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Certain investigations of sequential warped product submanifolds on cosymplectic manifolds

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Abstract

In a special class of almost contact metric manifolds known as cosymplectic manifolds, the current study aims to establish the existence result and a few inequalities for sequential warped product submanifolds. These results and inequalities represent fruitful connections between the primary intrinsic and extrinsic invariants. Furthermore, findings related to Dirichlet energy have been addressed. Finally, some exceptional cases resulting in several inequities are examined.

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1 Introduction

The Nash theorem was created to ensure that Riemannian manifolds are always perceived as Riemannian submanifolds [23]. Gromov in [16], however, noted that this hope had not come true. The fundamental cause of this is that the extrinsic aspects of the submanifolds are not within the control of the known intrinsic facts. The establishment of meaningful connections between the primary intrinsic and extrinsic invariants of submanifolds is a fundamental concern in submanifold theory. There is growing interest in stabilizing this concern by developing several kinds of geometric equalities and inequalities that admit invariants. The research along this path gained momentum after Chen [7–9], employed the warped product technique as a method to develop inequalities involving invariants and deduced some significant information. After that, several differential geometers established geometric inequalities to analyze the relationships between extrinsic and intrinsic parameters not only for warped products but also for their generalizations, i.e., double and multiple warped products in different ambient spaces (see [10, 12, 13, 24, 32, 33]). The sequential warped product (briefly, SWP), whose base remains a warped product manifold, is a new generalization of warped products (in short, WP) [5]. Shenawy revealed SWP in [31]. Thereafter, De, Shenawy, and Unal did a full investigation of SWP in [14]. With certain examples, Pahan and Pal [25], Karaca and Ozgür [18] extended the study of sequential warped space with (quasi-) Einstein's structure. Sahin [28] discovered the presence of SWP submanifolds in Kähler manifolds and presented geometric inequalities and equalities such that the associations between the intrinsic and extrinsic invariants are illustrated. In [19], the author recently generalized the Chen inequality involving invariants in nearly Kähler manifolds. Recently, Perktas–Blaga [27] continued the study on Sasakian manifolds and provided some nonexistence results. Motivated by these developments, in this article, the authors first present a numerical example in support of the existence of SWP of the form $(\Omega_T \times_f \Omega_\perp) \times_h \Omega_\theta$ on cosymplectic manifolds: an odd-dimensional counterpart of a Kaehler manifold, and then generalized Chen's type inequality and equalities involved Dirichlet energy and curvature to analyze the geometric invariants using the SWP technique for such type of SWP on a cosymplectic manifold. Other types of SWP on a cosymplectic manifold may be thought of as open problems. A succinct summary of the article is provided below. We first recall some key concepts and definitions of an almost contact, cosymplectic, and their submanifolds in Sect. 2 and Sect. 3. Then, in Sect. 4 we derive and review a few results related to SWP for future use. Finally, in Sect. 5 we establish some important results that extend inequalities for various warped products, inequalities involving Dirichlet energy and curvature in cosymplectic manifolds. We also look at the specific instances and corresponding equality.

2 Preliminaries

Let $\bar{\Omega}$ be a $2m+1$-dimensional smooth manifold and $\mathfrak{X}(\bar{\Omega})$ be the Lie algebra of smooth vector fields on $\bar{\Omega}^{2m+1}$. Then, (φ, ξ, η) is said to have an almost contact structure on $\bar{\Omega}^{2m+1}$ [22], if there exist an endomorphism φ of type $(1,1)$, a smooth global vector field ξ and a 1-form η satisfying

$$\varphi^2 = -\mathcal{I} + \eta \otimes \xi \quad \text{and} \quad \eta\varphi = 0, \tag{1}$$

where $\mathcal{I}$ is the identity map. If an almost contact structure admits Riemann metric g such that

$$g(\varphi\cdot, \varphi\cdot) = g(\cdot,\cdot) - \eta(\cdot)\eta(\cdot), \tag{2}$$

$$\eta(\xi) = 1, \qquad \varphi\xi = 0, \quad \text{and} \quad g(\cdot, \varphi\cdot) + g(\varphi\cdot, \cdot) = 0, \tag{3}$$

then it is called an almost contact Riemann structure (φ, ξ, η, g) to $\bar{\Omega}^{2m+1}$. Furthermore, $\bar{\Omega}^{2m+1}$ associated with the structure (φ, ξ, η, g) is called an almost contact metric manifold $\bar{\Omega}(\varphi, \xi, \eta, g)$.

Also, $g(\cdot, \varphi\cdot) = \Phi(\cdot,\cdot)$ where Φ is termed the fundamental 2-form of $\bar{\Omega}^{2m+1}$. If both η and Φ are closed then $\bar{\Omega}(\varphi, \xi, \eta, g)$ is an almost cosymplectic manifold.

Definition 1 An almost contact Riemannian manifold $\bar{\Omega}^{2m+1}$ is *cosymplectic* [22] if Φ is parallel, that is, $\bar{\nabla}\Phi = 0$. In this context, $\bar{\nabla}$ stands for the connection Levi–Civita on $\bar{\Omega}^{2m+1}$ w.r.t. g.

From the above definition and (1), it is easy to obtain that

$$\bar{\nabla}_U \xi = (\bar{\nabla}_U \eta)V = 0, \quad \text{for all } U, V \in \mathfrak{X}(\bar{\Omega}). \tag{4}$$

Let $\bar{\Omega}(k)$ be a cosymplectic space form, then the Riemannian curvature on $\bar{\Omega}(k)$ for arbitrary $U, V, W, X \in \mathfrak{X}(\bar{\Omega})$ is given by

$$\begin{aligned}
&\bar{R}(U,V,W,X)\\
&\quad= \frac{k}{4}\big(g(V,W)g(U,X) - g(U,W)g(V,X) - \eta(V)\eta(W)g(U,X)\\
&\qquad + \eta(U)\eta(W)g(V,X) + \eta(V)\eta(X)g(U,W) - \eta(U)\eta(X)g(V,W)\\
&\qquad + g(\varphi V,W)g(\varphi U,X) - g(\varphi U,W)g(\varphi V,X) + 2g(\varphi U,V)g(\varphi W,X)\big). \qquad (5)
\end{aligned}$$

3 Geometry of submanifolds

Now, we review some important formulas and definitions for future usage concerning submanifolds Ω of dimension d, immersed isometrically in $\bar{\Omega}^{2m+1}$. (For details refer to [6].) The formulas of Gauss–Weingarten are defined by the expressions:

$$\bar{\nabla}_U V = \nabla_U V + \sigma(U,V), \qquad (6)$$

$$\bar{\nabla}_U N = \nabla^{\perp}_U N - A_N U, \qquad (7)$$

$\forall U, V \in \mathfrak{X}(\Omega)$: a space tangent to Ω and $N \in \mathfrak{X}(\Omega^{\perp})$: a space normal to Ω. The induced tangent and normal connections on $\mathfrak{X}(\Omega)$ and $\mathfrak{X}(\Omega^{\perp})$ are denoted by the symbols ∇ and $\nabla^{\perp}$, respectively. Then, the shape operator and the second fundamental form (abbreviated SFF) are represented by the A_N at N and σ, respectively, in such a way that

$$g(A_N U, V) = g\big(\sigma(U,V), N\big). \qquad (8)$$

If p is any point in Ω and $\{x_1,\dots,x_d,x_{d+1},\dots,x_{2m+1}\}$ is an orthonormal frame of the tangent space $\mathfrak{X}_p(\bar{\Omega}) = \mathfrak{X}_p(\Omega) \oplus \mathfrak{X}_p(\Omega^{\perp})$ such that $\{x_1,\dots,x_d\} \in \mathfrak{X}_p(\Omega)$ and $\{x_{d+1},\dots,x_{2m+1}\} \in \mathfrak{X}_p(\Omega^{\perp})$, then $\mathcal{H}(p) = \frac{1}{d}\mathrm{trace}\sigma$, gives the mean curvature vector of Ω and $\|\sigma\|^2$ is computed by

$$\|\sigma\|^2 = \sum_{a,b=1}^{d} g\big(\sigma(x_a,x_b), \sigma(x_a,x_b)\big). \qquad (9)$$

By setting $\sigma^c_{ab} = g(\sigma(x_a,x_b),x_c)$, $a,b \in \{x_1,\dots,x_d\}$, $c \in \{x_{d+1},\dots,x_{2m+1}\}$, (9) can be represented as

$$\|\sigma\|^2 = \sum_{c=d+1}^{2m+1}\sum_{a,b=1}^{d} g\big(\sigma(x_a,x_b),x_c\big). \qquad (10)$$

