mu}}{(b-a)^{\theta\mu}}\left[{}^{\mu}\mathcal{N}^{\theta}_{(\frac{a+b}{2})+}\mathfrak{M}(b)+{}^{\mu}\mathcal{N}^{\theta}_{(\frac{a+b}{2})-}\mathfrak{M}(a)\right]\\ &\le \frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}.\end{aligned} \tag{10}$$

3 Main outcomes

First, in order to investigate the error estimates of the Hermite–Hadamard inequality, the following integral identity is established:

Lemma 3.1 *Let $a,b\in\mathbf{R}$ with $a<b$ and $\mathfrak{M}:[a,b]\to\mathbb{R}$ be a function, also let $\mathfrak{Z}:[a,b]\to\mathbb{R}$ be a strictly monotone function such that $\mathfrak{M}\circ\mathfrak{Z}^{-1}$ is differentiable and $(\mathfrak{M}\circ\mathfrak{Z}^{-1})'\in L[a,b]$. Then, the following identity holds*:

$$\begin{aligned}&\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^{\mu}}-\frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta}}\left({}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(a)^+}\mathfrak{M}(b)+{}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\\ &\quad=\frac{\mathfrak{Z}(b)-\mathfrak{Z}(a)}{2}\int_0^1\left(\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu}-\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right)\\ &\qquad\times\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)'\left(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b)\right)dt.\end{aligned} \tag{11}$$

Proof First, we calculate the next integral:

$$\begin{aligned}&\int_0^1\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)'\left(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b)\right)dt\\ &\quad=\frac{-\mathfrak{M}(a)}{\theta^{\mu}(\mathfrak{Z}(b)-\mathfrak{Z}(a))}\\ &\qquad+\frac{\mu}{(\mathfrak{Z}(b)-\mathfrak{Z}(a))}\int_0^1\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu-1}\\ &\qquad\times\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)\left(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b)\right)\frac{dt}{(1-t)^{1-\theta}}\\ &\quad=\frac{-\mathfrak{M}(a)}{\theta^{\mu}(\mathfrak{Z}(b)-\mathfrak{Z}(a))}\\ &\qquad+\frac{\mu}{(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta+1}}\int_{\mathfrak{Z}(b)}^{\mathfrak{Z}(a)}\left[\frac{(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\theta}-(u-\mathfrak{Z}(a))^{\theta}}{\theta}\right]^{\mu-1}\\ &\qquad\times\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)(u)\frac{du}{(u-\mathfrak{Z}(a))^{1-\theta}}\\ &\quad=\frac{-\mathfrak{M}(a)}{\theta^{\mu}(\mathfrak{Z}(b)-\mathfrak{Z}(a))}+\frac{\Gamma(\mu+1)}{(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta+1}}J^{\mu}_{\mathfrak{Z}(a)^+}\mathfrak{M}(b).\end{aligned} \tag{12}$$

In a similar vein, the equation that follows can be obtained from integration by parts:

$$\begin{aligned}&\int_0^1\left[\frac{1-t^\theta}{\theta}\right]^\mu(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\,dt\\&\quad=\frac{\mathfrak{M}(b)}{\theta^\mu(\mathfrak{Z}(b)-\mathfrak{Z}(a))}-\frac{\Gamma(\mu+1)}{(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta+1}}{}^\mu\mathcal{N}^\theta_{\mathfrak{Z}(b)^-}\mathfrak{M}(a).\end{aligned}\tag{13}$$

The left-hand side can be obtained by employing (12) and (13) in the right-hand side of (11). □

Theorem 3.1 *Let $a,b\in\mathbf{R}$ with $a<b$ and $\mathfrak{M}:[a,b]\to\mathbb{R}$ be a function, also let $\mathfrak{Z}:[a,b]\to\mathbb{R}$ be a strictly monotone function such that $\mathfrak{M}\circ\mathfrak{Z}^{-1}$ is differentiable and $(\mathfrak{M}\circ\mathfrak{Z}^1)'\in L[a,b]$. If $|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'|$ is convex, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^\mu}-\frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta}}\left({}^\mu\mathcal{N}^\theta_{\mathfrak{Z}(a)^+}\mathfrak{M}(b)+{}^\mu\mathcal{N}^\theta_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\right|\\&\quad\le\frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2\theta^{(\mu+1)}}\left((\mathfrak{M}\circ\mathfrak{Z}^{-1})'\mathfrak{Z}(a)+(\mathfrak{M}\circ\mathfrak{Z}^{-1})'\mathfrak{Z}(b)\right)\\&\quad\times\left(\beta\left(\frac{1}{2^\theta},\frac{1}{\theta},\mu+1\right)-\beta\left(1-\frac{1}{2^\theta},\mu+1,\frac{1}{\theta}\right)\right).\end{aligned}\tag{14}$$

