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# On Rayleigh–Taylor instability in Navier–Stokes–Korteweg equations

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**Abstract**

This paper focuses on the Rayleigh–Taylor instability in the two-dimensional system of equations of nonhomogeneous incompressible viscous fluids with capillarity effects in a horizontal periodic domain with infinite height. First, we use the modified variational method to construct (linear) unstable solutions for the linearized capillary Rayleigh–Taylor problem. Then, motivated by the Grenier's idea in (Grenier in Commun. Pure Appl. Math. 53(9):1067–1091, 2000), we further construct approximate solutions with higher-order growing modes to the capillary Rayleigh–Taylor problem and derive the error estimates between both the approximate solutions and nonlinear solutions of the capillary Rayleigh–Taylor problem. Finally, we prove the existence of escape points based on the bootstrap instability method of Hwang–Guo in (Hwang and Guo in Arch. Ration. Mech. Anal. 167(3):235–253, 2003), and thus obtain the nonlinear Rayleigh–Taylor instability result. Our instability result presents that the Rayleigh–Taylor instability can occur in the fluids with capillarity effects for any capillary coefficient $\kappa > 0$ if the critical capillary coefficient is infinite. In particular, it improves the previous Zhang's result in (Zhang in J. Math. Fluid Mech. 24(3):70–23, 2022) with the assumption of smallness of the capillary coefficient.

**Keywords:** Incompressible viscous fluids with capillarity effects; Rayleigh–Taylor instability; Incompressible Navier–Stokes–Korteweg equations

## 1 Introduction

The two-dimensional (2D) motion equations of a nonhomogeneous incompressible viscous fluid with capillarity effects in the presence of a uniform gravitational field in a domain $D \subset \mathbb{R}^2$ are given as follows:

$$\begin{cases} \rho_t + V \cdot \nabla \rho = 0, \\ \rho V_t + \rho V \cdot \nabla V + \nabla P - \mu \Delta V = -\rho g e_2 - \kappa \nabla \rho \Delta \rho, \\ \operatorname{div} V = 0, \end{cases} \tag{1.1}$$

where the unknowns $\rho := \rho(x,t)$, $V := V(x,t)$ and $P := P(x,t)$ denote the density, velocity and the pressure, respectively, $\mu > 0$ stands for the coefficient of shear viscosity, $g > 0$ is the gravitational constant. The positive constant $\kappa > 0$ represents for the capillary coefficient, $e_2 := (0,1)^T$ the vertical unit vector, the superscript T the transposition and $-ge_2$

the gravitational force. Here and in what follows $x \in \mathbb{R}^2$ and $t \geq 0$ are the spacial and temporal variables, respectively. In the system (1.1), the equation $(1.1)_1$ describes the law of conservation of mass and $(1.1)_2$ the law of conservation of momentum. We call (1.1) the inhomogeneous incompressible Navier–Stokes–Korteweg equations. We mention that the general capillary tensor $K$ is written as

$$K = \big(\rho\,\mathrm{div}\big(\kappa(\rho)\nabla\rho\big) + \big(\kappa(\rho) - \rho\kappa'(\rho)\big)|\nabla\rho|^2/2\big)\mathbb{I} - \kappa(\rho)\nabla\rho\otimes\nabla\rho, \tag{1.2}$$

where $\mathbb{I}$ denotes the identity matrix. However, we assume that the capillarity function $\kappa$ is a positive constant for the sake of the simplicity, and thus $\mathrm{div}\,K = -\kappa\nabla\rho\Delta\rho$.

The well-posedness problem of Euler–Korteweg/Navier–Stokes–Korteweg equations has been widely investigated, see [4–16] and the references cited therein. In this paper, we are interested in the Rayleigh–Taylor (RT) instability problem in the inhomogeneous incompressible fluids with capillarity effects.

It is well-known that the equilibrium of a heavier fluid on top of a lighter one, subject to gravity, is unstable. In this case, the equilibrium state is unstable to sustain small disturbances, and this disturbance grow and leads to the release of potential energy, as the heavier fluid moves down under the gravitational force, and the lighter one is displaced upwards. This phenomenon was first studied by Rayleigh [17] and then Taylor [18], and thus is called the Rayleigh–Taylor (RT) instability. In the last decades, this phenomenon has been extensively investigated from mathematical, physical, and numerical aspects, see [19–24] for examples. Moreover, the RT instability also has been investigated under other physical factors, such as internal surface tension [25–29], the elasticity [22, 30–37], magnetic fields [38–44], rotation [45, 46] and so on. Next we further introduce the nonlinear RT instability results, which are closely related to our results in this paper, on the inhomogeneous incompressible fluids.

In 2003, Hwang–Guo [2] proved the existence of classical solutions of (nonlinear) RT instability in the sense of $L^2$-norm for a 2D inhomogeneous incompressible inviscid pure fluid. Then Jiang–Jiang further showed the existence of strong solutions of RT instability for the nonhomogeneous incompressible viscous pure fluids in the sense of $L^2$-norm [47]. Similar instability result was also established in the inhomogeneous incompressible magnetohydrodynamics fluids [48]. Later, Jiang–Wu–Zhong [23] also investigated the RT instability in the inhomogeneous incompressible viscoelastic fluids and surprisingly found that the elasticity can inhibit RT instability.

Recently, Zhang proved the RT instability for viscous incompressible fluids with capillarity effects for small enough capillary coefficient [3], in which the fluid domain is $(2\pi\mathbb{T})^2 \times \mathbb{R}$. Motivated by Zhang's result, we further investigate the RT instability for the system (1.1) with any given capillary coefficient by a new method, where the fluid domain is given as follows:

$$D := 2\pi\mathbb{T} \times \mathbb{R}. \tag{1.3}$$

Obviously, we automatically obtain the RT instability solutions for the three-dimensional inhomogeneous incompressible Navier–Stokes–Korteweg equations in the presence of a uniform gravitational field, defined on $(2\pi\mathbb{T})^2 \times \mathbb{R}$, if the ones for the system (1.1) defined on $(2\pi\mathbb{T})^2 \times \mathbb{R}$ are constructed, see (1.12) for the details.

Now we consider the RT equilibrium state $(\rho_0, 0, P_0)$ of the system (1.1), where the density profile $\rho_0$ is independent of $x_1$ and satisfies

$$\rho_0' \in C_0^\infty(\mathbb{R}), \qquad \inf_{x_2\in\mathbb{R}}\{\rho_0\} > 0, \qquad \inf_{x_2\in\mathbb{R}}\{\rho_0'\} \geq 0, \tag{1.4}$$

$$\rho_0'(x_2) = O\big(|x_2 - \bar{x}_{02}|^{2+l_0}\big) \quad \text{as } x_2 \to \bar{x}_{02} \text{ for a point } \bar{x}_{02} \in \mathbb{R}. \tag{1.5}$$

Here and in what follows $' := \mathrm{d}/\mathrm{d}x_2$, $l_0 > 0$ is a fixed constant and $x_2$ denotes the second component of $x \in D$. Then the corresponding equilibrium pressure can be computed out by the following relation of hydrostatics:

$$\nabla P_0 = -\rho_0 g e_2 - \kappa \nabla \rho_0 \Delta \rho_0. \tag{1.6}$$

We remark that the second condition in (1.4) prevents us from treating vacuum, while the last condition in (1.4) implies that the steady density profile has larger density with increasing height $x_2$, thus may lead to the classical RT instability, as shown below.

We now consider a perturbation around the RT equilibrium state $(\rho_0, 0, P_0)$ by

$$\sigma = \rho - \rho_0, \qquad v = V - 0, \qquad p = P - P_0, \tag{1.7}$$

then, the triple $(\sigma, v, p)$ satisfies the following equations:

$$\begin{cases} \sigma_t + \nabla(\sigma + \rho_0) \cdot v = 0, \\ (\sigma + \rho_0)v_t + (\sigma + \rho_0)v \cdot \nabla v + \nabla p - \mu \Delta v \\ \quad = -\kappa \nabla \sigma \Delta \sigma - \kappa \nabla \rho_0 \Delta \sigma - \kappa \nabla \sigma \Delta \rho_0 - \sigma g e_2, \\ \operatorname{div} v = 0. \end{cases} \tag{1.8}$$

To complete the statement of the perturbed problem, we specify the initial and boundary conditions:

$$(\sigma, v)|_{t=0} = (\sigma_0, v_0) \quad \text{in } D, \tag{1.9}$$

$$\lim_{|x|\to\infty} v = 0 \quad \text{for any } t > 0. \tag{1.10}$$

From now on, we call the initial-boundary value problem (1.8)–(1.10) the CRT problem for the sake of simplicity.

In order to study the instability of RT equilibrium state, it seems to be convenient to start with the linearized (perturbation) equations, because the linearized perturbation equations not only enable us to understand the physical and mathematical mechanisms of capillarity, but also provide a beginning for the study of the nonlinear case. Hence, we omit the nonlinear terms in (1.8) and thus get the linearized CRT equations

$$\begin{cases} \sigma_t + \rho_0' v_2 = 0, \\ \rho_0 v_t + \nabla p - \mu \Delta v = -\kappa \nabla \rho_0 \Delta \sigma - \kappa \nabla \sigma \Delta \rho_0 - \sigma g e_2, \\ \operatorname{div} v = 0. \end{cases} \tag{1.11}$$

The linearized system (1.11) with the initial-boundary conditions (1.9)–(1.10) constitutes the linearized CRT problem.

Now we state the main result of this paper.

**Theorem 1.1** *For any $\kappa > 0$, the steady state $(\rho_0, 0, P_0)$ of system (1.1) is unstable under the assumption (1.4)–(1.5), that is, there is positive constant $\varepsilon_0 > 0$, such that for any small $\delta > 0$ there exists a family of classical solutions $(\rho^\delta(t,x), V^\delta(t,x), P^\delta(t,x))$ to (1.1) such that*

$$\left\|(\rho^\delta(0,\cdot) - \rho_0(\cdot)\right\|_{H^4(D)} + \left\|V^\delta(0,\cdot)\right\|_{H^3(D)} \le \delta,$$

*but for $T^\delta = O(|\ln\delta|)$*

$$\sup_{0\le t\le T^\delta}\left\{\left\|\rho^\delta(t,\cdot) - \rho_0(\cdot)\right\|_{L^2(D)} + \left\|V^\delta(t,\cdot)\right\|_{L^2(D)}\right\} \ge \varepsilon_0.$$

The proof of Theorem 1.1 is based on the bootstrap instability method, which has its origin in Guo and Strauss's articles [49, 50]. Later, various versions of bootstrap instability approaches were established by many authors, see [51–54] for instance. Recently, Zhang used the bootstrap instability method with the energy inequality of Gronwall-type in [54] to prove the RT instability for the inhomogeneous incompressible viscous fluids with capillarity effects for small enough capillary coefficient [3]. However, in this paper we use the other version of bootstrap instability method established by [2] to prove Theorem 1.1 for any given capillary coefficient. Such bootstrap instability method was also used to investigate the nonlinear instability of Hele–Shaw flows with smooth viscous profiles by Daripa–Hwang [55]. In addition, we automatically obtain the RT instability solutions for the three-dimensional inhomogeneous incompressible Navier–Stokes–Korteweg equations defined on $(2\pi\mathbb{T})^2 \times \mathbb{R}$. In fact, let

$$\varrho^\delta(t,x_0,x_1,x_2) = \varrho^\delta, \qquad U^\delta(t,x_0,x_1,x_2) = \left(0, V^\delta\right), \qquad Q^\delta(t,x_0,x_1,x_2) = P^\delta, \tag{1.12}$$

where $(\rho^\delta(t,x), U^\delta(t,x), Q^\delta(t,x))$ is the instability solution in Theorem 1.1. It is easy to check that the solution $(\varrho^\delta, U^\delta, Q^\delta)$ is the RT instability solution for the three-dimensional inhomogeneous incompressible Navier–Stokes–Korteweg equations in the presence of a uniform gravitational field, and such instability result presents that the smallness condition of capillary coefficient in Zhang's result can be removed.

