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Schatten class operators on exponential weighted Bergman spaces

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Abstract

In this paper, we study Toeplitz and Hankel operators on exponential weighted Bergman spaces. For $0 < p < \infty$, we obtain sufficient and necessary conditions for Toeplitz and Hankel operators to belong to Schatten-p class by the averaging functions of symbols. For a continuous increasing convex function h, the Schatten-h class Toeplitz and Hankel operators are also characterized.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $dA(z) = \frac{dxdy}{\pi}$ be the normalized Lebesgue area measure on $\mathbb{D}$. Let $\mathcal{L}$ denote a class (see [2, 13] for more details about the class). A function $\rho(z)$ is said to be in $\mathcal{L}$ if $\rho(z)$ is positive on $\mathbb{D}$ satisfying the following conditions:

(a) For any $z \in \mathbb{D}$, there is a constant $c_1 > 0$ such that $\rho(z) \le c_1(1 - |z|)$.

(b) There is a constant $c_2 > 0$ such that $|\rho(z) - \rho(w)| \le c_2|z - w|$, where $z, w \in \mathbb{D}$.

Write $A \lesssim B$ for two quantities A and B if there is a constant $C > 0$ such that $A \le CB$. Furthermore, $A \asymp B$ means that both $A \lesssim B$ and $B \lesssim A$ are satisfied. A subharmonic function $\varphi(z) \in C^2(\mathbb{D})$ satisfying $(\Delta\varphi(z))^{-1/2} \asymp \rho(z)$ is called $\varphi \in \mathcal{L}^*$, where $\rho(z) \in \mathcal{L}$ and Δ is the standard Laplace operator.

The Lebesgue space L^p_φ $(0 < p < \infty)$ consists of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{\varphi,p} = \left(\int_{\mathbb{D}} \left|f(z)e^{-\varphi(z)}\right|^p dA(z)\right)^{1/p} < \infty.$$

In particular, L^∞_φ consists of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{\varphi,\infty} = \operatorname{esssup}_{z\in\mathbb{D}} \left|f(z)e^{-\varphi(z)}\right| < \infty.$$

Now let $\mathcal{H}(\mathbb{D})$ be the space of analytic functions in the unit disk $\mathbb{D}$. The exponential weighted Bergman spaces $A^p_\varphi = L^p_\varphi \cap \mathcal{H}(\mathbb{D})$. When $1 \le p \le \infty$, A^p_φ is a Banach space, and A^p_φ is a Fréchet space if $0 < p < 1$.

Notice that A^2_φ is a reproducing kernel Hilbert space, and hence there is a function $K_{\varphi,z} \in A^2_\varphi$ such that the orthogonal projection P from L^2_φ to A^2_φ can be represented as

$$P(f)(z) = \int_{\mathbb{D}} f(w)\overline{K_{\varphi,z}(w)}e^{-2\varphi(w)}\,dA(w), \quad z \in \mathbb{D}.$$

See [3, 13]. The function $K_{\varphi,z}(\cdot)$ is called the reproducing kernel of Bergman space A^2_φ and has the property that $K_{\varphi,z}(w) = \overline{K_{\varphi,w}(z)}$ for every $z, w \in \mathbb{D}$. It follows from [3, Theorems 4.1 and 4.2] that, for $\varphi \in \mathcal{E}$ and $1 \leq p \leq \infty$, the Bergman projection $P : L^p_\varphi \to A^p_\varphi$ is bounded.

For a positive Borel measure μ on $\mathbb{D}$ and a measurable function f, the Toeplitz operator and Hankel operator are defined respectively by

$$T_\mu(g)(z) = \int_{\mathbb{D}} g(w)K_\varphi(z,w)e^{-2\varphi(w)}\,d\mu(w), \quad g \in A^p_\varphi$$

and

$$H_f(g)(z) = \int_{\mathbb{D}} \big(f(z)g(w) - f(w)g(w)\big)K_\varphi(z,w)e^{-2\varphi(w)}\,dA(w), \quad g \in A^p_\varphi.$$

The pioneering work on this class of exponential weighted Bergman spaces was done by Oleinik and Perelman [14]. Throughout this paper, we call these spaces $\mathcal{OPS}$. Later, has attracted much attention. In [12], Lin and Rochberg characterized the boundedness and compactness of Hankel operators on exponential weighted Bergman spaces. To further study these spaces, Lin and Rochberg [13] gave the necessary and sufficient conditions for Schatten-p class Toeplitz (or Hankel) operators when $1 \leq p < \infty$. Furthermore, for $0 < p < 1$, the sufficient condition for Schatten class membership of the Toeplitz operator was obtained as well. In [3, 4], Arroussi and Pau studied the dual space and estimates of the reproducing kernel.

Borichev, Dhuez, and Kellay [5] introduced another exponential weighted Bergman spaces. The authors, in [2], showed the Schatten class membership of the Toeplitz operator on spaces introduced by [5]. Hu, Lv, and Schuster [8] characterized a new kind of space, which contains these exponential weighted Bergman spaces considered in [5], write $\mathcal{HLS}$ for simplicity. Indeed, the spaces $\mathcal{HLS}$ differ from the spaces in this paper, see [8]. In [9], Hu and Pau gave bounded and compact Hankel operators associated with general symbols. Zhang, Wang and Hu [17] showed the boundedness and compactness of Toeplitz operators with positive symbols acting between different spaces $\mathcal{HLS}$, and Schatten-p class membership. Recently, in [16], the authors studied the sufficient and necessary conditions for Schatten-p class membership of Hankel operators associated with general symbols on $\mathcal{HLS}$.

For $0 < p < \infty$, by using averaging functions, we obtain the sufficient and necessary conditions for Schatten-p class membership of Toeplitz operators with positive symbols and Hankel operators with general symbols on $\mathcal{OPS}$. These results fill the research gap of [13]. Generally speaking, the difficulty in such problems lies in the characterization of $0 < p < 1$. For this goal, we need more tools than [13]. Schatten-h class membership of operators is an important generalization of Schatten-p class operators, and it is interesting to study

Schatten-h class membership. We refer to [1] and the relevant references therein for a brief account on Schatten-h class. In this paper, we explore Schatten-h class Toeplitz and Hankel operators on the spaces. Such properties of Hankel operators are not yet known in the existing literature.

By [8, Theorem 3.2], the following estimate holds for the reproducing kernel in this space: there exist constants $C, \sigma > 0$ such that

$$\left|K(z,w)\right| \le C\frac{e^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)}e^{-\sigma\, d_\rho(z,w)}, \quad z,w \in \mathbb{D},$$

where $d_\rho(z,w)$ is the Bergman metric induced by reproducing kernel. However, the reproducing kernel in $\mathcal{OPS}$ does not have the similar estimate, which brings more obstacles to the research in this paper.

The paper is organized as follows. In Sect. 2, we give some basic notation and lemmas. In Sect. 3, we show the sufficient and necessary conditions for Schatten-p class membership of Toeplitz operators with positive symbols, and give the characterization for Schatten-h class membership of Toeplitz operators induced by continuous increasing convex functions. Finally, in Sect. 4, we investigate membership in Schatten-p class Hankel operators with general symbols, and also obtain Schatten-h class properties of Hankel operators.

2 Preliminaries

We begin with giving some basic notation and lemmas. For $z \in \mathbb{D}$ and $r > 0$, let $D(z,r) = \{w : |w-z| < r\}$ be the Euclidean disk with radius r and center z. Also, we use $D^r(z) = D(z, r\rho(z))$ to denote the disk with radius $r\rho(z)$ and center z.

The following lemma is from [3, (2.1)].

Lemma 2.1 *Suppose $\rho \in \mathcal{L}$, $z \in \mathbb{D}$ and $w \in D^\alpha(z)$, where $0 < \alpha < m_\rho = \frac{\min\{1, c_1^{-1}, c_2^{-1}\}}{4}$. Then*

$$\frac{1}{2}\rho(w) < \rho(z) < 2\rho(w). \tag{2.1}$$

It is from [3, Lemma A] that we have the following pointwise estimate.

Lemma 2.2 *Suppose $\varphi \in \mathcal{L}^*$, $0 < p < \infty$, $\beta \in \mathbb{R}$ and $z \in \mathbb{D}$. Then there exists a constant $M \ge 1$, for $f \in \mathcal{H}(\mathbb{D})$ and small enough $\delta > 0$, such that*

$$\left|f(z)\right|^p e^{-\beta\varphi(z)} \le \frac{M}{\delta^2\rho(z)^2}\int_{D^\delta(z)}\left|f(\zeta)\right|^p e^{-\beta\varphi(\zeta)}\,dA(\zeta). \tag{2.2}$$

As we known, the covering lemma is useful for studying Bergman spaces, so does exponential weighted Bergman spaces. The following lemma comes from [2, Lemma B].

Lemma 2.3 *Suppose $\rho \in \mathcal{L}$ and $0 < r < m_\rho$. Then there exists a sequence $\{a_j\}_{j=1}^\infty \subseteq \mathbb{D}$ satisfying*

(a) *$a_j \notin D^r(a_k)$, $k \neq j$.*

(b) *$\mathbb{D} = \bigcup_{j=1}^\infty D^r(a_j)$.*

(c) *$\tilde{D}^r(a_j) \subseteq D^{3r}(a_j)$, where $\tilde{D}^r(a_j) = \bigcup_{z\in D^r(a_j)} D^r(z)$.*

(d) $\{D^{3r}(a_j)\}_{j=1}^{\infty}$ *is a covering of* $\mathbb{D}$ *of finite multiplicity, that is, for any* $z \in \mathbb{D}$,

$$1 \le \sum_{j=1}^{\infty} \chi_{D^{3r}(a_j)}(z) \le N, \tag{2.3}$$

where N *is a positive constant integer.*

A sequence $\{a_j\}_{j=1}^{\infty}$ satisfying the above lemma is called the (ρ, r)-lattice. Furthermore, the conditions (a) and (c) indicate there is a $s > 0$ such that

$$D^{sr}(a_j) \cap D^{sr}(a_k) = \emptyset, \quad j \neq k.$$

It is important to investigate pointwise and norm estimates of the reproducing kernels $K_{\varphi,z}$ on A^2_{φ}. The following results are from [3, Lemma B, Theorem 3.1 and (3.1)].