Proof By involving the property of the function of the absolute value in Lemma 3.1, one can obtain the following inequality:

$$\begin{aligned}&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^\mu}-\frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta}}\left({}^\mu\mathcal{N}^\theta_{\mathfrak{Z}(a)^+}\mathfrak{M}(b)+{}^\mu\mathcal{N}^\theta_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\right|\\&\quad\le\frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2}\int_0^1\left|\left[\frac{1-t^\theta}{\theta}\right]^\mu-\left[\frac{1-(1-t)^\theta}{\theta}\right]^\mu\right|\\&\quad\times\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\right|dt.\end{aligned}\tag{15}$$

The following inequality can be obtained by employing the convexity of $(\mathfrak{M}\circ\mathfrak{Z}^{-1})'$ on the right-hand side of the preceding inequality (15):

$$\begin{aligned}&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^\mu}-\frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta-1}}\left({}^\mu\mathcal{N}^\theta_{\mathfrak{Z}(a)^+}\mathfrak{M}(b)+{}^\mu\mathcal{N}^\theta_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\right|\\&\quad\le\frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2}\int_0^1\left|\left[\frac{1-t^\theta}{\theta}\right]^\mu-\left[\frac{1-(1-t)^\theta}{\theta}\right]^\mu\right|\\&\quad\times\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\right|dt\\&\quad\le\frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2}\left(\int_0^{\frac12}\left[\frac{1-t^\theta}{\theta}\right]^\mu-\left[\frac{1-(1-t)^\theta}{\theta}\right]^\mu\right.\\&\quad\times\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\right|dt\\&\quad+\left.\int_{\frac12}^1\left[\frac{1-(1-t)^\theta}{\theta}\right]^\mu-\left[\frac{1-t^\theta}{\theta}\right]^\mu\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\right|dt\right)\\&\quad\le\frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2}\left((\mathfrak{M}\circ\mathfrak{Z}^{-1})'\mathfrak{Z}(a)\left(\int_0^{\frac12}t\left[\frac{1-t^\theta}{\theta}\right]^\mu dt-\int_0^{\frac12}t\left[\frac{1-(1-t)^\theta}{\theta}\right]^\mu dt\right.\right.\end{aligned}\tag{16}$$

$$
\begin{aligned}
&+\int_{\frac{1}{2}}^{1} t\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu} dt-\int_{\frac{1}{2}}^{1} t\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} dt\Big) \\
&+\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)'\mathfrak{Z}(b)\Big(\int_{0}^{\frac{1}{2}}(1-t)\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} dt-\int_{0}^{\frac{1}{2}}(1-t)\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu} dt \\
&+\int_{\frac{1}{2}}^{1}(1-t)\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu} dt-\int_{\frac{1}{2}}^{1}(1-t)\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} dt\Big)\Big) \\
=&\frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2\theta^{(\mu+1)}}\left(\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)'\mathfrak{Z}(a)+\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)'\mathfrak{Z}(b)\right) \\
&\times\left(\beta\left(\frac{1}{2^{\theta}},\frac{1}{\theta},\mu+1\right)-\beta\left(1-\frac{1}{2^{\theta}},\mu+1,\frac{1}{\theta}\right)\right).
\end{aligned}
$$

This completes the proof. □

Corollary 3.1 *If $\theta = 1$ in the inequality (14), then one can obtain the next inequality*:

$$
\begin{aligned}
&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}-\frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu}}\left({}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(a)^{+}}\mathfrak{M}(b)+{}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(b)^{-}}\mathfrak{M}(a)\right)\right| \\
&\quad\le\frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2(\mu+1)}\left(1-\frac{1}{2^{\mu}}\right)\left(\left|\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)'\mathfrak{Z}(a)\right|+\left|\left(\mathfrak{M}\circ\mathfrak{Z}^{-1}\right)'\mathfrak{Z}(b)\right)\right|.
\end{aligned}
\tag{17}
$$

