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On fractional biparameterized Newton-type inequalities

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Abstract

In this work, we present a novel biparameterized identity that yields a family of one-, two-, three-, and four-point Newton-type formulas. Subsequently, we establish some new Newton-type inequalities for functions whose first derivatives are α-convex. The investigation is concluded with numerical examples accompanied by graphical representations to substantiate the accuracy of the obtained results.

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1 Introduction

Convexity is a fundamental notion in mathematics with broad-ranging applications in pure and applied fields. It plays a key role in the development of inequality theory, which has been an active area of research for many years. Integral inequality theory has found applications in many scientific fields, including physics, economics, and engineering. For some works related to Newton-type inequalities via different types of convexity, we refer the readers to [1–6].

Orlicz [7] introduced the class of α-convex functions as follows.

Definition 1.1 A function $h : I \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be α-convex for some fixed $0 < \alpha \le 1$ if for all $u, v \in I$ and $t \in [0,1]$,

$$h\bigl(tu + (1-t)v\bigr) \le t^{\alpha} h(u) + \bigl(1 - t^{\alpha}\bigr)h(v).$$

This notion, also called s-convexity in the first sense, represents a generalization of classical convexity, which can be recovered for $\alpha = 1$.

Recently, fractional calculus has emerged as a powerful tool for modeling various physical and real-world phenomena. The idea of fractional derivatives and integrals has been around for centuries, but it is only in the last few decades that fractional calculus has received significant attention from mathematicians and scientists. The notions of fractional derivatives and integrals have been used in many areas of mathematics, including differential equations, complex analysis, and numerical analysis. For a comprehensive overview of the applications and recent developments in fractional calculus, we refer the readers to [8–10].

Many researchers have devoted their efforts to the study of integral inequalities via different fractional operators, and their works have led to significant advances in various areas of mathematics and beyond. In [11] the authors investigated Hermite–Hadamard-type inequalities using the Riemann–Liouville fractional integrals. Hamida et al. [12] established the fractional Bullen-type inequality. Sarikaya et al. [13] derived the fractional Simpson-type inequalities based on convexity, whereas Chen et al. [14] extended them to s-convexity. For further related works, we refer to [15–21] and the references therein.

More recently, Ali et al. [22] explored parameterized inequalities of Simpson 3/8 type for differentiable convex functions and established numerous classical and fractional inequalities of Newton type, trapezoidal type, and others.

In this paper, motivated by the above-mentioned results, we investigate a class of fractional inequalities, the biparametrized Newton-type inequalities for differentiable α-convex mappings. Our goal is to explore the general framework of quadrature formulas with one, two, three, and four points. To achieve this, we introduce a biparameterized identity involving the Riemann–Liouville fractional integrals, which enabled us to establish several new results for α-convex derivatives. We also review the existing literature on this topic and highlight the significance of our contributions. This study provides a plethora of new results, as well as some well-known ones, which can be applied in various fields of science and engineering.

Let us consider the following biparameterized four-point quadrature formula:

$$\begin{aligned}\mathcal{P}(\upsilon, x, \varrho; \delta; \lambda) &= \frac{\lambda(x-\upsilon)^{\delta}}{\rho-\upsilon} h(\upsilon) \\ &\quad + \frac{2^{\delta}(1-\lambda)(x-\upsilon)^{\delta} + (\upsilon+\rho-2x)^{\delta}}{2^{\delta}(\rho-\upsilon)} \big(h(x) + h(\upsilon+\rho-x)\big) \\ &\quad + \frac{\lambda(x-\upsilon)^{\delta}}{\rho-\upsilon} h(\rho), \qquad (1)\end{aligned}$$

where $x \in [\upsilon, \frac{\upsilon+\varrho}{2}]$ and $\lambda \in [0,1]$.

2 Preliminaries

In this section, we review some well-known definitions and essential tools related to fractional calculus.

Definition 2.1 ([23]) The beta function is defined for complex numbers u and v such that $Re(u) > 0$ and $Re(v) > 0$ by

$$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1}\, dt.$$

Definition 2.2 ([23]) The incomplete beta function is defined for complex numbers u and v such that $Re(u) > 0$ and $Re(v) > 0$ and for $0 < c < 1$ by

$$B(c; u, v) = \int_0^c t^{u-1}(1-t)^{v-1}\, dt.$$

Definition 2.3 ([23]) The hypergeometric function is defined for complex numbers u, v, w, and z such that $Re(w) > Re(v) > 0$ and $|z| < 1$ by

$$ {}_2F_1(u,v,w;z) = \frac{1}{B(v,w-v)} \int_0^1 t^{v-1}(1-t)^{w-v-1}(1-zt)^{-u}\,dt, $$

where $B(\cdot,\cdot)$ is the beta function.

Definition 2.4 ([23]) Given $h \in L^1[\upsilon,\varrho]$, the Riemann–Liouville fractional integrals of order $\delta > 0$ with $\upsilon \geq 0$, denoted by $I^{\delta}_{\upsilon^+}h$ and $I^{\delta}_{\varrho^-}h$, are defined as follows:

$$ \begin{aligned} I^{\delta}_{\upsilon^+}h(x) &= \frac{1}{\Gamma(\delta)} \int_{\upsilon}^{x} (x-t)^{\delta-1}h(t)\,dt, \quad x > \upsilon, \\ I^{\delta}_{\varrho^-}h(x) &= \frac{1}{\Gamma(\delta)} \int_{x}^{\varrho} (t-x)^{\delta-1}h(t)\,dt, \quad \varrho > x, \end{aligned} $$

respectively, where $\Gamma(\delta) = \int_0^{\infty} e^{-t}t^{\delta-1}\,dt$ is the gamma function, and $I^{0}_{\upsilon^+}f(x) = I^{0}_{\varrho^-}f(x) = f(x)$.

The paper is organized as follows. In Sect. 3, we introduce and prove a biparametric fractional identity, from which we establish a multitude of Newton-type inequalities with one, two, three, and four points for functions with α-convex derivatives. In Sect. 4, we provide illustrative examples along with graphical representations to confirm the accuracy of the obtained results and showcase some applications. The study concludes with a final section summarizing our findings and providing a comprehensive conclusion.

3 Main results

Our approach begins by introducing the following identity, which plays a central role in establishing our results.

Lemma 3.1 *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\upsilon, \varrho \in I^{\circ}$ with $\upsilon < \varrho$, and $h' \in L^1[\upsilon,\varrho]$. Then we have the following equality for real numbers $\lambda \in [0,1]$, $x \in [\upsilon, \frac{\upsilon+\varrho}{2}]$, and* $0 < \delta \leq 1$:

$$ \begin{aligned} &\mathcal{P}(\upsilon,x,\varrho;\delta;\lambda) - \Lambda(\upsilon,x,\varrho;\delta;\lambda) \\ &\quad= \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon} \int_0^1 \bigl(t^{\delta}-\lambda\bigr)h'\bigl((1-t)\upsilon+tx\bigr)\,dt \\ &\qquad- \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)} \int_0^1 (1-t)^{\delta}h'\left((1-t)x+t\frac{\upsilon+\varrho}{2}\right)dt \\ &\qquad+ \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)} \int_0^1 t^{\delta}h'\left((1-t)\frac{\upsilon+\varrho}{2}+t(\upsilon+\varrho-x)\right)dt \\ &\qquad- \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon} \int_0^1 \bigl((1-t)^{\delta}-\lambda\bigr)h'\bigl((1-t)(\upsilon+\varrho-x)+t\varrho\bigr)\,dt, \end{aligned} $$

where $\mathcal{P}(\upsilon, x, \varrho; \delta; \lambda)$ *is given by* (1), *and*

$$\Lambda(\upsilon, x, \varrho; \delta) = \frac{\Gamma(\delta+1)}{\varrho-\upsilon}\left(I_{x^-}^{\delta} h(\upsilon) + I_{x^+}^{\delta} h\left(\frac{\upsilon+\varrho}{2}\right)\right.$$
$$\left. + I_{(\upsilon+\varrho-x)^-}^{\delta} h\left(\frac{\upsilon+\varrho}{2}\right) + I_{(\upsilon+\varrho-x)^+}^{\delta} h(\varrho)\right). \tag{2}$$