If $\varphi \in \mathcal{L}^*$, $0 < r < m_{\rho}$ and $w \in D^r(z)$, then we have

$$\left|K_{\varphi,z}(w)\right| \asymp \|K_{\varphi,z}\|_{\varphi,2}\|K_{\varphi,w}\|_{\varphi,2}. \tag{2.4}$$

Lemma 2.4 *Suppose* $\varphi \in \mathcal{L}^*$ *and function* ρ *satisfies that, if there exist* $b > 0$ *and* $0 < t < 1$, *for* $z, w \in \mathbb{D}$ *and* $|z - w| > b\rho(w)$, *such that*

$$\rho(z) \le \rho(w) + t|z - w|,$$

then

$$\|K_{\varphi,z}\|^2_{\varphi,2} \asymp e^{2\varphi(z)}\rho^{-2}(z). \tag{2.5}$$

Definition 2.5 The weight $\varphi \in \mathcal{L}^*$ is called $\varphi \in \mathcal{E}$ if the function ρ satisfies, for any $m \ge 1$, there exist constants $b_m > 0$ and $0 < t_m < 1/m$, when $|z - w| > b_m\rho(w)$, such that

$$\rho(z) \le \rho(w) + t_m|z - w|.$$

Theorem 2.6 *If* $\varphi \in \mathcal{E}$, *then for any* $M \ge 1$ *there is a constant* $C > 0$ *such that*

$$\left|K_{\varphi,w}(z)\right| \le Ce^{\varphi(z)}e^{\varphi(w)}\frac{1}{\rho(z)}\frac{1}{\rho(w)}\left(\frac{\min\{\rho(z), \rho(w)\}}{|z - w|}\right)^M, \quad z, w \in \mathbb{D}. \tag{2.6}$$

Proof See [3, Theorem 3.1]. □

With the help of estimates for the reproducing kernels, we get the following atomic decomposition.

Lemma 2.7 *Suppose* $\varphi \in \mathcal{E}$ *and* $\{a_j\}_{j=1}^{\infty}$ *is a* (ρ, r)-*lattice, where* $0 < r \le m_{\rho}$. *Then, if* $\{\lambda_j\}_{j=1}^{\infty} \in l^2$, *we have* $F(z) = \sum_{j=1}^{\infty}\lambda_j k_{\varphi,a_j}(z) \in A^2_{\varphi}$ *and*

$$\left\|\sum_{j=1}^{\infty}\lambda_j k_{\varphi,a_j}\right\|_{\varphi,2} \le C\left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{l^2},$$

where $k_{\varphi,w}(z) = \frac{K_{\varphi}(z,w)}{\|K_{\varphi,w}\|_{\varphi,2}}$ *is called normalized reproducing kernel.*

Proof By (2.5) and Hölder's inequality, we have

$$\begin{aligned}\|F(z)\|_{\varphi,2}^2 &\lesssim \int_{\mathbb{D}} \left(\sum_{j=1}^{\infty} |\lambda_j| e^{-\varphi(a_j)} \rho(a_j) \left|K_{\varphi,a_j}(z)\right|\right)^2 e^{-2\varphi(z)}\, dA(z) \\ &\lesssim \int_{\mathbb{D}} \left(\sum_{j=1}^{\infty} |\lambda_j|^2 e^{-\varphi(a_j)} \left|K_{\varphi,a_j}(z)\right|\right) M(z) e^{-2\varphi(z)}\, dA(z), \end{aligned} \tag{2.7}$$

where

$$M(z) = \sum_{j=1}^{\infty} \rho(a_j)^2 \left|K_{\varphi,a_j}(z)\right| e^{-\varphi(a_j)}.$$

It follows from (2.2), (2.5), and [3, Lemma 3.3] that

$$M(z) \lesssim \sum_{j=1}^{\infty} \int_{D^r(a_j)} \left|K_{\varphi,z}(w)\right| e^{-\varphi(w)}\, dA(w) \lesssim \int_{\mathbb{D}} \left|K_{\varphi,z}(w)\right| e^{-\varphi(w)}\, dA(w) \lesssim e^{\varphi(z)}. \tag{2.8}$$

This together with (2.7), (2.8), and (2.5) implies that

$$\begin{aligned}\|F(z)\|_{\varphi,2}^2 &\lesssim \int_{\mathbb{D}} \left(\sum_{j=1}^{\infty} |\lambda_j|^2 e^{-\varphi(a_j)} \left|K_{\varphi,a_j}(z)\right|\right) e^{-\varphi(z)}\, dA(z) \\ &\lesssim \sum_{j=1}^{\infty} |\lambda_j|^2 e^{-\varphi(a_j)} \int_{\mathbb{D}} \left|K_{\varphi,a_j}(z)\right| e^{-\varphi(z)}\, dA(z) \\ &\lesssim \sum_{j=1}^{\infty} |\lambda_j|^2 = \left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{l^2}^2, \end{aligned}$$

which ends the proof. □

To describe the Schatten-p membership of Hankel operators, we need some auxiliary conclusions. For $z, w \in \mathbb{D}$, we write

$$d_\rho(z,w) = \frac{|z-w|}{\min(\rho(z), \rho(w))}.$$

Lemma 2.8 ([2, Lemma 4.4]) *Let $\rho \in \mathcal{L}$ and $\{a_j\}_j$ be a (ρ, r)-lattice on $\mathbb{D}$. Then for any $w \in \mathbb{D}$, the set*

$$D_m(w) = \left\{z \in \mathbb{D} \mid d_\rho(z,w) < 2^m r\right\}$$

contains at most K points of the lattice, where K depends on the positive integer m, but not on the point w.

Lemma 2.9 ([2, Lemma 4.5]) *Let $\rho \in \mathcal{L}$, $r \in (0, m_\rho]$ and $k \in \mathbb{N}^+$. Any (ρ,r)-lattice $\{a_j\}_{j=1}^{\infty}$ on $\mathbb{D}$, can be partitioned into M subsequences such that, if a_i and a_j are different points in the same subsequence, then $|a_i - a_j| \geq 2^m r \min\{\rho(a_i), \rho(a_j)\}$.*

Given a positive Borel measure μ on $\mathbb{D}$ and $r > 0$, the averaging function $\hat{\mu}_r$ with respect to measure μ is defined by

$$\widehat{\mu}_r(z) = \frac{\int_{D^r(z)} d\mu}{|D^r(z)|}.$$

Lemma 2.10 *If μ is a positive Borel measure, $0 < p < \infty$ and $r \in (0, m_\rho]$, then*

$$\int_{\mathbb{D}} \left|g(z)e^{-\varphi(z)}\right|^p d\mu(z) \lesssim \int_{\mathbb{D}} \left|g(z)e^{-\varphi(z)}\right|^p \widehat{\mu}_r(z)\, dA(z), \tag{2.9}$$

where $g \in \mathcal{H}(\mathbb{D})$.

Proof See [7, Lemma 2.4]. □

3 Schatten class Toeplitz operators

In this section, for $0 < p < \infty$, we investigate the sufficient and necessary conditions for Schatten-p class membership of Toeplitz operators with positive measure symbols on $\mathcal{OPS}$. Also, we give the characterization for Schatten-h class membership of Toeplitz operators where h is a continuous increasing convex function.

Let $T : H_1 \to H_2$ be a bounded linear operator, and write $s_j(T)$ for the singular values of T, where

$$s_j(T) = \inf\left\{\|T - K\| : K : H_1 \to H_2,\ \text{rank}\,(K) \le j\right\}.$$

Here rank(K) means the rank of operator K. Recall that the operator T is compact if and only if $s_j(T) \to 0$ whenever $j \to \infty$. For $0 < p < \infty$, it is called T is in S_p if

$$\|T\|_{S_p}^p = \sum_{j=1}^{\infty} s_j(T)^p < \infty,$$

and we write $T \in S_p(H_1, H_2)$. Futhermore, $\|\cdot\|_{S_p}$ is actually a norm when $1 \le p < \infty$ and $\|\cdot\|_{S_p}$ is not, if $0 < p < 1$.

Using

$$\|S + T\|_{S_p} \le \|S\|_{S_p} + \|T\|_{S_p}, \quad 1 \le p < \infty, \tag{3.1}$$

and

$$\|S + T\|_{S_p}^p \le \|S\|_{S_p}^p + \|T\|_{S_p}^p, \quad 0 < p < 1, \tag{3.2}$$

it is easy to see $T \in S_p$ if and only if $T^*T \in S_{\frac{p}{2}}$.

As we known, the Schatten class of Toeplitz operators with positive measure symbols is an important problem in operator theory, which has been described in many papers (see, for example, [2, 13, 17]). The following theorem is closely related to the main result [2, Theorem 1.2]. To Study the Schatten class of Toeplitz operators, we define the measure $d\lambda_\rho$ by

$$d\lambda_\rho(z) = \frac{dA(z)}{\rho(z)^2}, \quad z \in \mathbb{D}.$$

Theorem 3.1 *Suppose $\varphi \in \mathcal{E}$, $0 < p < \infty$, and μ is a finite positive Borel measure on $\mathbb{D}$. Then following statements are equivalent*:

(a) $T_\mu \in \mathcal{S}_p(A^2_\varphi)$.

(b) $\widehat{\mu}_\delta \in L^p(\mathbb{D}, d\lambda_\rho)$, *where* $\delta \in (0, \alpha_m]$.

(c) $\{\widehat{\mu}_r(w_n)\}_n \in l^p$, *where* $\{\widehat{\mu}_r(w_n)\}_n$ *is a* (ρ, r)*-lattice with* $r \in (0, \alpha_m]$.

(d) $\widetilde{\mu} \in L^p(\mathbb{D}, d\lambda_\rho)$, *where* $\widetilde{\mu}(w) = \int_{\mathbb{D}} |k_{\varphi,w}(z)|^2\, d\mu(z)$ *is the Berezin transform of* μ.

Proof The proof of (b) ⇔ (c) ⇔ (d) is similar to [17, Proposition 2.5], and we omit the details here. Indeed, this proof indicates the L^p behavior of averaging function $\hat{\mu}_r$ is independent of r. (That is, for small enough r, $\|\hat{\mu}_\delta\|_{L^p} \asymp \|\hat{\mu}_r\|_{L^p}$ with small enough δ.) The rest part is an analogue of [17, Theorem 5.1], and for the convenience of readers, we give the proof for implication (a) ⇒ (c) when $0 < p < 1$.

Assume the Toeplitz operator T_μ is in $\mathcal{S}_p(A^2_\varphi)$. Let $\{w_n\}$ be a (ρ, r)-lattice with $r \in (0, m_\rho]$ sufficiently small. Set a large enough integer $m \geq 2$, by Lemma 2.9, the lattice $\{w_n\}$ can be devided into Γ subsequences such that

$$|w_i - w_j| \geq 2^m r \min\bigl(\rho(w_i), \rho(w_j)\bigr),$$

where w_i and w_j are in the same subsequence. Let $\{a_n\}$ be such a subsequence, and measure ν be defined by

$$d\nu = \left(\sum_n \chi_n\right) d\mu,$$

where χ_n is the characteristic function of $D^r(a_n)$. Disks $D^r(a_n)$ are pairwise disjoints since $m \geq 2$. Note that $T_\mu \in \mathcal{S}_p(A^2_\varphi)$ and $0 \leq \nu \leq \mu$, thus $0 \leq T_\nu \leq T_\mu$, and then $T_\nu \in \mathcal{S}_p(A^2_\varphi)$ and $\|T_\nu\|_{\mathcal{S}_p(A^2_\varphi)} \leq \|T_\mu\|_{\mathcal{S}_p(A^2_\varphi)}$.