If $\sigma(U,V)$ equals $g(U,V)\mathcal{H}$ (0), then Ω is umbilical (geodesic). For $\mathcal{H} = 0$, Ω is minimal. Moreover, if

$$\varphi U = tU + nU, \qquad (11)$$

$$\varphi N = t'N + n'N, \qquad (12)$$

for all $U \in \mathfrak{X}(\Omega)$, where tU ($t'N$) and nU ($n'N$) are tangential (normal) parts of φU (φN), then employing (1), (2), and (11), we obtain

$$g(U, tV) = -g(tU, V) \quad \text{for all } U, V \in \mathfrak{X}(\Omega). \tag{13}$$

In view of Definition 1 and (6) and (12), we have

$$(\nabla_U t)V = A_{nV}U + t'\sigma(U, V), \tag{14}$$

$$(\nabla_U n)V = -\sigma(U, tV) + n'\sigma(U, V), \tag{15}$$

where $(\nabla_U \varphi)V$, $(\nabla_U t)V$ and $(\nabla_U n)V$ are defined by

$$(\nabla_U \varphi)V = \nabla_U \varphi V - \varphi \nabla_U V, \tag{16}$$

$$(\nabla_U t)V = \nabla_U tV - t\nabla_U V, \tag{17}$$

$$(\nabla_U n)V = \nabla_U^{\perp} nV - n\nabla_U V, \tag{18}$$

for all $U, V \in \mathfrak{X}(\Omega)$. Next, the Gauss–Codazzi equations are defined, respectively, by,

$$\bar{R}(U, V, W, X) = g\big(\sigma(U, W), \sigma(V, X)\big) - g\big(\sigma(U, X), \sigma(V, W)\big) + R(U, V, W, X), \tag{19}$$

$$\big(\bar{R}(U, V)W\big)^{\perp}) = (\nabla_U \sigma)(V, W) - (\nabla_V \sigma)(U, W), \tag{20}$$

for every U, V, W and $X \in \mathfrak{X}(\Omega)$, where R and $\bar{R}$ denote the Riemann curvature on Ω and $\bar{\Omega}^{2m+1}$. Next, the gradient of the smooth function f on Ω is given by

$$g(\nabla f, U) = Uf \quad \text{and} \quad \|\nabla f\|^2 = \sum_{a=1}^{d} \big(U_a(f)\big)^2, \tag{21}$$

for any $U_a \in \mathfrak{X}(\Omega)$.

Here, we refresh certain key definitions and findings from [26] for further usage.

Definition 2 Let Ω be a submanifold of $\bar{\Omega}^{2m+1}$. If at a point p in Ω, any tangent vector field $W/\{0\}$ not proportional to the characteristic vector field $\xi \in \mathfrak{X}(\Omega)$, the *angle* symbolized by θ, between $\varphi(W)$ and $\mathfrak{X}_p(\Omega)$ does not depend on p nor W, then Ω is *pointwise slant*. Consequently, θ is identified as a function on Ω, and thus known as the *slant function*.

Remark 1 For θ constant on Ω, Ω is simply called *a slant submanifold*, specifically, *invariant* when $\theta = 0$ and *antinvariant* when $\theta = \pi/2$.

Next, from the definition and [20, 30], we write the iff condition for a submanifold Ω of $\bar{\Omega}^{2m+1}$ to be pointwise slant, given as follows for any vector field:

$$t^2 W = \cos^2(\theta)\varphi^2 W: \quad W \in \mathfrak{X}(\Omega). \tag{22}$$

Additionally, we have for $W_1, W_2 \in \mathfrak{X}(\Omega)$ that

$$g(tW_1, tW_2) = \cos^2 g(\varphi W_1, \varphi W_2), \tag{23}$$

$$g(nW_1, nW_2) = \sin^2 g(\varphi W_1, \varphi W_2). \tag{24}$$

Definition 3 A submanifold Ω is pointwise *semislant* (resp., *pseudoslant*), if there are distributions $\mathfrak{D}_T$: an invariant and $\mathfrak{D}_\theta$: a pointwise slant on Ω (resp., $\mathfrak{D}_\perp$: an antiinvariant and $\mathfrak{D}_\theta$) with slant function θ and characteristic vector field ξ such that

i. The tangent space $\mathfrak{X}(\Omega)$ admits the orthogonal direct decomposition $\mathfrak{X}(\Omega) = \mathfrak{D}_T \oplus \mathfrak{D}_\theta \oplus < \xi >$ (resp., $\mathfrak{X}(\Omega) = \mathfrak{D}_\perp \oplus \mathfrak{D}_\theta \oplus < \xi >$);

ii. The distribution $\mathfrak{D}_T$ is invariant, i.e., $\varphi(\mathfrak{D}_T) \subseteq \mathfrak{D}_T$ (resp., $\mathfrak{D}_\perp$ is antiinvariant, i.e., $\varphi(\mathfrak{D}_\perp) \subseteq \mathfrak{X}(\Omega)^\perp$).

4 Sequential warped product submanifolds

Shenawy in [31] and De et al. in [14] defined SWP as: consider three Riemannian manifolds B, F_1, and F_2. If $f : B \to \mathbb{R}^+$ and $h : B \times F_1 \to \mathbb{R}^+$, then the SWP $(B \times_f F_1) \times_h F_2$ of B, F_1 and F_2 is the product manifold $\bar{\Omega} = B \times F_1 \times F_2$ endowed with Riemannian metric $\bar{g} = (g_B \oplus_{f^2} g_1) \oplus_{h^2} g_2$. $\bar{\Omega}$ is a Riemannian triple product for f, h constants; $\bar{\Omega}$ is a WP with a product base manifold for exactly one of f, h constant. $\bar{\Omega}$ is SWP, if f, h are nonconstants.

Sahin recently analyzed the possibility of warped products of type $(\Omega_T \times_f \Omega_\perp) \times_h \Omega_\theta$ for Kaehler manifolds and discovered some significant findings [28]. Motivated by the work of Sahin, in this section we continue the study for cosymplectic manifolds: an odd-dimensional counterpart of a Kaehler manifold, and derive several geometric characterizations for such types of submanifolds in cosymplectic manifold.

For a vector field on a factor manifold and its lift to the sequential warped product manifold, we use the same notation. We start by keeping in mind the following statements about SWP manifolds for future use.

Proposition 1 [14] *For $U_a, V_a, W_a \in \mathfrak{X}(\Omega_a)$, $a \in \{1,2,3\}$, we have on $\Omega = (\Omega_1 \times_f \Omega_2) \times_h \Omega_3$ that*

1. $\nabla_{U_a} U_3 = \nabla_{U_3} U_a = U_3(\ln h) U_a = \frac{U_3(h)}{h} U_a$, $a \in \{1,2\}$;
2. $\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = U_1(\ln f) U_2 = \frac{U_1(f)}{f} U_2$;
3. $R(U_a, V_3) W_b = \frac{-1}{h} H^h(U_a, W_b) V_3$, $a, b \in \{1,2\}$.

Proposition 2 [14] *In a SWP $\Omega = (\Omega_1 \times_f \Omega_2) \times_h \Omega_3$, Ω_1 and $\Omega_1 \times_f \Omega_2$ are totally geodesic submanifolds in $\Omega_1 \times_f \Omega_2$ and Ω. Also, Ω_2 and Ω_3 are totally umbilical in $\Omega_1 \times_f \Omega_2$ and Ω.*

Now, we present a nonexistence result or SWP submanifolds by including ξ in the second or the third factor manifolds.

Theorem 1 *If $\Omega = (\Omega_T \times_f \Omega_\perp) \times_h \Omega_\theta$ is a SWP immersion in cosymplectic manifold $\bar{\Omega}^{2m+1}$. This means*

1. *if $\xi \in \mathfrak{X}(\Omega_\perp)$, then Ω is a CR-slant WP submanifold*;
2. *if $\xi \in \mathfrak{X}(\Omega_\theta)$, then Ω is a single WP submanifold.*

Proof Consider $\xi \in \mathfrak{X}(\Omega_\perp)$, then by the use of Proposition 1, we attain

$$\nabla_U \xi = U(\ln f)\xi, \quad \forall U \in \mathfrak{X}(\Omega_T).$$

In view of (4), (6), and (7), we have $U(\ln f) = 0$. This shows that f is constant on Ω_T. From the above discussion, we conclude that Ω is a CR-slant warped product submanifold. On the other hand, $\xi \in \mathfrak{X}(\Omega_\theta)$, then by the use of Proposition 1, we achieve

$$\nabla_U \xi = U(\ln h)\xi \quad \text{and} \quad \nabla_W \xi = W(\ln h)\xi, \tag{25}$$

for every $U \in \mathfrak{X}(\Omega_T)$ and $V \in \mathfrak{X}(\Omega_\perp)$. In light of (4), (6), (7), and (25), we observe that $U(\ln h) = V(\ln h) = 0$. This shows that h is constant on both the factors Ω_T and $\Omega_\perp$. This completes the proof. □

Next, we first present a numerical example that shows the existence of SWP of the form $(\Omega_T \times_f \Omega_\perp) \times_h \Omega_\theta$ and then continue the study by presenting several important results for such submanifolds.