Corollary 3.2 *If $\mathfrak{Z}(x) = \frac{1}{x}$ in the inequality (14), then one can obtain the next inequality*:

$$
\begin{aligned}
&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^{\mu}}-\frac{\Gamma(\mu+1)}{2}\left(\frac{ab}{b-a}\right)^{\mu\theta}\left({}^{\mu}\mathcal{N}^{\theta}_{(\frac{1}{a})^{+}}\mathfrak{M}\circ g\left(\frac{1}{b}\right)+{}^{\mu}\mathcal{N}^{\theta}_{(\frac{1}{b})^{-}}\mathfrak{M}\circ g\left(\frac{1}{a}\right)\right)\right| \\
&\quad\le\frac{b-a}{2ab\theta^{\mu+1}}\left(a^{2}\mathfrak{M}'(a)+b^{2}\mathfrak{M}'(b)\right)\left(\beta\left(\frac{1}{2^{\theta}},\frac{1}{\theta},\mu+1\right)-\beta\left(1-\frac{1}{2^{\theta}},\mu+1,\frac{1}{\theta}\right)\right),
\end{aligned}
\tag{18}
$$

where $g(s) = \frac{1}{s}$.

Corollary 3.3 *If $\mathfrak{Z}(x) = \frac{1}{x}$ and $\theta = 1$ in the inequality (14), then one can obtain the next inequality*:

$$
\begin{aligned}
&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}-\frac{\Gamma(\mu+1)}{2}\left(\frac{ab}{b-a}\right)^{\mu-1}\left({}^{\mu}\mathcal{N}^{\theta}_{(\frac{1}{a})^{+}}\mathfrak{M}\circ g\left(\frac{1}{b}\right)+{}^{\mu}\mathcal{N}^{\theta}_{(\frac{1}{b})^{-}}\mathfrak{M}\circ g\left(\frac{1}{a}\right)\right)\right| \\
&\quad\le\frac{b-a}{2ab}\left(a^{2}\mathfrak{M}'(a)+b^{2}\mathfrak{M}'(b)\right)\left(\beta\left(\frac{1}{2},1,\mu+1\right)-\beta\left(\frac{1}{2},\mu+1,1\right)\right),
\end{aligned}
\tag{19}
$$

where $g(s) = \frac{1}{s}$.

Corollary 3.4 *If $\mathfrak{Z}(x) = \frac{1}{x}$, $\mu = 1$ and $\theta = 1$ in the inequality (14), then one can obtain the next inequality*:

$$
\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2}-\frac{ab}{(b-a)}\int_{\frac{1}{b}}^{\frac{1}{a}}\mathfrak{M}\circ g(s)\,ds\right|\le\frac{b-a}{8ab}\left\{\left|a^{2}\mathfrak{M}'(a)+b^{2}\mathfrak{M}'(b)\right|\right\},
\tag{20}
$$

where $g(t) = \frac{1}{t}$.

Theorem 3.2 *Let $a, b \in r$ with $a < b$ and $\mathfrak{M} : [a,b] \to \mathbb{R}$ be a function, also let $\mathfrak{Z} : [a,b] \to \mathbb{R}$ be a strictly monotone function such that $\mathfrak{M} \circ \mathfrak{Z}^{-1}$ is differentiable and $(\mathfrak{M} \circ \mathfrak{Z}^{1})' \in L[a,b]$. If $|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'|^q, q > 1$ is convex, then the following inequality holds*:

$$\begin{aligned} &\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^{\mu}} - \frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta}}\left({}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(a)^+}\mathfrak{M}(b) + {}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\right| \\ &\quad \leq \frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2^{\frac{1}{q}}\theta^{(\mu+1)}}\left(\beta\left(\frac{1}{2^{\theta}}, \frac{1}{\theta}, \mu+1\right) - \beta\left(1-\frac{1}{2^{\theta}}, \mu+1, \frac{1}{\theta}\right)\right) \\ &\qquad \times \left(\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(\mathfrak{Z}(a))\right|^q + \left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(\mathfrak{Z}(b))\right|^q\right)^{\frac{1}{q}}. \end{aligned} \tag{21}$$

Proof The proof can be divided into two cases. Case 1: $q = 1$. By using the property of the absolute value function and the convexity of $|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'|$ in Lemma 3.1, inequality (14) is obtained.

