Uzor et al. *Journal of Inequalities and Applications* (2023) 2023:141
https://doi.org/10.1186/s13660-023-03043-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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Outer approximated projection and contraction method for solving variational inequalities

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Abstract

In this paper we focus on solving the classical variational inequality (VI) problem. Most common methods for solving VIs use some kind of projection onto the associated feasible set. Thus, when the involved set is not simple to project onto, then the applicability and computational effort of the proposed method could be arguable. One such scenario is when the given set is represented as a finite intersection of sublevel sets of convex functions. In this work we develop an outer approximation method that replaces the projection onto the VI's feasible set by a simple, closed formula projection onto some "superset". The proposed method also combines several known ideas such as the inertial technique and self-adaptive step size.

Under standard assumptions, a strong minimum-norm convergence is proved and several numerical experiments validate and exhibit the performance of our scheme.

Mathematics Subject Classification: 47H10; 47J25; 47J40; 90C25; 90C52

Keywords: Projection and contraction method; Sublevel sets; Non-Lipschitz variational inequalities; Inertial technique; Relaxation technique; Self-adaptive step size

1 Introduction

Let H be a real Hilbert space with a nonempty, closed, and convex set $C \subseteq H$. Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the induced norm and inner product on H, respectively, and let $F : H \to H$ be a single-valued mapping. The variational inequality (VI) problem formulated by (1) is an age-old problem in mathematical analysis with present relevance. It was introduced independently by Fichera [15] and Stampacchia [40], and since then, numerous researchers have developed various methods for solving the VIs with applications in diverse fields, such as the sciences, engineering, medicine, cryptography, image processing, signal processing, optimal control, etc., see [2, 3, 13, 14, 17, 18, 21, 22, 27, 29, 33], for more details. The VI, with solution set denoted by $VI(C,F)$, is defined as finding a point $p \in C$ such that

$$\langle Fp, z - p\rangle \geq 0, \quad \forall z \in C. \tag{1}$$

Two main known methods for solving VIs are the projection and regularization methods. The foremost projection method is the gradient method (GM) that generates a sequence $\{x_n\}$ according to the following rule:

$$x_{n+1} = P_C(x_n - \nu F x_n), \tag{2}$$

where P_C is the metric projection of H onto the feasible set C. Although the GM has a simple structure, it has two major drawbacks. The first is the quite strong monotonicity assumption required for its convergence and the second is the need for computing the projection onto the feasible set C, per iteration.

As a way to overcome the first GM's monotonicity limitation, Korpelevich [31] (Antipin [5] independently) proposed the extragradient method (EGM) that, on the one hand, converges under a weaker monotonicity assumption but requires the evaluation of two projections onto C, per iteration. Censor et al. [9] introduced the subgradient extragradient method (SEGM) in which one of the projections is replaced by an easy-closed formula projection onto a "superset" containing C. Other modifications in this directions can be found, for example in [32, 47].

Other relevant EGM extensions are Tseng's extragradient method (TEGM) [49], and the projection and contraction method (PCM) [41], see also [10, 13, 19, 50]. Both methods use only one projection onto C, per iteration. The PCM, for example, generates $\{x_n\}$ according to the following rule:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \xi F x_n), \\ d(x_n, y_n) := (x_n - y_n) - \xi(F x_n - F y_n), \\ x_{n+1} = x_n - \rho\beta_n d(x_n, y_n), \end{cases} \tag{3}$$

where $\rho \in (0,2)$, $\xi \in (0, \frac{1}{L}), L$ is the Lipschitz constant of F, $\beta_n := \frac{\alpha(x_n,y_n)}{\|d(x_n,y_n)\|^2}$, $\alpha(x_n, y_n) := \langle x_n - y_n, d(x_n, y_n)\rangle$, $\forall n \geq 1$.

As the implementation of the above methods (SEGM, TEGM, and PCM) still requires the computation of P_C for each iteration, a need for a "free-projection" method encouraged many researchers to come up with some creative ideas. One such idea is the two-subgradient method (TSEGM) of Censor et al. [9]. Suppose that the closed and convex set C can be represented as a sublevel set of some convex function $c : H \to \mathbb{R}$, that is

$$C = \{x \in H : c(x) \leq 0\}. \tag{4}$$

Denote by $\partial c(x)$, the subdifferential of the convex function $c(\cdot)$ at x. The TSEGM generates $\{x_n\}$ according to the following rule:

$$\begin{cases} x_1 \in H, \\ y_n = P_{C_n}(x_n - \lambda F x_n), \\ C_n := \{x \in H : c(x_n) + \langle \zeta_n, x - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(x_n - \lambda F y_n), \end{cases} \tag{5}$$

where $\zeta_n \in \partial c(x_n)$. Observe that if $\zeta_n = 0$ then $C_n = H$ and otherwise it is a half-space containing the set C. The convergence of (5) was raised as an open problem in [9].

Recently, Cao and Guo [8] as well as Ma and Wang [34] partially answered this open question by proposing an inertial two-subgradient extragradient method (ITSEGM), and self-adaptive TSEGM for solving a Lipschitz continuous and monotone variational inequality problem with weak convergence properties.

Other relevant works related to the subgradient extragradient method are of He and Wu [24], in which a line search is involved and the two projections onto the set C are replaced by projections onto two particular half-spaces. He et al. [23] proposed a relaxed projection and contraction method where again the projections onto the set C are replaced by projection onto a particular constructible half-space.

In this paper, we are interested in studying VIs where the feasible set C is given as a finite intersection of sublevel sets of convex functions defined as follows:

$$C := \bigcap_{i=1}^{k} C^i := \{z \in H : c_i(z) \leq 0\}, \tag{6}$$

where k is a positive integer and $c_i : H \to \mathbb{R}$ for all $i \in I := \{1, 2, \dots, k\}$ are convex functions.

A very recent result for solving VIs defined over sets of the form (6) is the He et al. [25] totally relaxed self-adaptive subgradient extragradient.

Remark 1.1 Although all the above results, He and Wu [24], He et al. [23], He et al. [25], Cao and Guo [8], and Ma and Wang [34] managed to replace successfully the projections onto C by some closed-formula projections onto some set, there are still some limitations. First, we note that their proposed methods either require knowledge of the Lipschitz constants of F and the Gâteaux differential $c'(\cdot)$ of $c(\cdot)$ (which are often unknown or very difficult to estimate) or employed a line-search procedure, which is known to be time consuming to implement. In addition, all results obtain weak convergence, that is known to be a drawback when solving optimization problems, see, e.g., Bauschke [6].

Following the above methods and results, in this paper we establish a totally relaxed, inertial, self-adaptive projection and contraction method (TRISPCM) for solving VI (1) defined over a finite intersection of some closed, convex sublevel sets (as seen in (6)). Our method employs projections onto some constructable "supersets" and inertial ([4, 10, 20, 37, 42, 50, 52])) and relaxation [28] techniques are incorporated to speed up the convergence rate of our method. Although we assume that F is Gâteaux differentiable, and $c_i'(\cdot)$ of $c_i(\cdot)$ are Lipschitz continuous, our method does not require any line-search procedure, rather we employ a more efficient self-adaptive step-size technique that generates a nonmonotonic sequence of step sizes. Moreover, under suitable conditions we prove strong convergence to a minimum-norm solution of the problem. Relevant numerical experiments at the end of this paper clearly display the efficiency of our methods over those in the literature.

The remainder of this paper is organized as follows. Section 2 contains definitions and existing results relevant to our analysis. In Sect. 3, the proposed algorithm is presented and its strong convergence is established in Sect. 4. Numerical experiments and comparisons with related methods are given in Sect. 5, illustrating the performance of our scheme. Finally, some concluding remarks on our work are presented in Sect. 6.

2 Preliminaries

In this section, we review basic definitions and important lemmas, vital in proving our main results.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Also, throughout this paper, we let the strong and weak convergence of a sequence $\{x_n\}$ to a point $x^* \in H$ be denoted by $x_n \to x^*$ and $x_n \rightharpoonup x^*$, respectively. The set of weak limits of $\{x_n\}$, denoted by $w_\omega(x_n)$, is defined by

$$w_\omega(x_n) := \{x^* \in H : x_{n_k} \rightharpoonup x^* \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}. \tag{7}$$

The *metric projection* $P_C : H \to C$ ([1]) is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

It is known that P_C is nonexpansive (see [4, 38]). For more interesting features of the metric projection, see Lemma 2.1.

Lemma 2.1 [25, 30] *Let H be a real Hilbert space, and I be the identity map on H. Let C be a nonempty, closed, and convex subset of H. We have the following results for any $x \in H$ and $f, g \in C$:*

(i) $g = P_C x \Longleftrightarrow \langle x - g, g - f\rangle \geq 0$;
(ii) $\langle (I - P_C)x - (I - P_C)f, x - f\rangle \geq \|(I - P_C)x - (I - P_C)f\|^2$;
(iii) $\|f - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - f\|^2$;
(iv) $\langle x - f, P_C x - P_C f\rangle \geq \|P_C x - P_C f\|^2$;
(v) *Let $D = \{u \in H : \langle x, u - d\rangle \leq 0\}$ be a half-space, where $x \neq 0$, and $d \in \mathbb{R}$. Then, for $a \in H$,*

$$P_D(a) = a - \max\left\{0, \frac{\langle x, a - d\rangle}{\|x\|^2}\right\}x. \tag{8}$$

Note that (8) is the explicit formula for the orthogonal projection onto the half-space D.