Proof Let

$$I = \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon} I_1 - \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)} I_2 + \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)} I_3 - \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon} I_4, \tag{3}$$

where

$$I_1 = \int_0^1 \left(t^{\delta} - \lambda\right) h'\left((1-t)\upsilon + tx\right) dt,$$
$$I_2 = \int_0^1 (1-t)^{\delta} h'\left((1-t)x + t\frac{\upsilon+\varrho}{2}\right) dt,$$
$$I_3 = \int_0^1 t^{\delta} h'\left((1-t)\frac{\upsilon+\varrho}{2} + t(\upsilon+\varrho-x)\right) dt,$$

and

$$I_4 = \int_0^1 \left((1-t)^{\delta} - \lambda\right) h'\left((1-t)(\upsilon+\varrho-x) + t\varrho\right) dt.$$

Integrating I_1 by parts, we get

$$\begin{aligned}
I_1 &= \int_0^1 \left(t^{\delta} - \lambda\right) h'\left((1-t)\upsilon + tx\right) dt \\
&= \frac{1}{x-\upsilon}\left(t^{\delta} - \lambda\right) h\left((1-t)\upsilon + tx\right)\Big|_{t=0}^{t=1} - \frac{\delta}{x-\upsilon}\int_0^1 t^{\delta-1} h\left((1-t)\upsilon + tx\right) dt \\
&= \frac{1-\lambda}{x-\upsilon} h(x) + \frac{\lambda}{x-\upsilon} h(\upsilon) - \frac{\delta}{(x-\upsilon)^{\delta+1}}\int_{\upsilon}^{x} (u-\upsilon)^{\delta-1} h(u)\, du \\
&= \frac{1-\lambda}{x-\upsilon} h(x) + \frac{\lambda}{x-\upsilon} h(\upsilon) - \frac{\Gamma(\delta+1)}{(x-\upsilon)^{\delta+1}} I_{x^-}^{\delta} h(\upsilon).
\end{aligned} \tag{4}$$

Likewise, we get

$$\begin{aligned}
I_2 &= \int_0^1 (1-t)^{\delta} h'\left((1-t)x + t\frac{\upsilon+\varrho}{2}\right) dt \\
&= \frac{2}{\upsilon+\varrho-2x}(1-t)^{\delta} h\left((1-t)x + t\frac{\upsilon+\varrho}{2}\right)\Big|_{t=0}^{t=1} \\
&\quad + \frac{2\delta}{\upsilon+\varrho-2x}\int_0^1 (1-t)^{\delta-1} h\left((1-t)x + t\frac{\upsilon+\varrho}{2}\right) dt \\
&= -\frac{2}{\upsilon+\varrho-2x} h(x) + \frac{2^{\delta+1}\delta}{(\upsilon+\varrho-2x)^{\delta+1}}\int_x^{\frac{\upsilon+\varrho}{2}} \left(\frac{\upsilon+\varrho}{2} - u\right)^{\delta-1} h(u)\, du \\
&= -\frac{2}{\upsilon+\varrho-2x} h(x) + \frac{2^{\delta+1}\Gamma(\delta+1)}{(\upsilon+\varrho-2x)^{\delta+1}} I_{x^+}^{\delta} h\left(\frac{\upsilon+\varrho}{2}\right),
\end{aligned} \tag{5}$$

$$
\begin{aligned}
I_3 &= \int_0^1 t^\delta h'\left((1-t)\frac{\upsilon+\varrho}{2} + t(\upsilon+\varrho-x)\right) dt \\
&= \frac{2}{\upsilon+\varrho-2x} t^\delta h\left((1-t)\frac{\upsilon+\varrho}{2} + t(\upsilon+\varrho-x)\right)\Big|_{t=0}^{t=1} \\
&\quad - \frac{2\delta}{\upsilon+\varrho-2x}\int_0^1 t^{\delta-1} h\left((1-t)\frac{\upsilon+\varrho}{2} + t(\upsilon+\varrho-x)\right) dt \\
&= \frac{2}{\upsilon+\varrho-2x} h(\upsilon+\varrho-x) - \frac{2^{\delta+1}\delta}{(\upsilon+\varrho-2x)^{\delta+1}}\int_{\frac{\upsilon+\varrho}{2}}^{\upsilon+\varrho-x}\left(u-\frac{\upsilon+\varrho}{2}\right)^{\delta-1} h(u)\, du \\
&= \frac{2}{\upsilon+\varrho-2x} h(\upsilon+\varrho-x) - \frac{2^{\delta+1}\Gamma(\delta+1)}{(\upsilon+\varrho-2x)^{\delta+1}} I^{\delta}_{(\upsilon+\varrho-x)^-} h\left(\frac{\upsilon+\varrho}{2}\right),
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1 \left((1-t)^\delta - \lambda\right) h'\left((1-t)(\upsilon+\varrho-x) + t\varrho\right) dt \\
&= \frac{1}{x-\upsilon}\left((1-t)^\delta - \lambda\right) h\left((1-t)(\upsilon+\varrho-x) + t\varrho\right)\Big|_{t=0}^{t=1} \\
&\quad + \frac{\delta}{x-\upsilon}\int_0^1 (1-t)^{\delta-1} h\left((1-t)(\upsilon+\varrho-x) + t\varrho\right) dt \\
&= -\frac{\lambda}{x-\upsilon} h(\varrho) - \frac{1-\lambda}{x-\upsilon} h(\upsilon+\varrho-x) + \frac{\delta}{(x-\upsilon)^{\delta+1}}\int_{\upsilon+\varrho-x}^{\varrho} (\varrho-u)^{\delta-1} h(u)\, du \\
&= -\frac{\lambda}{x-\upsilon} h(\varrho) - \frac{1-\lambda}{x-\upsilon} h(\upsilon+\varrho-x) + \frac{\Gamma(\delta+1)}{(x-\upsilon)^{\delta+1}} I^{\delta}_{(\upsilon+\varrho-x)^+} h(\varrho).
\end{aligned} \tag{7}
$$

By substituting equations (4)–(7) into equation (3) we obtain the intended outcome. □

Theorem 3.2 *Let $h : [\upsilon, \varrho] \to \mathbb{R}$ be a differentiable function on $[\upsilon, \varrho]$ such that $h' \in L^1[\upsilon, \varrho]$ with $0 \le \upsilon < \varrho$. If $|h'|$ is α-convex on $[\upsilon, \varrho]$, then we have*

$$
\begin{aligned}
&\left|\mathcal{P}(\upsilon, x, \varrho; \delta; \lambda) - \Lambda(\upsilon, x, \varrho; \delta)\right| \\
&\quad \le \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\Big[\left(\mathcal{C}_1(\lambda,\delta) - \mathcal{C}_2(\lambda,\delta,\alpha)\right)\left|h'(\upsilon)\right| + \mathcal{C}_2(\lambda,\delta,\alpha)\left|h'(x)\right| \\
&\quad\quad + \left(\mathcal{C}_1(\lambda,\delta) - \mathcal{C}_3(\lambda,\delta,\alpha)\right)\left|h'(\upsilon+\varrho-x)\right| + \mathcal{C}_3(\lambda,\delta,\alpha)\left|h'(\varrho)\right|\Big] \\
&\quad\quad + \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)}\left[\left(\frac{1}{\delta+1} - B(\delta+1,\alpha+1)\right)\left|h'(x)\right| + \frac{1}{\alpha+\delta+1}\left|h'(\upsilon+\varrho-x)\right|\right. \\
&\quad\quad \left. + \left(B(\delta+1,\alpha+1) + \frac{\alpha}{(\delta+1)(\alpha+\delta+1)}\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right],
\end{aligned}
$$

where $B(\cdot,\cdot)$ and $B(\cdot;\cdot,\cdot)$ are beta and incomplete beta functions, respectively, and $\mathcal{C}_i$ ($i = 1, 2, 3$) are given by

$$
\mathcal{C}_1(\lambda,\delta) = \frac{2\delta\lambda^{\frac{\delta+1}{\delta}} - \lambda(\delta+1) + 1}{\delta+1}, \tag{8}
$$

$$
\mathcal{C}_2(\lambda,\delta,\alpha) = \frac{2\delta\lambda^{\frac{\alpha+\delta+1}{\delta}}}{(\alpha+1)(\alpha+\delta+1)} + \frac{1}{\alpha+\delta+1} - \frac{\lambda}{\alpha+1}, \tag{9}
$$

and

$$\mathcal{C}_3(\lambda,\delta,\alpha) = B(\delta+1,\alpha+1) - 2B\left(\lambda^{1/\delta};\delta+1,\alpha+1\right) - \frac{2\lambda(1-\lambda^{1/\delta})^{\alpha+1}-\lambda}{\alpha+1}. \tag{10}$$