Let $\{e_n\}$ be an orthonormal basis for A^2_φ. Consider an operator G on A^2_φ as

$$Gf = \sum_n \langle f, e_n\rangle_{A^2_\varphi} k_{\varphi,a_n}, \quad f \in A^2_\varphi. \tag{3.3}$$

It follows from Lemma 2.7 that G is bounded on A^2_φ, then $T = G^* T_\nu G$ is in $\mathcal{S}_p(A^2_\varphi)$ and

$$\|T\|_{\mathcal{S}_p(A^2_\varphi)} \leq \|G\|^2 \cdot \|T_\nu\|_{\mathcal{S}_p(A^2_\varphi)} \lesssim \|T_\mu\|_{\mathcal{S}_p(A^2_\varphi)}. \tag{3.4}$$

By (3.3) and

$$\langle Tf, g\rangle_{A^2_\varphi} = \langle T_\nu Gf, Gg\rangle_{A^2_\varphi}, \quad f, g \in A^2_\varphi,$$

we have

$$Tf = \sum_{n,j} \langle T_\nu k_{\varphi,a_n}, k_{\varphi,a_j}\rangle_{A^2_\varphi} \langle f, e_n\rangle_{A^2_\varphi} e_j, \quad f \in A^2_\varphi.$$

We now take a decomposition of the operator T as $T = T_1 + T_2$, where T_1 is the diagonal operator defined by

$$T_1 f = \sum_n \langle T_\nu k_{\varphi,a_n}, k_{\varphi,a_n}\rangle_{A^2_\varphi} \langle f, e_n\rangle_{A^2_\varphi} e_n, \quad f \in A^2_\varphi,$$

and $T_2 = T - T_1$ is the non-diagonal part. Using Rotfel'd inequality (see [15]), we see

$$\|T\|^p_{\mathcal{S}_p(A^2_\varphi)} \geq \|T_1\|^p_{\mathcal{S}_p(A^2_\varphi)} - \|T_2\|^p_{\mathcal{S}_p(A^2_\varphi)}. \tag{3.5}$$

Notice that T_1 is a positive diagonal operator, this together with the definition of ν, (2.1), (2.4), and (2.5) gives

$$\begin{aligned}\|T_1\|^p_{\mathcal{S}_p(A^2_\varphi)} &= \sum_n \langle T_\nu k_{\varphi,a_n}, k_{\varphi,a_n}\rangle^p_{A^2_\varphi} = \sum_n \left(\int_{\mathbb{D}} \left|k_{\varphi,a_n}(z)\right|^2 e^{-2\varphi(z)}\,d\nu(z)\right)^p \\ &\gtrsim \sum_n \left(\int_{D^r(a_n)} \frac{1}{\rho(z)^2}\,d\mu(z)\right)^p \gtrsim \sum_n \widehat{\mu}_r(a_n)^p.\end{aligned} \tag{3.6}$$

For $0 < p < 1$, [18, Proposition 1.29] and Lemma 2.3 show

$$\begin{aligned}\|T_2\|^p_{\mathcal{S}_p(A^2_\varphi)} &\leq \sum_n \sum_k \langle T_2 e_n, e_k\rangle^p_{A^2_\varphi} = \sum_{k\neq n} \langle T_\nu k_{\varphi,a_n}, k_{\varphi,a_k}\rangle^p_{A^2_\varphi} \\ &\leq \sum_{k\neq n} \left(\int_{\mathbb{D}} \left|k_{\varphi,a_n}(\xi)\right|\left|k_{\varphi,a_k}(\xi)\right| e^{-2\varphi(\xi)}\,d\nu(\xi)\right)^p \\ &\leq \sum_{k\neq n} \left(\sum_j \int_{D^r(a_j)} \left|k_{\varphi,a_n}(\xi)\right|\left|k_{\varphi,a_k}(\xi)\right| e^{-2\varphi(\xi)}\,d\mu(\xi)\right)^p.\end{aligned} \tag{3.7}$$

If $n \neq k$, then $|a_n - a_k| \geq 2^m r \min(\rho(a_n), \rho(a_k))$. Hence, for $\xi \in D^r(a_j)$, we get either

$$|a_n - \xi| \geq 2^{m-2} r \min\big(\rho(a_n), \rho(\xi)\big) \quad \text{or} \quad |\xi - a_k| \geq 2^{m-2} r \min\big(\rho(\xi), \rho(a_k)\big).$$

Therefore, for any $\xi \in D^r(a_j)$, we may assume $|a_n - \xi| \geq 2^{m-2} r \min(\rho(a_n), \rho(\xi))$.

For any $n, k \in \mathbb{N}^+$, set

$$J_{nk}(\mu) = \sum_j \int_{D^r(a_j)} \left|k_{\varphi,a_n}(\xi)\right|\left|k_{\varphi,a_k}(\xi)\right| e^{-2\varphi(\xi)}\,d\mu(\xi).$$

This, combined with (3.7), yields

$$\|T_2\|^p_{\mathcal{S}_p(A^2_\varphi)} \leq \sum_{n,k:k\neq n} J_{nk}(\mu)^p. \tag{3.8}$$

Let M be large enough. Here M is from Theorem 2.6. Applying $|a_n - \xi| \geq 2^{m-2} r \times \min(\rho(a_n), \rho(\xi))$, we have

$$\left|k_{a_n}(\xi)\right| e^{-\varphi(\xi)} \lesssim \frac{1}{\rho(\xi)} \left(\frac{\min(\rho(a_n), \rho(\xi))}{|a_n - \xi|}\right)^M \lesssim \frac{1}{\rho(\xi)} 2^{-Mm}.$$

And hence,

$$\left|k_{\varphi,a_n}(\xi)\right| = \left|k_{\varphi,a_n}(\xi)\right|^{1/2}\left|k_{\varphi,a_n}(\xi)\right|^{1/2} \lesssim 2^{-Mm/2} \frac{e^{\varphi(\xi)/2}}{\rho(\xi)^{1/2}} \left|k_{\varphi,a_n}(\xi)\right|^{1/2}. \tag{3.9}$$

It follows from (2.4), (2.5), and (2.6) that

$$\left|k_{\varphi,a_k}(\xi)\right| = \frac{|K_\varphi(\xi,a_k)|^{1/2}}{\|K_{\varphi,a_k}\|_{A^2_\varphi}^{1/2}}\left|k_{\varphi,a_k}(\xi)\right|^{1/2} \lesssim \frac{e^{\varphi(\xi)/2}}{\rho(\xi)^{1/2}}\left|k_{\varphi,a_k}(\xi)\right|^{1/2}. \tag{3.10}$$

By joining (3.9), (3.10), and Lemma 2.1, we obtain

$$J_{nk}(\mu) \lesssim 2^{\frac{-Mm}{2}} \sum_j \frac{1}{\rho(a_j)} \int_{D^r(a_j)} \left|k_{\varphi,a_n}(\xi)\right|^{1/2}\left|k_{\varphi,a_k}(\xi)\right|^{1/2} e^{-\varphi(\xi)}\, d\mu(\xi).$$

Applying Lemmas 2.1, 2.2, and 2.3 (c), for $\xi \in D^r(a_j)$, we conclude

$$\left|k_{\varphi,a_n}(\xi)\right|^{1/2} e^{-\varphi(\xi)/2} \lesssim \left(\frac{1}{\rho(\xi)^2}\int_{D^r(\xi)} \left|k_{\varphi,a_n}(z)\right|^{p/2} e^{-p\varphi(z)/2}\, dA(z)\right)^{1/p} \lesssim \rho(a_j)^{-2/p} S_n(a_j)^{1/p},$$

where

$$S_n(\cdot) = \int_{D^{3\alpha}(\cdot)} \left|k_{\varphi,a_n}(z)\right|^{p/2} e^{-p\varphi(z)/2}\, dA(z).$$

The analogous reasons indicate

$$\left|k_{\varphi,a_k}(\xi)\right|^{1/2} e^{-\varphi(\xi)/2} \lesssim \rho(a_j)^{-2/p} S_k(a_j)^{1/p}.$$

So, for M large enough, we have

$$\begin{aligned} J_{nk}(\mu) &\lesssim 2^{-Mm/2} \sum_j \frac{\rho(a_j)^{-4/p}}{\rho(a_j)} S_n(a_j)^{1/p} S_k(a_j)^{1/p} \mu\big(D^r(a_j)\big) \\ &\le 2^{-m} \sum_j \rho(a_j)^{1-4/p} S_n(a_j)^{1/p} S_k(a_j)^{1/p} \widehat{\mu}_r(a_j). \end{aligned}$$

And hence, for $0 < p < 1$,

$$J_{nk}(\mu)^p \lesssim 2^{-mp} \sum_j \rho(a_j)^{p-4} S_n(a_j) S_k(a_j) \widehat{\mu}_r(a_j)^p.$$

Now (3.8) can be estimated further as

$$\|T_2\|^p_{S_p(A^2_\varphi)} \lesssim 2^{-mp} \sum_j \rho(a_j)^{p-4} \widehat{\mu}_r(a_j)^p \left(\sum_n S_n(a_j)\right)\left(\sum_k S_k(a_j)\right). \tag{3.11}$$

On the other hand, by the definition of $S_k(a_j)$, we see

$$\sum_k S_k(a_j) = \int_{D^{3\alpha}(a_j)} \left(\sum_k \left|k_{\varphi,a_k}(z)\right|^{p/2}\right) e^{-p\varphi(z)/2}\, dA(z). \tag{3.12}$$

We claim that

$$\sum_k \left|k_{\varphi,a_k}(z)\right|^{p/2} \lesssim e^{p\varphi(z)/2} \rho(z)^{-p/2}. \tag{3.13}$$

For this goal, by (2.4), (2.1), and Lemma 2.3 (d), for some $r_0 > 0$, we get

$$\sum_{a_k \in D^{r_0}(z)} \left|k_{\varphi,a_k}(z)\right|^{p/2} \lesssim e^{p\varphi(z)/2} \sum_{a_k \in D^{r_0}(z)} \rho(a_k)^{-p/2} \lesssim e^{p\varphi(z)/2} \rho(z)^{-p/2}. \tag{3.14}$$