Definition 2.2 [44, 51] Let $F : H \to H$ be a mapping defined on a real Hilbert space H. Then, F is said to be:

(i) *L-Lipschitz continuous*, where $L > 0$, if

$$\|Fu - Fv\| \leq L\|u - v\|, \quad \forall u, v \in H.$$

A *contraction* if $L \in [0, 1)$, and *nonexpansive*, if $L = 1$;

(ii) λ- strongly monotone, if there exists $\lambda > 0$ such that

$$\langle u - v, Fu - Fv\rangle \geq \lambda\|u - v\|^2, \quad \forall u, v \in H;$$

(iii) *monotone*, if

$$\langle Fu - Fv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

Lemma 2.3 [43, 50] *Let H be a real Hilbert space. Then, the following results hold, for all $x, y \in H$ and $\zeta \in \mathbb{R}$:*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;

(ii) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;

(iii) $\|\zeta x + (1-\zeta)y\|^2 = \zeta\|x\|^2 + (1-\zeta)\|y\|^2 - \zeta(1-\zeta)\|x-y\|^2$.

Definition 2.4 [36] Let $c : H \to \mathbb{R}$ be a real-valued function. Then,

(i) c is said to be *Gâteaux differentiable* at $z \in H$, if there exists an element in H, denoted by $c'(z)$, such that

$$\lim_{t\to 0} \frac{c(z+th) - c(z)}{t} = \langle h, c'(z)\rangle, \quad \forall h \in H, \tag{9}$$

where $c'(z)$ (also written as $\nabla c(z)$), is known as the Gâteaux differential (or gradient) of c at z.

(ii) If c is convex, then c is said to be *subdifferentiable* at point $z \in H$, if $\partial c(z)$ is nonempty, where $\partial c(z)$ is defined as follows:

$$\partial c(z) := \{x \in H : c(y) \ge c(z) + \langle x, y-z\rangle \ \forall y \in H\}. \tag{10}$$

c is said to be *subdifferentiable* on H, if for each $z \in H$, c is subdifferentiable at z.

(iii) c is said to be *weakly lower semicontinuous (w-lsc)* at $z \in H$, if $z_n \rightharpoonup z$ implies

$$c(z) \le \liminf_{n\to\infty} c(z_n). \tag{11}$$

c is said to be w-lsc on H if for each $z \in H$, c is w-lsc at z.

Remark 2.5 We note the following from Definition 2.4:

(i) Each element in $\partial c(z)$ is referred to as a subgradient of c at z. Also, (10) is said to be the subdifferential inequality of c at z, where $\partial c(z)$ is the subdifferential of c at z.

(ii) It is also known that if c is Gâteaux differentiable at z, then c is subdifferentiable at z, and $\partial c(z) = \{c'(z)\}$, in particular, $\partial c(z)$ is a singleton set (see [25]).

Lemma 2.6 [7] *Let $c : H \to \mathbb{R} \cup \{+\infty\}$ be convex. Then, the following results are equivalent:*

(i) *c is weakly sequential lower semicontinuous;*

(ii) *c is lower semicontinuous.*

Lemma 2.7 [45] *Let $\{\xi_n\}$ and $\{\mu_n\}$ be two nonnegative real sequences such that*

$$\xi_{n+1} \le \xi_n + \mu_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \mu_n < +\infty$, then $\lim_{n\to\infty} \xi_n$ exists.

Lemma 2.8 [12] *Suppose C is a nonempty, closed, and convex subset of H, and suppose $F : C \to H$ is a continuous monotone mapping, with $x \in C$, then*

$$x \in VI(C,F) \quad \Longleftrightarrow \quad \langle Fy, y-x\rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.9 [39] *Suppose $\{x_n\}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n = +\infty$ and $\{z_n\}$ is a sequence of real numbers. Let*

$$x_{n+1} \leq (1-\alpha_n)x_n + \alpha_n z_n, \quad \textit{for all } n \geq 1,$$

if $\limsup_{k\to\infty} z_{n_k} \leq 0$ *for every subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *satisfying* $\liminf_{k\to\infty}(x_{n_{k+1}} - x_{n_k}) \geq 0$, *then* $\lim_{n\to\infty} x_n = 0$.

Lemma 2.10 [26, 35] *Let C be a set defined as in (6), and let $F: h \to H$ be an operator. Suppose the solution set $VI(C,F)$ is nonempty. Then, the following alternating theorem holds for the solution of the VI(C,F), that is, given $\hat{z} \in C$, $\hat{z} \in VI(C,F)$ if and only if one of the following holds.*

(i) *$F\hat{z} = 0$; or*

(ii) *$\hat{z} \in bd(C)$, and there exist $\beta_{\hat{z}} > 0$ (depending on the point $\hat{z}$), and $\kappa \in \operatorname{conv}\{c_i'(\hat{z}) : i \in I_{\hat{z}}^*\}$ such that $F(\hat{z}) = -\beta_{\hat{z}}\kappa$, where $bd(C)$ denotes the boundary of the set C, $I_\kappa^* = \{i \in I : c_i(\hat{z}) = 0\}$ and $\operatorname{conv}\{c_i'(\hat{z}) : i \in I_{\hat{z}}^*\}$ is the convex hull of the set $\{c_i'(\hat{z}) : i \in I_{\hat{z}}^*\}$.*

3 Proposed method

Here, we present our algorithm: A totally relaxed inertial self-adaptive projection and contraction method (TRISPCM) for solving the monotone variational inequality problem defined over the feasible set (6). Our results are based on the following assumptions:

Assumption A

(A1) $F: H \to H$ is monotone and $\mathcal{J}-$ Lipschitz continuous on H.

(A2) The solution set $VI(C,F)$ is nonempty.

(A3) For all $i \in H$, the family of functions $c_i : H \to \mathbb{R}$ satisfy the following conditions:

- (i) Any $c_i (i \in I)$ is convex on H.
- (ii) Any $c_i (i \in I)$ is weakly lower semicontinuous on H.
- (iii) Any $c_i (i \in I)$ is Gâteaux differentiable and $c_i' (i \in I)$ is L_i- Lipschitz on H.
- (iv) There exists a positive constant K such that for all $\hat{z} \in bd(C)$, the following holds:

$$\|F\hat{z}\| \leq K \inf\Big\{ \big\|m(\hat{z})\big\| : m(\hat{z}) \in \operatorname{con}\big\{c_i'(\hat{z}) : i \in I_{\hat{z}}^*\big\}\Big\},$$

where $I_{\hat{z}}^*$ is defined as in Lemma 2.10.

Assumption B

(B1) Let $\tau > 0$, $\gamma_1 > 0$, $\ell \in (0,2)$, $\delta \in (0, \frac{2-\ell}{2-\ell+2K})$;

(B2) $\alpha_n \in (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = +\infty$, $\{\theta_n\} \subset \mathbb{R}_+$ such that $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$;

(B3) Let μ_n be a nonnegative sequence such that $\sum_{n=1}^{\infty}\mu_n < +\infty$.

We show our algorithm below:

Algorithm 3.1

Step 0. Set $n = 1$, and let $x_0, x_1 \in H$ be two arbitrary initial points.

Step 1. Given the $(n-1)th$ and nth iterates, choose τ_n such that $0 \leq \tau_n \leq \hat{\tau}_n$ with $\hat{\tau}_n$ defined by

$$\hat{\tau}_n = \begin{cases} \min\{\tau, \frac{\theta_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \tau, & \text{otherwise.} \end{cases} \tag{12}$$

Step 2. Compute

$$w_n = (1-\alpha_n)\big(x_n + \tau_n(x_n - x_{n-1})\big). \tag{13}$$

Step 3. Given the current iterate w_n, construct the family of half-spaces

$$C_n^i = \big\{w \in H : c_i(w_n) + \big\langle c_i'(w_n), w - w_n\big\rangle \leq 0\big\}, i \in I. \tag{14}$$

Set,

$$C_n := \bigcap_{i \in I} C_n^i \tag{15}$$

and compute:

$$y_n := P_{C_n}(w_n - \gamma_n F w_n). \tag{16}$$

If $w_n = y_n$, then stop, $w_n \in SOL(C,F)$, otherwise, proceed to *Step 4.*

Step 4. Compute:

$$x_{n+1} = w_n - \ell \psi_n d_n,$$

where, $d_n := w_n - y_n - \gamma_n(Fw_n - Fy_n)$, and

$$\psi_n = \begin{cases} \frac{\langle w_n - y_n, d_n \rangle}{\|d_n\|^2}, & \text{if } d_n \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Update:

$$\gamma_{n+1} = \begin{cases} \min\{\frac{\delta\|w_n - y_n\|}{\|Fw_n - Fy_n\| + \|c_{i_n}'(w_n) - c_{i_n}'(y_n)\|}, \gamma_n + \mu_n\}, \\ \quad \text{if } \|Fw_n - Fy_n\| + \|c_{i_n}'(w_n) - c_{i_n}'(y_n)\| \neq 0, \\ \gamma_n + \mu_n, \quad \text{otherwise,} \end{cases} \tag{18}$$

where $\|c_{i_n}'(w_n) - c_{i_n}'(y_n)\| = \max_{i \in I}\{\|c_i'(w_n) - c_i'(y_n)\|\}$. Set $n := n+1$ and return to *Step 1.*

Remark 3.2 We highlight below some of the key features of our proposed Algorithm 3.1.

(i) Observe that the feasible set is constructed as a finite intersection of sublevel sets, as seen in (6), which is more general than the feasible set adopted in [8, 19, 23, 24, 34]. Also, observe that in (14), if $c_i'(w_n) = 0$, then we have that $C_n^i = H$.