Proof From Lemma 3.1 we have

$$\begin{aligned}
&\left|\mathcal{P}(\upsilon,x,\varrho;\delta;\lambda) - \Lambda(\upsilon,x,\varrho;\delta)\right| \\
&\le \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\left(\int_0^1 \left|t^\delta-\lambda\right|\left|h'\left((1-t)\upsilon+tx\right)\right|dt\right. \\
&\quad\left.+\int_0^1 \left|(1-t)^\delta-\lambda\right|\left|h'\left((1-t)(\upsilon+\varrho-x)+t\varrho\right)\right|dt\right) \\
&\quad+\frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)}\left(\int_0^1 (1-t)^\delta\left|h'\left((1-t)x+t\frac{\upsilon+\varrho}{2}\right)\right|dt\right. \\
&\quad\left.+\int_0^1 t^\delta\left|h'\left((1-t)\frac{\upsilon+\varrho}{2}+t(\upsilon+\varrho-x)\right)\right|dt\right).
\end{aligned}$$

Using the α-convexity of $|h'|$, we get

$$\begin{aligned}
&\left|\mathcal{P}(\upsilon,x,\varrho;\delta;\lambda) - \Lambda(\upsilon,x,\varrho;\delta)\right| \\
&\le \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\left(\int_0^1 \left|t^\delta-\lambda\right|\left\{\left(1-t^\alpha\right)\left|h'(\upsilon)\right|+t^\alpha\left|h'(x)\right|\right\}dt\right. \\
&\quad\left.+\int_0^1 \left|(1-t)^\delta-\lambda\right|\left\{\left(1-t^\alpha\right)\left|h'(\upsilon+\varrho-x)\right|+t^\alpha\left|h'(\varrho)\right|\right\}dt\right) \\
&\quad+\frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)}\left(\int_0^1 (1-t)^\delta\left\{\left(1-t^\alpha\right)\left|h'(x)\right|+t^\alpha\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right\}dt\right. \\
&\quad\left.+\int_0^1 t^\delta\left\{\left(1-t^\alpha\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|+t^\alpha\left|h'(\upsilon+\varrho-x)\right|\right\}dt\right) \\
&= \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\left(\left|h'(\upsilon)\right|\int_0^1 \left|t^\delta-\lambda\right|\left(1-t^\alpha\right)dt+\left|h'(x)\right|\int_0^1\left|t^\delta-\lambda\right|t^\alpha\,dt\right. \\
&\quad\left.+\left|h'(\upsilon+\varrho-x)\right|\int_0^1\left|(1-t)^\delta-\lambda\right|\left(1-t^\alpha\right)dt+\left|h'(\varrho)\right|\int_0^1\left|(1-t)^\delta-\lambda\right|t^\alpha\,dt\right) \\
&\quad+\frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)}\left(\left|h'(x)\right)\right|\int_0^1(1-t)^\delta\left(1-t^\alpha\right)dt+\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\int_0^1(1-t)^\delta t^\alpha\,dt \\
&\quad\left.+\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\int_0^1 t^\delta\left(1-t^\alpha\right)dt+\left|h'(\upsilon+\varrho-x)\right|\int_0^1 t^{\delta+\alpha}\,dt\right) \\
&= \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\left[\left(\mathcal{C}_1(\lambda,\delta)-\mathcal{C}_2(\lambda,\delta,\alpha)\right)\left|h'(\upsilon)\right|+\mathcal{C}_2(\lambda,\delta,\alpha)\left|h'(x)\right|\right. \\
&\quad\left.+\left(\mathcal{C}_1(\lambda,\delta)-\mathcal{C}_3(\lambda,\delta,\alpha)\right)\left|h'(\upsilon+\varrho-x)\right|+\mathcal{C}_3(\lambda,\delta,\alpha)\left|h'(\varrho)\right|\right] \\
&\quad+\frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)}\left[\left(\frac{1}{\delta+1}-B(\delta+1,\alpha+1)\right)\left|h'(x)\right|+\frac{1}{\alpha+\delta+1}\left|h'(\upsilon+\varrho-x)\right|\right. \\
&\quad\left.+\left(B(\delta+1,\alpha+1)+\frac{\alpha}{(\delta+1)(\alpha+\delta+1)}\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right].
\end{aligned}$$

This deduction is based on the fact that

$$\int_0^1 \left|t^\delta - \lambda\right|\left(1 - t^\alpha\right) dt = \mathcal{C}_1(\lambda, \delta) - \int_0^1 \left|t^\delta - \lambda\right| t^\alpha \, dt = \mathcal{C}_1(\lambda, \delta) - \mathcal{C}_2(\lambda, \delta, \alpha), \tag{11}$$

$$\begin{aligned}\int_0^1 \left|(1-t)^\delta - \lambda\right|\left(1 - t^\alpha\right) dt &= \mathcal{C}_1(\lambda, \delta) - \int_0^1 \left|(1-t)^\delta - \lambda\right| t^\alpha \, dt \\ &= \mathcal{C}_1(\lambda, \delta) - \mathcal{C}_3(\lambda, \delta, \alpha),\end{aligned} \tag{12}$$

$$\int_0^1 (1-t)^\delta \left(1 - t^\alpha\right) dt = \frac{1}{\delta + 1} - \int_0^1 (1-t)^\delta t^\alpha \, dt = \frac{1}{\delta + 1} - B(\delta + 1, \alpha + 1), \tag{13}$$

and

$$\int_0^1 t^\delta \left(1 - t^\alpha\right) dt = \frac{1}{\delta + 1} - \int_0^1 t^{\delta + \alpha} \, dt = \frac{\alpha}{(\delta + 1)(\alpha + \delta + 1)} \tag{14}$$

with $\mathcal{C}_1(\lambda, \delta)$, $\mathcal{C}_2(\lambda, \delta, \alpha)$, and $\mathcal{C}_3(\lambda, \delta, \alpha)$ as defined in (8), (9), and (10), respectively.

Hence the proof is complete. □

Theorem 3.2 presents a diverse array of findings regarding one-, two-, three-, and four-point Newton-type inequalities, encompassing both original discoveries and preexisting results. Notably, by scrutinizing select parameter values of x and λ, a wealth of new inequalities arises for various categories of convexity, and certain known ones are also reencountered, some of them even improved. We will enumerate a selection of such results.

Corollary 3.3 *Setting $\alpha = 1$ in Theorem 3.2, we obtain the following fractional biparameterized four-point Newton-type inequality for convex functions*:

$$\begin{aligned}&\left|\mathcal{P}(\upsilon, x, \varrho; \delta; \lambda) - \Lambda(\upsilon, x, \varrho, \delta)\right| \\ &\quad\leq \frac{(x - \upsilon)^{\delta+1}}{\varrho - \upsilon}\left(\left(\frac{1}{(\delta + 1)(\delta + 2)} + \frac{2\delta}{\delta + 1}\lambda^{\frac{\delta+1}{\delta}} - \frac{\delta}{\delta + 2}\lambda^{\frac{\delta+2}{\delta}} - \frac{\lambda}{2}\right)\left(\left|h'(\upsilon)\right| + \left|h'(\varrho)\right|\right)\right. \\ &\quad\quad\left. + \left(\frac{1}{\delta + 2} + \frac{\delta}{\delta + 2}\lambda^{\frac{\delta+2}{\delta}} - \frac{\lambda}{2}\right)\left(\left|h'(x)\right| + \left|h'(\upsilon + \varrho - x)\right|\right)\right) \\ &\quad\quad + \frac{(\upsilon + \varrho - 2x)^{\delta+1}}{2^{\delta+1}(\varrho - \upsilon)(\delta + 2)}\left(\left|h'(x)\right| + \frac{2}{\delta + 1}\left|h'\left(\frac{\upsilon + \varrho}{2}\right)\right| + \left|h'(\upsilon + \varrho - x)\right|\right).\end{aligned}$$