Case 2: $q > 1$. By using the property of the absolute value function and power mean inequality on the right-hand side of inequality (11), the following inequality is established:

$$\begin{aligned} &\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^{\mu}} - \frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta}}\left({}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(a)^+}\mathfrak{M}(b) + {}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\right| \\ &\quad \leq \frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2}\left(\int_0^1\left|\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right|\right)^{1-\frac{1}{q}} \\ &\qquad \times \left(\int_0^1\left|\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right|\right. \\ &\qquad \left. \times \left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{22}$$

We also have that

$$\begin{aligned} &\int_0^1\left|\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right| dt \\ &\quad = \int_0^{\frac{1}{2}}\left(\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right) dt \\ &\qquad + \int_{\frac{1}{2}}^1\left(\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-t^{\theta}}{\theta}\right]^{\mu}\right) dt \\ &\quad = \frac{2}{\theta^{\mu+1}}\left(\beta\left(\frac{1}{2^{\theta}}, \frac{1}{\theta}, \mu+1\right) - \beta\left(1-\frac{1}{2^{\theta}}, \mu+1, \frac{1}{\theta}\right)\right). \end{aligned} \tag{23}$$

Also, by the convexity of $|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'|^q$, we obtain the following inequality:

$$\begin{aligned} &\int_0^1\left|\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right|\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\right|^q dt \\ &\quad \leq \int_0^{\frac{1}{2}}\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu} \\ &\qquad \times \left(t\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(\mathfrak{Z}(a))\right|^q + (1-t)\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(\mathfrak{Z}(b))\right|^q\right) dt \\ &\qquad + \int_{\frac{1}{2}}^1\left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} \end{aligned}$$

$$
\begin{aligned}
&\times \big(t\big|(\mathfrak{M}\circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(a))\big|^q+(1-t)\big|(\mathfrak{M}\circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(b))\big|^q\big)\,dt\\
&\leq \big|(\mathfrak{M}\circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(a))\big|^q\biggl(\int_0^{\frac{1}{2}} t\biggl[\frac{1-t^\theta}{\theta}\biggr]^\mu\,dt-\int_0^{\frac{1}{2}} t\biggl[\frac{1-(1-t)^\theta}{\theta}\biggr]^\mu\,dt \qquad (24)\\
&+\int_{\frac{1}{2}}^1 t\biggl[\frac{1-(1-t)^\theta}{\theta}\biggr]^\mu\,dt-\int_{\frac{1}{2}}^1 t\biggl[\frac{1-t^\theta}{\theta}\biggr]^\mu\,dt\biggr)\\
&+\big|(\mathfrak{M}\circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(b))\big|^q\biggl(\int_0^{\frac{1}{2}} (1-t)\biggl[\frac{1-t^\theta}{\theta}\biggr]^\mu\,dt-\int_0^{\frac{1}{2}} (1-t)\biggl[\frac{1-(1-t)^\theta}{\theta}\biggr]^\mu\,dt\\
&+\int_{\frac{1}{2}}^1 (1-t)\biggl[\frac{1-(1-t)^\theta}{\theta}\biggr]^\mu\,dt-\int_{\frac{1}{2}}^1 (1-t)\biggl[\frac{1-t^\theta}{\theta}\biggr]^\mu\,dt\biggr)\\
&=\frac{1}{\theta^{\mu+1}}(\big|(\mathfrak{M}\circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(a))\big|^q\\
&+\big|(\mathfrak{M}\circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(b))\big|^q)\biggl(\beta\biggl(\frac{1}{2^\theta},\frac{1}{\theta},\mu+1\biggr)-\beta\biggl(1-\frac{1}{2^\theta},\mu+1,\frac{1}{\theta}\biggr)\biggr).
\end{aligned}
$$

Using (23) and (24) in (22), the inequality (21) can be obtained. □

Remark 3.1 If $\theta = 1$ in the inequality (21), then one can obtain Theorem 4 in [22].