(ii) We note that our proposed algorithm completely avoids projection onto the feasible set, but rather allows only one projection onto some half-space, as seen in (14)–(16). This obviously ensures easier computation, since projection onto half-spaces can be calculated using an explicit formula, see (8).

(iii) We note that our Algorithm 3.1 employs the relaxation and inertial techniques to improve its rate of convergence. Also, we observe that *Step 1* of our proposed algorithm is easily implemented, since we have prior knowledge of the estimate $\|x_n - x_{n-1}\|$ before choosing τ_n.

(iv) We emphasize that while the cost operator is Lipschitz continuous, our algorithm does not require any line-search procedure (unlike the methods of [23–25]). Instead, we adopt a more efficient self-adaptive step-size technique that generates nonmonotonic sequence of step sizes (as seen in (18)).

(v) We also point out that unlike the results in [8, 23–25, 34], our proposed algorithm generates a strong convergence sequence, which converges to a minimum-norm solution of the VI.

Remark 3.3 By applying condition (B2), from (12) we have that

$$\lim_{n\to\infty} \tau_n\|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

4 Convergence analysis

In this section, we carry out the convergence analysis of our proposed algorithm. First, we establish some lemmas that are needed to prove the strong convergence theorem for the proposed algorithm.

Lemma 4.1 *Let C and C_n be the sets defined by* (6) *and* (15), *respectively, then we have that $C \subset C_n$, $\forall n \geq 1$.*

Proof For all $i \in I$, let $C^i := \{x \in H : c_i(x) \leq 0\}$. Thus,
we see that $C = \bigcap_{i\in I} C^i$. Then, for each $i \in I$ and any $x \in C^i$, by the subdifferential inequality, it follows that

$$c_i(w_n) + \langle c_i'(w_n), x - w_n\rangle \leq c_i(x) \leq 0. \tag{19}$$

By definition of the sets C_n^i (14), we see that $x \in C_n^i$. It then follows that $C^i \subset C_n^i$, $\forall n \geq 1$, $i \in I$. Hence, $C \subset C_n$, $\forall n \geq 1$ as required. □

Lemma 4.2 *If $y_n = w_n$ for some $n \geq 1$ in Algorithm* 3.1, *then $w_n \in VI(C,F)$.*

Proof Suppose $y_n = w_n$ for some $n \geq 1$. Then, by (16), we obtain

$$w_n = P_{C_n}(w_n - \gamma_n F w_n). \tag{20}$$

From (20), it follows that $w_n \in C_n$, in particular, $w_n \in C_n^i$ for each $i \in I$ and $n \geq 1$. Then, we have that $c_i(w_n) + \langle c_i'(w_n), w_n - w_n\rangle \leq 0$, by the definition of C_n^i. From this, we obtain $c_i(w_n) \leq 0$ for each $i \in I$. Thus, $w_n \in C$.

Next, by (20) and (2), we have $w_n \in VI(C_n, F)$, which implies that

$$\langle Fw_n, z - w_n \rangle \geq 0, \quad \forall z \in C_n. \tag{21}$$

Therefore, from the fact that $w_n \in C \in C_n$ and (21), the conclusion follows. □

Lemma 4.3 *Let* $\{\gamma_n\}$ *be the sequence generated by* (18). *Then,* $\{\gamma_n\}$ *is well defined and* $\lim_{n\to\infty} \gamma_n = \gamma$, *where* $\gamma \in [\min\{\frac{\delta}{M}, \gamma_1\}, \gamma_1 + \Phi]$, *for some constants* $M > 0$ *and* $\Phi = \sum_{n=1}^{\infty} \Phi_n$.

Proof Since c_i' and F are both Lipschitz continuous, considering the case $\|Fw_n - Fy_n\| + \|c_{i_n}'(w_n) - c_{i_n}'(y_n)\| \neq 0$ in (18), we obtain for all $n \geq 1$, that

$$\begin{aligned} \frac{\delta\|w_n - y_n\|}{\|Fw_n - Fy_n\| + \|c_{i_n}'(w_n) - c_{i_n}'(y_n)\|} &\geq \frac{\delta\|w_n - y_n\|}{\mathcal{J}\|w_n - y_n\| + L\|w_n - y_n\|} \\ &\geq \frac{\delta\|w_n - y_n\|}{(\mathcal{J} + L)\|w_n - y_n\|} = \frac{\delta}{M}, \end{aligned}$$

where $M := (\mathcal{J} + L) > 0$, $L = \max\{L_i : i \in I\}$. Thus, by the definition of γ_{n+1}, it is obvious that the sequence $\{\gamma_n\}$ has upper bound and lower bound $\gamma_1 + \Phi$ and $\min\{\frac{\delta}{M}, \gamma_1\}$, respectively. Hence, by Lemma 2.7, we have that $\lim_{n\to\infty} \gamma_n$ exists, and we denote by $\lim_{n\to\infty} \gamma_n = \gamma$. Clearly, $\gamma \in [\min\{\frac{\delta}{M}, \gamma_1\}, \gamma_1 + \Phi]$. □

Lemma 4.4 *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.1. *Suppose Assumptions A and B are satisfied, then the following inequalities hold for all* $p \in VI(C, F)$.

$$\|w_n - y_n\|^2 \leq \frac{1}{\ell^2}\left(\frac{\gamma_{n+1} + \delta\gamma_n}{\gamma_{n+1} - \delta\gamma_n}\right)^2 \|w_n - x_{n+1}\|^2 \tag{22}$$

and

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \left[\frac{2 - \ell}{\ell} - \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1} - \delta\gamma_n)}\right]\|w_n - x_{n+1}\|^2. \tag{23}$$

Proof From (18), we obtain

$$\begin{aligned} \gamma_{n+1} &= \min\left\{\frac{\delta\|w_n - y_n\|}{\|Fw_n - Fy_n\| + \|c_{i_n}'(w_n) - c_{i_n}'(y_n)\|}, \gamma_n + \mu_n\right\} \\ &\leq \frac{\delta\|w_n - y_n\|}{\|Fw_n - Fy_n\| + \|c_{i_n}'(w_n) - c_{i_n}'(y_n)\|}, \end{aligned} \tag{24}$$

which implies that

$$\|Fw_n - Fy_n\| + \left\|c_{i_n}'(w_n) - c_{i_n}'(y_n)\right\| \leq \frac{\delta}{\gamma_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 1. \tag{25}$$

Since both terms on the left-hand side of (25) are positive terms, then it implies that

$$\left\|c_{i_n}'(w_n) - c_{i_n}'(y_n)\right\| \leq \frac{\delta}{\gamma_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 1. \tag{26}$$

Next, we proceed to prove the first inequality (22). From the definition of ψ_n, if $d_n \neq 0$ and by applying (25), we have

$$\begin{aligned}\psi_n \|d_n\|^2 = \langle w_n - y_n, d_n\rangle &\geq \|w_n - y_n\|^2 - \gamma_n \|Fw_n - Fy_n\| \|w_n - y_n\| \\ &\geq \|w_n - y_n\|^2 - \gamma_n \left(\frac{\delta}{\gamma_{n+1}}\right) \|w_n - y_n\|^2 \\ &= \left(1 - \frac{\delta\gamma_n}{\gamma_{n+1}}\right) \|w_n - y_n\|^2. \end{aligned} \tag{27}$$

Also,

$$\begin{aligned}\|d_n\| &\leq \|w_n - y_n\| + \gamma_n \|Fw_n - Fy_n\| \\ &\leq \|w_n - y_n\| + \frac{\delta\gamma_n}{\gamma_{n+1}} \|w_n - y_n\| \\ &= \left(1 + \frac{\delta\gamma_n}{\gamma_{n+1}}\right) \|w_n - y_n\|. \end{aligned} \tag{28}$$

From (27), we obtain

$$\left(\psi_n \|d_n\|^2\right)^2 \geq \left(1 - \frac{\delta\gamma_n}{\gamma_{n+1}}\right)^2 \|w_n - y_n\|^4.$$

From the last inequality and by applying (28), we obtain

$$\begin{aligned}\psi_n^2 \|d_n\|^2 &\geq \left(1 - \frac{\delta\gamma_n}{\gamma_{n+1}}\right)^2 \frac{\|w_n - y_n\|^4}{\|d_n\|^2} \\ &\geq \frac{(1 - \frac{\delta\gamma_n}{\gamma_{n+1}})^2 \|w_n - y_n\|^4}{(1 + \frac{\delta\gamma_n}{\gamma_{n+1}})^2 \|w_n - y_n\|^2} \\ &= \frac{(1 - \frac{\delta\gamma_n}{\gamma_{n+1}})^2}{(1 + \frac{\delta\gamma_n}{\gamma_{n+1}})^2} \|w_n - y_n\|^2. \end{aligned} \tag{29}$$

Observe that (29) still holds when $d_n = 0$. Hence, from (29) and the definition of x_{n+1}, we have that

$$\begin{aligned}\|x_{n+1} - w_n\|^2 = \|w_n - \ell\psi_n d_n - w_n\|^2 = \|\ell\psi_n d_n\|^2 &= \ell^2 \psi_n^2 \|d_n\|^2 \\ &\geq \ell^2 \frac{(1 - \frac{\delta\gamma_n}{\gamma_{n+1}})^2}{(1 + \frac{\delta\gamma_n}{\gamma_{n+1}})^2} \|w_n - y_n\|^2. \end{aligned}$$

Hence,

$$\|w_n - y_n\|^2 \leq \frac{1}{\ell^2} \left(\frac{\gamma_{n+1} + \delta\gamma_n}{\gamma_{n+1} - \delta\gamma_n}\right)^2 \|w_n - x_{n+1}\|^2,$$

as required.