Corollary 3.4 *Setting $x = \upsilon$ in Corollary 3.3, we obtain the following fractional trapezoidal-type inequality for convex functions*:

$$\begin{aligned}&\left|\frac{h(\upsilon) + h(\varrho)}{2} - \frac{2^{\delta-1}\Gamma(\delta + 1)}{(\varrho - \upsilon)^\delta}\left(I^\delta_{\upsilon^+} h\left(\frac{\upsilon + \varrho}{2}\right) + I^\delta_{\varrho^-} h\left(\frac{\upsilon + \varrho}{2}\right)\right)\right| \\ &\quad\leq \frac{\varrho - \upsilon}{4(\delta + 2)}\left[\left|h'(\upsilon)\right| + \frac{2}{\delta + 1}\left|h'\left(\frac{\upsilon + \varrho}{2}\right)\right| + \left|h'(\varrho)\right|\right].\end{aligned}$$

Moreover, using the convexity of $|h'|$, *we obtain*

$$\left|\frac{h(\upsilon)+h(\varrho)}{2}-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\varrho-\upsilon)^{\delta}}\left(I_{\upsilon^+}^{\delta}h\left(\frac{\upsilon+\varrho}{2}\right)+I_{\varrho^-}^{\delta}h\left(\frac{\upsilon+\varrho}{2}\right)\right)\right| \le \frac{\varrho-\upsilon}{4(\delta+1)}\left[\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right].$$

This improves the result obtained in Corollary 3.6 *of* [17].

Corollary 3.5 *Setting* $x=\frac{\upsilon+\varrho}{2}$ *in Corollary* 3.3, *we obtain the following fractional parameterized three-point Newton–Cotes-type inequality for convex functions*:

$$\begin{aligned}&\left|\frac{\lambda}{2}h(\upsilon)+(1-\lambda)h\left(\frac{\upsilon+\varrho}{2}\right)+\frac{\lambda}{2}h(\varrho)-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\varrho-\upsilon)^{\delta}}\left(I_{(\frac{\upsilon+\varrho}{2})^-}^{\delta}h(\upsilon)+I_{(\frac{\upsilon+\varrho}{2})^+}^{\delta}h(\varrho)\right)\right|\\&\quad\le\frac{\varrho-\upsilon}{4}\left(\left(\frac{1}{(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}\lambda^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}\lambda^{\frac{\delta+2}{\delta}}-\frac{\lambda}{2}\right)\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)\right.\\&\qquad\left.+2\left(\frac{1}{\delta+2}+\frac{\delta}{\delta+2}\lambda^{\frac{\delta+2}{\delta}}-\frac{\lambda}{2}\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right).\end{aligned}$$

Remark 3.6 Consider some particular cases in Corollary 3.5.

- If $\lambda=0$, then we obtain the following fractional midpoint inequality for convex functions:

$$\left|h\left(\frac{\upsilon+\varrho}{2}\right)-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\varrho-\upsilon)^{\delta}}\left(I_{(\frac{\upsilon+\varrho}{2})^-}^{\delta}h(\upsilon)+I_{(\frac{\upsilon+\varrho}{2})^+}^{\delta}h(\varrho)\right)\right| \le\frac{\varrho-\upsilon}{4(\delta+2)}\left(\frac{1}{\delta+1}\left|h'(\upsilon)\right|+2\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|+\frac{1}{\delta+1}\left|h'(\varrho)\right|\right).$$

  This result improves that of Sarikaya et al. [11] (Theorem 5 for $q=1$), which can be derived by using the convexity of $|h'|$ in the preceding inequality.
- If $\lambda=1$, then we obtain another variant of fractional trapezoidal-type inequality for convex functions,

$$\left|\frac{h(\upsilon)+h(\varrho)}{2}-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\varrho-\upsilon)^{\delta}}\left(I_{(\frac{\upsilon+\varrho}{2})^-}^{\delta}h(\upsilon)+I_{(\frac{\upsilon+\varrho}{2})^+}^{\delta}h(\varrho)\right)\right| \le\frac{\varrho-\upsilon}{4}\left(\frac{\delta^2+3\delta}{2(\delta+1)(\delta+2)}\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)+\frac{\delta}{\delta+2}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right).$$

  Moreover, using the convexity of $|h'|$, we can recover the result established in Corollary 3.4.
- If $\lambda=\frac{1}{2}$, then we obtain the following fractional Bullen-type inequality for convex functions:

$$\begin{aligned}&\left|\frac{1}{4}\left(h(\upsilon)+2h\left(\frac{\upsilon+\varrho}{2}\right)+h(\varrho)\right)-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\varrho-\upsilon)^{\delta}}\left(I_{(\frac{\upsilon+\varrho}{2})^-}^{\delta}h(\upsilon)+I_{(\frac{\upsilon+\varrho}{2})^+}^{\delta}h(\varrho)\right)\right|\\&\quad\le\frac{\varrho-\upsilon}{4}\left(\left(\frac{1}{(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}\left(\frac{1}{2}\right)^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}\left(\frac{1}{2}\right)^{\frac{\delta+2}{\delta}}-\frac{1}{4}\right)\right.\\&\qquad\times\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)\end{aligned}$$

$$+2\left(\frac{1}{\delta+2}+\frac{\delta}{\delta+2}\left(\frac{1}{2}\right)^{\frac{\delta+2}{\delta}}-\frac{1}{4}\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\bigg).$$

- If $\lambda = \frac{1}{3}$, then we obtain the following fractional Simpson-type inequality for convex functions:

$$\begin{aligned}&\left|\frac{1}{6}\left(h(\upsilon)+4h\left(\frac{\upsilon+\varrho}{2}\right)+h(\varrho)\right)-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\varrho-\upsilon)^{\delta}}\left(I^{\delta}_{(\frac{\upsilon+\varrho}{2})^-}h(\upsilon)+I^{\delta}_{(\frac{\upsilon+\varrho}{2})^+}h(\varrho)\right)\right|\\&\quad\le\frac{\varrho-\upsilon}{4}\Bigg(\left(\frac{1}{(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}\left(\frac{1}{3}\right)^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}\left(\frac{1}{3}\right)^{\frac{\delta+2}{\delta}}-\frac{1}{6}\right)\\&\quad\quad\times\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)\\&\quad\quad+2\left(\frac{1}{\delta+2}+\frac{\delta}{\delta+2}\left(\frac{1}{3}\right)^{\frac{\delta+2}{\delta}}-\frac{1}{6}\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\Bigg).\end{aligned}$$

- If $\lambda = \frac{1}{4}$, then we obtain the following fractional Simpson-type inequality for convex functions:

$$\begin{aligned}&\left|\frac{1}{8}\left(h(\upsilon)+6h\left(\frac{\upsilon+\varrho}{2}\right)+h(\varrho)\right)-\frac{2^{\delta-1}\Gamma(\delta+1)}{(\varrho-\upsilon)^{\delta}}\left(I^{\delta}_{(\frac{\upsilon+\varrho}{2})^-}h(\upsilon)+I^{\delta}_{(\frac{\upsilon+\varrho}{2})^+}h(\varrho)\right)\right|\\&\quad\le\frac{\varrho-\upsilon}{4}\Bigg(\left(\frac{1}{(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}\left(\frac{1}{4}\right)^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}\left(\frac{1}{4}\right)^{\frac{\delta+2}{\delta}}-\frac{1}{8}\right)\\&\quad\quad\times\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)\\&\quad\quad+2\left(\frac{1}{\delta+2}+\frac{\delta}{\delta+2}\left(\frac{1}{4}\right)^{\frac{\delta+2}{\delta}}-\frac{1}{8}\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\Bigg).\end{aligned}$$