Taking M in Theorem 2.6 such that $Mp/2 - 2 > 0$, then

$$\begin{aligned}\sum_{a_k \notin D^{r_0}(z)} \left|k_{\varphi,a_k}(z)\right|^{p/2} &\lesssim e^{p\varphi(z)/2} \rho(z)^{Mp/2-p/2-2} \sum_{a_k \notin D^{r_0}(z)} \frac{\rho(a_k)^2}{|z-a_k|^{Mp/2}} \\ &= e^{p\varphi(z)/2} \rho(z)^{Mp/2-p/2-2} \sum_{j=0}^{\infty} \sum_{a_k \in R_j(z)} \frac{\rho(a_k)^2}{|z-a_k|^{Mp/2}},\end{aligned}$$

where

$$R_j(z) = \left\{\zeta \in \mathbb{D} : 2^j r_0 \rho(z) \le |\zeta - z| < 2^{j+1} r_0 \rho(z)\right\}, \quad j = 0, 1, 2 \ldots.$$

By Lemma 2.3, for $j = 0, 1, 2, \ldots$, when $a_k \in D^{r_0 2^{j+1}}(z)$, we obtain

$$D^{r_0}(a_k) \subset D^{20 r_0 2^j}(z).$$

So

$$\sum_{a_k \in R_j(z)} \rho(a_k)^2 \lesssim \left|D^{20 r_0 2^j}(z)\right| \lesssim 2^{2j} \rho(z)^2,$$

and hence (3.13) holds by (3.14) and the following estimate

$$\begin{aligned}\sum_{a_k \notin D^{r_0}(z)} \left|k_{\varphi,a_k}(z)\right|^{p/2} &\lesssim e^{p\varphi(z)/2} \rho(z)^{-p/2-2} \sum_{j=0}^{\infty} 2^{-Mpj/2} \sum_{a_k \in R_j(z)} \rho(a_k)^2 \\ &\lesssim e^{p\varphi(z)/2} \rho(z)^{-p/2} \sum_{j=0}^{\infty} 2^{\frac{(4-Mp)}{2} j} \lesssim e^{p\varphi(z)/2} \rho(z)^{-p/2}.\end{aligned}$$

Bearing in mind (3.13), (3.12) can be estimated as

$$\sum_k S_k(a_j) \lesssim \rho(a_j)^{2-p/2}. \tag{3.15}$$

Similarly,

$$\sum_n S_n(a_j) \lesssim \rho(a_j)^{2-p/2}. \tag{3.16}$$

By joining (3.15), (3.16), and (3.11), for integer $m > 0$ large enough, we get

$$\|T_2\|^p_{S_p(A^2_\varphi)} \lesssim 2^{-mp} \sum_j \widehat{\mu}_r(a_j)^p \le \frac{1}{2} \sum_j \widehat{\mu}_r(a_j)^p.$$

This together with (3.6) and (3.5) yields

$$\sum_j \widehat{\mu}_r(a_j)^p \lesssim \|T\|^p_{\mathcal{S}_p(A^2_\varphi)}.$$

Since the above estimate holds for each of the Γ subsequences $\{w_n\}$, we finally obtain

$$\sum_n \widehat{\mu}_r(w_n)^p \lesssim M\|T\|^p_{\mathcal{S}_p(A^2_\varphi)} \lesssim M\|T_\mu\|^p_{\mathcal{S}_p(A^2_\varphi)} < \infty$$

by (3.4), which finishes this proof. □

We are going to describe the Schatten-h class Toeplitz operators. See [1] and the references therein for details about the Schatten-h class. We give first the following analogous definition.

Definition 3.2 Let T be a compact operator and $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing convex function. We say that $T \in S_h$ if there is a positive constant c such that

$$\sum_{j=1}^{\infty} h\big(c \cdot s_j(T)\big) < \infty.$$

Similar to [1], we get the following consequence.

Theorem 3.3 *Suppose $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing convex function, and μ is a positive Borel measure such that Toeplitz operator $T_\mu : A^2_\varphi \to A^2_\varphi$ is compact. Then $T_\mu \in S_h$ if and only if there exists a constant $c > 0$ such that*

$$\int_{\mathbb{D}} h\big(c\tilde{\mu}(z)\big)\rho^{-2}(z)\,dA(z) < \infty.$$

Proof Assume that $T_\mu \in S_h$. Then there exists $c > 0$ such that

$$\sum_{j=1}^{\infty} h\big(cs_j(T)\big) < \infty.$$

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for A^2_φ, and

$$T_\mu f = \sum_{k=1}^{\infty} s_k \langle f, e_k\rangle_{A^2_\varphi} e_k,$$

where s_k is the singular value sequence of T_μ. With the help of

$$\sum_{k=1}^{\infty} \big|\langle K_{\varphi,z}, e_k\rangle_{A^2_\varphi}\big|^2 = 1,$$

the convexity of h, Jensen's inequality, (2.4), and (2.5), we have

$$\begin{aligned}
\int_{\mathbb{D}} h\big(c\widetilde{\mu}(z)\big)\rho^{-2}(z)\,dA(z) &= \int_{\mathbb{D}} h\big(c\langle T_\mu k_{\varphi,z}, k_{\varphi,z}\rangle_{A^2_\varphi}\big)\rho^{-2}(z)\,dA(z)\\
&= \int_{\mathbb{D}} h\left(\sum_{k=1}^{\infty} cs_k\big|\langle k_{\varphi,z}, e_k\rangle_{A^2_\varphi}\big|^2\right)\rho^{-2}(z)\,dA(z)\\
&\le \int_{\mathbb{D}} \sum_{k=1}^{\infty} h(cs_k)\big|\langle k_{\varphi,z}, e_k\rangle_{A^2_\varphi}\big|^2\rho^{-2}(z)\,dA(z)\\
&= \int_{\mathbb{D}} \sum_{k=1}^{\infty} h(cs_k)\|K_{\varphi,z}\|^{-2}_{\varphi,2}\big|e_k(z)\big|^2\rho^{-2}(z)\,dA(z)\\
&\lesssim \sum_{k=1}^{\infty} h(cs_k)\int_{\mathbb{D}}\big|e_k(z)\big|^2 e^{-2\varphi(z)}\,dA(z)\\
&= \sum_{k=1}^{\infty} h(cs_k) < \infty.
\end{aligned}$$

Conversely, if there exists $c > 0$ such that $\int_{\mathbb{D}} h(c\tilde{\mu}(z))\rho^{-2}(z)\,dA(z) < \infty$, then it follows from (2.4) and (2.5) that

$$\begin{aligned}
\widehat{\mu}_r(z) &= \int_{D^r(z)} \rho^{-2}(z)\,d\mu(w)\\
&\asymp \int_{D^r(z)} \big|k_{\varphi,z}(w)\big|^2 e^{-2\varphi(w)}\,d\mu(w) \le \widetilde{\mu}(z).
\end{aligned}$$

Notice that

$$\begin{aligned}
\langle T_\mu e_k, e_k\rangle_{A^2_\varphi} &= \int_{\mathbb{D}} \big|e_k(z)\big|^2 e^{-2\varphi(z)}\,d\mu(z)\\
&\lesssim \int_{\mathbb{D}} \widehat{\mu}_r(z)\big|e_k(z)\big|^2 e^{-2\varphi(z)}\,dA(z)\\
&\lesssim \int_{\mathbb{D}} \widetilde{\mu}(z)\big|e_k(z)\big|^2 e^{-2\varphi(z)}\,dA(z),
\end{aligned}$$

then by Jensen's inequality again we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} h\big(c\langle T_\mu e_k, e_k\rangle_{A^2_\varphi}\big)\\
&\quad\le \int_{\mathbb{D}} h\big(c\widetilde{\mu}(z)\big)\left(\sum_{k=1}^{\infty}\big|e_k(z)\big|^2\right)e^{-2\varphi(z)}\,dA(z)\\
&\quad= \int_{\mathbb{D}} h\big(c\widetilde{\mu}(z)\big)\|K_{\varphi,z}\|^2_{\varphi,2}e^{-2\varphi(z)}\,dA(z)\\
&\quad= \int_{\mathbb{D}} h\big(c\widetilde{\mu}(z)\big)\rho^{-2}(z)\,dA(z) < \infty,
\end{aligned}$$

which gives $T_\mu \in S_h$. This completes the proof. □

4 Schatten class Hankel operators

This section devotes to studying membership in Schatten ideals of Hankel operators with general symbols. First, when $0 < p < \infty$, we get the sufficient and necessary conditions for Hankel operators are in Schatten-p class. Here we mainly discuss case $0 < p < 1$, see case $1 \le p < \infty$ in [13]. Next, for a continuous increasing convex function h, we obtain the sufficient and necessary conditions for Hankel operators to be in Schatten-h class. This kind of problem is new for Hankel operators.

Lemma 4.1 *If A and B are bounded linear operators, $p \in (0,1)$, then*

$$\|AB\|_{S_p}^p \le ||B||^p||A||_{S_p}^p \quad \text{and} \quad ||AB||_{S_p}^p \le ||A||^p||B||_{S_p}^p. \tag{4.1}$$

Proof See [6]. □

Let $L^2_{\text{loc}}(\mathbb{D})$ denote the space consists of locally square integrable Lebesgue measurable functions on $\mathbb{D}$. If $f \in L^2_{\text{loc}}(\mathbb{D})$ and $z \in \mathbb{D}$, $G_r(f)(z)$ is defined by

$$G_r(f)(z) = \inf\left\{\left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f-h|^2\,dA\right)^{1/2} : h \in \mathcal{H}\big(D^r(z)\big)\right\},$$

where $\mathcal{H}(D^r(z))$ is the analytic functions space on $D^r(z)$. For $z \in \mathbb{D}$, $f \in L^2(D^r(z), dA)$ and $r > 0$, the averaging function of $|f|$ on $D^r(z)$ is defined by

$$M_r(f)(z) = \left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f|^2\,dA\right)^{1/2}.$$

Indeed, $M_r(f)(z) = (\widehat{|f|^2}_r)^{1/2}$.

Lemma 4.2 *For $z \in \mathbb{D}$, $f \in L^2(D^r(z), dA)$, and $r > 0$, there exists an $h \in \mathcal{H}(D^r(z))$ such that*

$$M_r(f-h)(z) = G_r(f)(z). \tag{4.2}$$

Proof The proof is similar to [11, Lemma 3.3]. □

For $z \in \mathbb{D}$ and $r > 0$, let

$$A^2\big(D^r(z), dA\big) = L^2\big(D^r(z), dA\big) \cap \mathcal{H}\big(D^r(z)\big)$$

denote the Bergman space on $D^r(z)$. Let $B_{z,r}$ denote Bergman projection induced by the reproducing kernel of $A^2(D^r(z), dA)$. As we known, $B_{z,r}$ is bounded and $B_{z,r}h = h$, where $h \in A^2(D^r(z), dA)$. The following consequence is similar to [11, Lemma 3.4] with $q = 2$.