Next, we proceed to prove the second inequality (23). Now, if there exists $n^* \geq 1$ such that $d_{n^*} = 0$, then $x_{n^*+1} = w_{n^*}$, and hence, (23) holds. Hence, we consider the nontrivial case, where $d_n \neq 0$, for each $n \geq 1$.

Let $p \in VI(C,F)$. Then, from (17), we have that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|w_n - \ell\psi_n d_n - p\|^2 \\ &= \|w_n - p\|^2 - 2\ell\psi_n\langle w_n - p, d_n\rangle + \ell^2\psi_n^2\|d_n\|^2. \end{aligned} \tag{30}$$

By the definition of d_n, we obtain

$$\begin{aligned} \langle w_n - p, d_n\rangle &= \langle w_n - y_n, d_n\rangle + \langle y_n - p, d_n\rangle \\ &= \langle w_n - y_n, d_n\rangle + \big\langle y_n - p, w_n - y_n - \gamma_n(Fw_n - Fy_n)\big\rangle \\ &= \langle w_n - y_n, d_n\rangle + \langle y_n - p, w_n - y_n - \gamma_n Fw_n\rangle \\ &\quad + \gamma_n\langle y_n - p, Fy_n\rangle. \end{aligned} \tag{31}$$

By the monotonicity of F, we have that $\langle y_n - p, Fy_n - Fp\rangle \geq 0$, which implies that

$$\langle y_n - p, Fy_n\rangle \geq \langle y_n - p, Fp\rangle. \tag{32}$$

Also, since $y_n = P_{C_n}(w_n - \gamma_n Fw_n)$ and by Lemma 2.1, we have

$$\langle w_n - y_n - \gamma_n Fw_n, y_n - p\rangle \geq 0. \tag{33}$$

By (31), (32), and (33), we obtain

$$\langle w_n - p, d_n\rangle \geq \langle w_n - y_n, d_n\rangle + \gamma_n\langle y_n - p, Fp\rangle. \tag{34}$$

Now, we consider the following *two cases*:

Case 1: $Fp = 0$. If $Fp = 0$, then from (34) we obtain

$$\langle w_n - p, d_n\rangle \geq \langle w_n - y_n, d_n\rangle. \tag{35}$$

Then, it follows from (30), (35), the definition of ψ_n, and the conditions imposed on the control parameter that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - 2\ell\psi_n\langle w_n - y_n, d_n\rangle + \ell^2\psi_n^2\|d_n\|^2 \\ &= \|w_n - p\|^2 - 2\ell\psi_n^2\|d_n\|^2 + \ell^2\psi_n^2\|d_n\|^2 \\ &= \|w_n - p\|^2 - \frac{2-\ell}{\ell}\|\ell\psi_n d_n\|^2 \\ &= \|w_n - p\|^2 - \frac{2-\ell}{\ell}\|w_n - x_{n+1}\|^2. \end{aligned} \tag{36}$$

Hence, the desired inequality (23) follows from (36).

Case 2: $Fp \neq 0$. By applying Lemma 2.10, we have that $p \in bd(C)$ and we have

$$Fp = -\beta_p \sum_{i \in I_p^*} \alpha_i c_i'(p), \tag{37}$$

where β_p is some positive constant, $I_p^* = \{i \in I : c_i(p) = 0\}$, and $\{\alpha_i\}_{i \in I_p^*}$ are nonnegative numbers satisfying $\sum_{i \in I_p^*} \alpha_i = 1$. Then, by the subdifferential inequality, we obtain

$$c_i(p) + \langle c_i'(p), y_n - p \rangle \leq c_i(y_n), \quad \forall n \geq 0, i \in I_p^*. \tag{38}$$

Since $p \in bd(C)$, we have that $c_i(p) = 0$, for each $i \in I_p^*$, and then

$$\langle c_i'(p), y_n - p \rangle \leq c_i(y_n), \quad \forall n \geq 0, i \in I_p^*. \tag{39}$$

We have from (37) and (39) that

$$\langle -Fp, y_n - p \rangle \leq \beta_p \sum_{i \in I_p^*} \alpha_i c_i(y_n). \tag{40}$$

Since $y_n \in C_n = \bigcap_{i \in I} C_n^i$, we have

$$c_i(w_n) + \langle c_i'(w_n), y_n - w_n \rangle \leq 0. \tag{41}$$

Then, by the differential inequality, we obtain

$$c_i(y_n) + \langle c_i'(y_n), w_n - y_n \rangle \leq c_i(w_n), \quad \forall n \geq 0, i \in I_p^*. \tag{42}$$

From (41) and (42), and by applying (26) we have

$$\begin{aligned}
& c_i(y_n) + \langle c_i'(y_n), w_n - y_n \rangle \leq -\langle c_i'(w_n), y_n - w_n \rangle \\
\implies \quad & c_i(y_n) \leq \langle c_i'(w_n) - c_i'(y_n), w_n - y_n \rangle \\
& \qquad = \langle c_i'(y_n) - c_i'(w_n), y_n - w_n \rangle \\
& \qquad \leq \| c_i'(y_n) - c_i'(w_n) \| \| y_n - w_n \| \\
& \qquad \leq \frac{\delta}{\gamma_{n+1}} \| y_n - w_n \|^2.
\end{aligned} \tag{43}$$

Observe that by condition $(A3)(iv)$, we have

$$\beta_p \leq K. \tag{44}$$

Hence, from (40) and by applying (43), (44), and the condition on γ_n, we obtain

$$\begin{aligned}
& \langle -Fp, y_n - p \rangle \leq K \frac{\delta}{\gamma_{n+1}} \| y_n - w_n \|^2 \\
\implies \quad & \langle Fp, y_n - p \rangle \geq -K \frac{\delta}{\gamma_{n+1}} \| y_n - w_n \|^2
\end{aligned}$$

$$\implies \quad \gamma_n\langle Fp, y_n - p\rangle \geq -K\frac{\delta\gamma_n}{\gamma_{n+1}}\|y_n - w_n\|^2. \tag{45}$$

By substituting (45) into (34), we obtain

$$\langle w_n - p, d_n\rangle \geq \langle w_n - y_n, d_n\rangle - K\frac{\delta\gamma_n}{\gamma_{n+1}}\|y_n - w_n\|^2. \tag{46}$$

Then, substituting (46) into (30), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - 2\ell\psi_n\left[\langle w_n - y_n, d_n\rangle - K\frac{\delta\gamma_n}{\gamma_{n+1}}\|y_n - w_n\|^2\right] \\
&\quad + \ell^2\psi_n^2\|d_n\|^2 \\
&= \|w_n - p\|^2 - 2\ell\psi_n\langle w_n - y_n, d_n\rangle + \ell^2\psi_n^2\|d_n\|^2 \\
&\quad + 2\ell\psi_n K\frac{\delta\gamma_n}{\gamma_{n+1}}\|y_n - w_n\|^2 \\
&= \|w_n - p\|^2 - \frac{2-\ell}{\ell}\|\ell\psi_n d_n\|^2 + 2\ell\psi_n K\frac{\delta\gamma_n}{\gamma_{n+1}}\|y_n - w_n\|^2 \\
&= \|w_n - p\|^2 - \frac{2-\ell}{\ell}\|w_n - x_{n+1}\|^2 + 2\ell\psi_n K\frac{\delta\gamma_n}{\gamma_{n+1}}\|y_n - w_n\|^2.
\end{aligned} \tag{47}$$

From (27), we obtain

$$\|w_n - y_n\|^2 \leq \frac{\psi_n\|d_n\|^2}{(1 - \frac{\delta\gamma_n}{\gamma_{n+1}})}. \tag{48}$$

Hence, using (48) and the definition of x_{n+1}, we have

$$\begin{aligned}
2\ell\psi_n K\frac{\delta\gamma_n}{\gamma_{n+1}}\|y_n - w_n\|^2 &\leq 2\ell\psi_n K\frac{\delta\gamma_n}{\gamma_{n+1}}\frac{\psi_n\|d_n\|^2}{(1 - \frac{\delta\gamma_n}{\gamma_{n+1}})} \\
&= \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1} - \delta\gamma_n)}\|\ell\psi_n d_n\|^2 \\
&= \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1} - \delta\gamma_n)}\|w_n - x_{n+1}\|^2.
\end{aligned} \tag{49}$$

By substituting (49) into (47), we obtain

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \left[\frac{2-\ell}{\ell} - \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1} - \delta\gamma_n)}\right]\|w_n - x_{n+1}\|^2,$$

which is the required inequality. □

Since the limit of $\{\gamma_n\}$ exists, we have that $\lim_{n\to\infty}\gamma_n = \lim_{n\to\infty}\gamma_{n+1}$. Hence, by the conditions imposed on the control parameters, we have that

$$\lim_{n\to\infty}\left[\frac{2-\ell}{\ell} - \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1} - \delta\gamma_n)}\right] = \left[\frac{2-\ell}{\ell} - \frac{2K}{\ell}\frac{\delta}{(1-\delta)}\right] > 0.$$

Thus, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, we have

$$\left[\frac{2-\ell}{\ell} - \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1}-\delta\gamma_n)}\right] > 0. \tag{50}$$