Corollary 3.7 *Setting $\lambda = 0$ in Corollary 3.3, we obtain the following fractional symmetrical two-point type inequality for convex functions*:

$$\begin{aligned}&\left|\frac{2^{\delta}(x-\upsilon)^{\delta}+(\upsilon+\varrho-2x)^{\delta}}{2^{\delta}(\varrho-\upsilon)}\big(h(x)+h(\upsilon+\varrho-x)\big)-\Lambda(\upsilon,x,\varrho;\delta)\right|\\&\quad\le\frac{(x-\upsilon)^{\delta+1}}{(\varrho-\upsilon)(\delta+2)}\left(\frac{1}{\delta+1}\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)+\left(\left|h'(x)\right|+\left|h'(\upsilon+\varrho-x)\right|\right)\right)\\&\quad\quad+\frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)(\delta+2)}\left(\left|h'(x)\right|+\frac{2}{\delta+1}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|+\left|h'(\upsilon+\varrho-x)\right|\right).\end{aligned}$$

Moreover, taking $\delta = 1$, we obtain the following companion Ostrowski-type inequality for convex functions:

$$\begin{aligned}&\left|\frac{h(x)+h(\upsilon+\varrho-x)}{2}-\frac{1}{\varrho-\upsilon}\int_{\upsilon}^{\varrho}h(z)\,dz\right|\\&\quad\le\frac{(x-\upsilon)^{2}}{3(\varrho-\upsilon)}\left(\frac{1}{2}\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)+\left(\left|h'(x)\right|+\left|h'(\upsilon+\varrho-x)\right|\right)\right)\\&\quad\quad+\frac{(\upsilon+\varrho-2x)^{2}}{12(\varrho-\upsilon)}\left(\left|h'(x)\right|+\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|+\left|h'(\upsilon+\varrho-x)\right|\right).\end{aligned}$$

Corollary 3.8 *Setting $x=\frac{2\upsilon+\varrho}{3}$ and $\lambda=\frac{3}{8}$ in Corollary 3.3, we obtain the following fractional four-point type inequality for convex functions*:

$$\begin{aligned}&\left|\frac{(\varrho-\upsilon)^{\delta-1}}{8\times 3^{\delta-1}}h(\upsilon)+\frac{(5\times 2^{\delta}+8)(\varrho-\upsilon)^{\delta-1}}{8\times 6^{\delta}}\left(h\left(\frac{2\upsilon+\varrho}{3}\right)+h\left(\frac{\upsilon+2\varrho}{3}\right)\right)\right.\\&\quad\left.+\frac{(\varrho-\upsilon)^{\delta-1}}{8\times 3^{\delta-1}}h(\varrho)-\Lambda\left(\upsilon,\frac{2\upsilon+\varrho}{3},\varrho;\delta\right)\right|\\&\leq\frac{(\varrho-\upsilon)^{\delta}}{3^{\delta+1}}\left[\left(\frac{1}{(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}\left(\frac{3}{8}\right)^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}\left(\frac{3}{8}\right)^{\frac{\delta+2}{\delta}}-\frac{3}{16}\right)\right.\\&\quad\times\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)\\&\quad+\left(\frac{1}{\delta+2}+\frac{\delta}{\delta+2}\left(\frac{3}{8}\right)^{\frac{\delta+2}{\delta}}-\frac{3}{16}+\frac{1}{2^{\delta+1}(\delta+2)}\right)\left(\left|h'\left(\frac{2\upsilon+\varrho}{3}\right)\right|+\left|h'\left(\frac{\upsilon+2\varrho}{3}\right)\right|\right)\\&\quad\left.+\frac{1}{2^{\delta}(\delta+1)(\delta+2)}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right].\end{aligned}$$

Moreover, taking $\delta=1$, we obtain the following Simpson-type 3/8 inequality for convex functions:

$$\begin{aligned}&\left|\frac{1}{8}h(\upsilon)+\frac{3}{8}\left(h\left(\frac{2\upsilon+\varrho}{3}\right)+h\left(\frac{\upsilon+2\varrho}{3}\right)\right)+\frac{1}{8}h(\varrho)-\frac{1}{\varrho-\upsilon}\int_{\upsilon}^{\varrho}h(z)\,dz\right|\\&\leq\frac{\varrho-\upsilon}{9}\left[\frac{157}{1536}\left(\left|h'(\upsilon)\right|+\left|h'(\varrho)\right|\right)+\frac{379}{1536}\left(\left|h'\left(\frac{2\upsilon+\varrho}{3}\right)\right|+\left|h'\left(\frac{\upsilon+2\varrho}{3}\right)\right|\right)\right.\\&\quad\left.+\frac{1}{12}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right].\end{aligned}$$

Remark 3.9 Using the convexity of $|h'|$, i.e., $|h'(\frac{\upsilon+\varrho}{2})|\leq\frac{1}{2}(|h'(\frac{3\upsilon+\varrho}{4})|+|h'(\frac{\upsilon+3\varrho}{4})|)$, the second inequality of Corollary 3.8 recaptures Corollary 3.8 from [24].

Corollary 3.10 *In Theorem 3.2, taking $\delta=1$, we obtain the following biparameterized Newton-type inequality via α-convexity*:

$$\begin{aligned}&\left|\frac{\lambda(x-\upsilon)}{\varrho-\upsilon}h(\upsilon)+\frac{2(1-\lambda)(x-\upsilon)+\upsilon+\varrho-2x}{2(\varrho-\upsilon)}\left(h(x)+h(\upsilon+\varrho-x)\right)\right.\\&\quad\left.+\frac{\lambda(x-\upsilon)}{\varrho-\upsilon}h(\varrho)-\frac{1}{\varrho-\upsilon}\int_{\upsilon}^{\varrho}h(z)\,dz\right|\\&\leq\frac{(x-\upsilon)^2}{\varrho-\upsilon}\left(\left(\frac{2\lambda^2-2\lambda+1}{2}+\frac{\lambda(\alpha+2)-(\alpha+1)-2\lambda^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right)\left|h'(\upsilon)\right|\right.\\&\quad+\left(\frac{2\lambda^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+2}-\frac{\lambda}{\alpha+1}\right)\left|h'(x)\right|\\&\quad+\left(\frac{2\lambda^2-2\lambda+1}{2}-\frac{2(1-\lambda)^{\alpha+2}+\lambda(\alpha+2)-1}{(\alpha+1)(\alpha+2)}\right)\left|h'(\upsilon+\varrho-x)\right|\\&\quad\left.+\frac{2(1-\lambda)^{\alpha+2}+\lambda(\alpha+2)-1}{(\alpha+1)(\alpha+2)}\left|h'(\varrho)\right|\right)\end{aligned}$$

$$+\frac{(\upsilon+\varrho-2x)^2}{4(\varrho-\upsilon)}\left(\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\left|h'(x)\right|+\frac{1}{\alpha+2}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|\right.$$
$$\left.+\frac{\alpha^2+\alpha+2}{2(\alpha+1)(\alpha+2)}\left|h'(\upsilon+\varrho-x)\right|\right).$$

Corollary 3.11 *In Corollary 3.10, taking $\alpha=1$, we obtain the following biparameterized Newton-type inequality for convex functions*:

$$\left|\frac{\lambda(x-\upsilon)}{\varrho-\upsilon}h(\upsilon)+\frac{2(1-\lambda)(x-\upsilon)+\upsilon+\varrho-2x}{2(\varrho-\upsilon)}\big(h(x)+h(\upsilon+\varrho-x)\big)\right.$$
$$\left.+\frac{\lambda(x-\upsilon)}{\varrho-\upsilon}h(\varrho)-\frac{1}{\varrho-\upsilon}\int_{\upsilon}^{\varrho}h(z)\,dz\right|$$
$$\leq\frac{(x-\upsilon)^2}{\varrho-\upsilon}\left(\frac{1-3\lambda+6\lambda^2-2\lambda^3}{6}\left|h'(\upsilon)\right|+\frac{2-3\lambda+2\lambda^3}{6}\left|h'(x)\right|\right.$$
$$\left.+\frac{2-3\lambda+2\lambda^3}{6}\left|h'(\upsilon+\varrho-x)\right|+\frac{1-3\lambda+6\lambda^2-2\lambda^3}{6}\left|h'(\varrho)\right|\right)$$
$$+\frac{(\upsilon+\varrho-2x)^2}{12(\varrho-\upsilon)}\left(\left|h'(x)\right|+\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|+\left|h'(\upsilon+\varrho-x)\right|\right).$$

Remark 3.12 In Corollary 3.11:

- If we take $x=\upsilon$, then we obtain Corollary 2 from [4]. Moreover, if we use the convexity of $|h'|$, then we obtain Theorem 2.2 from [3].
- If we take $x=\frac{\upsilon+\varrho}{2}$ and $\lambda=0$, then we obtain Corollary 2 (for $q=1$) from [6]. Moreover, if we use the convexity of $|h'|$, then we obtain Theorem 2.2 from [5].
- If we take $x=\frac{\upsilon+\varrho}{2}$ and $\lambda=\frac{1}{3}$, then using the convexity of $|h'|$, we obtain Theorem 5 from [13].
- If we take $x=\frac{\upsilon+\varrho}{2}$ and $\lambda=\frac{1}{2}$, then using the convexity of $|h'|$, we obtain Corollary 3.2 from [12].
- If we take $\lambda=0$, then using the convexity of $|h'|$, we obtain Theorem 5 from [2].