Example 1 Consider a 13-dimensional Euclidean space $\mathbb{E}^{13}$ with coordinates $(x_1, \ldots, x_6, y_1, \ldots, y_6, t)$ and Euclidean metric g. The almost contact structure (φ, ξ, η) on $\mathbb{E}^{13}$ is described as

$$\varphi\left(\frac{\partial}{\partial x_a}\right) = -\frac{\partial}{\partial y_a}, \qquad \varphi\left(\frac{\partial}{\partial y_a}\right) = \frac{\partial}{\partial x_a}, \qquad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \tag{26}$$

$$\xi = \frac{\partial}{\partial t}, \qquad \eta = dt, \qquad g = dt^2 + \sum_{a=1}^{12} dx_a^2. \tag{27}$$

One can easily verify that the Euclidean space $\mathbb{E}^{13}$ with almost contact structure is a cosymplectic manifold. Consider a subset $\Omega \subset \mathbb{E}^{13}$ immersed as a submanifold by the following immersion

$$x_1 = u\cos\alpha, \quad x_2 = v\cos\beta, \quad x_3 = v\sin\alpha, \quad x_4 = u\sin\beta, \quad x_5 = \alpha\sin\beta,$$
$$x_6 = w, \quad y_1 = v\cos\alpha, \quad y_2 = u\cos\beta, \quad y_3 = u\sin\alpha, \quad y_4 = v\sin\beta,$$
$$y_5 = \alpha\cos\beta, \quad y_6 = \beta, \quad t = t.$$

The tangent subspace of Ω at each point is spanned by the basis

$$Z_\alpha = -u\sin\alpha\frac{\partial}{\partial x_1} + v\cos\beta\frac{\partial}{\partial x_3} + \sin\beta\frac{\partial}{\partial x_5} - v\sin\alpha\frac{\partial}{\partial y_1} + u\cos\alpha\frac{\partial}{\partial y_3} + \cos\beta\frac{\partial}{\partial y_5},$$
$$Z_\beta = -v\sin\beta\frac{\partial}{\partial x_2} + u\cos\beta\frac{\partial}{\partial x_4} + \alpha\cos\beta\frac{\partial}{\partial x_5} - u\sin\beta\frac{\partial}{\partial y_2} + v\cos\beta\frac{\partial}{\partial y_4} - \alpha\sin\beta\frac{\partial}{\partial y_5} + \frac{\partial}{\partial y_6},$$
$$Z_u = \cos\alpha\frac{\partial}{\partial x_1} + \sin\beta\frac{\partial}{\partial x_4} + \cos\beta\frac{\partial}{\partial y_2} + \sin\alpha\frac{\partial}{\partial y_3}, \qquad Z_w = \frac{\partial}{\partial x_6},$$
$$Z_v = \cos\beta\frac{\partial}{\partial x_2} + \sin\alpha\frac{\partial}{\partial x_3} + \cos\alpha\frac{\partial}{\partial y_1} + \sin\beta\frac{\partial}{\partial y_4}, \qquad Z_t = \xi.$$

By straightforward computation, we observed that the distribution $\mathfrak{D}_T = \operatorname{span}\{Z_u, Z_v, Z_t\}$ is invariant, the distribution $\mathfrak{D}_\perp = \operatorname{span}\{Z_\alpha\}$ is antiinvariant and distribution $\mathfrak{D}_\theta =$

span$\{Z_\beta, Z_w\}$ is pointwise slant with slant angle

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{1+u^2+v^2+\alpha^2}}\right).$$

The induced metric on Ω is given by

$$g_\Omega = du^2 + dv^2 + dt^2 + f^2\, d\alpha^2 + h^2\left(d\beta^2 + dw^2\right).$$

This shows that Ω is a *SWP* manifold of $\mathbb{E}^{13}$ with warping functions $f = \sqrt{u^2+v^2+1}$ and $h = \sqrt{u^2+v^2+\alpha^2+1}$.

Remark 2 From here onwards, we consider and study the SWP submanifolds of the type $(\Omega_T \times_f \Omega_\perp) \times_h \Omega_\theta$ when the characteristic vector field ξ is tangent to Ω_T.

Proposition 3 *Let* $(\Omega_T \times_f \Omega_\perp) \times_h \Omega_\theta$ *be a SWP submanifold of cosymplectic manifold* $\bar{\Omega}^{2m+1}$. *Then,* $\xi(\ln f) = \xi(\ln h) = 0$.

Proof Consider $U \in \mathfrak{X}(\Omega_T)$ and $V \in \mathfrak{X}(\Omega_\perp)$, then by consequence of the first part of Proposition 1, we have $\nabla_U V = \nabla_V U = U(\ln f)V = U(f)/f.V$. Since $\xi \in \mathfrak{U}(\Omega_T)$, then the previous equation becomes $\nabla_V \xi = \xi(\ln f)V$. Hence, applying (4) and (6) in the above expression, we obtain the first part. By the use of the second part of Proposition 1, we derive $\nabla_U W = U(f)/f * W$, $W \in \mathfrak{X}(\Omega_\theta)$. With the help of (4) and (6) in the last relation, we obtain the second part. □

Here, we present some crucial findings for later use.

Lemma 1 *Assume* $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ *is a SWP submanifold of cosymplectic manifold* $\bar{\Omega}^{2m+1}$. *Then, for all* $U_1, U_2 \in \mathfrak{X}(\Omega_T)$ *and* $V_1, V_2 \in \mathfrak{X}(\Omega_\perp)$ *we have*

i. $g(\sigma(U_1,U_2), \varphi V_1) = 0$;
ii. $g(\sigma(U_1,V_1), \varphi V_2) = -\varphi U_1(f)/f.g(V_1,V_2)$.

Proof By the utilization of (3), (6), (16), and Definition 1, we concede that

$$g\left(\sigma(U_1,U_2), \varphi V_1\right) = -g(\bar{\nabla}_{U_1}\varphi V, V_1). \tag{28}$$

Inserting the covariant derivative's characteristic into (28), we arrive at

$$g\left(\sigma(U_1,U_2), \varphi V_1\right) = g(\varphi V, \bar{\nabla}_{U_1} V_1) - Ug(\varphi V, V_1). \tag{29}$$

By the use of the first part of Proposition 1 in (29), we obtain the first part. For the second part, we utilize (3), (6), and (11) to obtain $g(\sigma(U_1,V_1), \varphi V_2) = -g(\varphi \bar{\nabla}_{V_1} U_1, V_2)$. With the help of the first part of Proposition 1, we obtain the second part. □

Lemma 2 *If* $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ *is a SWP of cosymplectic manifold* $\bar{\Omega}^{2m+1}$, *then we attain for all* $U_1, U_2 \in \mathfrak{X}(\Omega_T)$ *and* $W_1, W_2 \in \mathfrak{X}(\Omega_\theta)$ *that*

i. $g(\sigma(U_1,U_2), nW_1) = 0$;
ii. $g(\sigma(U_1,W_1), nW_2) = -\varphi U_1(h)/h.g(W_1,W_2) + U_1(h)/h.g(tW_1,W_2)$.

Proof By the result of (6) and (11), we obtain $g(\sigma(U_1, U_2), nW_1) = g(\bar{\nabla}_{U_1} U_2, \varphi W_1 - g(\bar{\nabla}_{U_1} U_2, tW_1)$. The previous expression simplifies to the following form when (3) is used:

$$g\big(\sigma(U_1, U_2), nW_1\big) = -g(\bar{\nabla}_{U_1} \varphi U_2, W_1) - g(\bar{\nabla}_{U_1} U_2, tW_1).$$

Applying Definition 1 and the second part of Proposition 1 into the above expression, we obtain the first part. By reusing (6) and (11), we have $g(\sigma(U_1, W_1), nW_2) = g(\bar{\nabla}_{W_1} U_1, \varphi W_2 - g(\bar{\nabla}_{W_1} U_2, tW_2)$. Next, employing (3) and Proposition 1(ii), we obtain the second part. □

Moreover, employing (1) in Lemma 2, we obtain that

$$g\big(\sigma(\varphi U_1, U_2), nW_1\big) = 0, \tag{30}$$

$$g\big(\sigma(\varphi U_1, W_1), nW_2\big) = U_1(h)/h.g(W_1, W_2) + \varphi U_1(h)/h.g(tW_1, W_2). \tag{31}$$

Replacing W_1 by tW_1 in Lemma 2, then utilizing (22), we obtain

$$g\big(h(U_1, tW_1), nW_2\big) = -U_1(h)/h.g(tW_1, W_2) - \cos^2\theta\varphi U_1(h)/h.g(W_1, W_2). \tag{32}$$