Hence, from (23), we have that for all $n \geq n_0$,

$$\|x_{n+1}-p\| \leq \|w_n - p\|. \tag{51}$$

Lemma 4.5 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that Assumptions A and B hold. Then, $\{x_n\}$ is bounded.*

Proof Let $p \in VI(C,F)$. Then, by the definition of w_n, we have

$$\begin{aligned}
\|w_n - p\| &= \big\|(1-\alpha_n)\big(x_n + \tau_n(x_n - x_{n-1}) - p\big)\big\| \\
&= \big\|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\tau_n(x_n - x_{n-1}) - \alpha_n p\big\| \\
&\leq (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\tau_n\|x_n - x_{n-1}\| + \alpha_n\|p\| \\
&= (1-\alpha_n)\|x_n - p\| + \alpha_n\left[(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|\right].
\end{aligned} \tag{52}$$

From Remark 3.3, we obtain that $\lim_{n\to\infty}[(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|] = \|p\|$. Thus, there exists $M_1 > 0$ such that

$$(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\| \leq M_1, \quad \forall n \geq 1. \tag{53}$$

Combining (52) and (53), we obtain

$$\|w_n - p\| \leq (1-\alpha_n)\|x_n - p\| + \alpha_n M_1. \tag{54}$$

Now, using (54) together with (51), we have

$$\begin{aligned}
\|x_{n+1}-p\| \leq \|w_n - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n M_1 \\
&\leq \max\big\{\|x_n - p\|, M_1\big\} \\
&\ \ \vdots \\
&\leq \big\{\|x_{n_0} - p\|, M_1\big\}.
\end{aligned}$$

Therefore, we have that the sequence $\{x_n\}$ is bounded. Consequently, $\{w_n\}$ and $\{y_n\}$ are both bounded. □

Lemma 4.6 *Suppose $\{x_n\}$ is a sequence generated by Algorithm 3.1 under Assumptions A and B. Then, for all $p \in VI(C,F)$ the following inequality holds:*

$$\left[\frac{2-\ell}{\ell} - \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1}-\delta\gamma_n)}\right]\|w_n - x_{n+1}\|^2 \leq \|x_n - p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n M_2.$$

Proof Let $p \in VI(C,F)$. Then, we see from (54) that

$$\begin{aligned} \|w_n - p\|^2 &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n(1-\alpha_n)M_1\|x_n - p\| + \alpha_n^2 M_1^2 \\ &\leq \|x_n - p\|^2 + \alpha_n\big(2(1-\alpha_n)M_1\|x_n - p\| + \alpha_n M_1^2\big) \\ &\leq \|x_n - p\|^2 + \alpha_n M_2, \end{aligned} \tag{55}$$

where $M_2 = \sup_{n\in\mathbb{N}}\{2(1-\alpha_n)M_1\|x_n - p\| + \alpha_n M_1^2\} > 0$.

Then, by substituting (55) into (23), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + \alpha_n M_2 \\ &\quad - \left[\frac{2-\ell}{\ell} - \frac{2K\delta}{\ell}\frac{\gamma_n}{(\gamma_{n+1} - \delta\gamma_n)}\right]\|w_n - x_{n+1}\|^2. \end{aligned}$$

The desired result follows from the last inequality. □

Lemma 4.7 *Assume that $\{x_n\}$ is a sequence generated by Algorithm 3.1 under Assumptions A and B. Then, for all $p \in VI(C,F)$, the following inequality holds:*

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 \\ &\quad + \alpha_n\bigg[2(1-\alpha_n)\|x_n - p\|\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\quad + \tau_n\|x_n - x_{n-1}\|\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\quad + 2\|p\|\|w_n - x_{n+1}\| + 2\langle p, p - x_{n+1}\rangle\bigg]. \end{aligned}$$

Proof Using Lemma 2.3 and (51), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 \\ &= \big\|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\tau_n(x_n - x_{n-1}) - \alpha_n p\big\|^2 \\ &\leq \big\|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\tau_n(x_n - x_{n-1})\big\|^2 \\ &\quad + 2\alpha_n\langle -p, w_n - p\rangle \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2(1-\alpha_n)\tau_n\|x_n - p\|\|x_n - x_{n-1}\| \\ &\quad + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle p, p - w_n\rangle \\ &= (1-\alpha_n)^2\|x_n - p\|^2 + 2(1-\alpha_n)\tau_n\|x_n - p\|\|x_n - x_{n-1}\| \\ &\quad + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle p, x_{n+1} - w_n\rangle + 2\alpha_n\langle p, p - x_{n+1}\rangle \\ &\leq (1-\alpha_n)\|x_n - p\|^2 \\ &\quad + \alpha_n\bigg[2(1-\alpha_n)\|x_n - p\|\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\quad + \tau_n\|x_n - x_{n-1}\|\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| \end{aligned}$$

$$+ 2\|p\|\|w_n - x_{n+1}\| + 2\langle p, p - x_{n+1}\rangle\Bigg],$$

which is the required inequality. □

Lemma 4.8 *Let $\{w_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 3.1 such that Assumptions A and B hold. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup x^* \in H$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, then $x^* \in VI(C,F)$.*

Proof Assume that $\{w_n\}$ and $\{y_n\}$ are two sequences generated by Algorithm 3.1 with subsequences $\{w_{n_k}\}$ and $\{y_{n_k}\}$, respectively, such that $w_{n_k} \rightharpoonup x^*$. By the hypothesis of the lemma, we have $y_{n_k} \rightharpoonup x^*$. Since $y_{n_k} \in C_{n_k}$ by the definition of C_n, we have

$$c_i(w_{n_k}) + \langle c_i'(w_{n_k}), y_{n_k} - w_{n_k}\rangle \leq 0. \tag{56}$$

By applying the Cauchy–Schwarz inequality, from (56) we obtain

$$c_i(w_{n_k}) \leq \|c_i'(w_{n_k})\|\|y_{n_k} - w_{n_k}\|. \tag{57}$$

By the Lipschitz continuity of $c_i'(\cdot)$ and the fact that $\{w_{n_k}\}$ is bounded, then $\{c_i'(w_{n_k})\}$ is bounded. This implies that there exists a constant $M_4 > 0$, such that $\|c_i(w_{n_k})\| \leq M_4$, $\forall k \geq 0$. Hence, we see from (57) that

$$c_i(w_{n_k}) \leq M_4\|y_{n_k} - w_{n_k}\|. \tag{58}$$

Since $c_i(\cdot)$ is continuous, then it is lower semicontinuous. Also, since $c_i(\cdot)$ is convex, then by Lemma 2.6, $c_i(\cdot)$ is weakly lower semicontinuous. Then, we have from (58) and the definition of weakly lower semicontinuity that

$$c_i(x^*) \leq \liminf_{k\to\infty} c_i(w_{n_k}) \leq \lim_{k\to\infty} M_4\|y_{n_k} - w_{n_k}\|. \tag{59}$$

Hence, by the hypothesis of the lemma it follows from (59) that $x^* \in C$. By the property of the projection map (see Lemma 2.1), we have

$$\langle y_{n_k} - w_{n_k} + \gamma_{n_k} F w_{n_k}, p - y_{n_k}\rangle \geq 0, \quad \forall p \in C \subset C_{n_k}.$$

Then, by the monotonicity of F, we obtain

$$\begin{aligned}
0 &\leq \langle y_{n_k} - w_{n_k}, p - y_{n_k}\rangle + \gamma_{n_k}\langle F w_{n_k}, p - y_{n_k}\rangle \\
&= \langle y_{n_k} - w_{n_k}, p - y_{n_k}\rangle + \gamma_{n_k}\langle F w_{n_k}, p - w_{n_k}\rangle + \gamma_{n_k}\langle F w_{n_k}, w_{n_k} - y_{n_k}\rangle \\
&\leq \langle y_{n_k} - w_{n_k}, p - y_{n_k}\rangle + \gamma_{n_k}\langle F p, p - w_{n_k}\rangle + \gamma_{n_k}\langle F w_{n_k}, w_{n_k} - y_{n_k}\rangle.
\end{aligned} \tag{60}$$

Since $\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0$ and $\lim_{k\to\infty} \gamma_{n_k} = \gamma > 0$, then by letting $k \to \infty$ in (60), we have

$$\langle Fp, p - x^*\rangle \geq 0, \quad \forall p \in C.$$

Hence, by Lemma 2.8, we have that $x^* \in VI(C,F)$.

At this juncture, we proceed to prove the strong convergence theorem of our proposed Algorithm 3.1. □

Theorem 4.9 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumptions A and B. Then, $\{x_n\}$ converges strongly to an element $\hat{x} \in VI(C,F)$, where $\|\hat{x}\| = \min\{\|p\| : p \in VI(C,F)\}$.*

Proof Since $\|\hat{x}\| = \min\{\|p\| : p \in VI(C,F)\}$, then we have $\hat{x} = P_{VI(C,F)}(0)$. From Lemma 4.7 we obtain

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 \leq{}& (1-\alpha_n)\|x_n - \hat{x}\|^2 \\ & + \alpha_n \biggl[2(1-\alpha_n)\|x_n - \hat{x}\| \frac{\tau_n}{\alpha_n} \|x_n - x_{n-1}\| \\ & + \tau_n \|x_n - x_{n-1}\| \frac{\tau_n}{\alpha_n} \|x_n - x_{n-1}\| \\ & + 2\|\hat{x}\| \|w_n - x_{n+1}\| + 2\langle \hat{x}, \hat{x} - x_{n+1}\rangle \biggr] \qquad (61) \\ ={}& (1-\alpha_n)\|x_n - \hat{x}\|^2 + \alpha_n d_n, \qquad (62) \end{aligned}$$

where, $d_n = [2(1-\alpha_n)\|x_n - \hat{x}\| \frac{\tau_n}{\alpha_n} \|x_n - x_{n-1}\| + \tau_n \|x_n - x_{n-1}\| \frac{\tau_n}{\alpha_n} \|x_n - x_{n-1}\| + 2\|\hat{x}\| \|w_n - x_{n+1}\| + 2\langle \hat{x}, \hat{x} - x_{n+1}\rangle]$.