Corollary 3.5 *If* $\mathfrak{Z}(x) = x$ *in the inequality* (21), *then one can obtain the next inequality*:

$$
\begin{aligned}
&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^\mu}-\frac{\Gamma(\mu+1)}{2(b-a)^{\mu\theta}}\big({}^\mu\mathcal{N}^\theta_{(a)^+}\mathfrak{M}(b)+{}^\mu\mathcal{N}^\theta_{(b)^-}\mathfrak{M}(a)\big)\right|\\
&\leq \frac{b-a}{2^{\frac{1}{q}}\theta^{(\mu+1)}}\biggl(\beta\biggl(\frac{1}{2^\theta},\frac{1}{\theta},\mu+1\biggr)-\beta\biggl(1-\frac{1}{2^\theta},\mu+1,\frac{1}{\theta}\biggr)\biggr) \qquad (25)\\
&\times\Big(\big|\mathfrak{M}'(a)\big|^q+\big|(\mathfrak{M}'(b)\big|^q\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.6 *If* $\mathfrak{Z}(x) = \frac{1}{x}$ *in the inequality* (21), *then one can obtain the next inequality*:

$$
\begin{aligned}
&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^\mu}-\frac{\Gamma(\mu+1)}{2}\biggl(\frac{ab}{b-a}\biggr)^{\mu\theta}\biggl({}^\mu\mathcal{N}^\theta_{(\frac{1}{a})^+}\mathfrak{M}\circ g\biggl(\frac{1}{b}\biggr)+{}^\mu\mathcal{N}^\theta_{(\frac{1}{b})^-}\mathfrak{M}\circ g\biggl(\frac{1}{a}\biggr)\biggr)\right|\\
&\leq \frac{b-a}{2^{\frac{1}{q}}ab\theta^{(\mu+1)}}\biggl(\beta\biggl(\frac{1}{2^\theta},\frac{1}{\theta},\mu+1\biggr)-\beta\biggl(1-\frac{1}{2^\theta},\mu+1,\frac{1}{\theta}\biggr)\biggr) \qquad (26)\\
&\times\Big(a^{2q}\big|\mathfrak{M}'\mathfrak{Z}(a)\big|^q+\big|(b^{2q}\mathfrak{M}'\mathfrak{Z}(b)\big|^q\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.7 *If* $\mathfrak{Z}(x) = \ln x$ *in the inequality* (21), *then one can obtain the next inequality*:

$$
\begin{aligned}
&\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^\mu}-\frac{\Gamma(\mu+1)}{2(\ln b-\ln a)^{\mu\theta}}\big({}^\mu\mathcal{N}^\theta_{\ln a^+}\mathfrak{M}(b)+{}^\mu\mathcal{N}^\theta_{\ln b^-}\mathfrak{M}(a)\big)\right|\\
&\leq \frac{\ln b-\ln a}{2^{\frac{1}{q}}\theta^{(\mu+1)}}\biggl(\beta\biggl(\frac{1}{2^\theta},\frac{1}{\theta},\mu+1\biggr)-\beta\biggl(1-\frac{1}{2^\theta},\mu+1,\frac{1}{\theta}\biggr)\biggr) \qquad (27)\\
&\times\Big(a^{q}\big|\mathfrak{M}'\mathfrak{Z}(a)\big|^q+\big|(b^{q}\mathfrak{M}'\mathfrak{Z}(b)\big|^q\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.8 *If $\mathfrak{Z}(x) = x^r, r \neq 0$, in the inequality (21), then one can obtain the next inequality*:

$$\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^{\mu}} - \frac{r^{\mu\theta}\Gamma(\mu+1)}{2(b^r-a^r)^{\mu\theta}}\left({}^{\mu}\mathcal{N}^{\theta}_{(a)^+}\mathfrak{M}(b) + {}^{\mu}\mathcal{N}^{\theta}_{(b)^-}\mathfrak{M}(a)\right)\right| \leq \frac{|b^r-a^r|}{2^{\frac{1}{q}}\theta^{(\mu+1)}}\left(\beta\left(\frac{1}{2^{\theta}},\frac{1}{\theta},\mu+1\right) - \beta\left(1-\frac{1}{2^{\theta}},\mu+1,\frac{1}{\theta}\right)\right) \times \left(a^{(1-r)q}\left|\mathfrak{M}'(a)\right|^q + b^{(1-r)q}\left|\left(\mathfrak{M}'(b)\right|^q\right)^{\frac{1}{q}}. \tag{28}$$

The next lemma is helpful to prove the upcoming theorem.