Replacing W_1 by tW_1 in (28) and (29), and then employing (22) in the obtained expression, we arrive at

$$g\big(h(\varphi U_1, tW_1), nW_2\big) = U_1(h)/h.g(tW_1, W_2) - \cos^2\theta\varphi U_1(h)/h.g(W_1, W_2). \tag{33}$$

Replacing W_2 by tW_2 in (29), (32), and (33) and using (23), yields

$$g\big(h(\varphi U_1, W_1), ntW_2\big) = U_1(h)/h.g(W_1, tW_2) + \cos^2\theta\varphi U_1(h)/h.g(W_1, W_2). \tag{34}$$

Replacing U_1 by φU_1 in (34), we achieve that

$$g\big(h(U_1, W_1), ntW_2\big) = -\varphi U_1(h)/h.g(W_1, tW_2) - \cos^2\theta U_1(h)/h.g(W_1, W_2). \tag{35}$$

Lemma 3 *If $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ is a SWP of cosymplectic manifold $\bar{\Omega}^{2m+1}$, then we derive for all $U \in \mathfrak{X}(\Omega_T)$, $V \in \mathfrak{X}(\Omega_\perp)$ and $W \in \mathfrak{X}(\Omega_\theta)$ that*

i. *$g(\sigma(U, V), nU)$ vanishes;*
ii. *$g(\sigma(U, W), \varphi V)$ vanishes;*
iii. *$\sigma(V, W) = \sigma(W, W)$.*

Proof As a result of (6) and (11), we obtain $g(\sigma(U, V), nW) = g(\bar{\nabla}_V U, \varphi W) - g(\nabla_V U, tW)$. Now, employing the first part of Lemma 2 into the last expression, we achieve the first part. Likewise, we can prove the second part. By the consequence of (3), (6), and (11), we obtain

$$g\big(\sigma(V, tW), nW\big) = -g(\varphi\bar{\nabla}_V tW, W) - g(\nabla_V tW, tW).$$

With the help of Definition 1 and the third part of Proposition 2, we have

$$g\big(\sigma(V, tW), nW\big) = -g(\bar{\nabla}_V \varphi tW, W) - V(h)/h.g(tW, tW).$$

Applying (11), (22), and (23), we achieve the third part. □

Lemma 4 *If $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ is a SWP of cosymplectic manifold $\bar{\Omega}^{2m+1}$, then we deduce for all $V \in \mathfrak{X}(\Omega_\perp)$ and $W \in \mathfrak{X}(\Omega_\theta)$ that*

$$g(\sigma(W,V),ntW) = \cos^2\theta V(h)/h.g(W,W) + g(\sigma(W,tW),\varphi V). \tag{36}$$

Proof By use of (3) and (6), we obtain $g(\sigma(W,tW),\varphi V) = -g(\varphi\bar{\nabla}_W tW, V)$. Now, utilizing (11) and Definition 1 in the last expression, we observe that

$$g(\sigma(W,tW),\varphi V) = -g(\bar{\nabla}_W t^2 W, V) - g(\bar{\nabla}_W ntW, V).$$

By virtue of (8) and (22), we concede that

$$g(\sigma(W,tW),\varphi V) = -\cos^2\theta g(\bar{\nabla}_W W, V) + g(\sigma(W,V),ntW).$$

Using the second part of Proposition 1 and the property of the covariant derivative, we achieve (36). □

Lemma 5 *If $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ is a SWP of cosymplectic manifold $\bar{\Omega}^{2m+1}$, then we have for all $U \in \mathfrak{X}(\Omega_T)$, $V \in \mathfrak{X}(\Omega_\perp)$ and $W \in \mathfrak{X}(\Omega_\theta)$ that*

$$\|\sigma_\nu(U,V)\|^2 = g(\sigma(\varphi U,V),\varphi\sigma(U,V)), \tag{37}$$

$$\|\sigma_\nu(U,W)\|^2 = 2\cos^2\theta U(h)^2/h.g(W,W) + g(\sigma(\varphi U,W),\varphi\sigma(U,W)). \tag{38}$$

Proof By the use of (16) and (6), we have $\varphi\nabla_V U - \nabla_V\varphi U = \sigma(\varphi U,V) - \varphi\sigma(U,V)$. Employing the fact that Ω_T is an invariant submanifold, then by the application of Proposition 1(i) the above expression reduces to the following form:

$$U(h)/h.\varphi V - \varphi U(h)/h.V = \sigma(\varphi U,V) - \varphi\sigma(U,V). \tag{39}$$

Now, by taking an inner product with $\varphi\sigma(U,V)$ into (39), we achieve the first part. Reusing (16) and (6), we have $\varphi\nabla_W U - \nabla_W\varphi U = \sigma(\varphi U,W) - \varphi\sigma(W,V)$. By the application of Proposition 1(i) in the above expression, we obtain

$$U(h)/h.\varphi W - \varphi U(h)/h.W = \sigma(\varphi U,W) - \varphi\sigma(U,W). \tag{40}$$

Now, by taking the inner product with $\varphi\sigma(U,W)$ into (39), we have

$$U(h)/h.g(\varphi\sigma(U,W),nW) = g(\sigma(\varphi U,W),\varphi\sigma(U,W)) - \|\sigma_\nu(U,W)\|^2. \tag{41}$$

In view of (35) and (41), we obtain the second part. □

5 Main results

In this section, we show several significant findings and their geometric applications. Let Ω be a SWP submanifold of cosymplectic manifold $\bar{\Omega}^{2m+1}$.

Then, we can express $\mathfrak{X}(\bar{\Omega}) = \mathfrak{X}(\Omega) \oplus \mathfrak{X}(\Omega)^\perp$ with $\mathfrak{X}(\Omega) = \mathfrak{D}_T \oplus \mathfrak{D}_\perp \oplus \mathfrak{D}_\theta$ and $\mathfrak{X}(\Omega)^\perp = n\mathfrak{D}_\perp \oplus n\mathfrak{D}_\theta \oplus \nu$.

Take orthonormal frames as follows. For tangent distributions $\mathfrak{D}_T$, $\mathfrak{D}_\perp$, and $\mathfrak{D}_\theta$:

$$\{x_1,\dots,x_p,x_{p+1}=\varphi x_1,\dots,x_{2p}=\varphi x_p,\xi\},$$
$$\{x_{2p+1}=x_1^*,x_{2p+2}=x_2^*,\dots,x_{2p+q}=x_q^*\}$$

and

$$\{x_{2p+q+1}=\hat{x}_1,\dots,x_{2p+q+r}=\hat{x}_r,x_{2p+q+r+1}=\sec\theta t\hat{x}_1,\dots,x_{2p+q+2r}=\sec\theta t\hat{x}_r\}$$

and for normal distributions $n\mathfrak{D}_\perp$, $n\mathfrak{D}_\theta$, and ν:

$$\{\tilde{x}_1=\varphi x_1^*,\tilde{x}_2=\varphi x_2^*,\dots,\tilde{x}_q=\varphi x_q^*\},$$
$$\{\tilde{x}_{q+1}=\csc\theta n\hat{x}_1,\dots,\tilde{x}_{q+r}=\csc\theta\hat{x}_r,\tilde{x}_{q+r+1}=\csc\theta\sec\theta nt\hat{x}_1,\dots,\tilde{x}_{q+2r}=\csc\theta\sec\theta nt\hat{x}_r\}$$

and

$$\{\tilde{x}_{q+2r}=\overline{x}_1,\dots,\tilde{x}_{q+2r+s}=\overline{x}_s,\tilde{x}_{q+2r+s+1}=\overline{x}_{s+1},\dots,\tilde{x}_{q+2r+2s}=\overline{x}_{2s}\}.$$

Then, we deduce $\dim(\mathfrak{D}_T)=2p+1$, $\dim(\mathfrak{D}_\perp)=q$, $\dim(\mathfrak{D}_\theta)=2r$, $\dim(n\mathfrak{D}_\perp)=q$, $\dim(n\mathfrak{D}_\theta)=2r$ and $\dim(\nu)=2s$. Hence, Ω is $d=2p+q+2r+1$-dimensional and $\Omega^\perp$ is $2m-d+1=2s+q+2r$-dimensional.

Here, we present our first main result that represents a relationship between the warping function and the second fundamental form and acts as a generalization for earlier existing literature in this regard.

Theorem 2 *Assume $\Omega=\Omega_T\times_f\Omega_\perp\times_h\Omega_\theta$ is a d-dimensional SWP submanifold of cosymplectic manifold $\bar{\Omega}^{2m+1}$, then*

$$\|\sigma\|^2\geq 2q\left\|\frac{\nabla^T(f)}{f}\right\|^2+4r\left(1+2\cot^2\theta\right)\left\|\frac{\nabla^T(h)}{h}\right\|^2+2r\cot^2\theta\left\|\frac{\nabla^\perp(h)}{h}\right\|^2 \tag{42}$$

holds, where σ symbolizes the SFF, $\nabla^T(f)$: gradient(f) on Ω_T, $\nabla^T(h)$ and $\nabla^\perp(h)$: gradients(h) on Ω_T and $\Omega_\perp$.