Now, we claim that $\lim_{n\to\infty} \|x_n - \hat{x}\| = 0$. To verify this claim, it suffices to show by Lemma 2.9 that $\lim_{k\to\infty} d_{n_k} \leq 0$, for every subsequence $\{\|x_{n_k} - \hat{x}\|\}$ of $\{\|x_n - \hat{x}\|\}$ satisfying

$$\liminf_{k\to\infty} \bigl(\bigl(\|x_{n_{k+1}} - \hat{x}\| - \|x_{n_k} - \hat{x}\|\bigr) \geq 0. \qquad (63)$$

We assume that $\{\|x_{n_k} - \hat{x}\|\}$ is a subsequence of $\{\|x_n - \hat{x}\|\}$ such that (63) holds. Then, from Lemma 4.6 we have

$$\begin{aligned} \biggl[\frac{2-\ell}{\ell} - \frac{2K\delta}{\ell} \frac{\gamma_{n_k}}{(\gamma_{n_k+1} - \delta\gamma_{n_k})} \biggr] \|w_{n_k} - x_{n_k+1}\|^2 \leq{}& \|x_{n_k} - \hat{x}\|^2 \\ & - \|x_{n_k+1} - \hat{x}\|^2 + \alpha_{n_k} M_2. \qquad (64) \end{aligned}$$

By applying (63) and the fact that $\lim_{k\to\infty} \alpha_{n_k} = 0$, we obtain

$$\biggl[\frac{2-\ell}{\ell} - \frac{2K\delta}{\ell} \frac{\gamma_{n_k}}{(\gamma_{n_k+1} - \delta\gamma_{n_k})} \biggr] \|w_{n_k} - x_{n_k+1}\|^2 \to 0, \quad k \to \infty.$$

Hence, by (50) we obtain

$$\|w_{n_k} - x_{n_k+1}\| \to 0, \quad k \to \infty. \qquad (65)$$

Also, from (22) and (65), we obtain

$$\|w_{n_k} - y_{n_k}\| \to 0, \quad k \to \infty. \qquad (66)$$

By Remark (3.3), the definition of w_n, and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we have

$$\begin{aligned}\|w_{n_k}-x_{n_k}\| &= \big\|(1-\alpha_{n_k})\big(x_{n_k}+\tau_{n_k}(x_{n_k}-x_{n_k-1})-x_{n_k}\big)\big\| \\ &= \big\|(1-\alpha_{n_k})(x_{n_k}-x_{n_k})+(1-\alpha_{n_k})\tau_{n_k}(x_{n_k}-x_{n_k-1})-\alpha_{n_k}x_{n_k}\big\| \\ &\le (1-\alpha_{n_k})\|x_{n_k}-x_{n_k}\|+(1-\alpha_{n_k})\tau_{n_k}\|x_{n_k}-x_{n_k-1}\|+\alpha_{n_k}\|x_{n_k}\| \\ &\to 0, \quad k\to\infty. \end{aligned} \tag{67}$$

From (65) and (67), we obtain

$$\|x_{n_k}-x_{n_k+1}\| \le \|x_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k+1}\|\to 0, \quad k\to\infty. \tag{68}$$

To complete the proof, we need to show that $w_\omega(x_n)\subset VI(C,F)$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)\neq\emptyset$. Now, let $x^*\in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. It follows from (67) that $w_{n_k}\rightharpoonup x^*$ as $k\to\infty$. Then, by Lemma 4.8 together with (66), we see that $x^*\in VI(C,F)$. Hence, $w_\omega(x_n)\subset VI(C,F)$ since $x^*\in w_\omega(x_n)$ was chosen arbitrarily.

Again, since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, such that $x_{n_{k_j}}\rightharpoonup z$, and

$$\lim_{j\to\infty}\langle\hat{x},\hat{x}-x_{n_{k_j}}\rangle=\limsup_{k\to\infty}\langle\hat{x},\hat{x}-x_{n_k}\rangle. \tag{69}$$

Since $\hat{x}=P_{VI(C,F)}(0)$, then by the property of the projection map and (69), we obtain

$$\limsup_{k\to\infty}\langle\hat{x},\hat{x}-x_{n_k}\rangle=\lim_{j\to\infty}\langle\hat{x},\hat{x}-x_{n_{k_j}}\rangle=\langle\hat{x},\hat{x}-z\rangle\le 0. \tag{70}$$

By combining (68) and (70), we obtain

$$\limsup_{k\to\infty}\langle\hat{x},\hat{x}-x_{n_k+1}\rangle=\limsup_{k\to\infty}\langle\hat{x},\hat{x}-x_{n_k}\rangle=\langle\hat{x},\hat{x}-z\rangle\le 0. \tag{71}$$

Using (65), Remark 3.3 together with (71), we clearly see that $\limsup_{k\to\infty}d_{n_k}\le 0$. Then, by applying Lemma 2.9 to (61), we easily conclude that $\lim_{n\to\infty}\|x_n-\hat{x}\|=0$ as required. This completes the proof. □

Remark 4.10 We note that our strong convergence analysis completely avoids the "two-cases" approach, which is usually employed by researchers in the proof of strong convergence theorems (see [32, 46]). However, we adopted a much simpler and more straightforward approach in our proofs.

5 Numerical examples

Here, we perform some numerical experiments to illustrate the performance of our method, Algorithm 3.1 (Proposed Alg.), in comparison with Algorithm A.1 proposed by Ma and Wang (Ma and Wang Alg.), Algorithm A.2 proposed by He *et al.* (He *et al.* Alg.), Algorithm A.3 by He, Wu *et al.* (He, Wu *et al.* Alg.), Algorithm A.4 by Thong and Gibali

(Thong and Gibali Alg.) and Algorithm A.5 by Thong and Gibali (Thong and Gibali Alg.). Our experiments were carried out on MATLAB R2021(b).

The choice of values for our parameters are as follows: In Algorithm 3.1, we chose $\tau = 0.88$, $\theta_n = (\frac{2}{3n+1})^2$, $\alpha_n = \frac{2}{3n+1}$, $\gamma_1 = 0.99$, $\mu_n = \frac{30}{(3n+4)^2}$, $\ell = 0.25$. Also, we chose $\gamma_{-1} = 0.0017$, $\xi = 0.76$, $\nu = 0.87$ in Algorithm A.1, $\chi = 0.97$ in Algorithm A.2 $\sigma = 0.02$, $\omega = 0.05$ in Algorithm A.3, $g = 0.66$, $\lambda = 1.2$, $\delta_n = \frac{2}{3n+1}$, $\beta_n = \frac{1-\delta_n}{2}$, in Algorithm A.4 and $f(x) = \frac{1}{4}x$ in Algorithm A.5.

Our numerical experiments will be conducted using the following examples below:

Example 5.1 Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x_1, x_2) = (6h(x_1), 3x_1 + x_2)$ on the feasible set $C := C^1 \cap C^2 \subseteq \mathbb{R}^2$, where

$$C^1 := \{(x_1, x_2) \in \mathbb{R}^2 : c_1(x_1, x_2) := x_1^2 + x_2^2 - 2 \le 0\},$$
$$C^2 := \{(x_1, x_2) \in \mathbb{R}^2 : c_2(x_1, x_2) := x_1^2 - x_2 \le 0\}, \quad \text{and,}$$
$$h(t) := \begin{cases} e(t-1) + e & \text{if } t > 1, \\ e^t & \text{if } -1, \\ e^{-1}(t+1) + e^{-1} & \text{if } t < -1. \end{cases}$$

We see from Lemma 2.10 that $VI(C,F)$ is nonempty, in particular, the solution, $VI(C,F)$ of the VI 5.1, is the set $\{(-1,1)\}$. We note that F is monotone and Lipschitz continuous. The functions $c_i'(\cdot)$ are also Lipschitz continuous, for $i = 1,2$, with constants $L_1 = L_2 = 2$, where $L = \max\{L_1, L_2\}$. Also, $K = 3\sqrt{e^2+1}$, and $K' = 6\sqrt{e^2+1}$, see [25]. For this example we choose $\delta = 0.068$ and set $M = K$.