Lemma 4.3 *For $z \in \mathbb{D}$ and $r > 0$, if $f \in L^2(D^r(z), dA)$, then we have*

$$M_r\big(f - B_{z,r}(f)\big)(z) \asymp G_r(f)(z). \tag{4.3}$$

Proof Taking h from Lemma 4.2, we have $h \in A^2(D^r(z), dA)$ since $f \in L^2_{\mathrm{loc}}(\mathbb{D})$. Then $B_{z,r}h = h$. By trigonometric inequality and Lemma 4.2,

$$\begin{aligned} M_r\big(f - B_{z,r}(f)\big)(z) &\le M_r(f-h)(z) + M_r\big(h - B_{z,r}(f)\big)(z) \\ &= M_r(f-h)(z) + M_r\big(B_{z,r}(h-f)\big)(z) \\ &\lesssim M_r(f-h)(z) = G_r(f)(z). \end{aligned}$$

It is obvious that $G_r(f)(z) \le M_r(f - B_{z,r}(f))(z)$, and hence this proof is complete. □

Given $r > 0$, let $\{a_j\}_{j=1}^{\infty}$ be a $(\rho, r/3)$-lattice, $J_z = \{j : z \in D^r(a_j)\}$, and $|J_z|$ be the number of elements of J_z. By (2.3), $1 \le |J_z| \le N$. Let $\{\psi_j\}_{j=1}^{\infty}$ denote the unit decomposition induced by $\{D^{r/3}(a_j)\}_{j=1}^{\infty}$, that is,

$$\psi_j \in C^\infty(\mathbb{D}), \quad \operatorname{supp}\psi_j \subseteq D^{r/3}(a_j), \qquad |\bar\partial\psi_j| \le C\rho(a_j)^{-1}, \qquad \sum_{j=1}^{\infty}\psi_j = 1, \quad \psi_j \ge 0.$$

By (2.1), it is easy to see

$$\rho(z)\big|\bar\partial\psi_j(z)\big| \le C, \quad \text{for any } j = 1, 2, \dots \text{ and } z \in \mathbb{D}.$$

Given $f \in L^2_{\mathrm{loc}}(\mathbb{D})$, for $j = 1, 2, \dots$, taking $h_j \in \mathcal{H}(D^r(a_j))$ in Lemma 4.2 such that

$$M_r(f - h_j) = G_r(f)(a_j).$$

Definition 4.4 By the decomposition above, we define

$$f_1 = \sum_{j=1}^{\infty} h_j\psi_j \quad \text{and} \quad f_2 = f - f_1. \tag{4.4}$$

Note that $f_1(z)$ is actually a finite summation for any $z \in \mathbb{D}$, and by $\operatorname{supp}\psi_j \subseteq D^{r/3}(a_j) \subseteq D^r(a_j)$, then f_1 is well-defined.

Lemma 4.5 *Let $f \in L^2_{\mathrm{loc}}(\mathbb{D})$ and $r > 0$. By (4.4), f admits a decomposition $f = f_1 + f_2$. Then $f_1 \in C^1(\mathbb{D})$ and*

$$\big|\rho(z)\bar\partial f_1(z)\big| + M_{r/9}(\rho\bar\partial f_1)(z) + M_{r/9}(f_2)(z) \le CG_{9r}(f)(z), \tag{4.5}$$

where $z \in \mathbb{D}$ and $C > 0$ is independent of f.

Proof Since $h_j \in \mathcal{H}(D^r(a_j))$ and $\psi_j \in C^\infty(\mathbb{D})$, $f_1 \in C^1(\mathbb{D})$. For $z \in \mathbb{D}$, without loss of generality, we may assume $z \in D^{r/3}(a_1)$. It is easy to check that $D^{r/9}(z) \subseteq D^r(a_j)$ whenever $z \in D^{r/3}(a_j)$. By $\sum_{j=1}^{\infty}\bar\partial\psi_j(z) = 0$ and the subharmonic property of $|h_j - h_1|$ on $D^{r/9}(z) \subseteq$

$D^r(a_j)$,

$$\begin{aligned}\left|\rho(z)\bar\partial f_1(z)\right| &= \left|\sum_{j=1}^{\infty}\bigl(h_j(z)-h_1(z)\bigr)\rho(z)\bar\partial\psi_j(z)\right|\\ &\le \sum_{j=1}^{\infty}\left|h_j(z)-h_1(z)\right|\left|\rho(z)\bar\partial\psi_j(z)\right|\\ &\le C\sum_{\{j:z\in D^{r/3}(a_j)\}} M_{r/9}(h_j-h_1)(z)\\ &\le C\sum_{\{j:z\in D^{r/3}(a_j)\}}\bigl[M_{r/9}(f-h_j)(z)+M_{r/9}(f-h_1)(z)\bigr]\\ &\lesssim \sum_{\{j:z\in D^{r/3}(a_j)\}} G_r(f)(a_j).\end{aligned}$$

If $z\in D^{r/3}(a_j)$, then we have $D^r(a_j)\subseteq D^{9r}(z)$, and

$$G_r(f)(a_j)\le CG_{9r}(f)(z).$$

Hence,

$$\left|\rho(z)\bar\partial f_1(z)\right|\le CG_{9r}(f)(z). \tag{4.6}$$

If $w\in D^{r/9}(z)$, then $D^{3r}(w)\subseteq D^{9r}(z)$. Thus, similar to (4.6),

$$\begin{aligned}M_{r/9}(\rho\bar\partial f_1)(z)^2 &\le C\rho(z)^{-2}\int_{D^{r/9}(z)} G_{3r}(f)(w)^2\,dA(w)\\ &\le CG_{9r}(f)(z)^2,\end{aligned} \tag{4.7}$$

since $G_{3r}(f)(w)\le CG_{9r}(f)(z)$ for $w\in D^{r/9}(z)$.

Using Cauchy–Schwarz inequality,

$$\left|f_2(z)\right|^2\le\sum_{j=1}^{\infty}\left|f(z)-h_j(z)\right|^2\psi_j(z).$$

Therefore,

$$\begin{aligned}M_{r/9}(f_2)(z)^2 &\le \sum_{j=1}^{\infty}\frac{1}{|D^{r/9}(z)|}\int_{D^{r/9}(z)}|f-h_j|^2\psi_j\,dA\\ &\le C\sum_{\{j:z\in D^{r/3}(a_j)\}}\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)}|f-h_j|^2\,dA\\ &= C\sum_{\{j:z\in D^r(a_j)\}} G_r(f)(a_j)^2\\ &\le CG_{9r}(f)(z)^2.\end{aligned}$$

This finishes the proof. □

Lemma 4.6 *Let* $0 < p < \infty$ *and* $f \in L^2_{\mathrm{loc}}(\mathbb{D})$. *Then following statements are equivalent*:

(a) *For some (or any)* $r \le m_\rho$, $M_r(f)(z) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$.

(b) *For some (or any)* $r \le m_\rho$, $\{a_j\}_{j=1}^\infty$ *is a* (ρ,δ)*-lattice with* $\delta \le r$, *then the sequence* $\{M_\delta(f)(a_j)\}_{j=1}^\infty \in l^p$, *and*

$$\|M_r(f)\|_{L^p(\mathbb{D},\rho^{-2}\, dA)} \asymp \big\|\{M_\delta(f)(a_j)\}_{j=1}^\infty\big\|_{l^p}. \tag{4.8}$$

Proof The proof is an analogue of [17, Proposition 2.4]. □

For $z \in \mathbb{D}$ and $r > 0$, we denote $L^2(D^r(z), e^{-2\varphi}\, dA) = L^2_\varphi(D^r(z))$ and $A^2_\varphi(D^r(z)) = L^2_\varphi(D^r(z)) \cap \mathcal{H}(D^r(z))$. Let $P_{z,r} : L^2_\varphi(D^r(z)) \to A^2_\varphi(D^r(z))$ be the projection. Given $f \in L^2_\varphi(D^r(z))$, we may assume $P_{z,r}(f)(w) = 0$, when $w \in \mathbb{D} \setminus D^r(z)$, it follows that $P_{z,r}(f)$ is a natural extension on $\mathbb{D}$. If $f, g \in L^2_\varphi$, then it is easy to see $f, g \in L^2_\varphi(D^r(z))$. Then, for $f, g \in L^2_\varphi$, we have $P^2_{z,r}(f) = P_{z,r}(f)$ and $\langle f, P_{z,r}(g)\rangle = \langle P_{z,r}(f), g\rangle$. Also, if $h \in A^2_\varphi$, then $P_{z,r}(h) = \chi_{D^r(z)} h$, and hence

$$\langle h, \chi_{D^r(z)} g\rangle = \langle \chi_{D^r(z)} h, g\rangle = \big\langle P_{z,r}(h), g\big\rangle = \big\langle h, P_{z,r}(g)\big\rangle, \quad g \in L^2_\varphi.$$

Equivalently,

$$\big\langle h, \chi_{D^r(z)} g - P_{z,r}(g)\big\rangle = 0. \tag{4.9}$$

Lemma 4.7 *If* $f, g \in L^2_\varphi$, *then*

$$\big\langle f - P(f), \chi_{D^r(z)} g - P_{z,r}(g)\big\rangle = \big\langle \chi_{D^r(z)} f - P_{z,r}(f), \chi_{D^r(z)} g - P_{z,r}(g)\big\rangle.$$

Proof See [10, Lemma 5.1]. □

By [13], $H_f : A^2_\varphi \to L^2_\varphi$ is bounded if and only if $G_r(f) \in L^\infty$. In fact, $G_r(f) \in L^\infty$ is independent of r. Further, $\|G_r(f)\|_{L^\infty} \asymp \|G_\delta(f)\|_{L^\infty}$. Suppose $G_r(f) \in L^\infty$, it is from Lemma 4.5 that

$$||M_r(f_2)||_{L^\infty} \lesssim ||G_r(f)||_{L^\infty}. \tag{4.10}$$

Hence, the condition $G_r(f) \in L^\infty$ is natural in the study of Schatten class membership of Hankel operators.