Proof By the use of (10), we have

$$\|\sigma\|^2=\sum_{a,b=1}^{d}\sum_{c=1}^{2m+1-d}g\left(\sigma(x_a,x_b),\tilde{x}_c\right)^2.$$

Since $\mathfrak{X}(\Omega^\perp)=\mathfrak{X}(n\mathfrak{D}_\perp)\oplus\mathfrak{X}(n\mathfrak{D}_\theta)\oplus\nu$, then the preceding expression is simplified into its subsequent form as

$$\begin{aligned}\|\sigma\|^2={}&\sum_{a,b=1}^{d}\sum_{c=1}^{q}g\left(\sigma(x_a,x_b),\tilde{x}_c\right)^2+\sum_{a,b=1}^{d}\sum_{c=1}^{2r}g\left(\sigma(x_a,x_b),\tilde{x}_{q+c}\right)^2\\&+\sum_{c=1}^{2s}\sum_{a,b=1}^{d}g\left(\sigma(x_a,x_b),\bar{x}_c\right)^2.\end{aligned} \tag{43}$$

The $n\mathfrak{D}_\perp$ and $n\mathfrak{D}_\theta$ components are found in the first and second terms, while ν-components are found in the final term in the above expression. Further, we only use the frame fields described above to calculate the components in the first and second terms as follows:

$$\|\sigma\|^2 \geq \sum_{a,b=1}^{d}\sum_{c=1}^{q} g\big(\sigma(x_a,x_b),\tilde{x}_c\big)^2 + \sum_{a,b=1}^{d}\sum_{c=1}^{2r} g\big(\sigma(x_a,x_b),\tilde{x}_{q+c}\big)^2. \tag{44}$$

By using the above-mentioned frame field, we concede that

$$\begin{aligned}
\|\sigma\|^2 \geq{}& \sum_{a,b=1}^{2p}\sum_{c=1}^{q} g\big(\sigma(x_a,x_b),\tilde{x}_c\big)^2 + \sum_{c=1}^{q} g\big(\sigma(\xi,\xi),\tilde{x}_c\big)^2 + \sum_{a,b=1}^{2p}\sum_{c=1}^{2r} g\big(\sigma(x_a,x_b),\tilde{x}_{q+c}\big)^2 \\
&+ \sum_{a,b,c=1}^{q} g\big(\sigma(x_a^*,x_b^*),\tilde{x}_c\big)^2 + \sum_{c=1}^{2r} g\big(\sigma(\xi,\xi),\tilde{x}_{q+c}\big)^2 + \sum_{a,b=1}^{q}\sum_{c=1}^{2r} g\big(\sigma(x_a^*,x_b^*),\tilde{x}_{q+c}\big)^2 \\
&+ \sum_{a,b,c=1}^{2r} g\big(\sigma(\hat{x}_a,\hat{x}_b),\tilde{x}_{q+c}\big)^2 + \sum_{a,b=1}^{2r}\sum_{c=1}^{q} g\big(\sigma(\hat{x}_a,\hat{x}_b),\tilde{x}_c\big)^2 \\
&+ 2\sum_{a,c=1}^{q}\sum_{b=1}^{2r} g\big(\sigma(x_a^*,\hat{x}_b),\tilde{x}_c\big)^2 + 2\sum_{a=1}^{q}\sum_{b,c=1}^{2r} g\big(\sigma(x_a^*,\hat{x}_b),\tilde{x}_{q+c}\big)^2 \\
&+ 2\sum_{a=1}^{2p}\sum_{b=1}^{2r}\sum_{c=1}^{q} g\big(\sigma(x_a,\hat{x}_b),\tilde{x}_c\big)^2 + 2\sum_{a=1}^{2p}\sum_{b=1}^{q}\sum_{c=1}^{2r} g\big(\sigma(x_a,x_b^*),\tilde{x}_{q+c}\big)^2 \\
&+ 2\sum_{a=1}^{2p}\sum_{b,c=1}^{2r} g\big(\sigma(x_a,\hat{x}_b),\tilde{x}_{q+c}\big)^2 + 2\sum_{a=1}^{2p}\sum_{b,c=1}^{q} g\big(\sigma(x_a,x_b^*),\tilde{x}_c\big)^2.
\end{aligned} \tag{45}$$

Now, employing the first and second parts of Lemma 1 and the first and second parts of Lemma 3 in the above expression

$$\begin{aligned}
\|\sigma\|^2 \geq{}& \sum_{c=1}^{q} g\big(\sigma(\xi,\xi),\tilde{x}_c\big)^2 + \sum_{a,b,c=1}^{q} g\big(\sigma(x_a^*,x_b^*),\tilde{x}_c\big)^2 + \sum_{c=1}^{2r} g\big(\sigma(\xi,\xi),\tilde{x}_{q+c}\big)^2 \\
&+ \sum_{a,b=1}^{q}\sum_{c=1}^{2r} g\big(\sigma(x_a^*,x_b^*),\tilde{x}_{q+c}\big)^2 + \sum_{a,b,c=1}^{2r} g\big(\sigma(\hat{x}_a,\hat{x}_b),\tilde{x}_{q+c}\big)^2 \\
&+ 2\sum_{a=1}^{q}\sum_{b,c=1}^{2r} g\big(\sigma(x_a^*,\hat{x}_b),\tilde{x}_{q+c}\big)^2 \\
&+ \sum_{a,b=1}^{2r}\sum_{c=1}^{q} g\big(\sigma(\hat{x}_a,\hat{x}_b),\tilde{x}_c\big)^2 + 2\sum_{a,c=1}^{q}\sum_{b=1}^{2r} g\big(\sigma(x_a^*,\hat{x}_b),\tilde{x}_c\big)^2 \\
&+ 2\sum_{a=1}^{2p}\sum_{b,c=1}^{2r} g\big(\sigma(x_a,\hat{x}_b),\tilde{x}_{q+c}\big)^2 + 2\sum_{a=1}^{2p}\sum_{b,c=1}^{q} g\big(\sigma(x_a,x_b^*),\tilde{x}_c\big)^2.
\end{aligned} \tag{46}$$

With the help of (4) and (6), we have $\sigma(\xi,\xi) = 0$. In order to reduce the complexity of the computation, we extract the final three terms from the equation above, and by using the orthonormal frame previously defined, we reach

$$\begin{aligned}
\|\sigma\|^2 \geq{}& 2\sum_{a=1}^{p}\sum_{b,c=1}^{q} g\big(\sigma\left(x_a, x_b^*\right), ne_c^*\big)^2 + 2\sum_{a=1}^{p}\sum_{b,c=1}^{q} g\big(\sigma\left(\varphi x_a, x_b^*\right), ne_c^*\big)^2 \\
&+ 2\csc^2\theta\left(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma\left(x_a, \hat{x}_b\right), n\hat{x}_c\big)^2 + g\big(\sigma\left(\varphi x_a, \hat{x}_b\right), n\hat{x}_c\big)^2\right) \\
&+ 2\sec^2\theta\csc^2\theta\left(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma\left(x_a, \hat{x}_b\right), nt\hat{x}_c\big)^2 + g\big(\sigma\left(\varphi x_a, \hat{x}_b\right), nt\hat{x}_c\big)^2\right) \\
&+ 2\sec^2\theta\csc^2\theta\left(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma\left(x_a, t\hat{x}_b\right), n\hat{x}_c\big)^2 + g\big(\sigma\left(\varphi x_a, t\hat{x}_b\right), n\hat{x}_c\big)^2\right) \\
&+ 2\sec^4\theta\csc^2\theta\left(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma\left(x_a, t\hat{x}_b\right), nt\hat{x}_c\big)^2 + g\big(\sigma\left(\varphi x_a, t\hat{x}_b\right), nt\hat{x}_c\big)^2\right) \\
&+ 2\sec^2\theta\csc^2\theta\left(\sum_{b,c=1}^{2r}\sum_{a=1}^{q} g\big(\sigma\left(x_a^*, \hat{x}_b\right), nt\hat{x}_c\big)^2\right. \\
&\left.+ \sum_{b,b=1}^{2r}\sum_{a=1}^{q} g\big(\sigma\left(x_a^*, t\hat{x}_b\right), n\hat{x}_c\big)^2\right). \qquad (47)
\end{aligned}$$