We test the algorithms for four different initial points as follows:

Case I: $x_0 = (5,5)$, $x_1 = (-2,-1)$;

Case II: $x_0 = (2,6)$, $x_1 = (1,-2)$;

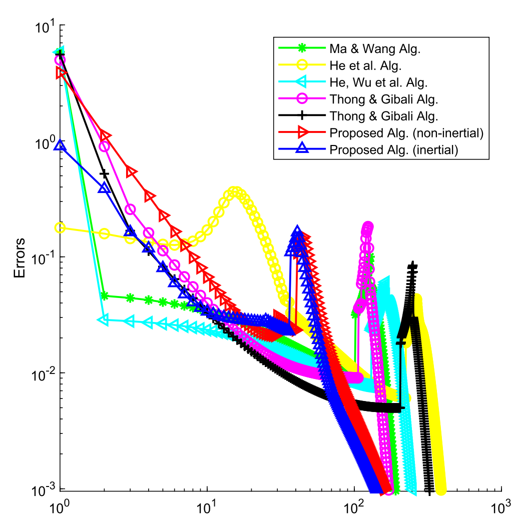

Figure 1 Example 5.1: Case I

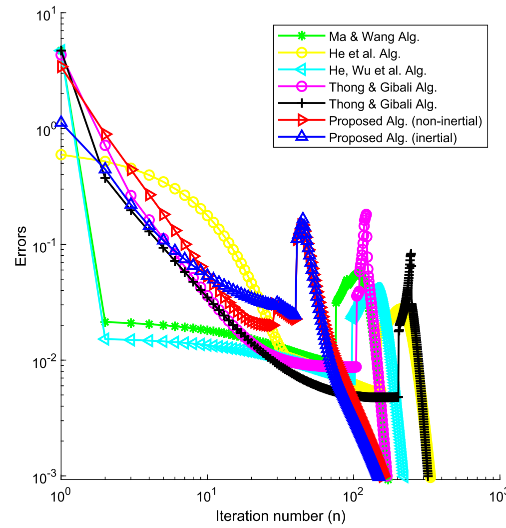

Figure 2 Example 5.1: Case II

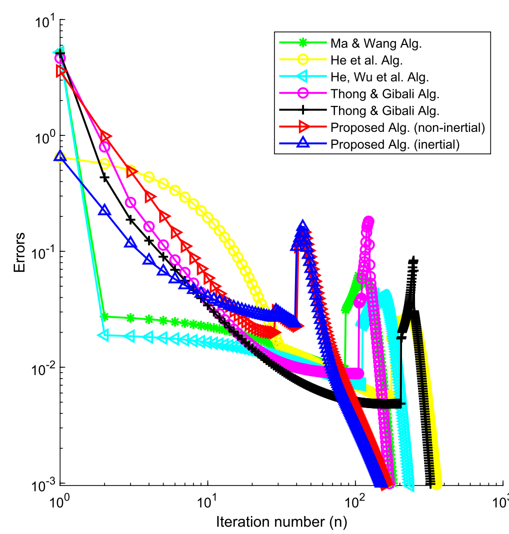

Figure 3 Example 5.1: Case III

Case III: $x_0 = (3,6)$, $x_1 = (1,0)$;
Case IV: $x_0 = (5,5)$, $x_1 = (-1,1)$.

The stopping criterion used for this example is $\|x_{n+1} - x_n\| < 10^{-3}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figs. 1–4 and Table 1.

Next, we provide an example in infinite-dimensional spaces for the experiment of our strong convergence result.

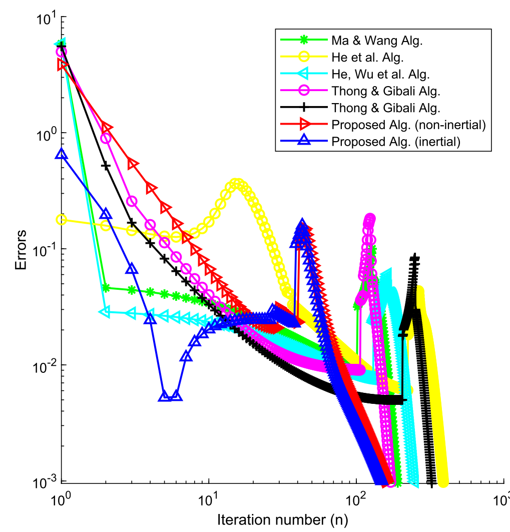

Figure 4 Example 5.1: Case IV

Table 1 Numerical results for Example 5.1

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Ma and Wang Alg.	172	0.0090	172	0.0080	180	0.0080	191	0.0079
He *et al. Alg.*	335	0.0168	335	0.0169	355	0.0158	389	0.0157
He, Wu *et al. Alg.*	222	0.0059	222	0.0061	235	0.0056	249	0.0056
Thong and Gibali Alg.	169	0.0064	169	0.0071	170	0.0067	171	0.0065
Thong and Gibali Alg.	322	0.0081	322	0.0078	323	0.0073	325	0.0079
Proposed Alg. 3.1 (noninertial)	159	0.0207	159	0.0213	160	0.0202	160	0.0202
Proposed Alg. 3.1	149	0.0171	149	0.0177	150	0.0170	149	0.0172

Example 5.2 Let $F(x) = 3x$, $\forall x \in H$, and let $C \subset H$ be the closed, convex, feasible set defined as follows:

$$C := \bigcap_{i=1}^{m} C^i := \bigcap_{i=1}^{m} \{x \in H : c_i(x) := \|x_i\|^2 - 2 \leq 0\}, \quad \text{for each } i = 1, \dots, m,$$

$$\text{where, } H = (\ell_2(\mathbb{R}), \|\cdot\|), \|x\|_2 = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}}, \langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k, \ \forall x \in \ell_2(\mathbb{R}),$$

$$\text{and, } \ell_2(\mathbb{R}) := \left\{x = (x_1, x_2, \dots, x_n, \dots), x_k \in \mathbb{R} : \sum_{k=1}^{\infty} |x_k|^2 < \infty\right\}.$$

Also, note that F is monotone and 3-Lipschitz continuous, c_i' is Lipschitz continuous, and $K = 1$. We chose $\delta = 0.14$ in this example.

We chose different initial values as follows:

Case I: $x_0 = (-4, 1, -\frac{1}{4}, \dots)$, $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$;

Case II: $x_0 = (3, 1, \frac{1}{3}, \dots)$, $x_1 = (1, 0.1, 0.01, \dots)$;

Case III: $x_0 = (5, 1, \frac{1}{5}, \dots)$, $x_1 = (1, -0.1, 0.01, \dots)$;

Case IV: $x_0 = (4, 1, \frac{1}{4}, \dots)$, $x_1 = (-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots)$.

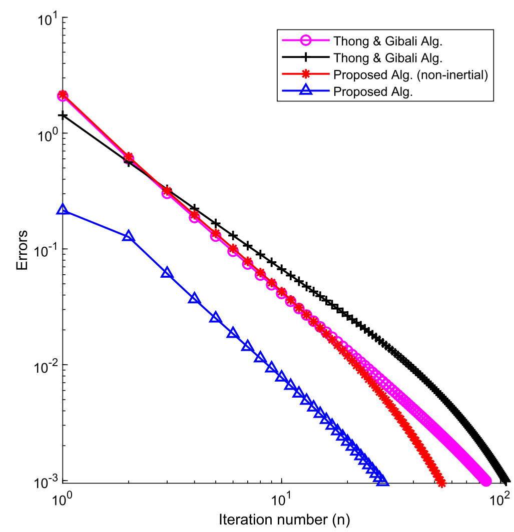

Figure 5 Example 5.2: Case I

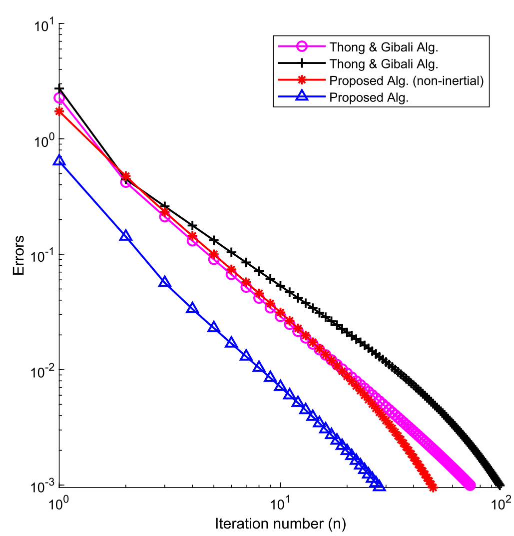

Figure 6 Example 5.2: Case II

The stopping criterion used for this example is $\|x_{n+1} - x_n\| < 10^{-3}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figs. 5–8 and Table 2.

6 Conclusion

In this paper, we studied the classical variational inequality problem defined over a finite intersection of sublevel sets of convex functions. We proposed a new iterative method called a "Totally relaxed inertial self-adaptive projection and contraction method" (TRISPCM), in which the projection onto the feasible set is replaced with a projection onto some half-space. Our method does not require any line-search procedure, rather it uses