In view of Hwang–Guo's bootstrap instability method in [2, 55], the proof of Theorem 1.1 can be divided into three steps. First, we use modified variational method to construct (linear) unstable solutions for the linearized CRT problem in Sect. 2. Then we further construct approximate solutions with higher order growing modes to the CRT problem in Sect. 3.1 as in Grenier's work [1] (also see [56, 57]); moreover we also derive the error estimates between both the approximate solutions and nonlinear solutions of the CRT problem in Sect. 3.2. Finally, we prove the existence of escape points based on the bootstrap instability approaches, and thus completes the proof of Theorem 1.1 in Sect. 3.3.

We end this section by introducing some abbreviations, which will be repeatedly used in the rest parts of this paper.

$$H^k := W^{k,2}(D), \qquad H^\infty := H^\infty(D), \qquad L^p := L^p(D),$$

$$\|\cdot\|_{H^k} := \|\cdot\|_{H^k(D)}, \qquad \|\cdot\|_{L^p} := \|\cdot\|_{L^p(D)}, \qquad \int := \int_D .$$

## 2 Linear instability

This section is devoted to constructing a family of unstable solutions to the linearized CRT problem (1.9)–(1.11), which have growing $H^k$-norm for any $k$. We will construct such solutions via Fourier synthesis by first constructing a growing mode for any but fixed spatial frequency.

### 2.1 Linear growing modes

We make the following ansatz of growing mode solutions to the linearized problem (1.9)–(1.11).

$$\big(\sigma(x,t), v(x,t), p(x,t)\big) = e^{\lambda t}\big(\tilde{\sigma}(x), \tilde{v}(x), \tilde{p}(x)\big) \quad \text{for some } \lambda > 0. \tag{2.1}$$

Substituting this ansatz into (1.11) and then eliminating $\tilde{\sigma}(x)$ by using the first equation, one obtains the following time-independent system:

$$\begin{cases} \lambda^2\rho_0\tilde{v} + \lambda\nabla\tilde{p} - \lambda\mu\Delta\tilde{v} = \kappa\nabla\rho_0\Delta(\rho_0'\tilde{v}_2) + \kappa\nabla(\rho_0'\tilde{v}_2)\Delta\rho_0 + \rho_0'\tilde{v}_2 g e_2, \\ \operatorname{div}\tilde{v} = 0 \end{cases} \tag{2.2}$$

with boundary-value condition

$$\lim_{|x|\to\infty}\tilde{v}(x) = 0. \tag{2.3}$$

We fix a spatial frequency $\xi \in (L^{-1}\mathbb{Z})$, and define the new unknowns $\varphi$, $\psi$ and $\varpi$, which depend on $x_2$ by the following relations

$$\tilde{v}_1(x) = -i\varphi(x_2)e^{ix_1\xi}, \qquad \tilde{v}_2(x) = \psi(x_2)e^{ix_1\xi}, \qquad \tilde{p}(x) = \varpi(x_2)e^{ix_1\xi}.$$

Inserting the above expressions into (2.2) and (2.3), it is easy to check that $\varphi$, $\psi$, $\varpi$, and $\lambda$ satisfy the following ODEs:

$$\begin{cases} -\lambda^2\rho_0\varphi + \lambda\xi\varpi - \lambda\mu(\xi^2\varphi - \varphi'') = \kappa\xi\rho_0'\rho_0''\psi, \\ \lambda^2\rho_0\psi + \lambda\varpi' + \lambda\mu(\xi^2\psi - \psi'') = -\kappa|\rho_0'|^2\xi^2\psi + \kappa\partial_{x_2}(\rho_0'\partial_{x_2}(\rho_0'\psi)) + \rho_0'\psi g, \\ \psi' + \xi\varphi = 0 \end{cases} \tag{2.4}$$

with boundary-value condition

$$\varphi(-\infty) = \psi(-\infty) = \varphi(+\infty) = \psi(+\infty) = 0. \tag{2.5}$$

Eliminating $\varpi$ in $(2.4)_2$ by using $(2.4)_1$ and $(2.4)_3$, we have the following ODE for $\psi$

$$\begin{aligned} -\lambda^2\big(\xi^2\rho_0\psi - (\rho_0\psi')'\big) = {} & \lambda\mu\big(\psi^{(4)} - 2\xi^2\psi'' + \xi^4\psi\big) + \kappa|\rho_0'|^2\xi^4\psi \\ & - \kappa\xi^2\big(|\rho_0'|^2\psi'\big)' - g\xi^2\rho_0'\psi, \end{aligned} \tag{2.6}$$

with boundary-value condition

$$\psi(-\infty) = \psi'(-\infty) = \psi(+\infty) = \psi'(+\infty) = 0. \tag{2.7}$$

Next, we use the modified variational method to construct a solution of (2.6)–(2.7). (For more details on this idea, check out Guo and Tice's paper on compressible viscous stratified flows [26]). We now fix a non-zero vector $\xi \in (L^{-1}\mathbb{Z})$ and $s > 0$, then we get a family of the following modified problems from (2.6)–(2.7).

$$\begin{aligned}-\lambda^2\left(\xi^2\rho_0\psi - \left(\rho_0\psi'\right)'\right) = s\mu\left(\psi^{(4)} - 2\xi^2\psi'' + \xi^4\psi\right) + \kappa|\rho_0'|^2\xi^4\psi \\ - \kappa\xi^2\left(|\rho_0'|^2\psi'\right)' - g\xi^2\rho_0'\psi.\end{aligned} \tag{2.8}$$

Multiplying (2.8) by $\psi \in H^2(\mathbb{R})$, integrating by parts and using the boundary-value conditions (2.7), we thus get

$$\begin{aligned}-\lambda^2\int_{\mathbb{R}}\left(\rho_0\left(\xi^2\psi^2 + \left|\psi'\right|^2\right)\mathrm{d}x_2 = \xi^2\int_{\mathbb{R}}\left(\kappa\xi^2\left|\rho_0'\right|^2\psi^2 + \kappa\left|\rho_0'\right|^2\left|\psi'\right|^2 - g\rho_0'\psi^2\right)\mathrm{d}x_2 \\ + s\mu\int_{\mathbb{R}}\left(\left|\psi''\right|^2 + 2\xi^2\left|\psi'\right|^2 + \xi^4\psi^2\right)\mathrm{d}x_2.\end{aligned} \tag{2.9}$$

Then the standard energy functional for the problem (2.8) is given by

$$E(\psi) = \xi^2E_1(\psi) + sE_2(\psi) \tag{2.10}$$

with an associated admissible set

$$\mathcal{A} = \left\{\psi \in H^2(\mathbb{R}) \,\middle|\, J(\psi) := \int_{\mathbb{R}}\rho_0\left(\xi^2\psi^2 + \left|\psi'\right|^2\right)\mathrm{d}x_2 = 1\right\}, \tag{2.11}$$

where

$$\begin{aligned}E_1(\psi) &= \int_{\mathbb{R}}\left(\kappa\xi^2\left|\rho_0'\right|^2\psi^2 + \kappa\left|\rho_0'\right|^2\left|\psi'\right|^2 - g\rho_0'\psi^2\right)\mathrm{d}x_2, \\ E_2(\psi) &= \mu\int_{\mathbb{R}}\left(\left|\psi''\right|^2 + 2\xi^2\left|\psi'\right|^2 + \xi^4\psi^2\right)\mathrm{d}x_2.\end{aligned}$$

Thus we can find a $-\lambda^2$ (depending on $\xi$) by minimizing

$$-\lambda^2(\xi) = \alpha(\xi) := \inf_{\psi\in\mathcal{A}} E(\psi). \tag{2.12}$$

In order to emphasize the dependence on $s \in (0, \infty)$, we will sometimes write

$$E(\psi, s) := E(\psi) \quad \text{and} \quad \alpha(s) := \inf_{\psi\in\mathcal{A}} E(\psi, s) < +\infty.$$

Before constructing the growth solutions, we shall introduce some preliminary results, which will be used later. In order to get a positive $\lambda(\xi)$ in the variational problem (2.12), let the critical capillary coefficient $\kappa_c$ and the critical frequency constant $|\xi_c|$ by the following variational forms

$$\kappa_c := \sup_{\psi\in H^2(\mathbb{R}),\psi\neq 0} \frac{\int_{\mathbb{R}} g\rho_0'\psi^2\,\mathrm{d}x_2}{\int_{\mathbb{R}} |\rho_0'|^2|\psi'|^2\,\mathrm{d}x_2},$$

and for $\kappa \in (0,\kappa_c)$,

$$|\xi_c| := \sup_{\psi\in H^2(\mathbb{R}),\psi\neq 0} \sqrt{\frac{\int_{\mathbb{R}} g\rho_0'\psi^2\,\mathrm{d}x_2 - \kappa\int_{\mathbb{R}} |\rho_0'|^2|\psi'|^2\,\mathrm{d}x_2}{\kappa\int_{\mathbb{R}} |\rho_0'|^2|\psi|^2\,\mathrm{d}x_2}}.$$

More precisely, we have the following conclusions:

**Proposition 2.1** *If $\rho_0$ satisfies the conditions* (1.4)–(1.5), *then $\kappa_c$ is infinite.*

*Proof* See Remark 1.2 in [58]. □

**Proposition 2.2** *If $\rho_0$ satisfies the conditions* (1.4)–(1.5) *and $\kappa \in (0,+\infty)$, then $|\xi_c|$ is infinite.*

*Proof* Since the proof is similar to Proposition 2.1, we omit the details here. □

Next we show that a minimizer of (2.12) exists and that the corresponding Euler–Lagrange equations are equivalent to (2.7)–(2.8).