In the following theorems, we will not provide specific cases, as these are left to the reader's curiosity.

Theorem 3.13 *Let $h:[\upsilon,\varrho]\to\mathbb{R}$ be a differentiable function on $[\upsilon,\varrho]$ such that $h'\in L^1[\upsilon,\varrho]$ with $0\leq\upsilon<\varrho$. If $|h'|^q$ is α-convex for $q>1$ and $p>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then we have*

$$\left|\mathcal{P}(\upsilon,x,\varrho;\delta;\lambda)-\Lambda(\upsilon,x,\varrho;\delta)\right|$$
$$\leq\frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\Omega^{\frac{1}{p}}\left\{\left(\frac{\alpha}{\alpha+1}\left|h'(\upsilon)\right|^q+\frac{1}{\alpha+1}\left|h'(x)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\alpha}{\alpha+1}\left|h'(\upsilon+\varrho-x)\right|^q+\frac{1}{\alpha+1}\left|h'(\varrho)\right|^q\right)^{\frac{1}{q}}\right\}$$
$$+\frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)(p\delta+1)^{1/p}}\left\{\left(\frac{\alpha}{\alpha+1}\left|h'(x)\right|^q+\frac{1}{\alpha+1}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\alpha}{\alpha+1}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q+\frac{1}{\alpha+1}\left|h'(\upsilon+\varrho-x)\right|^q\right)^{\frac{1}{q}}\right\},$$

where $\mathcal{P}$ *and* Λ *are defined by* (1) *and* (2), *respectively, and*

$$\Omega = \frac{\lambda^{p+\frac{1}{\delta}}}{\delta} B\left(\frac{1}{\delta}, p+1\right) + \frac{(1-\lambda)^{p+1}}{\delta(p+1)} {}_2F_1\left(1-\frac{1}{\delta}, 1, p+2; 1-\lambda\right), \tag{15}$$

with $B(\cdot,\cdot,\cdot)$ *and* ${}_2F_1(\cdot,\cdot,\cdot;\cdot)$ *are the beta and hypergeometric functions, respectively.*

Proof From modulus applied to Lemma 3.1, using the Hölder inequality, we have

$$\begin{aligned}
&\left|\mathcal{P}(\upsilon,x,\varrho;\delta;\lambda) - \Lambda(\upsilon,x,\varrho;\delta)\right| \\
&\leq \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\left(\left(\int_0^1 \left|t^\delta - \lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|h'\left((1-t)\upsilon + tx\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 \left|(1-t)^\delta - \lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|h'\left((1-t)(\upsilon+\varrho-x) + t\varrho\right)\right|^q dt\right)^{\frac{1}{q}}\right) \\
&\quad + \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)}\left(\left(\int_0^1 (1-t)^{\delta p}\, dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|h'\left((1-t)x + t\frac{\upsilon+\varrho}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 t^{\delta p}\, dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|h'\left((1-t)\frac{\upsilon+\varrho}{2} + t(\upsilon+\varrho-x)\right)\right|^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Using the α-convexity of $|h'|^q$, we get

$$\begin{aligned}
&\left|\mathcal{P}(\upsilon,x,\varrho;\delta;\lambda) - \Lambda(\upsilon,x,\varrho;\delta)\right| \\
&\leq \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\Omega^{\frac{1}{p}}\left(\left(\int_0^1 \left[(1-t^\alpha)\left|h'(\upsilon)\right|^q + t^\alpha\left|h'(x)\right|^q\right] dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 \left[(1-t^\alpha)\left|h'(\upsilon+\varrho-x)\right|^q + t^\alpha\left|h'(\varrho)\right|^q\right] dt\right)^{\frac{1}{q}}\right) \\
&\quad + \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)(p\delta+1)^{1/p}}\left(\left(\int_0^1 \left[\left(1-t^\alpha\right)\left|h'(x)\right)\right|^q + t^\alpha\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q\right] dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 \left[\left(1-t^\alpha\right)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q + t^\alpha\left|h'(\upsilon+\varrho-x)\right|^q\right] dt\right)^{\frac{1}{q}}\right) \\
&= \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\Omega^{\frac{1}{p}}\left\{\left(\frac{\alpha}{\alpha+1}\left|h'(\upsilon)\right|^q + \frac{1}{\alpha+1}\left|h'(x)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{\alpha}{\alpha+1}\left|h'(\upsilon+\varrho-x)\right|^q + \frac{1}{\alpha+1}\left|h'(\varrho)\right|^q\right)^{\frac{1}{q}}\right\} \\
&\quad + \frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)(p\delta+1)^{1/p}}\left\{\left(\frac{\alpha}{\alpha+1}\left|h'(x)\right|^q + \frac{1}{\alpha+1}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{\alpha}{\alpha+1}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q + \frac{1}{\alpha+1}\left|h'(\upsilon+\varrho-x)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\Omega = \int_0^1 \left|(1-t)^\delta - \lambda\right|^p dt = \int_0^1 \left|t^\delta - \lambda\right|^p dt$$

$$
\begin{aligned}
&= \int_0^{\lambda^{\frac{1}{\delta}}} \left(\lambda - t^{\delta}\right)^p dt + \int_{\lambda^{\frac{1}{\delta}}}^1 \left(t^{\delta} - \lambda\right)^p dt \\
&= \frac{1}{\delta}\left(\int_0^{\lambda} (\lambda - u)^p u^{\frac{1}{\delta}-1} du + \int_{\lambda}^1 (u - \lambda)^p u^{\frac{1}{\delta}-1} du\right) \\
&= \frac{\lambda^{p+\frac{1}{\delta}}}{\delta} \int_0^1 (1 - v)^p v^{\frac{1}{\delta}-1} dv + \frac{(1-\lambda)^{p+1}}{\delta} \int_0^1 (1 - v)^p \left(1 - (1-\lambda)v\right)^{\frac{1}{\delta}-1} dv \\
&= \frac{\lambda^{p+\frac{1}{\delta}}}{\delta} B\left(\frac{1}{\delta}, p+1\right) + \frac{(1-\lambda)^{p+1}}{\delta} B(1, p+1) {}_2F_1\left(1 - \frac{1}{\delta}, 1, p+2; 1-\lambda\right) \\
&= \frac{\lambda^{p+\frac{1}{\delta}}}{\delta} B\left(\frac{1}{\delta}, p+1\right) + \frac{(1-\lambda)^{p+1}}{\delta(p+1)} {}_2F_1\left(1 - \frac{1}{\delta}, 1, p+2; 1-\lambda\right).
\end{aligned}
$$