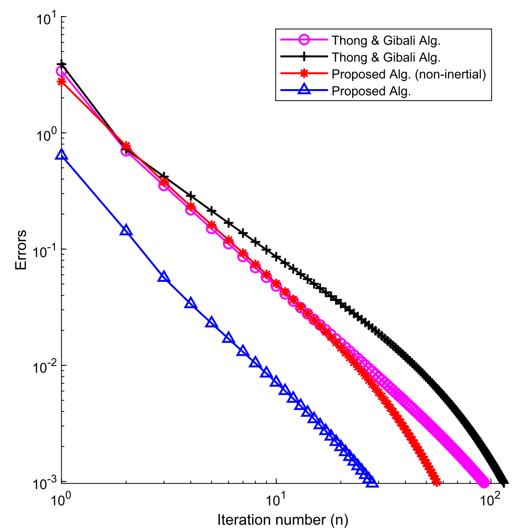

Figure 7 Example 5.2: Case III

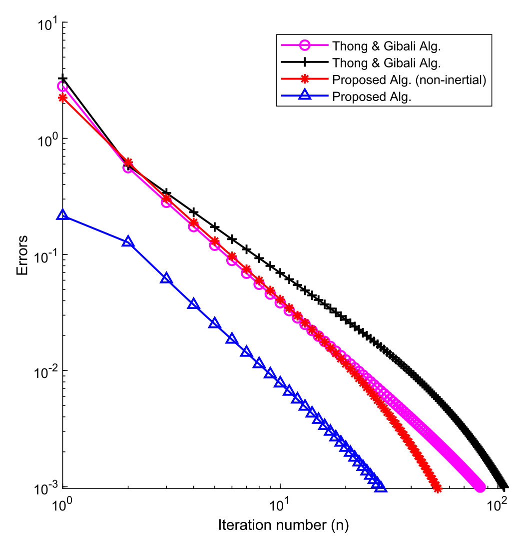

Figure 8 Example 5.2: Case IV

Table 2 Numerical results for Example 5.2

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Thong and Gibali Alg.	86	0.0147	76	0.0140	93	0.0140	83	0.0146
Thong and Gibali Alg.	106	0.0052	98	0.0056	115	0.0053	107	0.0053
Proposed Alg. 3.1 (noninertial)	54	0.0110	49	0.0118	56	0.0190	53	0.0074
Proposed Alg. 3.1	29	0.0096	28	0.0101	28	0.0091	29	0.0097

a more efficient self-adaptive step-size technique. We also employed the relaxation and inertial techniques to speed up the rate of convergence of our proposed method. Moreover, under some mild conditions we proved that the sequence generated by our proposed algorithm converges strongly to a minimum-norm solution of the problem. Lastly, we conducted some numerical experiments to clearly showcase the computational advantage of our proposed method over the existing methods in the literature.

Appendix

Algorithm A.1 Algorithm 2 in [34]

Step 0. Choose $x_{-1}, x_0, y_{-1} \in H$; $\xi, \nu \in [a,b] \subset (0,1)$, $\gamma_{-1} \in (0, \frac{1-\xi^2}{2\beta_p L_1}]$. Set $n = 0$.

Step 1. Given γ_{n-1}, y_{-1}, and x_{-1}. Let $p_{n_{n-1}} = x_{n-1} - y_{-1}$.

$$\gamma_n := \begin{cases} \gamma_{n-1}, & \gamma_{n-1}\|Fx_{n-1} - Fy_{n-1}\| \leq \xi \|p_{n-1}\|, \\ \gamma_{n-1}\nu, & \text{otherwise}. \end{cases}$$

Step 2. Compute

$$y_n = P_{C_n}(x_n - \gamma_n F x_n).$$

Step 3. Compute

$$x_{n+1} = P_{C_n}\big(y_n - \gamma_n(Fy_n - Fx_n)\big),$$

where,

$$C_n := \big\{x \in H : c(x_n) + \big\langle c_i'(x_n), x - x_n\big\rangle \leq 0\big\}.$$

Set $n := n + 1$ and return to *Step 1*. Where $F : H \to H$ is monotone and L_2– Lipschitz continuous, $c'(\cdot)$ is Lipschitz continuous, and β_p is the parameter in Lemma 2.10.

Algorithm A.2 Algorithm 1 in [23]

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ y_n = P_{C_n}(x_n - \gamma_n F x_n), \\ \text{If } x_n = y_n, \text{ stop, } x_n \text{ is a solution. Otherwise,} \\ d(x_n, y_n) = (x_n - y_n) - \gamma_n(Fx_n - Fy_n), \\ x_{n+1} = x_n - \ell\psi_n d(x_n, y_n), \\ \text{or,} \\ x^*_{n+1} = P_{T_n}[x_n - \ell\psi_n\gamma_n F y_n], \end{cases}$$

where C_n and T_n are half-spaces given by $C_n := \{w \in H : c(x_n) + \langle c'(x_n), w - x_n\rangle \leq 0\}$ and $T_n := \{w \in H : \langle x_n - y_n, \gamma_n F y_n - d(x_n, y_n)\rangle \geq 0\}$, respectively, and

$$\psi_n := \frac{\langle x_n - y_n, d(x_n, y_n)\rangle - M\gamma_n\|x_n - y_n\|^2}{\|d(x_n, y_n)\|^2},$$

$$\gamma_n = \frac{\chi}{M + \sqrt{M^2 + L_n^2}},$$

for some constant $M > 0$ and L_n satisfying certain conditions.

Algorithm A.3 (Algorithm 3.1 in [25])

Step 0. Set $L = \max\{L_i : i \in I\}$, $K' = KL$, where K is defined in Lemma 2.10, and $L_i (i \in I)$ is the Lipschitz constant. Choose arbitrarily, $x_0 \in H$, $\xi \in (0,1)$, and set $n = 0$.

Step 1. Given the current iterate x_n, construct the family of half-spaces

$$C_n^i = \left\{ w \in H : c_i(x_n) + \left\langle c_i'(x_n), w - x_n \right\rangle \le 0 \right\}, \quad i \in I.$$

Set,

$$C_n := \bigcap_{i \in I} C_n^i$$

and compute:

$$y_n := P_{C_n}(x_n - \gamma_n F x_n);$$

where, $\gamma_n = \sigma \omega^{g_n}, \sigma > 0, \omega \in (0,1)$,

and g_n is the smallest integer, such that

$$\gamma_n^2 \|Fx_n - Fy_n\|^2 + K' \gamma_n \|x_n - y_n\|^2 \le \xi \|x_n - y_n\|^2.$$

Step 2. If $w_n = y_n$, then stop, $x_n \in SOL(C,F)$, otherwise, calculate the next iterative step by:

$$x_{n+1} = P_{C_n}(x_n - \gamma_n F y_n),$$

or by,

$$x_{n+1} = P_{T_n}(x_n - \gamma_n F y_n),$$

where $T_n = \left\{ w \in H : \langle x_n - \gamma_n F x_n - y_n, w - y_n \rangle \le 0 \right\}$.

Set $n := n + 1$ and return to *Step 1*.

Algorithm A.4 (Algorithm 3.1 in [46])

Step 0. Given $\lambda > 0$, $g \in (0,1)$, $\nu \in (0,1)$, $\ell \in (0,2)$. Let $x_0 \in H$ be chosen arbitrarily. Given the current iterate x_n, calculate $x_n + 1$ as follows:

Step 1. Compute:

$$y_n = P_C(x_n - \gamma_n F x_n),$$

where γ_n is chosen to be the largest $\eta_n \in [\lambda, \lambda g, \lambda g^2, \dots]$ satisfying

$$\eta \|Fx_n - Fy_n\| \le \nu \|x_n - y_n\|.$$

If $w_n = y_n$, then stop, $x_n \in SOL(C,F)$, otherwise,

Step 2. Compute:

$$z_n = P_{T_n}\big(x_n - \ell \gamma_n F(y_n)\big),$$

$$\text{where } T_n := \left\{x \in H : \langle x_n - \gamma_n F x_n - y_n, x - y_n \rangle \le 0\right\}, \text{ and}$$
$$d_n := x_n - y_n - \gamma_n (F x_n - F y_n).$$

Step 3. Compute:

$$x_{n+1} = (1 - \delta_n - \beta_n) x_n + \beta_n z_n.$$

Set $n := n + 1$ and return to *Step 1.*

Algorithm A.5 (Algorithm 3.2 in [46])

Step 0. Given $\lambda > 0$, $g \in (0,1)$, $\nu \in (0,1)$, $\ell \in (0,2)$. Let $x_0 \in H$ be chosen arbitrarily. Given the current iterate x_n, calculate $x_n + 1$ as follows:

Step 1. Compute:

$$y_n = P_C(x_n - \gamma_n F x_n),$$

where γ_n is chosen to be the largest $\eta_n \in [\lambda, \lambda g, \lambda g^2, \dots]$ satisfying

$$\eta \| F x_n - F y_n \| \le \nu \| x_n - y_n \|.$$

If $w_n = y_n$, then stop, $x_n \in SOL(C, F)$, otherwise,

Step 2. Compute:

$$z_n = P_{T_n}(x_n - \ell \gamma_n F y_n),$$
$$\text{where } T_n := \left\{x \in H : \langle x_n - \gamma_n F x_n - y_n, x - y_n \rangle \le 0\right\}, \text{ and}$$
$$d_n := x_n - y_n - \gamma_n (F x_n - F y_n).$$

Step 3. Compute:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n.$$

Set $n := n + 1$ and return to *Step 1.*
Where $f : H \to H$ is a contraction with constant $\phi \in (0,1)$.

Acknowledgements
The authors sincerely thank the anonymous referees for their careful reading, constructive comments, and useful suggestions that improved the manuscript. The first author acknowledges with thanks the International Mathematical Union Breakout Graduate Fellowship (IMU-BGF) Award for his doctoral study. The second author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903) and DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa (Grant Number 2022-087-OPA). The third author is wholly supported by the University of KwaZulu-Natal, Durban, South Africa Postdoctoral Fellowship. He is grateful for the funding and financial support. The opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

Funding
The first author is funded by International Mathematical Union Breakout Graduate Fellowship (IMU-BGF). The second author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903) and DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa (Grant Number 2022-087-OPA). The third author is funded by the University of KwaZulu-Natal, Durban, South Africa Postdoctoral Fellowship.

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization of the article was given by OTM, TOA and AG, methodology by OTM and TOA, formal analysis, investigation and writing–original draft preparation by VAU, OTM, and TOA software and validation by OTM, writing–review and editing by VAU, OTM TOA, and AG, project administration and supervision by OTM and AG. All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

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Received: 13 April 2023 Accepted: 2 October 2023 Published online: 02 November 2023

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