**Proposition 2.3** *For any fixed $s > 0$ and $\xi$ with $\xi \neq 0$, the following assertions hold.*

(1) *$E(\psi)$ achieves its minimum on $\mathcal{A}$.*

(2) *Let $\psi_0$ be a minimizer and $-\lambda^2 := E(\psi_0)$, then the pair $(\psi_0,\lambda)$ satisfies the problem* (2.7)–(2.8). *In addition, $\psi_0 \in H^\infty(\mathbb{R}) := \bigcap_{k=0}^\infty H^k(\mathbb{R})$.*

*Proof* (1) Noting that for any $\psi \in \mathcal{A}$

$$E(\psi) \ge -\xi^2 g\int_{\mathbb{R}} \rho_0'\psi^2\,\mathrm{d}x_2 \ge -g\left\|\frac{\rho_0'}{\rho_0}\right\|_{L^\infty}\int_{\mathbb{R}} \rho_0\xi^2\psi^2\,\mathrm{d}x_2 \ge -g\left\|\frac{\rho_0'}{\rho_0}\right\|_{L^\infty}, \tag{2.13}$$

we see that $E(\psi)$ is bounded from below on $\mathcal{A}$ by virtue of (1.4), then $\inf_{\psi\in\mathcal{A}}E(\psi)$ is well defined and finite. We choose a minimizing sequence $\{\psi_n\}_{n=1}^\infty \subset \mathcal{A}$. It is easy to check that $E(\psi_n)$ is bounded. This fact, together with (2.10) and (2.13), implies that $\{\psi_n\}_{n=1}^\infty$ is bounded in $H^2$. Therefore, there exists a weak limit $\psi_0 \in H^2(\mathbb{R})$ and a subsequence (still denoted by $\psi_n$ for simplicity) such that

$$\psi_n \to \psi_0 \quad \text{weakly in } H^2(\mathbb{R})$$

and

$$\psi_n \to \psi_0 \quad \text{strongly in } H^1_{\mathrm{loc}}(\mathbb{R}).$$

Next, we show that $\psi_0$ is a minimizer and satisfies the constraint (2.11). By the lower semi-continuity and the properties of minimizing sequence, one has

$$E(\psi_0) \leq \liminf_{n\to+\infty} E(\psi_n) = \inf_{\psi\in\mathcal{A}} E(\psi) < 0 \quad \text{and} \quad 0 < J(\psi_0) \leq 1. \tag{2.14}$$

Suppose by contradiction that $J(\psi_0) < 1$, we could find an $\alpha > 1$ such that $J(\alpha\psi_0) = 1$ by the homogeneity of $J(\psi)$, i.e., $\alpha\psi_0 \in \mathcal{A}$, which implies that

$$E(\alpha\psi_0) \leq \alpha^2 \liminf_{n\to+\infty} E(\psi_n) = \alpha^2 \inf_{\psi\in\mathcal{A}} E(\psi) < \inf_{\psi\in\mathcal{A}} E(\psi) < 0, \tag{2.15}$$

leading to a contradiction. Thus $\psi_0$ is a minimizer satisfying the constraint (2.11). (2) By the same order homogeneity of $E(\psi)$ and $J(\psi)$, we can find that (2.12) is equivalent to

$$-\lambda^2 = \inf_{\psi\in H^2(\mathbb{R}),\psi\not\equiv 0} \frac{E(\psi)}{J(\psi)}. \tag{2.16}$$

For any $\tau \in \mathbb{R}$ and $\psi \in H^2(\mathbb{R})$, we define that $\tilde{\psi}(\tau) := \psi_0 + \tau\psi$. Then, by (2.16) we have

$$E(\tilde{\psi}(\tau)) + \lambda^2 J(\tilde{\psi}(\tau)) \geq 0. \tag{2.17}$$

If we set $I(\tau) = E(\tilde{\psi}(\tau)) + \lambda^2 J(\tilde{\psi}(\tau))$, then we have $I(\tau) \geq 0$ for all $\tau \in \mathbb{R}$ and $I(0) = 0$. This implies $I'(0) = 0$. Recalling the definitions of $E(\tilde{\psi}(\tau))$ and $J(\tilde{\psi}(\tau))$, a direct computation leads to

$$\begin{aligned} &-\lambda^2 \int_{\mathbb{R}} \rho_0 \left(\xi^2\psi_0\psi + \psi_0'\psi'\right) \mathrm{d}x_2 \\ &\quad = \kappa \int_{\mathbb{R}} \left(\xi^4 |\rho_0'|^2 \psi_0\psi + \xi^2 |\rho_0'|^2 \psi_0'\psi'\right) \mathrm{d}x_2 - \xi^2 \int_{\mathbb{R}} g\rho_0'\psi_0\psi \,\mathrm{d}x_2 \\ &\qquad + s\mu \int_{\mathbb{R}} \left(\xi^4\psi_0\psi + 2\xi^2\psi_0'\psi' + \psi_0''\psi''\right) \mathrm{d}x_2, \end{aligned} \tag{2.18}$$

which, together with the arbitrariness of $\psi$, shows that $\psi_0$ satisfies the equation (2.8) in the weak sense on $\mathbb{R}$ for the horizontal case. In order to improve the regularity of $\psi_0$, we rewrite (2.18) as

$$\begin{aligned} &\int_{\mathbb{R}} \psi_0''\psi'' \,\mathrm{d}x_2 \\ &\quad = \frac{1}{s\mu} \int_{\mathbb{R}} \left(\xi^2 g\rho_0'\psi_0\psi - \kappa\left(\xi^4 |\rho_0'|^2 \psi_0\psi + \xi^2 |\rho_0'|^2 \psi_0'\psi'\right)\right) \mathrm{d}x_2 \\ &\qquad - \int_{\mathbb{R}} \lambda^2 \rho_0 \left(\xi^2\psi_0\psi + \psi_0'\psi'\right)\left(\xi^4\psi_0\psi + 2\xi^2\psi_0'\psi'\right) \mathrm{d}x_2 := \int_{\mathbb{R}} f\psi \,\mathrm{d}x_2. \end{aligned} \tag{2.19}$$

For any $n \geq 1$, let $\psi_{1,n}, \psi_2 \in C_0^\infty(\mathbb{R})$ satisfying $\psi_{1,n}(x_2) \equiv 1$ for $|x_2| \leq n$. If we take $\psi = \psi_{1,n} \int_{-\infty}^{x_2} \psi_2 \,\mathrm{d}y$ in (2.19), then we get

$$\int_{\mathbb{R}} \left(\psi_{1,n}\psi_0''\right)\psi_2' \,\mathrm{d}x_2 = \int_{\mathbb{R}} \left(\int_{x_2}^{+\infty} \left(f\psi_{1,n} - \psi_{1,n}''\psi_0''\right) \mathrm{d}y - 2\psi_{1,n}'\psi_0''\right)\psi_2 \,\mathrm{d}x_2,$$

which together with $\psi_0 \in H^2(\mathbb{R})$ yields $\psi_0'' \in H^1_{\mathrm{loc}}(\mathbb{R})$ and

$$\psi_0''' = \left(\psi_{1,n}\psi_0''\right)' = \int_{x_2}^{+\infty} \left(f\psi_{1,n} - \psi_{1,n}''\psi_0''\right) \mathrm{d}y \quad \text{for any } |x_2| \le n.$$

Integrating by parts, we can rewrite (2.19) as follows

$$\begin{aligned}
&-\int_{\mathbb{R}} \psi_0'''\psi' \,\mathrm{d}x_2 \\
&\quad= \frac{1}{s\mu}\int_{\mathbb{R}} \left(\xi^2 g\rho_0'\psi_0\psi - \kappa\xi^4\left|\rho_0'\right|^2\psi_0\psi - \xi^2\left(\left|\rho_0'\right|^2\psi_0'\right)'\psi\right) \mathrm{d}x_2 \\
&\qquad- \int_{\mathbb{R}} \lambda^2\left(\xi^2\rho_0\psi_0 - \left(\rho_0\psi_0'\right)'\right)\psi + \left(\xi^4\psi_0 - 2\xi^2\psi_0''\right)\psi \,\mathrm{d}x_2,
\end{aligned}$$

which, keeping in mind that $\psi_0 \in H^2(\mathbb{R})$, yields $\psi_0^{(4)} \in L^2(\mathbb{R})$. Hence, $\psi_0 \in H^4_{\mathrm{loc}}(\mathbb{R}) \cap C^{3,1/2}_{\mathrm{loc}}(\mathbb{R})$, and $\psi_0'(\infty) = \psi_0''(\infty) = \psi_0'''(\infty) = 0$. Using these facts, Hölder's inequality, and integration by parts, we conclude that

$$\left\|\psi_0'''\right\|^2_{L^2(\mathbb{R})} = -\int_{\mathbb{R}} \psi_0''\psi_0'''' \,\mathrm{d}x_2 \le \left\|\psi_0''\right\|^2_{L^2(\mathbb{R})}\left\|\psi_0''''\right\|^2_{L^2(\mathbb{R})}. \tag{2.20}$$

Consequently, $\psi_0 \in H^4(\mathbb{R})$ solves (2.7) and (2.8). This immediately gives that $\psi_0 \in H^\infty(\mathbb{R})$ by applying the bootstrap method and the classical elliptic regularity theory to the formula (2.9). □

Next, we show that there exist a fixed point $s = \lambda$ such that $\sqrt{-\alpha(\lambda)} = \lambda$. For this purpose, we give some properties of $\alpha(s)$.

**Proposition 2.4** *The function $\alpha(s)$ defined on $(0,+\infty)$ enjoys the following properties*:

(1) $\alpha(s) \in C^{0,1}_{\mathrm{loc}}(0,+\infty)$ *is nondecreasing*;

(2) *For any $\xi \in (L^{-1}\mathbb{Z})$ with $\xi \neq 0$ and $\kappa > 0$, there are two positive constants $c_1$ and $c_2$ which depend on $g$, $\rho_0$, $\kappa$, $\mu$ and $\xi$ such that*

$$\alpha(s) \le -c_1 + c_2 s. \tag{2.21}$$

*Proof* (1) Let $\{v^n_{s_i}\}_{n=1}^{+\infty} \subset \mathcal{A}$ be a minimizing sequence of $\inf_{\psi\in\mathcal{A}} E(\psi, s_i) = \alpha(s_i)$ for $i = 1$ and 2. Then for any $0 < s_1 < s_2 < +\infty$

$$\alpha(s_1) \le \liminf_{n\to+\infty} E\left(\psi^n_{s_2}, s_1\right) \le \liminf_{n\to+\infty} E\left(\psi^n_{s_2}, s_2\right) = \alpha(s_2).$$

Hence, $\alpha(s)$ is nondecreasing on $(0,+\infty)$. Next, we verify that $\alpha(s)$ is a local Lipschitz continuous function. Let $I := [a,b] \subset (0,+\infty)$ be a bounded interval. In view of (2.13) and the monotonicity of $\alpha(s)$, we have

$$\left|\alpha(s)\right| \le \max\left\{\left|\alpha(a)\right|, g\left\|\rho_0'/\rho_0\right\|_{L^\infty(\mathbb{R})}\right\} < \infty. \tag{2.22}$$

On the other hand, for any $s \in I$, there exists a minimizing sequence $\{\psi^n_s\}_{n=1}^\infty \subset \mathcal{A}$ of $\inf_{\psi\in\mathcal{A}} E(\psi, s)$ such that

$$\left|\alpha(s) - E\left(\psi^n_s, s\right)\right| < 1. \tag{2.23}$$

Making use of (2.22) and (2.23), we infer that

$$\begin{aligned}0 &\le E_2\big(\psi_s^n\big)\\ &= \frac{E(\psi_s^n,s)}{s}+\frac{\xi^2}{s}\int_{\mathbb{R}} g\rho_0\big(\psi_s^n\big)^2\,\mathrm{d}x_2\\ &\quad -\kappa\int_{\mathbb{R}}\big(\xi^2\big|\rho_0'\psi_s^n\big|^2+\big|\rho_0'\big(\psi_s^n\big)'\big|^2\big)\,\mathrm{d}x_2\\ &\le \frac{1+\max\{|\alpha(a)|,g\|\rho_0'/\rho_0\|_{L^\infty(\mathbb{R})}\}}{a}+\frac{g}{a}\left\|\frac{\rho'{}_0}{\rho_0}\right\|_{L^\infty(\mathbb{R})}=:M.\end{aligned}\tag{2.24}$$

For $s_i \in I$ $(i = 1, 2)$, we further find that

$$\begin{aligned}\alpha(s_1) &\le \liminf_{n\to+\infty} E\big(\psi_{s_2}^n,s_1\big)\le \liminf_{n\to+\infty} E\big(\psi_{s_2}^n,s_2\big)+|s_1-s_2|\liminf_{n\to+\infty}E_2\big(\psi_{s_2}^n,s_2\big)\\ &\le \alpha(s_2)+M|s_1-s_2|,\end{aligned}\tag{2.25}$$

which yields

$$\alpha(s_1)-\alpha(s_2)\le M|s_1-s_2|.$$

Reversing the role of the indices 1 and 2 in the derivation of the inequality (2.25), one has

$$\big|\alpha(s_1)-\alpha(s_2)\big|\le M|s_1-s_2|,$$

which yields $\alpha(s) \in C^{0,1}_{\mathrm{loc}}(0+\infty)$.