Lemma 4.8 *Suppose* $\varphi \in \mathcal{E}$, $r \in (0, m_\rho]$, H_f *is densely defined satisfying* $G_r(f) \in L^\infty$ *and the decomposition* $f = f_1 + f_2$ *by Lemma* 4.5. *Then both* H_{f_1} *and* H_{f_2} *are bounded, and*

$$||H_{f_1}(g)||_{L^2_\varphi} \lesssim ||g\rho\bar\partial f_1||_{L^2_\varphi} \quad \text{and} \quad ||H_{f_2}(g)||_{L^2_\varphi} \lesssim ||f_2 g||_{L^2_\varphi}.$$

Proof See [13, Theorem 3.1]. □

Now we are ready for the characterization of Schatten class Hankel operators.

Theorem 4.9 *Suppose* $\varphi \in \mathcal{E}$, $0 < p < \infty$, $0 < r \le m_\rho$ *and* H_f *is densely defined satisfying* $G_r(f) \in L^\infty$. *Then following statements are equivalent*:

(a) *The Hankel operator H_f is in S_p.*

(b) *For some (or any) (ρ, r)-lattice $\{a_j\}_{j=1}^{\infty}$, $\{G_r(f)(a_j)\}_{j=1}^{\infty} \in l^p$.*

(c) *For some (or any) r, $G_r(f) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$.*

(d) *For some (or any) r, f admits a decomposition $f = f_1 + f_2$ such that $f_1 \in C^1(\mathbb{D})$, $M_r(\rho\bar{\partial}f_1) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$ and $M_r(f_2) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$.*

(e) *For some (or any) (ρ, r)-lattice $\{a_j\}_{j=1}^{\infty}$, f admits a decomposition $f = f_1 + f_2$ such that $f_1 \in C^1(\mathbb{D})$, $\{M_r(\rho\bar{\partial}f_1)(a_j)\}_{j=1}^{\infty} \in l^p$ and $\{M_r(f_2)(a_j)\}_{j=1}^{\infty} \in l^p$.*

Proof (a) $\Rightarrow$ (b). We give only the case $0 < p < 1$. Let $\{a_j\}_{j=1}^{\infty}$ be a (ρ, r)-lattice. By Lemma 2.9, $\{a_j\}_{j=1}^{\infty}$ can be devided into N subsequences, if a_i and a_j are in the same subsequence, then

$$|a_i - a_j| \geq 2^k r \min\bigl(\rho(a_i), \rho(a_j)\bigr). \tag{4.11}$$

In fact, just consider one of subsequences here. Without loss of generality, it is assumed that $\{a_j\}_{j=1}^{\infty}$. For any finite subset $J \subseteq \mathbb{N}^+$, let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for A^2_φ, and

$$A(g) = \sum_{j\in J} \langle g, e_j\rangle k_{\varphi,a_j}, \quad g \in A^2_\varphi.$$

Then, by Parseval's equality,

$$\sum_{j\in J} \bigl|\langle g, e_j\rangle\bigr|^2 \leq \sum_{j=1}^{\infty} \bigl|\langle g, e_j\rangle\bigr|^2 = \|g\|^2_\varphi.$$

It follows from Lemma 2.7 that A is bounded on A^2_φ.

If $\|\chi_{D^r(a_j)} g k_{\varphi,a_j} - P_{a_j,r}(g k_{\varphi,a_j})\|_{L^2_\varphi} \neq 0$, we let

$$h_j = \frac{\chi_{D^r(a_j)} f k_{\varphi,a_j} - P_{a_j,r}(f k_{\varphi,a_j})}{\|\chi_{D^r(a_j)} f k_{\varphi,a_j} - P_{a_j,r}(f k_{\varphi,a_j})\|_{L^2_\varphi}},$$

and $h_j = 0$ otherwise. It is easy to see $\|h_j\|^2_\varphi \leq 1$. Assume $D^r(a_i) \cap D^r(a_j) \neq \emptyset$, then $|a_i - a_j| \leq 3r \min\{\rho(a_i), \rho(a_j)\}$. For k large enough, we have $D^r(a_i) \cap D^r(a_j) = \emptyset$ whenever $i \neq j$. Hence, $\langle h_i, h_j\rangle = 0$ if $i \neq j$.

Let $\{c_j\}_{j\in J}$ denote nonnegative sequence, we define the operator B by

$$B(g) = \sum_{j\in J} c_j \langle g, h_j\rangle e_j.$$

It is easy to check that B is bounded on A^2_φ, and $\|B\| \leq \sup_{j\in J}\{c_j\}$. It follows that

$$\begin{aligned} BH_f A(g) &= \sum_{j\in J} c_j \bigl\langle H_f A(g), h_j\bigr\rangle e_j \\ &= \sum_{j\in J} \sum_{i\in J} c_j \langle H_f k_{\varphi,a_i}, h_j\rangle \langle g, e_i\rangle e_j. \end{aligned}$$

The application of Lemma 4.1 gives

$$\|BH_fA\|_{S_p}^p \leq ||B||^p||H_f||_{S_p}^p||A||^p \leq C\sup_{j\in J} c_j^p.$$

Taking a decomposition of the operator BH_fA as the diagonal part

$$Y(g) = \sum_{j\in J} c_j\langle H_f k_{\varphi,a_j}, h_j\rangle\langle g, e_j\rangle e_j$$

and the non-diagonal part

$$Z(g) = \sum_{j,i\in J:i\neq j} c_j\langle H_f k_{\varphi,a_i}, h_j\rangle\langle g, e_i\rangle e_j,$$

we have, by (3.2),

$$||Y||_{S_p}^p \lesssim ||BH_fA||_{S_p}^p + ||Z||_{S_p}^p.$$

By Lemma 2.4, there exists a constant $C > 0$ such that for $z \in D^r(a_j)$

$$\left|k_{\varphi,a_j}(z)\right| \geq Ce^{\varphi(z)}\rho(a_j)^{-1} > 0,$$

and hence $k_{\varphi,a_j}^{-1} \in \mathcal{H}(D^r(a_j))$. According to Lemma 4.7 and (2.5),

$$\begin{aligned}
\|Y\|_{S_p}^p &= \sum_{j\in J} c_j^p\left|\langle H_f k_{\varphi,a_j}, h_j\rangle\right|^p = \sum_{j\in J} c_j^p\left|\langle fk_{\varphi,a_j} - P(fk_{\varphi,a_j}), h_j\rangle\right|^p \\
&= \sum_{j\in J} c_j^p\left|\langle \chi_{D^r(a_j)} fk_{\varphi,a_j} - P_{a_j,r}(fk_{\varphi,a_j}), h_j\rangle\right|^p \\
&= \sum_{j\in J} c_j^p\left\|\chi_{D^r(a_j)} fk_{\varphi,a_j} - P_{a_j,r}(fk_{\varphi,a_j})\right\|_{L^2_\varphi}^p \\
&= \sum_{j\in J} c_j^p\left\{\int_{D^r(a_j)}\left|fk_{\varphi,a_j} - P_{a_j,r}(fk_{\varphi,a_j})\right|^2 e^{-2\varphi}\,dA\right\}^{p/2} \\
&= \sum_{j\in J} c_j^p\left\{\int_{D^r(a_j)}|k_{\varphi,a_j}|^2 e^{-2\varphi}\left|f - k_{\varphi,a_j}^{-1}P_{a_j,r}(fk_{\varphi,a_j})\right|^2 dA\right\}^{p/2} \\
&\asymp \sum_{j\in J} c_j^p\left\{\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)}\left|f - k_{\varphi,a_j}^{-1}P_{a_j,r}(fk_{\varphi,a_j})\right|^2 dA\right\}^{p/2} \\
&\geq \sum_{j\in J} c_j^p G_r(f)(a_j)^p.
\end{aligned}$$

This together with [18, Proposition 1.29], Lemma 4.7, and Cauchy–Schwarz inequality yields

$$\begin{aligned}
\|Z\|_{S_p}^p &\le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|\langle Ze_n, e_m\rangle\right|^p = \sum_{j,i\in J:i\neq j} c_j^p\left|\langle H_f k_{\varphi,a_i}, h_j\rangle\right|^p \\
&= \sum_{j,i\in J:i\neq j} c_j^p\left|\langle \chi_{D^r(a_j)} f k_{\varphi,a_i} - P_{a_j,r} f k_{\varphi,a_i}, h_j\rangle\right|^p \\
&\le \sum_{j,i\in J:i\neq j} c_j^p \|\chi_{D^r(a_j)} f k_{\varphi,a_i} - P_{a_j,r} f k_{\varphi,a_i}\|_{\varphi,2}^p \\
&= \sum_{j,i\in J:i\neq j} c_j^p\left\{\int_{D^r(a_j)}\left|f k_{\varphi,a_i} - P_{a_j,r}(f k_{\varphi,a_i})\right|^2 e^{-2\varphi}\,dA\right\}^{p/2} \\
&\le \sum_{j,i\in J:i\neq j} c_j^p\left\{\int_{D^r(a_j)}\left|f k_{\varphi,a_i} - k_{\varphi,a_i} B_{a_j,r}(f)\right|^2 e^{-2\varphi}\,dA\right\}^{p/2},
\end{aligned}$$

where $B_{z,r}$ is the projection from $L^2(D^r(z))$ to $A^2(D^r(z))$. Hence, by Lemma 4.3,

$$\begin{aligned}
\|Z\|_{S_p}^p &\le \sum_{j\in J} c_j^p\left\{\int_{D^r(a_j)}\sum_{i\in J:i\neq j}\left(|k_{\varphi,a_i}|^2 e^{-2\varphi}\right)\left|f - B_{a_j,r}(f)\right|^2 dA\right\}^{p/2} \\
&\lesssim \sum_{j\in J} c_j^p \sup_{z\in D^r(a_j)}\left(\sum_{i\in J:i\neq j}\left|k_{\varphi,a_i}(z)\right|^2 e^{-2\varphi(z)}\right)^{p/2}\rho(a_j)^p \\
&\quad\cdot\left\{\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)}\left|f - B_{a_j,r}(f)\right|^2 dA\right\}^{p/2} \\
&\asymp \sum_{j\in J} c_j^p G_r(f)(a_j)^p\rho(a_j)^p \sup_{z\in D^r(a_j)}\left(\sum_{i\in J:i\neq j}\left|k_{\varphi,a_i}(z)\right|^2 e^{-2\varphi(z)}\right)^{p/2}.
\end{aligned}\tag{4.12}$$