Lemma 3.2 *For $0 < \alpha < 1$ and $0 \leq a < b$, we have*

$$\left|a^{\alpha} - b^{\alpha}\right| \leq (b-a)^{\alpha}. \tag{29}$$

Theorem 3.3 *Let $a, b \in r$ with $a < b$ and $\mathfrak{M} : [a,b] \to \mathbb{R}$ be a function, also let $\mathfrak{Z} : [a,b] \to \mathbb{R}$ be a strictly monotone function such that $\mathfrak{M} \circ \mathfrak{Z}^{-1}$ is differentiable and $(\mathfrak{M} \circ \mathfrak{Z}^{1})' \in L[a,b]$. If $|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'|^q, q > 1$ is convex, then the following inequality holds*:

$$\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2} - \frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta}}\left({}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(a)^+}\mathfrak{M}(b) + {}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\right| \leq \frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2^{1+\frac{1}{q}}\theta^{\mu}(\theta\mu p+1)^{\frac{1}{p}}}\left(\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(\mathfrak{Z}(a))\right|^q + \left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(\mathfrak{Z}(b))\right|^q\right)^{\frac{1}{q}}, \tag{30}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof By using the property of the absolute value function and then Holder's inequality on the right-hand side of (2.1), the following inequality is obtained:

$$\left|\frac{\mathfrak{M}(a)+\mathfrak{M}(b)}{2\theta^{\mu}} - \frac{\Gamma(\mu+1)}{2(\mathfrak{Z}(b)-\mathfrak{Z}(a))^{\mu\theta}}\left({}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(a)^+}\mathfrak{M}(b) + {}^{\mu}\mathcal{N}^{\theta}_{\mathfrak{Z}(b)^-}\mathfrak{M}(a)\right)\right| \leq \frac{|\mathfrak{Z}(b)-\mathfrak{Z}(a)|}{2}\left(\int_0^1\left|\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right|^p\right)^{\frac{1}{p}} \times \left(\int_0^1\left|(\mathfrak{M}\circ\mathfrak{Z}^{-1})'(t\mathfrak{Z}(a)+(1-t)\mathfrak{Z}(b))\right|^q dt\right)^{\frac{1}{q}}. \tag{31}$$

Using Lemma 3.2, we have

$$\begin{aligned}\int_0^1\left|\left[\frac{1-t^{\theta}}{\theta}\right]^{\mu} - \left[\frac{1-(1-t)^{\theta}}{\theta}\right]^{\mu}\right|^p dt &\leq \frac{1}{\theta^{\mu p}}\int_0^1\left|(1-t)^{\theta}-t^{\theta}\right|^{\mu p} dt \\ &\leq \frac{1}{\theta^{\mu p}}\int_0^1|2t-1|^{\theta\mu p} dt = \frac{1}{\theta^{\mu p}}\int_0^{\frac{1}{2}}(1-2t)^{\theta\mu p} dt + \frac{1}{\theta^{\mu p}}\int_{\frac{1}{2}}^1(2t-1)^{\theta\mu p} dt \\ &= \frac{1}{\theta^{\mu p}(\theta\mu p+1)}.\end{aligned} \tag{32}$$

Also, by the convexity of $|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'|^q$, we obtain the following inequality:

$$\begin{aligned} &\int_0^1 \left|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'\big(t\mathfrak{Z}(a) + (1-t)\mathfrak{Z}(b)\big)\right|^q dt \\ &\leq \int_0^1 \left(t\left|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'\big(\mathfrak{Z}(a)\big)\right|^q + (1-t)\left|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'\big(\mathfrak{Z}(b)\big)\right|^q\right) dt \\ &= \frac{|(\mathfrak{M} \circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(a))|^q + |(\mathfrak{M} \circ \mathfrak{Z}^{-1})'(\mathfrak{Z}(b))|^q}{2}. \end{aligned} \tag{33}$$

Therefore, the inequality can be obtained using the aforementioned integral calculations (30). □