The proof is complete. □

Theorem 3.14 *Let $h : [\upsilon, \varrho] \to \mathbb{R}$ be a differentiable function on $[\upsilon, \varrho]$ such that $h' \in L^1[\upsilon, \varrho]$ with $0 \le \upsilon < \varrho$. If $|h'|^q$ is α-convex for $q > 1$, then we have*

$$
\begin{aligned}
&\left|\mathcal{P}(\upsilon, x, \varrho; \delta; \lambda) - \Lambda(\upsilon, x, \varrho; \delta)\right| \\
&\le \frac{(x-\upsilon)^{\delta+1}}{\varrho - \upsilon} \mathcal{C}_1^{1-\frac{1}{q}}(\lambda, \delta)\Big[\left(\left(\mathcal{C}_1(\lambda, \delta) - \mathcal{C}_2(\lambda, \delta, \alpha)\right)\left|h'(\upsilon)\right|^q + \mathcal{C}_2(\lambda, \delta, \alpha)\left|h'(x)\right|^q\right)^{\frac{1}{q}} \\
&\quad + \left(\left(\mathcal{C}_1(\lambda, \delta) - \mathcal{C}_3(\lambda, \delta, \alpha)\right)\left|h'(\upsilon + \varrho - x)\right|^q + \mathcal{C}_3(\lambda, \delta, \alpha)\left|h'(\varrho)\right|^q\right)^{\frac{1}{q}}\Big] \\
&\quad + \frac{(\upsilon + \varrho - 2x)^{\delta+1}}{2^{\delta+1}(\varrho - \upsilon)(\delta+1)^{1-\frac{1}{q}}} \\
&\quad \times \left[\left(\left(\frac{1}{\delta+1} - B(\delta+1, \alpha+1)\right)\left|h'(x)\right|^q + B(\delta+1, \alpha+1)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{\alpha}{(\delta+1)(\alpha+\delta+1)}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q + \frac{1}{\alpha+\delta+1}\left|h'(\upsilon+\varrho-x)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where $B(\cdot,\cdot)$ and $B(\cdot;\cdot,\cdot)$ are the beta and incomplete beta functions, respectively, and $\mathcal{C}_i$ $(i = 1, 2, 3)$ as defined in (8), (9), and (10).

Proof From modulus applied to Lemma 3.1, the power mean inequality, and the α-convexity of $|h'|^q$ we have

$$
\begin{aligned}
&\left|\mathcal{P}(\upsilon, x, \varrho; \delta; \lambda) - \Lambda(\upsilon, x, \varrho; \delta)\right| \\
&\le \frac{(x-\upsilon)^{\delta+1}}{\varrho - \upsilon}\left(\left(\int_0^1 \left|t^{\delta} - \lambda\right| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|t^{\delta} - \lambda\right|\left|h'\left((1-t)\upsilon + tx\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 \left|(1-t)^{\delta} - \lambda\right| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|(1-t)^{\delta} - \lambda\right|\left|h'\left((1-t)(\upsilon + \varrho - x) + t\varrho\right)\right|^q dt\right)^{\frac{1}{q}}\right) \\
&\quad + \frac{(\upsilon + \varrho - 2x)^{\delta+1}}{2^{\delta+1}(\varrho - \upsilon)} \\
&\quad \times \left(\left(\int_0^1 (1-t)^{\delta} dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^{\delta}\left|h'\left((1-t)x + t\frac{\upsilon+\varrho}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_0^1 t^\delta\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\delta\left|h'\left((1-t)\frac{\upsilon+\varrho}{2}+t(\upsilon+\varrho-x)\right)\right|dt\right)^{\frac{1}{q}}\Bigg)\\
\leq{}& \frac{(x-\upsilon)^{\delta+1}}{\varrho-\upsilon}\mathcal{C}_1^{1-\frac{1}{q}}(\lambda,\delta)\Big[\big((\mathcal{C}_1(\lambda,\delta)-\mathcal{C}_2(\lambda,\delta,\alpha))\big|h'(\upsilon)\big|^q+\mathcal{C}_2(\lambda,\delta,\alpha)\big|h'(x)\big|^q\big)^{\frac{1}{q}}\\
&+\big((\mathcal{C}_1(\lambda,\delta)-\mathcal{C}_3(\lambda,\delta,\alpha))\big|h'(\upsilon+\varrho-x)\big|^q+\mathcal{C}_3(\lambda,\delta,\alpha)\big|h'(\varrho)\big|^q\big)^{\frac{1}{q}}\Big]\\
&+\frac{(\upsilon+\varrho-2x)^{\delta+1}}{2^{\delta+1}(\varrho-\upsilon)(\delta+1)^{1-\frac{1}{q}}}\\
&\times\left[\left(\left(\frac{1}{\delta+1}-B(\delta+1,\alpha+1)\right)\big|h'(x)\big|^q+B(\delta+1,\alpha+1)\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{\alpha}{(\delta+1)(\alpha+\delta+1)}\left|h'\left(\frac{\upsilon+\varrho}{2}\right)\right|^q+\frac{1}{\alpha+\delta+1}\big|h'(\upsilon+\varrho-x)\big|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Here we have used (11)–(14) and the fact that

$$\int_0^1 \big|t^\delta-\lambda\big|\,dt=\int_0^1\big|(1-t)^\delta-\lambda\big|\,dt=\mathcal{C}_1(\lambda,\delta)$$

and

$$\int_0^1 (1-t)^\delta\,dt=\int_0^1 t^\delta\,dt=\frac{1}{\delta+1}.$$

The proof is complete. □

4 Examples and applications

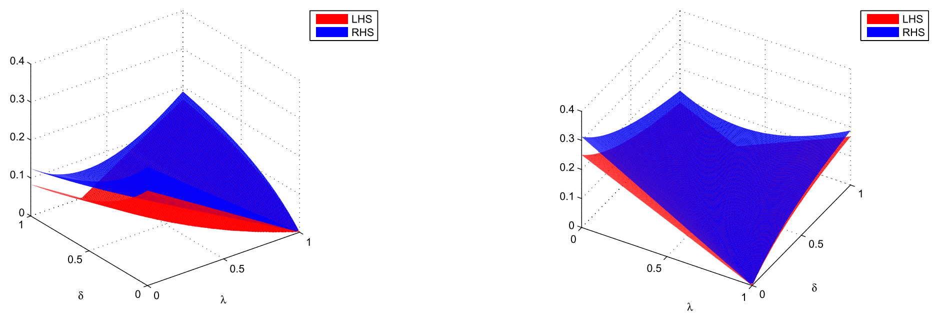
4.1 Illustrative examples

In this section, we presnt several two- and three-dimensional graphical representations to corroborate the accuracy of the results obtained in our study.

Note that the graphical representations in this section were created using Matlab software, and in all the figures, the red color is used to depict the left-hand side (*LHS*), whereas blue is used for the right-hand side (*RHS*), consistently across all results.

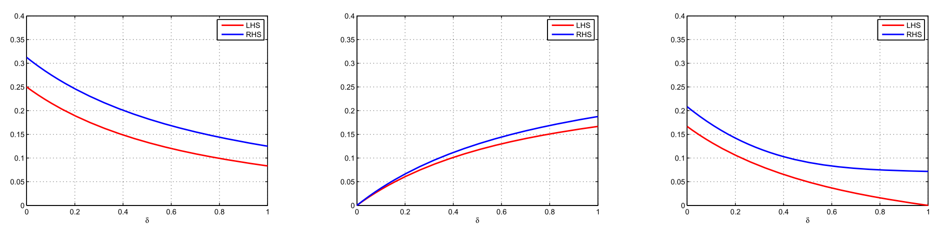
Example 4.1 In this particular example, we initiate the analysis by establishing the parameter values of $x=\frac{\upsilon+\varrho}{2}=\frac{1}{2}$ and $\alpha=1$, and subsequently exhibit the outcomes as functions of the remaining parameters λ and δ. The function h defined over the interval $[0,1]$ as $h(u)=u^2$ is considered for this purpose. Accordingly, $|h'(u)|=2u$ is a convex function over $[0,1]$, and hence Corollary 3.5 gives

$$
\begin{aligned}
&\left|\frac{\lambda}{2}h(0)+(1-\lambda)h\left(\frac{1}{2}\right)+\frac{\lambda}{2}h(1)-\Lambda\left(0,\frac{1}{2},1;\delta\right)\right|\\
&=\left|\frac{1-\lambda}{4}+\frac{\lambda}{2}-\frac{\delta(\delta+1)+(\delta+2)(\delta+3)+2}{2^3(\delta+1)(\delta+2)}\right|\\
&\leq\frac{1}{4}\left(2\left(\frac{1}{(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}\lambda^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}\lambda^{\frac{\delta+2}{\delta}}-\frac{\lambda}{2}\right)\right.\\
&\left.\quad+\frac{1}{2}\left(\frac{1}{\delta+2}+\frac{\delta}{\delta+2}\lambda^{1+\frac{2}{\delta}}-\frac{\lambda}{2}\right)\right).
\end{aligned}
$$



(A) View No. 1. (B) View No. 2.