Let $i,j\in J$ and $i\neq j$. Then there exists $w_{j,i}\in\overline{D^r(a_j)}$ such that

$$|w_{j,i}-a_i| = \inf_{z\in D^r(a_j)}|z-a_i|.$$

This combined with (2.6) and (2.1), for $z\in D^r(a_j)$, implies

$$\begin{aligned}
\left|k_{\varphi,a_i}(z)\right|^2 e^{-2\varphi(z)} &\le C\frac{1}{\rho(z)}\left(\frac{\min(\rho(z),\rho(a_i))}{|z-a_i|}\right)^N\left|k_{\varphi,a_i}(z)\right|e^{-\varphi(z)} \\
&\lesssim \frac{1}{\rho(a_j)}\left(\frac{\min(2\rho(w_{j,i}),\rho(a_i))}{|w_{j,i}-a_i|}\right)^N\left|k_{\varphi,a_i}(z)\right|e^{-\varphi(z)} \\
&\lesssim \frac{1}{\rho(a_j)}\left(\frac{\min(\rho(w_{j,i}),\rho(a_i))}{|w_{j,i}-a_i|}\right)^N\left|k_{\varphi,a_i}(z)\right|e^{-\varphi(z)}.
\end{aligned}$$

We claim that $|w_{j,i}-a_i|\ge 2^{k-2}r\min(\rho(w_{j,i}),\rho(a_i))$. If not, we assume $|w_{j,i}-a_i|\le 2^{k-2}r\times\min(\rho(w_{j,i}),\rho(a_i))$. By (2.1) and the trigonometric inequality,

$$|a_j-a_i|\le|a_j-w_{j,i}|+|w_{j,i}-a_i|\le r\rho(a_j)+2^{k-2}r\rho(w_{j,i})<2^k r\rho(a_j),$$

and

$$\begin{aligned}|a_j - a_i| &\le |a_j - w_{j,i}| + |w_{j,i} - a_i| \\ &\le r\rho(a_j) + 2^{k-2} r\rho(w_{j,i}) \\ &\le 2r\rho(w_{j,i}) + 2^{k-2} r\rho(w_{j,i}) \\ &\le 4r\rho(a_i) + 2^{k-1} r\rho(a_i) < 2^k r\rho(a_i).\end{aligned}$$

So $|a_j - a_i| < 2^k r \min(\rho(a_j), \rho(a_i))$, which causes a contradiction with (4.11). Thus, for $z \in D^r(a_j)$,

$$\begin{aligned}\left|k_{\varphi,a_i}(z)\right|^2 e^{-2\varphi(z)} &\lesssim \frac{1}{\rho(a_j)}\left(\frac{1}{2^{k-2}r}\right)^N \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)} \\ &\lesssim \frac{1}{\rho(a_j)}\left(\frac{1}{2}\right)^{Nk} \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)}.\end{aligned} \tag{4.13}$$

By joining (4.12) and (4.13), we obtain

$$\|Z\|_{S_p}^p \lesssim \left(\frac{1}{2}\right)^{\frac{Npk}{2}} \sum_{j\in J} c_j^p G_r(f)(a_j)^p \rho(a_j)^{\frac{p}{2}} \sup_{z\in D^r(a_j)} \left(\sum_{i\in J: i\neq j} \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)}\right)^{p/2}. \tag{4.14}$$

Set $r_0 = 3r$. Fix $j \in J$, then, for any $z \in D^r(a_j)$, we have

$$\begin{aligned}\sum_{i\in J: i\neq j} \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)} &\le \sum_{i=1: i\neq j}^{\infty} \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)} \\ &= \sum_{\{a_i: |a_j - a_i| \le r_0\rho(a_j)\}} \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)} \\ &\quad + \sum_{n=1}^{\infty} \sum_{\{a_i: 2^n r_0\rho(a_j) < |a_j - a_i| \le 2^{n+1} r_0 \rho(a_j)\}} \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)} \\ &= I + II.\end{aligned}$$

It is from (2.1) that, for any $z \in D^r(a_j)$,

$$I = \sum_{\{a_i: |a_j - a_i| \le r_0\rho(a_j)\}} \left|k_{\varphi,a_i}(z)\right| e^{-\varphi(z)} \lesssim \sum_{\{a_i: |a_j - a_i| \le r_0\rho(a_j)\}} \frac{1}{\rho(a_i)} \lesssim \frac{1}{\rho(a_j)}. \tag{4.15}$$

When $2^n r_0 \rho(a_j) < |a_j - a_i| \le 2^{n+1} r_0 \rho(a_j)$, we get

$$\begin{aligned}|w_{j,i} - a_i| &\ge |a_j - a_i| - |a_i - w_{j,i}| \\ &> 2^n r_0 \rho(a_j) - r\rho(a_j) \\ &= \left(2^n - \frac{1}{3}\right) r_0 \rho(a_j) \ge \frac{1}{2}\cdot 2^n r_0 \rho(a_j).\end{aligned}$$

For any $z \in D^r(a_j)$,

$$\begin{aligned} II &= \sum_{n=1}^{\infty} \sum_{\{a_i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \left|k_{\varphi,a_i}(z)\right|e^{-\varphi(z)} \\ &\lesssim \sum_{n=1}^{\infty} \sum_{\{a_i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \frac{1}{\rho(z)}\left(\frac{\min(\rho(a_i),\rho(z))}{|w_{j,i}-a_i|}\right)^N \\ &\lesssim \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{nN}\rho(a_j)^{-1-N+(N-2)} \sum_{\{a_i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \rho(a_i)^2. \end{aligned}$$

It is clear that for $n = 1, 2, \dots$, if $a_i \in D^{r_0 2^{n+1}}(a_j)$, we have

$$D^{r_0}(a_i) \subseteq D^{Cr_0 2^n}(a_j).$$

Hence

$$\sum_{\{a_i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \rho(a_i)^2 \lesssim \left|D^{Cr_0 2^n}(a_j)\right| \asymp 2^{2n}\rho(a_j)^2.$$

Choose $N-2>1$ such that

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n(N-2)} \le C.$$

So, for $z \in D^r(a_j)$,

$$\sum_{n=1}^{\infty} \sum_{\{a_i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \left|k_{\varphi,a_i}(z)\right|e^{-\varphi(z)} \lesssim \frac{1}{\rho(a_j)}. \tag{4.16}$$

For any $z \in D^r(a_j)$, by (4.15) and (4.16), we see

$$\sup_{z\in D^r(a_j)}\left(\sum_{i\in J:i\neq j}\left|k_{\varphi,a_i}(z)\right|e^{-\varphi(z)}\right)^{p/2} \lesssim \left(\frac{1}{\rho(a_j)}\right)^{\frac{p}{2}}.$$

Joining (4.14) and the above estimates, we get

$$\|Z\|_{S_p}^p \lesssim \left(\frac{1}{2}\right)^{\frac{Npk}{2}} \sum_{j\in J} c_j^p G_r(f)(a_j)^p. \tag{4.17}$$

Choose k large enough such that

$$\left(\frac{1}{2}\right)^{\frac{Npk}{2}} \to 0 \quad \text{as } k\to\infty.$$

Hence, for any J,

$$\sum_{j\in J} c_j^p G_r(f)(a_j)^p \lesssim \sup_{j\in J} c_j^p.$$

Therefore (a) $\Rightarrow$ (b) since l^∞ is the dual space of l^1.

(b) ⇒ (c). Notice that $\{a_j\}_{j=1}^{\infty}$ is a $(\rho, 3r)$-lattice if $\{a_j\}_{j=1}^{\infty}$ is a (ρ, r)-lattice. Assume $\sum_{j=1}^{\infty} G_{3r}(f)(a_j)^p < \infty$. Since $z \in D^r(a_j)$, $D^r(z) \subseteq D^{3r}(a_j)$, and hence

$$\begin{aligned}\int_{\mathbb{D}} G_r(f)(z)^p \rho(z)^{-2}\, dA(z) &\le \sum_{j=1}^{\infty} \int_{D^r(a_j)} G_r(f)(z)^p \rho(z)^{-2}\, dA(z) \\ &\lesssim \sum_{j=1}^{\infty} \sup_{z \in D^r(a_j)} G_r(f)(z)^p \\ &\lesssim \sum_{j=1}^{\infty} G_{3r}(f)(a_j)^p < \infty.\end{aligned}$$

(c) ⇒ (d). Suppose $G_r(f)(z) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$ and the decomposition $f = f_1 + f_2$ is from Lemma 4.5, then $f_1 \in C^1(\mathbb{D})$ and

$$\left|\rho(z)\bar{\partial} f_1(z)\right| + M_{r/81}(\rho\bar{\partial} f_1)(z) + M_{r/81}(f_2)(z) \le C G_r(f)(z).$$

By Lemma 4.6,

$$\left\|M_r(\rho\bar{\partial} f_1)\right\|_{L^p(\mathbb{D},\rho^{-2}\, dA)} \asymp \left\|M_{r/28}(\rho\bar{\partial} f_1)\right\|_{L^p(\mathbb{D},\rho^{-2}\, dA)} \le C\left\|G_r(f)\right\|_{L^p(\mathbb{D},\rho^{-2}\, dA)} < \infty$$

and

$$\left\|M_r(f_2)\right\|_{L^p(\mathbb{D},\rho^{-2}\, dA)} \asymp \left\|M_{r/28}(f_2)\right\|_{L^p(\mathbb{D},\rho^{-2}\, dA)} \le C\left\|G_r(f)\right\|_{L^p(\mathbb{D},\rho^{-2}\, dA)} < \infty.$$

(d) ⇔ (e). See Lemma 4.6.