Corollary 3.9 *If* $\mathfrak{Z}(x) = \frac{1}{x}$ *in the inequality* (30), *then one can obtain the next inequality*:

$$\begin{aligned} &\left|\frac{\mathfrak{M}(a) + \mathfrak{M}(b)}{2\theta^\mu} - \frac{\Gamma(\mu+1)}{2}\left(\frac{ab}{b-a}\right)^{\mu\theta}\left({}^\mu\mathcal{N}^\theta_{(\frac{1}{a})^+}\mathfrak{M} \circ g\left(\frac{1}{b}\right) + {}^\mu\mathcal{N}^\theta_{(\frac{1}{b})^-}\mathfrak{M} \circ g\left(\frac{1}{a}\right)\right)\right| \\ &\leq \frac{b-a}{2^{1+\frac{1}{q}}ab\theta^\mu(\theta\mu p+1)^{\frac{1}{p}}}\left(a^{2q}\left|\mathfrak{M}'\mathfrak{Z}(a)\right|^q + b^{2q}\left|\mathfrak{M}'\mathfrak{Z}(b)\right|^q\right)^{\frac{1}{q}}, \end{aligned} \tag{34}$$

where $g(x) = \frac{1}{x}$.

Corollary 3.10 *If* $\mathfrak{Z}(x) = \ln x$ *in the inequality* (30), *then one can obtain the next inequality*:

$$\begin{aligned} &\left|\frac{\mathfrak{M}(a) + \mathfrak{M}(b)}{2\theta^\mu} - \frac{\Gamma(\mu+1)}{2(\ln b - \ln a)^{\mu\theta}}\left({}^\mu\mathcal{N}^\theta_{(\ln a)^+}\mathfrak{M}(b) + {}^\mu\mathcal{N}^\theta_{(\ln b)^-}\mathfrak{M}(a)\right)\right| \\ &\leq \frac{\ln b - \ln a}{2^{1+\frac{1}{q}}\theta^\mu(\theta\mu p+1)^{\frac{1}{p}}}\left(a^q\left|\mathfrak{M}'(a)\right|^q + b^q\left|\mathfrak{M}'(b)\right|^q\right)^{\frac{1}{q}}. \end{aligned} \tag{35}$$

Corollary 3.11 *If* $\mathfrak{Z}(x) = x^r, r \neq 0$, *in the inequality* (30), *then one can obtain the next inequality*:

$$\begin{aligned} &\left|\frac{\mathfrak{M}(a) + \mathfrak{M}(b)}{2\theta^\mu} - \frac{r^{\mu\theta}\Gamma(\mu+1)}{2(b^r - a^r)^{\mu\theta}}\left({}^\mu\mathcal{N}^\theta_{(a)^+}\mathfrak{M}(b) + {}^\mu\mathcal{N}^\theta_{(b)^-}\mathfrak{M}(a)\right)\right| \\ &\leq \frac{|b^r - a^r|}{2^{1+\frac{1}{q}}\theta^\mu(\theta\mu p+1)^{\frac{1}{p}}|r|}\left(a^{(1-r)q}\left|\mathfrak{M}'(a)\right|^q + b^{(1-r)q}\left|(\mathfrak{M}'(b)\right|^q\right)^{\frac{1}{q}}. \end{aligned} \tag{36}$$

Remark 3.2 If $\theta = 1$ in the inequality (21), then one can obtain Theorem 5 in [22].

4 Conclusions

As is widely known, in nearly all fields of scientific study, inequalities are important in constructing mathematical frameworks and associated state functions. Convexity also has a significant impact on the optimization topic. This pushes us to address their expansions of inequalities in a variety of ways. With the use of the enlarged Riemann–Liouville integrals (5) and (6) and the convexity feature of the functions, we presented and established novel fractional inequalities in the current study. The technique proposed in this study, with fractional integral inequalities, is simple and straightforward, and can be applied to many different inequalities, like Ostrowski, Minkowski, midpoint, and Simpson inequalities. In addition, it is easy to recognize that each of the inequalities that have been constructed are encompassing and may be lowered to numerous other inequalities that were suggested earlier in the research literature.

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Availability of data and materials
The data used in this study will be made available by the corresponding author upon reasonable request.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Methodology, A.H, M.B., and A.H.S; Formal analysis, A.H, M.B., and A.H.S; Investigation,A.H, M.B., and A.H.S; Writing—original draft, A.H, M.B., and A.H.S; Writing—review & editing, A.H, M.B., and A.H.S. All authors have read and agreed to the published version of the manuscript.

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