(2) Recalling the virtue of Propositions 2.1 and 2.2, for any $\xi$ and $\kappa$, there exists a function $\psi_0 \in H^2(\mathbb{R})$

$$E_1(\psi_0)=\int_{\mathbb{R}}\big(\kappa\big(\xi^2\big|\rho_0'\psi_0\big|^2+\big|\rho_0'\psi_0'\big|^2\big)-g\rho_0'\psi_0^2\big)\,\mathrm{d}x_2<0.\tag{2.26}$$

On the other hand, $H^2(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, thus there is a function sequence $\{\psi_n\}_{n=1}^\infty \subset H^2(\mathbb{R})$, so that

$$\psi_n\to\psi_0\quad \text{stronly in } H^1(\mathbb{R}).$$

Furthermore, there exists a function $\psi_{n_0} \in \{\psi_n\}_{n=1}^\infty$ such that $\psi_{n_0} \not\equiv 0$ and

$$E_1(\psi_{n_0})=\int_{\mathbb{R}}\big(\kappa\big(\xi^2\big|\rho_0'\psi_{n_0}\big|^2+\big|\rho_0'\psi_{n_0}'\big|^2\big)-g\rho_0'\psi_{n_0}^2\big)\,\mathrm{d}x_2.\tag{2.27}$$

Thus, we have

$$\alpha(s)=\inf_{\psi\in H^2(\mathbb{R}),\psi\not\equiv0}\frac{E(\psi)}{J(\psi)}\le\frac{E(\psi_{n_0})}{J(\psi_{n_0})}=\xi^2\frac{E_1(\psi_{n_0})}{J(\psi_{n_0})}+s\frac{E_2(\psi_{n_0})}{J(\psi_{n_0})}:=-c_1+sc_2,\tag{2.28}$$

where the two positive constants $c_1$ and $c_2$ depend on $g$, $\kappa$, $\mu$, $\rho_0$, and $\xi$. □

Given $\xi \in (L^{-1}\mathbb{Z})$ with $\xi \neq 0$, by virtue of (2.21), there exists a constant $s_0 > 0$ depending on $g$, $\kappa$, $\mu$, $\rho_0$ and $\xi$, such that $\alpha(s) < 0$ for any $s \in (0, s_0]$. Hence, if we define

$$\mathfrak{C}_\xi := \sup\{s \mid \alpha(\tau) < 0 \text{ for any } \tau \in (0,s)\}, \tag{2.29}$$

then $\mathfrak{C}_\xi > 0$. This allows us to define $\lambda(s) = \sqrt{-\alpha(s)} > 0$ for any $s \in (0, \mathfrak{C}_\xi)$. Therefore, as a result of Proposition 2.3, we have the following existence result for the modified problem (2.7) and (2.8).

**Proposition 2.5** *For any $\xi \in (L^{-1}\mathbb{Z})$ with $\xi \neq 0$ and $s \in (0, \mathfrak{C}_\xi)$, there is a solution $\psi(\xi, x_2) \not\equiv 0$ with $\lambda = \lambda(\xi, s) > 0$ to the problem (2.7)–(2.8). Moreover, $\psi \in H^k(\mathbb{R})$ for any positive integer $k$.*

*Proof* Thanks to Proposition 2.4, (2.13) and (2.29), it is easy to check that $\lambda(s) \in C^{0,1}_{\mathrm{loc}}(0, \mathfrak{C}_\xi)$ is nonincreasing, $\lambda(s) \leq \|\sqrt{g\rho_0'/\rho_0}\|_{L^\infty}$ and $\lim_{s\to\mathfrak{C}_\xi} \lambda(s) = 0$ if $\mathfrak{C}_\xi < \infty$. Hence, we can employ a fixed point argument to find $s \in (0, \mathfrak{C}_\xi)$ such that $s = \lambda(\xi, s)$, and thus obtain a solution to the original problem (2.6)–(2.7). □

**Proposition 2.6** *For each $\xi \in (L^{-1}\mathbb{Z})$ with $\xi \neq 0$, then there exists a unique $s \in (0, \mathfrak{C}_\xi)$, such that $\lambda(\xi, s) = \sqrt{-\alpha(s)} > 0$ and $s = \lambda(\xi, s)$.*

*Proof* Please refer to Theorem 3.8 in [26] (or Lemma 3.7 in [59]) □

Therefore, in view of the Propositions 2.5 and 2.6, one immediately gets the following conclusion.

**Theorem 2.1** *For each $\xi \in (L^{-1}\mathbb{Z})$ with $\xi \neq 0$, there exist $\psi = \psi(\xi, x_2) \not\equiv 0$ and $\lambda(\xi) > 0$ satisfying (2.6)–(2.7). Moreover, $\psi \in H^k(\mathbb{R})$ for any positive integer $k$.*

We end this subsection by giving some properties for $\lambda(\xi)$, which shows that $\lambda$ is a bounded continuous function of $\xi$.

**Proposition 2.7** *The function $\lambda(\xi)$ is continuous and satisfies*

$$\Lambda := \sup_{\xi \in (L^{-1}\mathbb{Z}), \xi \neq 0} \lambda(\xi) \leq \sqrt{g\|\rho_0'/\rho_0\|_{L^\infty(\mathbb{R})}}. \tag{2.30}$$

*Proof* The boundedness of $\lambda(\xi)$ in (2.30) follows from (2.13). Now, we pay attention to the proof of the continuity of $\lambda(\xi)$. Since $\lambda(\xi) = \sqrt{-\alpha(\xi)}$, it suffices to prove the continuity of $\alpha(\xi)$. For any but fixed $\xi_0 \neq 0$, there exists an interval $(a, b) \subset (0, \infty)$ such that $|\xi_0| \in (a, b)$. Assume $|\xi| \to |\xi_0|$ with $|\xi| \in (a, b)$, and denote $\gamma = |\xi|^2 - |\xi_0|^2$, then $|\xi| \to |\xi_0|$ as $\gamma \to 0$. By Theorem 2.1, there exists $\psi_{|\xi|} \in \mathcal{A}$ satisfying (2.6)–(2.7) and

$$\begin{aligned}\alpha(\xi) = {}& \kappa \int_{\mathbb{R}} \left(\xi^4 |\rho_0'|^2 \psi_{|\xi|}^2 + \xi^2 |\rho_0'|^2 |\psi_{|\xi|}'|^2\right) \mathrm{d}x_2 - \xi^2 \int_{\mathbb{R}} g\rho_0' \psi_{|\xi|}^2 \,\mathrm{d}x_2 \\ & + s(\xi)\mu \int_{\mathbb{R}} \left(\xi^4 \psi_{|\xi|}^2 + 2\xi^2 |\psi_{|\xi|}'|^2 + |\psi_{|\xi|}''|^2\right) \mathrm{d}x_2 < 0, \end{aligned} \tag{2.31}$$

where $s(\xi) = \sqrt{-\alpha(\xi)}$. In order to make use of (2.31), we must show that $s(\xi)$ is bounded from below for any $|\xi| \in (a,b)$. By Proposition 2.4, there exists two positive constants $c_1$ and $c_2$, which depend on $a$, $b$, $g$, $\rho_0$, $\kappa$, and $\mu$, such that $\alpha(\xi) \le -c_1 + c_2 s(\xi)$, and $-\alpha(\xi) = s^2(\xi)$, so

$$0 \le s^2(\xi) + c_2 s(\xi) - c_1. \tag{2.32}$$

Thus, for any $|\xi| \in (a,b)$, $s(\xi)$ is uniformly bounded below by a positive constant. Then it follows from (2.31) and the fact that $\psi_\xi \in \mathcal{A}$ that $\psi_\xi$ is uniformly bounded in $H^2(\mathbb{R})$, i.e.

$$\|\psi_\xi\|_{H^2(\mathbb{R})} \le c_3 \tag{2.33}$$

for any $|\xi| \in (a,b)$, where $c_3$ depends on $g$, $\mu$, $\rho_0$, $a$, $b$ and $\kappa$. Inserting $|\xi|^2 = |\xi_0|^2 + \gamma$ into (2.31) results in

$$\alpha\big(|\xi|\big) \ge \alpha\big(|\xi_0|\big) + \gamma f(\gamma, \psi_{|\xi|}), \tag{2.34}$$

where

$$\begin{aligned} f(\gamma, \psi_{|\xi|}) = {}& \kappa \int_{\mathbb{R}} \big((2|\xi_0|^2 + \gamma)\big|\rho_0'\big|^2 \psi_{|\xi|}^2 + \big|\rho_0' \psi_{|\xi|}'\big|^2\big)\, \mathrm{d}x_2 - \int_{\mathbb{R}} g\rho_0' \psi_{|\xi|}^2 \,\mathrm{d}x_2 \\ & + s\big(|\xi|\big)\mu \int_{\mathbb{R}} \big(2\big|\psi_{|\xi|}'\big|^2 + \big(2|\xi_0|^2 + \gamma\big)\psi_{|\xi|}^2\big)\,\mathrm{d}x_2. \end{aligned}$$

By (2.33), we obtain that for any $|\xi| \in (a,b)$

$$\big|f(\gamma, \psi_{|\xi|})\big| \le c_4, \tag{2.35}$$

where $c_4$ depends on $g$, $\mu$, $\rho_0$, $a$, $b$, and $\kappa$. Similar to (2.34), we also have

$$\alpha\big(|\xi_0|\big) \ge \alpha\big(|\xi|\big) - \gamma f(-\gamma, \psi_{|\xi|}),$$

which together with (2.34) yields

$$\gamma f(\gamma, \psi_{|\xi|}) \le \alpha\big(|\xi|\big) - \alpha\big(|\xi_0|\big) \le \gamma f(-\gamma, \psi_{|\xi_0|}). \tag{2.36}$$

Furthermore, combining (2.35) and (2.36) and letting $\gamma \to 0$, we obtain

$$\lim_{|\xi| \to |\xi_0|} \alpha\big(|\xi|\big) = \alpha\big(|\xi_0|\big). \tag{2.37}$$

Therefore, $\lambda(|\xi|)$ is continuous. This completes the proof of this proposition. □

## 2.2 Construction of a solution to the ODEs system

Next, we will find a family of unstable solutions that satisfy (2.4)–(2.5), and provide an estimate for the $H^k$ norm of the constructed solutions.