(d) ⇒ (a). To finish this, we let M_{f_2} and $M_{\rho\bar{\partial} f_1}$ denote multiplication operators. Let ϕ be f_2 or $\rho\bar{\partial} f_1$. By $G_r(f)(z) \in L^\infty$ and Lemma 4.6, $M_r(\phi)(z) \in L^\infty$. We next show the operator M_ϕ is bounded from A^2_φ to L^2_φ. Indeed, by Lemma 2.10 with $p = 2$, then for $g \in A^2_\varphi$ we have

$$\begin{aligned}\|M_\phi g\|^2_{\varphi,2} &= \int_{\mathbb{D}} |g|^2 e^{-2\varphi} |\phi|^2\, dA \\ &\lesssim \int_{\mathbb{D}} \left|g(z)\right|^2 e^{-2\varphi(z)} \widehat{|\phi|^2}_r(z)\, dA(z) \\ &= \int_{\mathbb{D}} \left|g(z)\right|^2 e^{-2\varphi(z)} M_r(\phi)(z)^2\, dA(z) \\ &\le ||M_r(\phi)||^2_{L^\infty} ||g||^2_{L^2_\varphi}.\end{aligned}$$

For any $g, h \in A^2_\varphi$,

$$\left\langle M^*_\phi M_\phi g, h\right\rangle = \langle M_\phi g, M_\phi h\rangle = \langle T_{|\phi|^2} g, h\rangle.$$

This gives $M^*_\phi M_\phi = T_{|\phi|^2}$ on A^2_φ. Using [18, Theorem 1.26], we get $M_\phi \in S_p$ if and only if $M^*_\phi M_\phi = T_{|\phi|^2} \in S_{p/2}$. By Theorem 3.1, $T_{|\phi|^2} \in S_{p/2}$ if and only if $\widehat{|\phi|^2}_r(z) \in L^{p/2}(\mathbb{D}, \rho^{-2}\, dA)$ if and only if $M_r(\phi)(z) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$, and so $M_\phi \in S_p$. Since $\|H_{f_1}(g)\|_{\varphi,2} \lesssim \|g\rho\bar{\partial} f_1\|_{\varphi,2}$ and $\|H_{f_2}(g)\|_{\varphi,2} \lesssim \|f_2 g\|_{\varphi,2}$, both H_{f_1} and H_{f_2} are in S_p, therefore $H_f \in S_p$. This finishes the proof. □

Theorem 4.10 *Let $\varphi \in \mathcal{E}$ and $h(\sqrt{(\cdot)}) : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing convex function. Suppose H_f is densely defined satisfying $G_r(f) \in L^\infty$. Then following statements are equivalent*:

(a) *The Hankel operator H_f is in S_h.*

(b) *For some (or any) $0 < r < m_\rho$, there exists a constant $c > 0$ such that*

$$\int_{\mathbb{D}} h\bigl(cG_r(f)(z)\bigr)\rho^{-2}(z)\,dA(z) < \infty.$$

Proof (a) $\Rightarrow$ (b). Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for A^2_φ, define

$$T_{e_j} = \frac{\chi_{D^r(a_j)} H_f(k_{\varphi,a_j})}{(\int_{D^r(a_j)} |H_f(k_{\varphi,a_j})|^2 e^{-2\varphi}\,dA)^{\frac{1}{2}}} = t_j \chi_{D^r(a_j)} H_f(k_{\varphi,a_j})$$

where $\{a_j\}$ is a $(\rho, \frac{r}{3})$-lattice. It is clear that $\|T_g\|^2_{2,\varphi} \lesssim \|g\|^2_{2,\varphi}$, and hence T is bounded. By convexity of $h(\sqrt{(\cdot)})$, $h(\cdot)$ is a convex function. Let

$$A(g) = \sum_{j=1}^\infty \langle g, e_j\rangle k_{\varphi,a_j},$$

we have

$$\begin{aligned}
&\int_{\mathbb{D}} h\bigl(cG_{\frac{r}{3}}(f)(z)\bigr)\rho^{-2}(z)\,dA(z)\\
&\quad\leq \sum_{j=1}^\infty \int_{D^r(a_j)} h\bigl(cG_{\frac{r}{3}}(f)(z)\bigr)\rho^{-2}(z)\,dA(z)\\
&\quad\leq \sum_{j=1}^\infty \sup_{z\in D^{\frac{r}{3}}(a_j)} h\bigl(cG_r(f)(z)\bigr) \lesssim \sum_{j=1}^\infty h\bigl(c_1 G_r(f)(a_j)\bigr)\\
&\quad\lesssim \sum_{j=1}^\infty h\biggl(\biggl(\frac{c_2}{|D^r(a_j)|}\int_{D^r(a_j)} \biggl|f - \frac{1}{k_{\varphi,a_j}} P(fk_{\varphi,a,j})\biggr|^2\,dA(z)\biggr)^{\frac{1}{2}}\biggr)\\
&\quad\asymp \sum_{j=1}^\infty h\biggl(\biggl(c_3\int_{D^r(a_j)} \biggl|f - \frac{1}{k_{\varphi,a_j}} P(fk_{\varphi,a_j})\biggr|^2 |k_{\varphi,a_j}|^2 e^{-2\varphi(z)}\,dA(z)\biggr)^{\frac{1}{2}}\biggr)\\
&\quad= \sum_{j=1}^\infty h\biggl(\biggl(c_3\int_{D^r(a_j)} \bigl|H_f(k_{\varphi,a_j})\bigr|^2 e^{-2\varphi(z)}\,dA(z)\biggr)^{\frac{1}{2}}\biggr)\\
&\quad= \sum_{j=1}^\infty h\bigl(c_3\bigl|t_j\langle H_f k_{\varphi,a_j}, \chi_{D^r(a_j)} H_f k_{\varphi,a_j}\rangle\bigr|\bigr) \leq \sum_{j=1}^\infty h\bigl(c_3\bigl|\langle T^* H_f A e_j, e_j\rangle\bigr|\bigr)\\
&\quad\leq \sum_{j=1}^\infty h\bigl(c_4 s_j\bigl(T^* H_f A\bigr)\bigr) \lesssim \sum_{j=1}^\infty h\bigl(c_5 s_j(H_f)\bigr) < \infty.
\end{aligned}$$

(b) ⇒ (a). Suppose $\int_{\mathbb{D}} h(cG_r(f)(z))\rho^{-2}(z)\,dA(z) < \infty$, f admits a decomposition $f = f_1 + f_2$, where

$$f_1 = \sum_{j=1}^{\infty} h_j\psi_j \quad \text{and} \quad f_2 = f - f_1.$$

Here $\{\psi_j\}_{j=1}^{\infty}$ is the unit decomposition induced by $\{D^{\frac{r}{3}}(a_j)\}_{j=1}^{\infty}$. Choose $h_j \in \mathcal{H}(D^r(a_j))$ and $f \in L^p_{\text{loc}}(\mathbb{D})$, $j = 1, 2, \ldots$, such that

$$M_r(f - h_j) = G_r(f)(a_j),$$

then $f_1 \in C^1(\mathbb{D})$ and

$$\left|\rho(z)\bar{\partial} f_1(z)\right| + M_{\frac{r}{27}}(\rho\bar{\partial} f_1)(z) + M_{\frac{r}{27}}(f_2)(z) \le cG_r(f)(z).$$

Hence

$$\begin{aligned}\int_{\mathbb{D}} h\big(M_r(\rho\bar{\partial} f_1)\big)\rho(z)^{-2}\,dA(z) &< \int_{\mathbb{D}} h\big(cM_{\frac{r}{27}}(\rho\bar{\partial} f_1)\big)\rho(z)^{-2}\,dA(z)\\ &\le \int_{\mathbb{D}} h\big(cG_r(f)(z)\big)\rho(z)^{-2}\,dA(z) < \infty\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{D}} h(M_r(f_2)\rho(z)^{-2}\,dA(z) &< \int_{\mathbb{D}} h(cM_{\frac{r}{27}}(f_2)\rho(z)^{-2}\,dA(z)\\ &\le \int_{\mathbb{D}} h\big(cG_r(f)(z)\big)\rho(z)^{-2}\,dA(z) < \infty.\end{aligned}$$

Let θ be f_2 or $\rho\bar{\partial} f_1$, and M_θ be multiplication operator. By $G_r(f)(z) \in L^\infty$, $M_r(\theta)(z) \in L^\infty$, and so the operator M_θ is bounded from A^2_φ to L^2_φ. Note that, for $g, h \in A^2_\varphi$,

$$\big\langle M^*_\theta M_\theta g, h\big\rangle = \langle M_\theta g, M_\theta h\rangle = \langle T_{|\theta|^2} g, h\rangle.$$

Hence $M^*_\theta M_\theta = T_{|\theta|^2}$. Since $M_\theta \in S_h$ if and only if $M^*_\theta M_\theta = T_{|\theta|^2} \in S_{h(\sqrt{(\cdot)})}$. According to Theorem 3.3 and the convexity of $h(\sqrt{(\cdot)})$, $T_{|\theta|^2} \in S_{h(\sqrt{(\cdot)})}$ if and only if

$$\int_{\mathbb{D}} h\Big(c\big(\widetilde{|\theta|^2}(z)\big)^{\frac{1}{2}}\Big)\rho(z)^{-2}\,dA(z) < \infty.$$

It is easy to check that $\widehat{\mu}_r(z) \le \widetilde{\mu}(z)$, and we claim that

$$\int_{\mathbb{D}} h\big(\widetilde{\mu}(z)\big)\,dA(z) \le c\int_{\mathbb{D}} h\big(c\widehat{\mu}_r(w)\big)\,dA(w).$$

By Jensen's inequality, the convexity of h and $\widetilde{\mu}(z) \le \widetilde{\widehat{\mu}_r}(z)$,

$$\begin{aligned}\int_{\mathbb{D}} h\big(\widetilde{\mu}(z)\big)\,dA(z) &\le \int_{\mathbb{D}} h\big(c\widetilde{\widehat{\mu}_r}(z)\big)\,dA(z)\\ &\le \int_{\mathbb{D}} h\biggl(c\int_{\mathbb{D}}\big|k_{\varphi,z}e^{-\varphi(w)}\big|^2\widehat{\mu}_r(w)\,dA(w)\biggr)\\ &\le \int_{\mathbb{D}}\biggl(\int_{\mathbb{D}} h\big(c\widehat{\mu}_r(w)\big)\big|k_{\varphi,z}e^{-\varphi(w)}\big|^2\,dA(w)\biggr)\,dA(z)\\ &\le \int_{\mathbb{D}} h\big(c\widehat{\mu}_r(w)\big)\,dA(w)\int_{\mathbb{D}}\big|k_{\varphi,z}e^{-\varphi(w)}\big|^2\,dA(z)\\ &\le c\int_{\mathbb{D}} h\big(c\widehat{\mu}_r(w)\big)\,dA(w).\end{aligned}$$

Recall that

$$\int_{\mathbb{D}} h\big(cM_r(\theta)(z)\big)\rho(z)^{-2}\,dA(z) < \infty,$$

thus

$$\int_{\mathbb{D}} h\big(c\big(\widehat{|\theta|^2}_r(z)\big)^{\frac{1}{2}}\big)\rho(z)^{-2}\,dA(z) < \infty,$$

and hence

$$\int_{\mathbb{D}} h\big(c\big(\widetilde{|\theta|^2}(z)\big)^{\frac{1}{2}}\big)\rho(z)^{-2}\,dA(z) < \infty.$$

So $M_\theta \in S_h$. Since $\|H_{f_1}(g)\|_{L^2_\varphi} \lesssim \|g\rho\bar{\partial}f_1\|_{L^2_\varphi}$ and $\|H_{f_2}(g)\|_{L^2_\varphi} \lesssim \|f_2 g\|_{L^2_\varphi}$, we have that both H_{f_1} and H_{f_2} are in S_h, and therefore $H_f \in S_h$. This completes the proof. □

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Declarations

Competing interests
The authors declare no competing interests.

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X. Wang and J. Xia wrote the main manuscript text and Y. Liu collected literature. All authors reviewed the manuscript.

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