Next, utilizing (31)–(35) and the second parts of Lemmas 1 and 2, we have

$$\begin{aligned}
\|\sigma\|^2 \geq{}& 2\sum_{a=1}^{p}\sum_{b,c=1}^{q}\left(\left(\varphi\frac{x_a(f)}{f}\right)^2 + \left(\frac{x_a(f)}{f}\right)^2\right) g\big(x_b^*, x_c^*\big)^2 \\
&+ 4\csc^2\theta\sum_{a=1}^{p}\sum_{b,c=1}^{r}\left(\left(\varphi\frac{x_a(h)}{h}\right)^2 + \left(\frac{x_a(h)}{h}\right)^2\right) g(\hat{x}_b, \hat{x}_c)^2 \\
&+ 4\cot^2\theta\sum_{a=1}^{p}\sum_{b,c=1}^{r}\left(\left(\varphi\frac{x_a(h)}{h}\right)^2 + \left(\frac{x_a(h)}{h}\right)^2\right) g(\hat{x}_b, \hat{x}_c)^2 \\
&+ 2\cot^2\theta\sum_{b,c=1}^{r}\sum_{a=1}^{q}\frac{x_a^*(h)}{h} g(\hat{x}_b, \hat{x}_b)^2 \\
&+ 2\sec^2\theta\csc^2\theta\sum_{b,c=1}^{r}\sum_{a=1}^{q} g\big(\sigma\left(x_a^*, t\hat{x}_b\right), n\hat{x}_c\big)^2. \qquad (48)
\end{aligned}$$

Leaving the last term and by adding and subtracting the same quantity in (48), and by the application of Proposition 3, we have

$$\begin{aligned}
\|\sigma\|^2 \geq{}& 2\sum_{a=1}^{2p+1}\sum_{b,c=1}^{q}\left(\varphi\frac{x_a(f)}{f}\right)^2 g\big(x_b^*, x_c^*\big)^2 \\
&+ 4\csc^2\theta\sum_{a=1}^{2p+1}\sum_{b,c=1}^{r}\left(\frac{x_a(h)}{h}\right)^2 g(\hat{x}_b, \hat{x}_c)^2
\end{aligned}$$

$$+ 4\cot^2\theta \sum_{a=1}^{2p+1}\sum_{b,c=1}^{r}\left(\frac{x_a(h)}{h}\right)^2 g(\hat{x}_b,\hat{x}_c)^2$$

$$+ 2\cot^2\theta \sum_{b,c=1}^{r}\sum_{a=1}^{q}\left(\frac{x_a^*(f)}{f}\right)^2 g(\hat{x}_b,\hat{x}_b)^2. \tag{49}$$

Using (21) in the above expression, we derive (42). Hence, the theorem is proved. □

5.1 Particular cases

If in ineq. (42) of Theorem 2:

i. $r = 0$ and the norm of $\frac{\nabla^T(h)}{h}$ and $\frac{\nabla^\perp(h)}{h}$ vanishes, then SWP becomes a CR-warped product [34], and

$$\|\sigma\|^2 \geq 2q\left\|\frac{\nabla^T(f)}{f}\right\|^2.$$

ii. $q = 0$ and the norm of $\frac{\nabla^T(f)}{f}$ vanishes then SWP becomes WP $\Omega_T \times_h \Omega_\theta$ with

$$\|\sigma\|^2 \geq 4r\big(1 + 2\cot^2\theta\big)\left\|\frac{\nabla^T(h)}{h}\right\|^2.$$

It is to be noted that the same result was obtained at [26] for Sasakian manifolds, but now for cosymplectic ones.

iii. $p = 0$ and the norm of $\frac{\nabla^T(f)}{f}$ and $\frac{\nabla^\perp(h)}{h}$ vanishes, then SWP becomes pointwise pseudoslant WP $\Omega_\perp \times_h \Omega_\theta$ [2, Th. 4.1] and

$$\|\sigma\|^2 \geq 2r\cot^2\theta\left\|\frac{\nabla^\perp(h)}{h}\right\|^2.$$

iv. $\frac{\nabla^\perp(h)}{h}$ vanishes, then SWP becomes a biwarped product $\Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ [11] such that

$$\|\sigma\|^2 \geq 2q\left\|\frac{\nabla^T(f)}{f}\right\|^2 + 4r\big(1 + 2\cot^2\theta\big)\left\|\frac{\nabla^T(h)}{h}\right\|^2.$$

v. $\frac{\nabla^T(f)}{f}$ vanishes, then SWP becomes CR-slant WP $(\Omega_T \times \Omega_\perp) \times_h \Omega_\theta$ [1] satisfying

$$\|\sigma\|^2 \geq 4r\left(\big(\csc^2\theta + \cot^2\theta\big)\left\|\frac{\nabla^T(h)}{h}\right\|^2 + \cot^2\theta\left\|\frac{\nabla^\perp(h)}{h}\right\|^2\right).$$

Theorem 3 *Consider a d-dimensional SWP submanifold $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ of cosymplectic manifold $\bar{\Omega}^{2m+1}$. If the SFF σ satisfies*

$$\|\sigma\|^2 = 2q\left\|\frac{\nabla^T(f)}{f}\right\|^2 + 4r\big(1 + 2\cot^2\theta\big)\left\|\frac{\nabla^T(h)}{h}\right\|^2 + 2r\cot^2\theta\left\|\frac{\nabla^\perp(h)}{h}\right\|^2, \tag{50}$$

then, the claims that follow hold:

(i) *Ω is totally geodesic in $\bar{\Omega}^{2m+1}$;*

(ii) Ω *is* Ω_T–$\Omega_\perp$ (Ω_T–Ω_θ) *mixed totally geodesic*;
(iii) $\Omega_\perp \times_h \Omega_\theta$ (*resp.* $\Omega_\perp$) *is totally mixed geodesic* (*resp., umbilical*) *in* $\bar{\Omega}^{2m+1}$;
(iv) *The* $\mathcal{H} = -\nabla f$ *for* $\Omega_\perp$;
(v) Ω_θ *is totally umbilical in cosymplectic manifold* $\bar{\Omega}^{2m+1}$;
(vi) *The* $\mathcal{H} = -\nabla f$ *for* Ω_θ.

The present theorem can be proved using the same steps as Theorem 4.2 of [19].

Proof Suppose that σ satisfies (50), then by (44), we have

$$g\big(\sigma(T\Omega, T\Omega), \nu\big) = 0. \tag{51}$$

Hence, straightforwardly from (51), Lemma 1, Lemma 2, and Proposition 2, we obtain (1). Also from (46), we obtain that

$$g\big(\sigma(\mathfrak{D}_\perp, \mathfrak{D}_\perp), n\mathfrak{D}_\theta\big) = 0, \qquad g\big(\sigma(\mathfrak{D}_\perp, \mathfrak{D}_\perp), n\mathfrak{D}_\perp\big) = 0. \tag{52}$$

By the use of (46), we have

$$g\big(\sigma(\mathfrak{D}_\perp, \mathfrak{D}_\theta), n\mathfrak{D}_\theta\big) = 0, \qquad g\big(\sigma(\mathfrak{D}_\perp, \mathfrak{D}_\theta), n\mathfrak{D}_\perp\big) = 0. \tag{53}$$

From (51) and (53), we have

$$\sigma(\mathfrak{D}_\perp, \mathfrak{D}_\theta) = 0. \tag{54}$$

By the consequence (54), we accomplished the proof of the second part. By virtue of (51) and (52), we have

$$\big\|\sigma(\mathfrak{D}_\perp, \mathfrak{D}_\perp)\big\| = 0. \tag{55}$$

Since $\Omega_\perp$ is a totally umbilical submanifold in $\Omega_T \times_f \Omega_\perp$ using this fact in (55), we conclude that $\Omega_\perp$ is totally umbilical in $\overline{M}$. Similarly, we prove the fourth part. Let $\sigma^\perp$ be the second fundamental form on $\Omega_\perp$, then by the utilization of Proposition 1, we have $g(\sigma^\perp(Z_1, Z_2), X) = -g(\nabla_X Z_1, Z_2) = -X(\ln f)g(Z_1, Z_2)$. Now, assume σ and σ^θ are the second fundamental tensors of $\overline{M}$ and Ω_θ, respectively, then we have $\sigma(W_1, W_2) = \sigma(W_1, W_2) + \sigma^\theta(W_1, W_2)$. By the consequence of (49) and Proposition 2, we obtain $\sigma(W_1, W_2) = \sigma^\theta(W_1, W_2) = g(W_1, W_2)H^\theta$, where H^θ is the mean curvature of Ω_θ. Using (1), we obtain $g(\sigma^\theta(W_1, W_2), X) = g(\nabla_{W_1} W_2, X) = -X(\ln h)g(W_1, W_2)$. Now, by the definition of gradient, we achieve the last part. This completes the proof. □

Our next result presents one of the ways to analyze Dirichlet problems. Several authors in different senses have analyzed the existence of solutions for Dirichlet problems [3, 11]. Here, we investigate the relation involving Dirichlet energy of functions and the second fundamental form for SWP. Jackson et al. in [17] stated this as: The Dirichlet energy of smooth function ψ and compact submanifold Ω over its volume element dV is defined by

$$E(\psi) = \frac{1}{2}\int_\Omega \|\nabla\psi\|^2\, dV. \tag{56}$$

As a result, inspired by the publications mentioned above, we derive an important finding:

Theorem 4 *Let $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ be a d-dimensional connected, compact SWP submanifold of cosymplectic manifold $\bar{\Omega}^{2m+1}$. Then, the Dirichlet energy of f and h satisfies*

$$\begin{aligned} &4qE(\ln f) + 4r\cot^2\theta E(\ln h) + 4r\big(2 + 3\cot^2\theta\big)E^{\Omega_T}(\ln h) \\ &\quad\le \int_\Omega \|\sigma\|^2\,dV - 16r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\big(E^{\Omega_T}(\ln h)\big)\,dV \\ &\qquad - 4r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\big(E^{\Omega_\perp}(\ln h)\big)\,dV, \end{aligned} \tag{57}$$

where $E^{\Omega_T}(\ln h)$, $E^{\Omega_\perp}(\ln h)$ and $E(\ln f)$, $E(\ln h)$ are the Dirichlet energy of $\ln h$ on Ω_T, $\Omega_\perp$, and $\ln f$ and $\ln h$ on Ω, respectively.