**Theorem 2.2** *For any $\xi \in (L^{-1}\mathbb{Z})$ with $\xi \neq 0$, there exists a family of solutions $(\varphi, \psi, \varpi) := (\varphi, \psi, \varpi)(\xi, x_2)$ with $\lambda = \lambda(\xi) > 0$ to (2.4)–(2.5).*

*Proof* First of all, multiplying $(2.4)_1$ by $\xi$ and then utilizing $(2.4)_3$, we find that $\varphi$ and $\varpi$ can be expressed by $\psi$ and $\lambda$, i.e.,

$$\varphi = -\frac{\psi'}{\xi}, \tag{2.38}$$

$$\varpi = \frac{-\lambda^2\rho_0\psi' - \lambda\mu(\xi^2\psi' - \psi''')}{\lambda\xi^2} + \frac{\kappa\rho_0'\rho_0''\psi}{\lambda}. \tag{2.39}$$

Thus, thanks to Theorem 2.1, one finds that the solution $(\varphi, \psi, \varpi)$ constructed above satisfies (2.4)–(2.5). □

Next, we would provide an uniform estimate for the $H^k$ norm of the solutions $(\varphi, \psi, \varpi)$ constructed in Theorem 2.2, which would be used in the next subsection.

**Lemma 2.1** *Suppose $0 < a < b < \infty$ such that $|\xi| \in [a, b]$. Let $(\varphi, \psi, \varpi)$ be the solution constructed in Theorem 2.2. Then, for any nonnegative integer $k$, there exist positive constants $A_k$, $B_k$, and $C_k$ depending on the parameters $a$, $b$, $\rho_0$, $g$, $\kappa$, and $k$, such that*

$$\|\psi(\xi, x_2)\|_{L^2(\mathbb{R})} > 0, \tag{2.40}$$

$$\|\psi(\xi, x_2)\|_{H^k(\mathbb{R})} \le A_k, \tag{2.41}$$

$$\|\varphi(\xi, x_2)\|_{H^k(\mathbb{R})} \le B_k, \tag{2.42}$$

$$\|\varpi(\xi, x_2)\|_{H^k(\mathbb{R})} \le C_k. \tag{2.43}$$

*Proof* Throughout this proof, we denote by $\tilde{c}$ a generic positive constant which may vary from line to line, and may depend on $a$, $b$, $\rho_0$, $g$, $\kappa$ and $k$. Recalling the fact $\psi(\xi, x_2) \in \mathcal{A}$, we have (2.40), and there exists a positive constant $\tilde{c}$ such that

$$\|\psi(\xi)\|_{H^1(\mathbb{R})} \le \tilde{c}. \tag{2.44}$$

Next, we further derive the estimate (2.41), by virtue of the arguments in Proposition 2.7, we have

$$\lambda(\xi) \ge \tilde{c} \quad \text{for any } |\xi| \in [a, b]. \tag{2.45}$$

In addition, we rewrite (2.6) as,

$$\begin{aligned}\psi''''(\xi) = {}& \big(-\lambda^2\big(\xi^2\rho_0\psi - (\rho_0\psi')'\big) - \kappa|\rho_0'|^2\xi^4\psi + \kappa\xi^2\big(|\rho_0'|^2\psi'\big)' - g\xi^2\rho_0'\psi\big)/\lambda\mu \\ & + 2\xi^2\psi'' - \xi^4\psi.\end{aligned} \tag{2.46}$$

Multiplying (2.46) by $\psi(\xi)$ in $L^2(\mathbb{R})$, one has

$$\begin{aligned}\|\psi''(\xi)\|_{L^2(\mathbb{R})} = {}& (\lambda\mu)^{-1}\int_{\mathbb{R}} \big(-\lambda^2\big(\xi^2\rho_0\psi^2 + \rho_0|\psi'|^2\big) - \kappa|\rho_0'|^2\xi^4\psi^2\big)\,\mathrm{d}x_2 \\ & - \int_{\mathbb{R}} +\kappa\xi^2|\rho_0'|^2|\psi'|^2 - g\xi^2\rho_0'\psi^2 + \big(2\xi^2|\psi'|^2 + \xi^4\psi^2\big)\,\mathrm{d}x_2,\end{aligned} \tag{2.47}$$

which together with (2.44) and (2.45) yields

$$\|\psi''(\xi)\|_{L^2(\mathbb{R})} \leq \tilde{c}. \tag{2.48}$$

Thus, using Cauchy–Schwarz's inequality, (2.47), we deduce from (2.46) that

$$\|\psi'''(\xi)\|_{L^2(\mathbb{R})} \leq \tilde{c}. \tag{2.49}$$

Utilizing Gagliardo–Nirenberg interpolation inequality, we obtain

$$\|\psi''''(\xi)\|_{L^2} \leq \|\psi''(\xi)\|_{L^2}^{\frac{1}{2}} \|\psi'''(\xi)\|_{L^2}^{\frac{1}{2}} \leq \tilde{c}, \tag{2.50}$$

which, together with (2.44), (2.48), and (2.49), yields

$$\|\psi(\xi)\|_{H^4(\mathbb{R})} \leq \tilde{c}. \tag{2.51}$$

Now we proceed by induction on $k$. Suppose that the boundedness holds some $k \geq 1$, i.e.,

$$\|\psi(\xi)\|_{H^k(\mathbb{R})} \leq A_k. \tag{2.52}$$

Then by differentiating the equation (2.46) and using (2.51), we easily derive that there exists a constant $\tilde{c}$ depending on the various parameters so that

$$\|\psi(\xi)\|_{H^{k+1}(\mathbb{R})} \leq \tilde{c}\|\psi(\xi)\|_{H^k(\mathbb{R})} \leq \tilde{c}A_k = A_{k+1}. \tag{2.53}$$

It is easy to check that the bound holds for $k+1$, we thus find that (2.41) holds for any nonnegative integer $k$. Employing the expressions (2.38) and (2.39), we can also deduce that (2.42)–(2.43) holds. □

### 2.3 Instability of the linearized CRT problem

In this section, we will construct growing solutions to (1.9)–(1.11) by using the solutions to (2.4)–(2.5) constructed in Theorem 2.2.

**Theorem 2.3** *Let $f \in C_c^\infty(0,\infty)$ be a real-valued function. For any $\xi \in (L^{-1}\mathbb{Z})$ with $\xi \neq 0$, we define*

$$\hat{w}(\xi, x_2) = -i\varphi(\xi, x_2)e_1 + \psi(\xi, x_2)e_2, \tag{2.54}$$

*where $\varphi$, $\psi$, $\varpi$ are the solutions provided by Theorem 2.2. Let*

$$\sigma(x,t) = -\frac{1}{4\pi^2}\int_{(L^{-1}\mathbb{Z})} \rho_0'(x_2) f(\xi)\hat{w}_2(\xi,x_2)e^{\lambda(\xi)t}e^{ix_1\xi}\,\mathrm{d}\xi, \tag{2.55}$$

$$v(x,t) = \frac{1}{4\pi^2}\int_{(L^{-1}\mathbb{Z})} \lambda(\xi) f(\xi)\hat{w}(\xi,x_2)e^{\lambda(\xi)t}e^{ix_1\xi}\,\mathrm{d}\xi, \tag{2.56}$$

$$q(x,t) = \frac{1}{4\pi^2}\int_{(L^{-1}\mathbb{Z})} \lambda(\xi) f(\xi)\varpi(\xi,x_2)e^{\lambda(\xi)t}e^{ix_1\xi}\,\mathrm{d}\xi. \tag{2.57}$$

*Then, $(\sigma, v, q)$ is a real-valued solution to the linearized equations (1.9)–(1.11). For every $k \in \mathbb{N}$ we have the estimate*

$$\|\sigma(0)\|_{H^k} + \|v(0)\|_{H^k} + \|q(0)\|_{H^k} \le B_k \left( \int_{(L^{-1}\mathbb{Z})} \left(1 + |\xi|^2\right)^k \left|f(\xi)\right|^2 \right) \mathrm{d}\xi)^{1/2} < \infty \tag{2.58}$$

*for the constant $B_k > 0$ depending on the parameters $\rho_0$, $g$, $k$, $\kappa$. Moreover, for any $t > 0$ we have $(\sigma(t), v(t), q(t)) \in H^k$ and satisfies*

$$\begin{aligned} e^{t\lambda_f}\|\sigma(0)\|_{H^k} &\le \|\sigma(t)\|_{H^k} \le e^{t\Lambda}\|\sigma(0)\|_{H^k}, \\ e^{t\lambda_f}\|v(0)\|_{H^k} &\le \|v(t)\|_{H^k} \le e^{t\Lambda}\|v(0)\|_{H^k}, \\ e^{t\lambda_f}\|p(0)\|_{H^k} &\le \|p(t)\|_{H^k} \le e^{t\Lambda}\|p(0)\|_{H^k}, \end{aligned} \tag{2.59}$$

*where*

$$\lambda_f := \inf_{\xi \in \mathrm{supp}(f)} \lambda\left(|\xi|\right) > 0, \tag{2.60}$$

*and $\Lambda$ is given by (2.30).*

*Proof* We can easily establish the above conclusion by following the argument of Theorem 2.4 in [26], and thus the proof is omitted. □

## 3 Nonlinear RT instability

This section is devoted to the proof of the nonlinear RT instability. We first construct approximate solutions to the CRT problem (1.8)–(1.10) in Sect. 3.1. Then we formally derive error estimates between both the exact and approximate solutions Sect. 3.2. Finally, making use of approximate solutions and the error estimates, we prove the existence of escape points, and thus complete the proof of Theorem 1.1 in Sect. 3.3.

### 3.1 Construction of higher-order approximate solutions

In this section, we construct approximate solutions by using a similar method to [2] and make further energy estimates. Now we construct approximate solutions to (1.8), we choose and fix a $\xi$ with $\lambda = \lambda(\xi)$, such that

$$0 < \lambda < \Lambda. \tag{3.1}$$

We define $T^\delta$ by

$$T^\delta = \frac{1}{\lambda} \ln \frac{\theta}{\delta}, \tag{3.2}$$

where $\delta$ is an arbitrary small and positive parameter, $\theta$ is a small but fixed positive constant (independent of $\delta$).