Figure 1 The comparative analysis between the left and right sides of Corollary 3.5



(A) Case 1 (B) Case 2 (C) Case 3

Figure 2 Particular cases

The surface plot of this result is depicted in Fig. 1.

Some particular cases:

- By fixing $\lambda = 0$ we obtain the curve related to the fractional midpoint rule with respect to δ, as shown in Fig. 2 (Case 1).
- By fixing $\lambda = 1$ we obtain the curve related to the fractional trapezoidal rule with respect to δ, as shown in Fig. 2 (Case 2).
- By fixing $\lambda = \frac{1}{3}$ we obtain the curve related to the fractional Simpson rule with respect to δ, as shown in Fig. 2 (Case 3).

Example 4.2 In this example, we focus on the classical four-point Newton-type inequalities. Specifically, we fix $x = \frac{2\upsilon+\varrho}{3} = \frac{1}{3}$ and $\delta = 1$, and represent the resultant output with respect to λ and α. To accomplish this, we consider the function h defined over the interval $[0,1]$ as $h(u) = \frac{u^{\alpha+1}}{\alpha+1}$ with $0 < \alpha \leq 1$, whose derivative $|h'(u)| = u^{\alpha}$ is α-convex. Thus by Corollary 3.10 we obtain

$$\begin{aligned}
&\left|\frac{\lambda}{3}h(0) + \frac{3-2\lambda}{6}\left(h\left(\frac{1}{3}\right) + h\left(\frac{2}{3}\right)\right) + \frac{\lambda}{3}h(1) - \int_0^1 h(z)\,dz\right| \\
&= \left|\frac{3-2\lambda}{6(\alpha+1)}\left(\frac{1+2^{\alpha+1}}{3^{\alpha+1}}\right) + \frac{\lambda}{3(\alpha+1)} - \frac{1}{(\alpha+1)(\alpha+2)}\right| \\
&\leq \frac{1}{9}\left(\frac{1}{3^{\alpha}}\left(\frac{2\lambda^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{1}{\alpha+2} - \frac{\lambda}{\alpha+1}\right)\right. \\
&\quad \left. + \frac{2^{\alpha}}{3^{\alpha}}\left(\frac{2\lambda^2-2\lambda+1}{2} - \frac{2(1-\lambda)^{\alpha+2}+\lambda(\alpha+2)-1}{(\alpha+1)(\alpha+2)}\right) + \frac{2(1-\lambda)^{\alpha+2}+\lambda(\alpha+2)-1}{(\alpha+1)(\alpha+2)}\right)
\end{aligned}$$

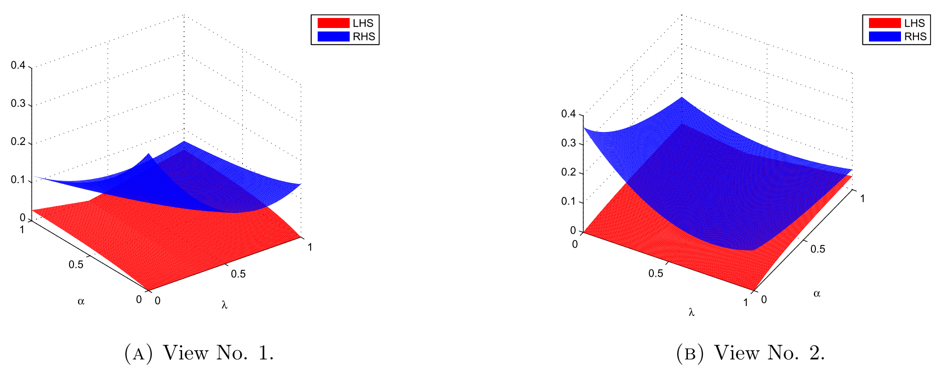
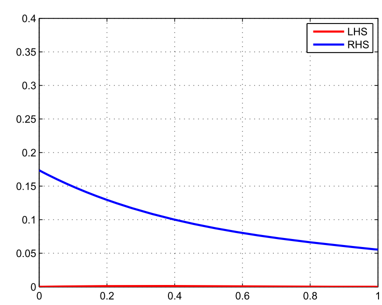


Figure 3 The comparative analysis between the left and right sides of Corollary 3.10

Figure 4 Particular case



$$+\frac{1}{36}\left(\frac{1}{3^{\alpha}}\left(\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\right)+\frac{1}{2^{\alpha}(\alpha+2)}+\frac{2^{\alpha}}{3^{\alpha}}\left(\frac{\alpha^2+\alpha+2}{2(\alpha+1)(\alpha+2)}\right)\right).$$

The surface plot of this result is depicted in Fig. 3.

Particular case:

- By fixing $\lambda = \frac{3}{8}$ we obtain the curve related to the Simpson-type 3/8 inequality via α-convexity, as shown in Fig. 4.

The graphical representation of the results of these examples demonstrates the accuracy of the findings obtained in this study.

4.2 Applications

For arbitrary real numbers $a_1, a_2, \dots, a_n$, we have:

The arithmetic mean $A(a_1, a_2, \dots, a_n) = \frac{a_1+a_2+\cdots+a_n}{n}$.

The harmonic mean $H(a_1, a_2, \dots, a_n) = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$, $a_1, a_2, \dots, a_n > 0$.

The geometric mean $G(a_1, a_2, \dots, a_n) = \sqrt[n]{a_1 a_2 \dots a_n}$

The logarithmic mean $L(a_1, a_2) = \frac{a_2-a_1}{\ln a_2 - \ln a_1}$, $a_1, a_2 > 0$, $a_1 \neq a_2$.

The p-logarithmic mean $L_p(a_1, a_2) = (\frac{a_2^{p+1}-a_1^{p+1}}{(p+1)(a_2-a_1)})^{\frac{1}{p}}$, $a_1 \neq a_2$, $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 4.3 *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then we have*

$$\begin{aligned}&\left|H(a,a,a,b)+H(a,b,b,b)-G^2(a,b)L^{-1}(a,b)\right|\\&\quad\leq\frac{b-a}{48ab}\left(\frac{a^2+b^2}{2}+2\left(\left(\frac{4ab}{3a+b}\right)^2+\left(\frac{4ab}{a+3b}\right)^2\right)+\left(\frac{2ab}{a+b}\right)^2\right).\end{aligned}$$

Proof The claim is a consequence of the second inequality in Corollary 3.7 applied to the function $h(z) = \frac{1}{z}$ on the interval $[\frac{1}{b}, \frac{1}{a}]$ with $x = \frac{3a+b}{4ab}$. □

Proposition 4.4 *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then we have*

$$\left|A\left(a^2, b^2\right) - L_2^2(a,b)\right| \le \frac{(b-a)\sqrt[3]{10}}{10}\left(\left(a^3 + \left(\frac{a+b}{2}\right)^3\right)^{\frac{1}{3}} + \left(\left(\frac{a+b}{2}\right)^3 + b^3\right)^{\frac{1}{3}}\right).$$

Proof The statement can be derived from Theorem 3.13 by setting $\alpha = \delta = 1$ and applying it to the function $h(z) = \frac{1}{2}z^2$ on the interval $[a, b]$ with $q = 3$ and $x = a$. □

5 Conclusions

We have investigated a class of fractional biparameterized four-point Newton-type inequalities via α-convexity. By introducing a general identity involving Riemann–Liouville integral operators we have derived several new results that complement and extend the existing literature on this topic. Our contributions provide a broad range of tools that can be used in diverse fields of science and engineering. Furthermore, our findings can potentially have practical implications in numerical integration, optimization, and other related areas.

Note that from a purely practical standpoint, this result is used to choose the value of λ and possibly that of x to achieve the minimum of the right-hand side, which are obviously specific to the function h.

Finally, we hope that this work will inspire further research in this direction and lead to new applications and developments.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The authors A.L and T. A prepared the original draft. The authors W.S and B. M edited the article. The authors W.S, A.L, T.A and B.M disscussed the mathematical results and validated them. All authors confirmed the last version.

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