Proof Integrating the relation (42) over Ω, we have

$$\begin{aligned} \int_\Omega \|\sigma\|^2\,dV &\ge 2q\int_\Omega \left\|\frac{\nabla^T(h)}{h}\right\|^2 dV + 4r\int_\Omega \big(1 + 2\cot^2\theta\big)\left\|\frac{\nabla^T(h)}{h}\right\|^2 dV \\ &\quad + 2r\int_\Omega \cot^2\theta\left\|\frac{\nabla^\perp(h)}{h}\right\|^2 dV. \end{aligned} \tag{58}$$

By using the property of integration in (58), we obtain

$$\begin{aligned} \int_\Omega \|\sigma\|^2\,dV &\ge 2q\int_\Omega \left\|\frac{\nabla^T(f)}{f}\right\|^2 dV + 4r\big(1 + 2\cot^2\theta\big)\int_\Omega \left\|\frac{\nabla^T(h)}{h}\right\|^2 dV \\ &\quad + 2r\cot^2\theta\int_\Omega \left\|\frac{\nabla^\perp(h)}{h}\right\|^2 dV \\ &\quad - 16r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\left(\int_\Omega \left\|\frac{\nabla^T(h)}{h}\right\|^2\right) dV \\ &\quad - 4r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\left(\int_\Omega \left\|\frac{\nabla^\perp(h)}{h}\right\|^2\right) dV. \end{aligned} \tag{59}$$

By utilization of $\nabla = \nabla^T + \nabla^\perp + \nabla^\theta$ and (56) in the above equation, we obtain (57). □

5.2 Particular cases

In inequality (57) of Theorem 4 if:

i. $r = 0$ and the norm of $\frac{\nabla^T(h)}{h}$, $\frac{\nabla^\perp(h)}{h}$ vanishes, then SWP is a *CR*-warped product with

$$4qE(\ln f) \le \int_\Omega \|\sigma\|^2\,dV.$$

ii. $q = 0$ and the norm of $\frac{\nabla^T(f)}{f}$ vanishes, then SWP is WP $\Omega_T \times_h \Omega_\theta$ [4] with

$$E(\ln h) \le \frac{1}{4r((1 + 2\cot^2\theta)}\left(\int_\Omega \|\sigma\|^2\,dV - 16r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\big(E(\ln h)\big)\,dV\right).$$

iii. $p = 0$ and the norm of $\frac{\nabla^T(f)}{f}$, $\frac{\nabla^T(h)}{h}$ vanishes, then SWP is pointwise pseudoslant WP $\Omega_\perp \times_h \Omega_\theta$ and

$$4r\cot^2\theta E(\ln h) \leq \int_\Omega \|\sigma\|^2\, dV - 4r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\big(E(\ln h)\big)\, dV.$$

iv. the norm of $\frac{\nabla^\perp(h)}{h}$ vanishes, then SWP is a biwarped product $\Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ satisfying

$$\begin{aligned}&4qE(\ln f) + 4r\big(1 + 2\cot^2\theta\big)E(\ln h)\\ &\quad\leq \int_\Omega \|\sigma\|^2\, dV - 16r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\big(E(\ln h)\big)\, dV.\end{aligned}$$

v. the norm of $\frac{\nabla^T(f)}{f}$ vanishes, then SWP is *CR*-slant WP $(\Omega_T \times \Omega_\perp) \times_h \Omega_\theta$ such that

$$\begin{aligned}&4r\cot^2\theta E(\ln h) + 4r\big(2 + 3\cot^2\theta\big)E^{\Omega_T}(\ln h)\\ &\quad\leq \int_\Omega \|\sigma\|^2\, dV - 16r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\big(E^{\Omega_T}(\ln h)\big)\, dV\\ &\qquad - 4r\int_\Omega \cos\theta\csc^3\theta\frac{d\theta}{dV}\big(E^{\Omega_\perp}(\ln h)\big)\, dV.\end{aligned}$$

Remark 3 The Dirichlet energy in the special cases (1), (2), and (4) can be easily reduced to *Corollary 6.1, Corollary 6.2, and Theorem 6.1* of [11] after certain computations.

Motivated by Sahin [28, 29], we give the Lawson–Simons-type inequality [21] (possesses important applications in the theory of integral currents [15] for SWP in a cosymplectic manifold with constant holomorphic sectional curvature k (briefly: Cosymplectic space form denoted by $(\bar{\Omega}^{2m+1}, k)$). We also extract exceptional cases for the same.

Theorem 5 *Let* $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ *be a* d*-dimensional SWP of cosymplectic space form* $(\bar{\Omega}^{2m+1}, k)$*, then*

$$\begin{aligned}&\sum_{b=1}^{2p}\sum_{a=1}^{2r}\big\|\sigma(x_a, \hat{x}_b)\big\|^2 + \sum_{a=1}^{2p}\sum_{b=1}^{q}\big\|\sigma(x_a, x_b^*)\big\|^2 + \sum_{a=1}^{q}\sum_{b=1}^{2r}\big\|\sigma(x_a^*, \hat{x}_b)\big\|^2\\ &\quad - \sum_{a=1}^{q}\sum_{b=1}^{2r} g\big(\sigma(x_a^*, x_a^*), \sigma(\hat{x}_b, \hat{x}_b)\big) \geq 2r\left(\frac{\Delta^\perp(h)}{h} - \frac{kq}{4}\right),\end{aligned} \tag{60}$$

where $\Delta^\perp(h)$ *denotes the Laplacian of* h *on* $\Omega_\perp$*. Moreover, if the equality holds then* $\Omega = \Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ *is single WP of* $(\bar{\Omega}^{2m+1}, k)$ *satisfying* $\sigma(\mathfrak{D}_T, \mathfrak{D}_\perp) \perp \nu$ *and* $\sigma(\mathfrak{D}_T, \mathfrak{D}_\theta) \perp \nu$.

Proof Substituting $U = V = x_a^*$ and $V = W = \hat{x}_b$ into (5), we have

$$\bar{R}\big(x_a^*, \hat{x}_b, x_a^*, \hat{x}_b\big) = -\frac{kqr}{2}. \tag{61}$$

By virtue of (19) and (61), we have

$$\sum_{a=1}^{q}\sum_{b=1}^{2r} g\big(R(x_a^*,\hat{x}_b)\hat{x}_b,x_a^*\big) - \sum_{a=1}^{q}\sum_{b=1}^{2r} g\big(\sigma(x_a^*,x_a^*),\sigma(\hat{x}_b,\hat{x}_b)\big) + \sum_{a=1}^{q}\sum_{b=1}^{2r}\|\sigma(x_a^*,\hat{x}_b\|^2 = -\frac{kqr}{2}. \tag{62}$$

Further, utilizing Proposition 1 in the above expression, we compute

$$\sum_{a=1}^{q}\sum_{b=1}^{2r}\|\sigma(x_a^*,\hat{x}_b\|^2 - \sum_{a=1}^{q}\sum_{b=1}^{2r} g\big(\sigma(x_a^*,x_a^*),\sigma(\hat{x}_b,\hat{x}_b)\big) = 2r\sum_{a=1}^{q}\frac{H^\lambda(x_a^*,x_a^*)}{h} - \frac{kqr}{2}. \tag{63}$$

By the definition of a Laplacian, the above expression reduces to the given form

$$\sum_{a=1}^{q}\sum_{b=1}^{2r}\|\sigma(x_a^*,\hat{x}_b\|^2 - \sum_{a=1}^{q}\sum_{b=1}^{2r} g\big(\sigma(x_a^*,x_a^*),\sigma(\hat{x}_b,\hat{x}_b)\big) = 2r\bigg(\frac{\Delta^\perp(h)}{h} - \frac{kqr}{2}\bigg). \tag{64}$$