**Lemma 3.1** *Let $\rho_0(x_2)$ be a smooth profile satisfying (1.4)–(1.5). Then, there is an approximate solution to (1.8):*

$$\begin{aligned} \sigma^a(t,x) &= \sum_{j=1}^{N} \delta^j \phi_j(t,x), \\ v^a(t,x) &= \sum_{j=1}^{N} \delta^j w_j(t,x), \\ p^a(t,x) &= \sum_{j=1}^{N} \delta^j q_j(t,x), \end{aligned} \tag{3.3}$$

*where*

$$\begin{aligned} &\sigma_t^a + \nabla\rho_0 \cdot v^a = -\nabla\sigma^a \cdot v^a + R_N^a, \\ &\rho_0 v_t^a + \nabla p^a + \sigma^a g e_2 + \kappa\nabla\rho_0\Delta\sigma^a + \kappa\nabla\sigma^a\Delta\rho_0 + \mu\Delta v^a \\ &\quad = -\sigma^a v_t^a - \big(\sigma^a + \rho_0\big) v^a \cdot \nabla v^a - \kappa\nabla\sigma^a\Delta\sigma^a + S_N^a, \\ &\operatorname{div} w_j = 0 \quad (1 \le j \le N). \end{aligned} \tag{3.4}$$

*Moreover, for any positive integer $s$, there is small $\theta > 0$ such that if $0 \le t \le T^\delta$ with $T^\delta$ defined by (3.2), then the $j$-th order coefficients $\phi_j(t,x)$, $w_j(t,x)$, $q_j(t,x)$ for $1 \le j \le N$ satisfy*

$$\big\|\phi_j(t)\big\|_{H^s} \le C\exp(j\lambda t), \tag{3.5}$$

$$\big\|\partial_t\phi_j(t)\big\|_{H^s} \le C\exp(j\lambda t), \tag{3.6}$$

$$\big\|\nabla\phi_j(t)\big\|_{H^s} \le C\exp(j\lambda t), \tag{3.7}$$

$$\big\|w_j(t)\big\|_{H^s} \le C\exp(j\lambda t), \tag{3.8}$$

$$\big\|\partial_t w_j(t)\big\|_{H^s} \le C\exp(j\lambda t), \tag{3.9}$$

$$\big\|\nabla q_j(t)\big\|_{H^s} \le C\exp(j\lambda t), \tag{3.10}$$

*where the $(N+1)$-th order remainders $R_N^a(t,x)$ and $S_N^a(t,x)$ satisfy*

$$\big\|R_N^a(t) + S_N^a(t)\big\|_{H^s} \le C\delta^{N+1}\exp(N+1)\lambda t. \tag{3.11}$$

*Proof* The construction of $\phi_j$, $w_j$, $q_j$, $R_j^a$, and $S_j^a$ will be made by introduction on $j$.

When $j = 1$, choose the the following growing mode solutions as in (2.1):

$$\begin{aligned} \phi_1 &= \tilde\sigma(x)\exp(\lambda t), \\ w_1 &= \tilde v(x)\exp(\lambda t), \\ q_1 &= \tilde p(x)\exp(\lambda t). \end{aligned} \tag{3.12}$$

Utilizing Hölder's inequality and Sobolev embedding for $\phi_1$, $w_1$, $q_1$, we see that the above expressions satisfy (3.5)–(3.10). Plugging $(\phi_1, w_1, q_1)$ into (3.4), we have

$$R_1^a = \nabla(\delta\phi_1)\cdot(\delta w_1),$$
$$S_1^a = (\delta\phi_1)(\delta w_{1t}) + (\delta\phi_1+\rho_0)\delta w_1\cdot\nabla(\delta w_1) + \kappa\nabla(\delta\phi_1)\Delta(\delta\phi_1).$$

Recalling Theorem 2.3, we have $(\phi_1, w_1, q_1)\in H^k$ for any $k > 0$, then it is easy to check that $R_1^a$ and $S_1^a$ satisfy (3.11).

Assume that we have constructed $\phi_j$, $w_j$, $q_j$, $R_j^a$, and $S_j^a$, which satisfy (3.5)–(3.11) for $j < N$, we now construct $\phi_{j+1}$, $w_{j+1}$, $q_{j+1}$, $R_{j+1}^a$, and $S_{j+1}^a$.

Let

$$\sigma_j = \sum_{k=1}^{j}\delta^k\phi_k,\qquad \nu_j = \sum_{k=1}^{j}\delta^k w_k,\qquad p_j = \sum_{j=1}^{j}\delta^k q_k.$$

We further define the nonlinear part of the system (3.4) substituted by $(\sigma_j, \nu_j, p_j)$ by

$$F_{j+1}(\delta) = \nabla\sigma_j\cdot\nu_j, \tag{3.13}$$
$$G_{j+1}(\delta) = \sigma_j\nu_{jt} + \sigma_j\nu_j\cdot\nabla\nu_j + \rho_0\nu_j\cdot\nabla\nu_j + \kappa\nabla\sigma_j\Delta\sigma_j. \tag{3.14}$$

Now, we expand $F_{j+1}(\delta)$ and $G_{j+1}(\delta)$ in terms of $\delta$ around $\delta = 0$, thus we have

$$\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!} = \sum_{j_1+j_2=j+1} A_{j_1,j_2}\nabla\phi_{j_1}\cdot w_{j_2}, \tag{3.15}$$
$$\begin{aligned}\frac{G_{j+1}^{(j+1)}(0)}{(j+1)!} &= \sum_{j_1+j_2=j+1} B_{j_1,j_2}\phi_{j_1}\partial_t w_{j_2}\\ &\quad+\sum_{j_1+j_2+j_3=j+1} C_{j_1,j_2,j_3}\phi_{j_1}w_{j_2}\cdot\nabla w_{j_3}\\ &\quad+\rho_0\sum_{j_1+j_2=j+1} D_{j_1,j_2}w_{j_1}\cdot\nabla w_{j_2}\\ &\quad+\kappa\sum_{j_1+j_2=j+1} E_{j_1,j_2}\nabla\phi_{j_1}\Delta\phi_{j_2},\end{aligned} \tag{3.16}$$

where $1\le j_k\le j$, and $A_{j_1,j_2}$, $B_{j_1,j_2}$, $C_{j_1,j_2,j_3}$, $D_{j_1,j_2}$, $E_{j_1,j_2}$ depend on $\rho_0(x_2)$ and $g$.

By the introduction hypothesis (3.5)–(3.10) for $\phi_k$, $\partial_t\phi_k$, $\nabla\phi_k$, $w_k$, $\partial_t w_k$, and $q_k$, $1\le k\le j$, we have, for all $s$,

$$\left\|\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!}\right\|_{H^s} \le Ce^{(j_1+j_2)\lambda t}\le Ce^{(j+1)\lambda t}, \tag{3.17}$$
$$\begin{aligned}\left\|\frac{G_{j+1}^{(j+1)}(0)}{(j+1)!}\right\|_{H^s} &\le C\big(e^{(j_1+j_2)\lambda t}+e^{(j_1+j_2+j_3)\lambda t}\big)\\ &\le Ce^{(j+1)\lambda t}.\end{aligned} \tag{3.18}$$

We now define the $(j+1)$-th order coefficients $\phi_{j+1}$, $w_{j+1,}$ and $q_{j+1}$ as solutions of the following inhomogeneous linear system:

$$\partial_t\phi_{j+1} + \nabla\rho_0\cdot w_{j+1} = -\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!}, \tag{3.19}$$

$$\rho_0\partial_t w_{j+1} + \nabla q_{j+1} + \phi_{j+1}ge_2 + \kappa\nabla\rho_0\Delta\phi_{j+1} + \kappa\nabla\phi_{j+1}\Delta\rho_0 - \mu\Delta w_{j+1} = -\frac{G_{j+1}^{(j+1)}(0)}{(j+1)!}, \tag{3.20}$$

$$\operatorname{div} w_{j+1} = 0 \tag{3.21}$$

with initial data $\phi_{j+1}(0,x) = 0$, $w_{j+1}(0,x) = (0,0)$.

For $s = 0$, multiplying the equations (3.19) and (3.20) by $\phi_{j+1}$, $w_{j+1,}$ respectively, and integrating over domain $D$, we arrive at

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int|\phi_{j+1}|^2\,\mathrm{d}x = -\int\nabla\rho_0\cdot w_{j+1}\phi_{j+1}\,\mathrm{d}x - \int\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!}\phi_{j+1}\,\mathrm{d}x, \tag{3.22}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\int|\sqrt{\rho_0}w_{j+1}|^2\,\mathrm{d}x = {}&-\int(\nabla q_{j+1} + \phi_{j+1}ge_2 + \kappa\nabla\rho_0\Delta\phi_{j+1})\cdot w_{j+1}\,\mathrm{d}x\\ &-\int\kappa\nabla\phi_{j+1}\Delta\rho_0\cdot w_{j+1}\,\mathrm{d}x + \mu\int\Delta w_{j+1}\cdot w_{j+1}\,\mathrm{d}x\\ &-\int\frac{G_{j+1}^{(j+1)}(0)}{(j+1)!}\cdot w_{j+1}\,\mathrm{d}x.\end{aligned} \tag{3.23}$$

As for the terms involving $\kappa$, using (3.19) and integrating by parts, we obtain

$$\begin{aligned}-\int\kappa\nabla\rho_0\Delta\phi_{j+1}\cdot w_{j+1}\,\mathrm{d}x &= +\int\kappa\nabla(\nabla\rho_0\cdot w_{j+1})\cdot\nabla\phi_{j+1}\,\mathrm{d}x\\ &= -\frac{1}{2}\frac{d}{\mathrm{d}t}\int\kappa|\nabla\phi_{j+1}|^2\,\mathrm{d}x\\ &\quad+\int\kappa\nabla\left(-\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!}\right)\cdot\nabla\phi_{j+1}\,\mathrm{d}x,\end{aligned} \tag{3.24}$$

$$-\kappa\int\nabla\phi_{j+1}\Delta\rho_0\cdot w_{j+1}\,\mathrm{d}x = \kappa\int\nabla\rho_0''\cdot w_{j+1}\phi_{j+1}\,\mathrm{d}x. \tag{3.25}$$

Putting (3.24) and (3.25) into (3.23) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}\int|\sqrt{\rho_0}w_{j+1}|^2\,\mathrm{d}x + \frac{1}{2}\frac{d}{\mathrm{d}t}\int\kappa|\nabla\phi_{j+1}|^2\,\mathrm{d}x + \mu\int\nabla w_{j+1}\cdot\nabla w_{j+1}\,\mathrm{d}x\\ &\quad= -\int(\nabla q_{j+1} + \phi_{j+1}ge_2)\cdot w_{j+1}\,\mathrm{d}x\\ &\qquad-\int\kappa\nabla\left(-\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!}\right)\cdot\nabla\phi_{j+1}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+ \int \kappa \nabla \rho_0'' \cdot w_{j+1} \phi_{j+1} \, \mathrm{d}x \\
&- \int \frac{G_{j+1}^{(j+1)}(0)}{(j+1)!} \cdot w_{j+1} \, \mathrm{d}x.
\end{aligned} \tag{3.26}
$$

Adding (3.26) and (3.22) together, then applying Hölder inequality and Cauchy–Schwarz's inequality to the resulting equation, coupled with the virtue of (3.17)–(3.18), we obtain

$$
\begin{aligned}
\frac{d}{dt}\big(\|\phi_{j+1}\|_{L^2}^2 + \|w_{j+1}\|_{L^2}^2 + \|\nabla\phi_{j+1}\|_{L^2}^2\big) &\le C\big(\|\phi_{j+1}\|_{L^2}^2 + \|w_{j+1}\|_{L^2}^2\big) \\
&\quad + C\big(\|\nabla\phi_{j+1}\|_{L^2}^2 + e^{2(j+1)\lambda t}\big).
\end{aligned} \tag{3.27}
$$

Applying Gronwall's inequality, one finds that

$$
\|\phi_{j+1}\|_{L^2} + \|\nabla\phi_{j+1}\|_{L^2} + \|w_{j+1}\|_{L^2} \le Ce^{(j+1)\lambda t}. \tag{3.28}
$$

Clearly, $\partial_t\phi_{j+1}$, $\partial_t w_{j+1}$ and $\nabla q_{j+1}$ also satisfy (3.6), (3.9) and (3.10) for $s = 0$, we thus verify our lemma for $s = 0$. A similar argument works for $s > 0$.

Now, assume that we have constructed all $\phi_j$, $w_j$, $q_j$ for all $1 \le j \le N$, we define

$$
\sigma^a = \sum_{j=1}^N \delta^j \phi_j, \qquad v^a = \sum_{j=1}^N \delta^j w_j, \qquad p^a = \sum_{j=1}^N \delta^j q_j.
$$

Clearly

$$
\begin{aligned}
&\sigma_t^a + \nabla\rho_0 \cdot v^a = -\sum_{j=1}^N \frac{\delta^{j+1} F_{j+1}^{(j+1)}(0)}{(j+1)!}, \\
&\rho_0 v_t^a + \nabla p^a + \sigma^a g e_2 + \kappa\nabla\rho_0\Delta\sigma^a + \kappa\nabla\sigma^a\Delta\rho_0 - \mu\Delta v^a = -\sum_{j=1}^N \frac{\delta^{j+1} G_{j+1}^{(j+1)}(0)}{(j+1)!},
\end{aligned} \tag{3.29}
$$

where

$$
F(\delta) = \nabla\sigma^a \cdot v^a, \tag{3.30}
$$

$$
G(\delta) = \sigma^a \partial_t v^a + \sigma^a v^a \cdot \nabla v^a + \rho_0 v^a \cdot \nabla v^a + \kappa\nabla\sigma^a\Delta\sigma^a. \tag{3.31}
$$

Compared to (3.4), we find that

$$
R_N^a = -\sum_{j=1}^N \frac{\delta^{j+1} F_{j+1}^{(j+1)}(0)}{(j+1)!} + F(\delta), \tag{3.32}
$$

$$
S_N^a = -\sum_{j=1}^N \frac{\delta^{j+1} G_{j+1}^{(j+1)}(0)}{(j+1)!} + G(\delta). \tag{3.33}
$$

Noticing that $R_N^a$ and $S_N^a$ are the sum of all higher terms than $N$ in the nonlinear part of the $\delta$-expansion, which clearly satisfy (3.11). Thus the proof is complete. □

### 3.2 Error estimates

This section is devoted to establish the error estimates between the exact solution $(\sigma, v)$ to (1.8) and the approximate solution given by Lemma 3.1. To begin with, we shall recall the local existence of the CRT problem (1.8)–(1.10).

**Proposition 3.1** (Local existence) *Assume that $\rho_0$ satisfies (1.4)–(1.5). For any given initial data $(\sigma_0, v_0) \in H^4 \times H^3$ and*

$$\inf_{x\in D} \rho(0) = \inf_{x\in D} \{\rho_0(x_2) + \sigma_0(x)\} > 0,$$

*then there exists a $T > 0$ and a unique solution $(\sigma, v) \in C([0,T]; H^4 \times H^3)$ to (1.8)–(1.10) satisfying*

$$\inf_{D\times(0,T)} \{\sigma + \rho_0\} > 0,$$

*where $T$ denotes the maximal time of existence of the solution $(\sigma, v)$.*

*Proof* We mention that the local existence of the strong solution to the incompressible Navier–Stokes–Korteweg equations has been established, see [60–63] for examples. In particular, by a slight modification of the arguments in [63] and using the expanding domain technique in [64], we can prove that there exists a unique strong solution $(\sigma, v) \in C([0,T]; H^4 \times H^3)$ to the CRT problem (1.8)–(1.10), and thus obtain Proposition 3.1. □

In what follows, the notation $a \lesssim b$ means that $a \le Cb$ for a universal constant $C > 0$, which may depend on some known physical parameters. $C(\varepsilon_0)$ means that the positive constant $C$ further depends on $\varepsilon_0$. We define

$$\mathcal{E}(t) := \mathcal{E}(\sigma^d, v^d)(t) := \sqrt{\|\sigma^d\|_{H^4}^2 + \|v^d\|_{H^3}^2} \le w/2 < 1, \tag{3.34}$$

$$\|u\| = \left(\int (\sigma + \rho_0)u^2 \,\mathrm{d}x\right)^{1/2}, \qquad \|u\|_s^2 = \sum_{|\alpha|\le s} \|\partial_\alpha u\|^2, \tag{3.35}$$

where $\alpha = (\alpha_1, \alpha_2)$ denotes a multi-index of order $|\alpha| = \alpha_1 + \alpha_2$.

In addition, we list some classical Sobolev embedding results, which will be repeated used in later.

$$\|u\|_{L^4} \lesssim \|u\|_{L^2}^{\frac{1}{4}} \|u\|_{H^1}^{\frac{3}{4}} \lesssim \|u\|_{H^1}, \tag{3.36}$$

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^2}^{\frac{1}{4}} \|u\|_{H^2}^{\frac{3}{4}} \lesssim \|u\|_{H^2}, \tag{3.37}$$

$$\|u\|_{L^6} \lesssim \|u\|_{W^{1,2}} = \|u\|_{H^1}. \tag{3.38}$$

Let $(\sigma(t,x), v(t,x)) \in C([0,T]; H^4 \times H^3)$ be a local solution as constructed in Proposition 3.1 and $(\sigma^a(t,x), v^a(t,x))$ the approximate solution provided by Lemma 3.1. Next, we shall establish the error estimate for $(\sigma^d, v^d)$, which is defined as follows

$$\sigma^d = \sigma - \sigma^a, \qquad v^d = v - v^a. \tag{3.39}$$

Noticing that $(\sigma^d, v^d)$ satisfies the following equations:

$$\begin{cases} \partial_t \sigma^d + v^d \cdot \nabla \rho + v^a \cdot \nabla \sigma^d = -R_N^a, \\ \rho v_t^d + \sigma^d v_t^a + \rho v^d \cdot \nabla(v^a + v^d) + \sigma^d v^a \cdot \nabla(v^a + v^d), \\ \qquad + (\sigma^a + \rho_0) v^a \cdot \nabla v^d + \nabla p^d - \mu \Delta v^d + \kappa \nabla \rho \Delta \sigma^d, \\ \qquad + \kappa \nabla \sigma^d \Delta(\sigma^a + \rho_0) + \kappa \nabla \rho_0 \Delta \sigma^d + \kappa \nabla \sigma^d \Delta \rho_0 + \sigma^d g e_2 \\ \quad = -S_N^a, \\ \operatorname{div} v^d = 0, \end{cases} \tag{3.40}$$

where $R_N^a$ and $S_N^a$ are defined in (3.4). Next, we make standard estimates for the error terms of density and velocity.

**Lemma 3.2** *Let* $(\sigma^a(t,x), v^a(t,x), p^a(t,x))$, $((R_N^a(t,x), S_N^a(t,x)) \in L^\infty_{\mathrm{loc}}(H^s)$ *as in Lemma* 3.1. *Assume that* $\|\sigma\|_{H^4} \le \frac{1}{2}\inf_{x_2\in\mathbb{R}}\{\rho_0\}$, *and* $\|\sigma^a\|_{H^4} \le \frac{1}{2}\inf_{x_2\in\mathbb{R}}\{\rho_0\}$. *Then, there exists a positive constant C depending on* $g$, $\rho_0$, $\kappa$, $\mu$ *such that*

$$\begin{aligned} &\frac{d}{dt}\left(\|v^d\|_3^2 + \|\sigma^d\|_{H^4}^2\right) \\ &\quad \le C\big(\|\sigma_t^a\|_{H^2}^2 + \|\sigma^a\|_{H^5}^2 + \|\sigma^a\|_{H^4}^2\|v^a\|_{H^4}^2 + \|\sigma^d\|_{H^4}^2 + \|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^4 \\ &\qquad + \|v_t^a\|_{H^4}^2 + 1\big)\left(\|v^d\|_3^2 + \|\sigma^d\|_{H^4}^2\right) + C\|R_N^a\|_{H^4}^2 + \|S_N^a\|_{H^4}^2. \end{aligned} \tag{3.41}$$

*Proof* We first establish some estimates of the difference $\sigma^d$ of perturbation density. Applying $\partial_\alpha$ with $|\alpha| = 4$ to $(3.40)_1$, and multiplying the resulting identity by $\partial_\alpha \sigma^d$ in $L^2$, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{0\le|\alpha|\le 4}\int \left|\partial_\alpha \sigma^d\right|^2 \mathrm{d}x \\ &\quad = -\sum_{0\le|\alpha|\le 4}\int \partial_a\left(v^a \cdot \nabla \sigma^d\right)\cdot \partial_\alpha \sigma^d \,\mathrm{d}x - \sum_{0\le|\alpha|\le 4}\int \partial_\alpha\left(v^d \cdot \nabla \rho\right)\cdot \partial_\alpha \sigma^d \,\mathrm{d}x \\ &\qquad - \sum_{0\le|\alpha|\le 4}\int \partial_a R_N^a \cdot \partial_\alpha \sigma^d \,\mathrm{d}x =: G_1 + G_2 + G_3. \end{aligned} \tag{3.42}$$

Using (3.36), (3.37), Hölder's and Young's inequalities, $G_1$ can be bounded as follows:

$$\begin{aligned} G_1 &\lesssim \|\nabla v^a\|_{L^4}\|\nabla\sigma^d\|_{L^4}\|\nabla\sigma^d\|_{L^2} + \big(\|\nabla^2 v^a\|_{L^4}\|\nabla\sigma^d\|_{L^4} \\ &\quad + \|\nabla v^a\|_{L^\infty}\|\nabla^2\sigma^d\|_{L^2}\big)\|\nabla^2\sigma^d\|_{L^2} + \big(\|\nabla^3 v^a\|_{L^4}\|\nabla\sigma^d\|_{L^\infty} \\ &\quad + \|\nabla^2 v^a\|_{L^4}\|\nabla^2\sigma^d\|_{L^4} + \|\nabla v^a\|_{L^\infty}\|\nabla^3\sigma^d\|_{L^2}\big)\|\nabla^3\sigma^d\|_{L^2} \\ &\quad + \big(\|\nabla^4 v^a\|_{L^2}\|\nabla\sigma^d\|_{L^\infty} + \|\nabla^3 v^a\|_{L^4}\|\nabla^2\sigma^d\|_{L^4} \\ &\quad + \|\nabla^2 v^a\|_{L^4}\|\nabla^3\sigma^d\|_{L^4} + \|\nabla v^a\|_{L^\infty}\|\nabla^4\sigma^d\|_{L^2}\big)\|\nabla^4\sigma^d\|_{L^2} \\ &\lesssim \left(\|v^a\|_{H^4}^2 + 1\right)\mathcal{E}^2. \end{aligned} \tag{3.43}$$

To bound $G_2$ and $G_3$, arguing in a way similar to that used for (3.43), thus, we can deduce from (3.42) that

$$\frac{1}{2}\frac{d}{dt}\|\sigma^d\|_{H^4}^2 \lesssim C(\varepsilon_0)\big(\|v^a\|_{H^4}^2+\|\sigma^a\|_{H^5}^2+1\big)\mathcal{E}^2+\varepsilon_0\|v^d\|_{H^4}^2+\|R_N^a\|_{H^4}^2. \tag{3.44}$$