Now, we compute the norm of $\sigma(\mathfrak{D}_T,\mathfrak{D}_\perp)$ and $\sigma(\mathfrak{D}_T,\mathfrak{D}_\theta)$. First, consider

$$\begin{aligned}&\sum_{a=1}^{2p}\sum_{b=1}^{q} g\big(\sigma(x_a,x_b^*),\sigma(x_a,x_b^*)\big)\\ &\quad= \sum_{a=1}^{2p}\sum_{b=1}^{q}\sum_{c=1}^{2s} g\big(\sigma(x_a,x_b^*),\tilde{x}_c\big)^2 + \sum_{a=1}^{2p}\sum_{b,c=1}^{q} g\big(\sigma(x_a,x_b^*),\tilde{x}_c\big)^2\\ &\qquad+ \sum_{a=1}^{2p}\sum_{b=1}^{q}\sum_{c=1}^{2s} g\big(\sigma(x_a,x_b^*),\tilde{x}_c\big)^2.\end{aligned} \tag{65}$$

In light of Lemmas 1 and 3, we have

$$\begin{aligned}&\sum_{a=1}^{2p}\sum_{b=1}^{q} g\big(\sigma(x_a,x_b^*),\sigma(x_a,x_b^*)\big)\\ &\quad= \sum_{a=1}^{p}\sum_{b,c=1}^{q}\bigg(\bigg(\varphi\frac{x_a(f)}{f}\bigg)^2 + \bigg(\frac{x_a(f)}{f}\bigg)^2\bigg) g\big(x_b^*,x_c^*\big)^2 + \sum_{a=1}^{2p}\sum_{b=1}^{q}\sum_{c=1}^{2s} g\big(\sigma(x_a,x_b^*),\tilde{x}_c\big)^2.\end{aligned}$$

This implies the following relation

$$\sum_{a=1}^{2p}\sum_{b=1}^{q} g\big(\sigma(x_a,x_b^*),\sigma(x_a,x_b^*)\big) = q\bigg\|\frac{\nabla^T(f)}{f}\bigg\|^2 + \sum_{a=1}^{2p}\sum_{b=1}^{q}\sum_{c=1}^{2s} g\big(\sigma(x_a,x_b^*),\tilde{x}_c\big)^2. \tag{66}$$

Next, consider $g(\sigma(x_a,\hat{x}_b),\sigma(x_a,\hat{x}_b))$ and using the adopt frame, we obtain

$$\begin{aligned}&\sum_{a=1}^{2p}\sum_{b=1}^{2s} g\big(\sigma(x_a,\hat{x}_b),\sigma(x_a,\hat{x}_b)\big)\\ &\quad= \csc^2\theta\bigg(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma(x_a,\hat{x}_b),n\hat{x}_c\big)^2 + g\big(\sigma(\varphi x_a,\hat{x}_b),n\hat{x}_c\big)^2\bigg)\end{aligned}$$

$$+\sec^2\theta\csc^2\theta\left(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma(x_a,\hat{x}_b),nt\hat{x}_c\big)^2+g\big(\sigma(\varphi x_a,\hat{x}_b),nt\hat{x}_c\big)^2\right)$$
$$+\sec^2\theta\csc^2\theta\left(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma(x_a,t\hat{x}_b),n\hat{x}_c\big)^2+g\big(\sigma(\varphi x_a,t\hat{x}_b),n\hat{x}_c\big)^2\right)$$
$$+\sec^4\theta\csc^2\theta\left(\sum_{a=1}^{p}\sum_{b,c=1}^{2r} g\big(\sigma(x_a,t\hat{x}_b),nt\hat{x}_c\big)^2+g\big(\sigma(\varphi x_a,t\hat{x}_b),nt\hat{x}_c\big)^2\right)$$
$$+\sum_{a=1}^{2p}\sum_{b=1}^{2r}\sum_{c=1}^{2s} g\big(\sigma(x_a,\hat{x}_b),\tilde{x}_c\big)^2+\sum_{a=1}^{2p}\sum_{b=1}^{2r}\sum_{c=1}^{q} g\big(\sigma(x_a,\hat{x}_b),n\tilde{x}_c\big)^2. \tag{67}$$

Utilizing Lemmas 1 and 2 in (67) and also utilizing (30)–(35) in (67), we receive that

$$\sum_{a=1}^{2p}\sum_{b=1}^{2r} g\big(\sigma(x_a,\hat{x}_b),\sigma(x_a,\hat{x}_b)\big)$$
$$=2\csc^2\theta\sum_{a=1}^{p}\sum_{b,c=1}^{r}\left(\left(\varphi\frac{x_a(h)}{h}\right)^2+\left(\frac{x_a(h)}{h}\right)^2\right)g(\hat{x}_b,\hat{x}_c)^2$$
$$+2\cot^2\theta\sum_{a=1}^{p}\sum_{b,c=1}^{r}\left(\left(\varphi\frac{x_a(h)}{h}\right)^2+\left(\frac{x_a(h)}{h}\right)^2\right)g(\hat{x}_b,\hat{x}_c)^2$$
$$+\sum_{a=1}^{2p}\sum_{b=1}^{2r}\sum_{c=1}^{2s} g\big(\sigma(x_a,\hat{x}_b),\tilde{x}_c\big)^2. \tag{68}$$

By the definition of gradient, we concede that

$$\sum_{b=1}^{2p}\sum_{a=1}^{2r} g\big(\sigma(x_a,\hat{x}_b),\sigma(x_a,\hat{x}_b)\big)$$
$$=4r\big(1+2\cot^2\big)\theta\left\|\frac{\nabla^T(h)}{h}\right\|^2+\sum_{a=1}^{2p}\sum_{b=1}^{2r}\sum_{c=1}^{2s} g\big(\sigma(x_a,\hat{x}_b),\tilde{x}_c\big)^2. \tag{69}$$

From (64), (66), and (69), we have (60). If equality holds then we have

$$4r\big(1+2\cot^2\theta\big)\left\|\frac{\nabla^T(h)}{h}\right\|^2+q\left\|\frac{\nabla^T(f)}{f}\right\|^2=0.$$

The above equation shows that the functions f and h are constant on Ω_T. Therefore, Ω reduces to $\Omega_T\times\Omega_\perp\times_h\Omega_\theta$ is a single warped product. By the direct consequence of (66) and (69), we conclude that $\sigma(\mathfrak{D}_T,\mathfrak{D}_\perp)\perp\nu$ and $\sigma(\mathfrak{D}_T,\mathfrak{D}_\theta)\perp\nu$. This completes the proof. □

5.3 Particular cases

In ineq. (42) of Theorem 5 if:

i. $r=0$ and the norm of $\frac{\nabla^T(h)}{h}$, $\frac{\nabla^\perp(h)}{h}$ vanishes then SWP is a CR-warped product with

$$\sum_{a=1}^{2p}\sum_{b=1}^{q}\left\|\sigma\left(x_a,x_b^*\right)\right\|^2\geq 0.$$

ii. $q = 0$ and $\frac{\nabla^T(f)}{f}$ vanishes, then SWP is WP $\Omega_T \times_h \Omega_\theta$ satisfying

$$\sum_{b=1}^{2p}\sum_{a=1}^{2r}\left\|\sigma(x_a,\hat{x}_b)\right\|^2 \geq 0.$$

iii. $p = 0$ and the norm of $\frac{\nabla^T(f)}{f}$, $\frac{\nabla^T(h)}{h}$ vanishes, then SWP is pointwise pseudoslant WP $\Omega_\perp \times_h \Omega_\theta$ with

$$\sum_{a=1}^{q}\sum_{b=1}^{2r}\left\|\sigma\left(x_a^*,\hat{x}_b\right)\right\|^2 - \sum_{a=1}^{q}\sum_{b=1}^{2r} g\left(\sigma\left(x_a^*,x_a^*\right),\sigma(\hat{x}_b,\hat{x}_b)\right) \geq 2s\frac{\Delta^\perp(h)}{h}.$$

iv. the norm of $\frac{\nabla^\perp(h)}{h}$ vanishes, then SWP is a biwarped product $\Omega_T \times_f \Omega_\perp \times_h \Omega_\theta$ with

$$\sum_{b=1}^{2p}\sum_{a=1}^{2r}\left\|\sigma(x_a,\hat{x}_b)\right\|^2 + \sum_{a=1}^{2p}\sum_{b=1}^{q}\left\|\sigma\left(x_a,x_b^*\right)\right\|^2 + \sum_{a=1}^{q}\sum_{b=1}^{2r}\|\sigma(x_a^*,\hat{x}_b\|^2 - \sum_{a=1}^{q}\sum_{b=1}^{2r} g\left(\sigma\left(x_a^*,x_a^*\right),\sigma(\hat{x}_b,\hat{x}_b)\right) \geq -\frac{cqr}{2}. \tag{70}$$

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