We proceed to derive higher-order estimates of the difference of velocity. Applying $\partial_\beta$ with $|\beta|=3$ to $(3.40)_2$, and multiplying the resulting identity by $\partial_\beta v^d$ in $L^2$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\partial_\beta v^d\|_{L^2}^2+\mu\|\partial_\beta\nabla v^d\|_{L^2}^2\\
&\quad=\frac{1}{2}\int\rho_t|\partial_\beta v^d|^2\,\mathrm{d}x-\int\Bigg(\sum_{\substack{\mu+\nu=\beta\\|\mu|\ge1}}\binom{\mu}{\nu}\partial_\mu\rho\partial_\nu v_t^d\Bigg)\cdot\partial_\beta v^d\,\mathrm{d}x\\
&\qquad-\int\partial_\beta\big(\sigma^d v_t^a\big)\cdot\partial_\beta v^d\,\mathrm{d}x-\int\partial_\beta\big(\rho v^d\cdot\nabla\big(v^a+v^d\big)\big)\cdot\partial_\beta v^d\,\mathrm{d}x\\
&\qquad-\int\partial_\beta\big(\sigma^d v^a\cdot\nabla\big(v^a+v^d\big)\big)\cdot\partial_\beta v^d\,\mathrm{d}x-\int\partial_\beta\big(\big(\sigma^a+\rho_0\big)v^a\cdot\nabla v^d\big)\cdot\partial_\beta v^d\,\mathrm{d}x\\
&\qquad-\int\partial_\beta\big(S_N^a\big)\cdot\partial_\beta v^d\,\mathrm{d}x-\kappa\int\partial_\beta\big(\nabla\sigma^d\Delta\big(\sigma^a+\rho_0\big)\big)\cdot\partial_\beta v^d\,\mathrm{d}x\\
&\qquad-\kappa\int\partial_\beta\big(\nabla\sigma^d\Delta\rho_0\big)\cdot\partial_\beta v^d\,\mathrm{d}x-\int\partial_\beta\big(\sigma^d ge_2\big)\cdot\partial_\beta v^d\,\mathrm{d}x\,\mathrm{d}x\\
&\qquad-\kappa\int\partial_\beta\big(\nabla\rho_0\Delta\sigma^d\big)\cdot\partial_\beta v^d\,\mathrm{d}x-\kappa\int\partial_\beta\big(\nabla\rho\Delta\sigma^d\big)\cdot\partial_\beta v^d\,\mathrm{d}x:=\sum_{j=1}^{12}I_j.
\end{aligned}\tag{3.45}$$

To bound $I_1$, we need to estimate the time-derivative of the difference $\sigma^d$ of the density. Using (3.36), Hölder's and Cauchy–Schwarz's inequalities, we can easily deduce from $(3.40)_1$ that

$$\|\sigma_t^d\|_{L^2}^2\lesssim\|v^d\|_{H^1}^2\big(\|\sigma^a\|_{H^2}^2+\|\sigma^d\|_{H^2}^2+1\big)+\|v^a\|_{H^1}^2\|\sigma^d\|_{H^2}^2+\|R_N^a\|_{L^2}^2, \tag{3.46}$$

$$\|\nabla\sigma_t^d\|_{L^2}^2\lesssim\|v^d\|_{H^2}^2\big(\|\sigma^a\|_{H^3}^2+\|\sigma^d\|_{H^3}^2+1\big)+\|v^a\|_{H^2}^2\|\sigma^d\|_{H^3}^2+\|R_N^a\|_{H^1}^2, \tag{3.47}$$

$$\|\Delta\sigma_t^d\|_{L^2}^2\lesssim\|v^d\|_{H^3}^2\big(\|\sigma^a\|_{H^4}^2+\|\sigma^d\|_{H^4}^2+1\big)+\|v^a\|_{H^3}^2\|\sigma^d\|_{H^4}^2+\|R_N^a\|_{H^2}^2. \tag{3.48}$$

Using (3.37), Hölder's and Cauchy's inequalities, $I_1$ can be bounded as follows:

$$\begin{aligned}
I_1&=\frac{1}{2}\int\rho_t|\partial_\alpha v^d|^2\,\mathrm{d}x\le\frac{1}{2}\|\big(\sigma_t^a+\sigma_t^d\big)\|_{L^\infty}\|\partial_\alpha v^d\|_{L^2}\|\partial_\alpha v^d\|_{L^2}\\
&\lesssim\|\sigma_t^a\|_{H^2}^2\|v^d\|_{H^3}^2+\|v^d\|_{H^3}^2+\|\sigma_t^d\|_{H^2}^2+\|v^d\|_{H^3}^4.
\end{aligned}\tag{3.49}$$

Recalling the virtue of $\mathcal{E}\le w/2<1$ and putting (3.46)–(3.48) into (3.49), we get

$$\begin{aligned}
I_1&\lesssim\big(\|\sigma_t^a\|_{H^2}^2+\|\sigma^a\|_{H^4}^2+\|\sigma^d\|_{H^4}^2+1\big)\|v^d\|_{H^3}^2\\
&\quad+\|v^a\|_{H^3}^2\|\sigma^d\|_{H^4}^2+\|R_N^a\|_{H^2}^2+\|v^d\|_{H^3}^4\\
&\lesssim\big(\|\sigma_t^a\|_{H^2}^2+\|\sigma^a\|_{H^4}^2+\|\sigma^d\|_{H^4}^2+\|v^a\|_{H^3}^2+1\big)\mathcal{E}^2+\|R_N^a\|_{H^2}^2.
\end{aligned}\tag{3.50}$$

Next we continue to derive more higher-order derivatives estimates of the difference $v^d$ of perturbation velocity. Multiplying $(3.40)_2$ by $v_t^d$ in $L^2$ and recalling the virtue of $\operatorname{div} v_t^d = 0$, one gets

$$\begin{aligned}
\|v_t^d\|_{L^2}^2 &\lesssim \|\sqrt{\rho} v_t^d\|_{L^2}^2 \\
&\lesssim \|v_t^a\|_{H^1}^2 \mathcal{E}^2 + \|\sigma^a\|_{H^1}^2 \mathcal{E}^4 + \|\sigma^a\|_{H^1}^2 \|v^a\|_{H^2}^2 \mathcal{E}^4 \\
&\quad + \|v^a\|_{H^2}^2 \mathcal{E}^4 + \|v^a\|_{H^2}^4 \mathcal{E}^2 + \|v^a\|_{H^2}^2 \mathcal{E}^2 \\
&\quad + \|\sigma^a\|_{H^3}^2 \mathcal{E}^2 + \mathcal{E}^2 + \mathcal{E}^4 + \mathcal{E}^6 + \|S_N^a\|_{L^2}^2 \\
&\lesssim \big(\|v_t^a\|_{H^1}^2 + \|\sigma^a\|_{H^3}^2 + \|\sigma^a\|_{H^1}^2 \|v^a\|_{H^2}^2 \\
&\quad + \|v^a\|_{H^2}^2 + \|v^a\|_{H^2}^4 + 1\big)\mathcal{E}^2 + \|S_N^a\|_{L^2}^2. 
\end{aligned} \tag{3.51}$$

Applying $\partial_i$ to $(3.40)_2$, multiplying the resulting equation by $\partial_i v_t^d$ in $L^2$, and using (3.36)–(3.38) and (3.51), we have

$$\begin{aligned}
\|\partial_i v_t^d\|_{L^2}^2 &\lesssim \|\sqrt{\rho}\partial_i v_t^d\|_{L^2}^2 \\
&\lesssim \big(\|v_t^a\|_{H^2}^2 + \|\sigma^a\|_{H^4}^2 + \|\sigma^a\|_{H^2}^2 \|v^a\|_{H^3}^2 \\
&\quad + \|v^a\|_{H^3}^2 + \|v^a\|_{H^3}^4 + 1\big)\mathcal{E}^2 + \|S_N^a\|_{H^1}^2 + \|v_t^d\|_{L^2}^2 \\
&\lesssim \big(\|v_t^a\|_{H^2}^2 + \|\sigma^a\|_{H^4}^2 + \|\sigma^a\|_{H^2}^2 \|v^a\|_{H^3}^2 \\
&\quad + \|v^a\|_{H^3}^2 + \|v^a\|_{H^3}^4 + 1\big)\mathcal{E}^2 + \|S_N^a\|_{H^1}^2.
\end{aligned} \tag{3.52}$$

Furthermore, applying $\partial_i\partial_j$ to $(3.40)_2$ yields

$$\begin{aligned}
\rho\partial_i\partial_j v_t^d = {}& -\partial_i^2 \rho v_t^d - \partial_i\rho\partial_j v_t^d - \partial_i\partial_j\big(\sigma^d v_t^a\big) - \partial_i\partial_j\big(\rho v^d \cdot \nabla\big(v^a + v^d\big)\big) \\
& - \partial_i\partial_j\big(\sigma^d v^a \cdot \nabla\big(v^a + v^d\big)\big) - \partial_i\partial_j\big(\big(\sigma^a + \rho_0\big)v^a \cdot \nabla v^d\big) - \partial_i\partial_j\nabla p^d \\
& - \mu\partial_i\partial_j\Delta v^d - \kappa\partial_i\partial_j\big(\nabla\rho\Delta\sigma^d\big) - \kappa\partial_i\partial_j\big(\nabla\sigma^d\Delta\big(\sigma^a + \rho_0\big)\big) \\
& - \kappa\partial_i\partial_j\big(\nabla\rho_0\Delta\sigma^d\big) - \kappa\partial_i\partial_j\big(\nabla\sigma^d\Delta\rho_0\big) - \partial_i\partial_j\sigma^d g e_2 - \partial_i\partial_j S_N^a.
\end{aligned} \tag{3.53}$$

Multiplying the (3.53) by $\partial_i\partial_j v_t^d$, integrating over $D$, and utilizing (3.51), (3.52), and (3.36)–(3.38), we have

$$\begin{aligned}
\|\partial_i\partial_j v_t^d\|_{L^2}^2 &\lesssim \|\sqrt{\rho}\partial_i\partial_j v_t^d\|_{L^2}^2 \\
&\lesssim \big(\|v_t^a\|_{H^3}^2 + \|\sigma^a\|_{H^5}^2 + \|\sigma^a\|_{H^3}^2 \|v^a\|_{H^4}^2 \\
&\quad + \|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^4 + 1\big)\mathcal{E}^2 + \|v^d\|_{H^4}^2 + \|S_N^a\|_{H^2}^2 + \|v_t^d\|_{H^1}^2 \\
&\lesssim \big(\|v_t^a\|_{H^3}^2 + \|\sigma^a\|_{H^5}^2 + \|\sigma^a\|_{H^3}^2 \|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^2 \\
&\quad + \|v^a\|_{H^4}^4 + 1\big)\mathcal{E}^2 + \|v^d\|_{H^4}^2 + \|S_N^a\|_{H^2}^2.
\end{aligned} \tag{3.54}$$

In particular, summing up the above three estimates, we conclude that

$$\begin{aligned}\|v_t^d\|_{H^2}^2 \lesssim &\big(\|v_t^a\|_{H^3}^2 + \|\sigma^a\|_{H^5}^2 + \|\sigma^a\|_{H^3}^2\|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^4 + 1\big)\mathcal{E}^2 \\ &+ \|v^d\|_{H^4}^2 + \|S_N^a\|_{H^2}^2. \end{aligned} \tag{3.55}$$

Therefore, the second term on the right-hand side of (3.45) can be bounded as follows. Using the embedding theorem, Hölder's, Cauchy–Schwarz's inequalities, and (3.55), we have

$$\begin{aligned} I_2 &= \int \sum_{\substack{\mu+\nu=\beta\\|\mu|\geq 1}} \binom{\mu}{\nu} \partial_\mu \rho \partial_\nu v_t^d \cdot \partial_\beta v^d \,\mathrm{d}x \\ &\lesssim \|\nabla\rho\|_{L^\infty}\|\Delta v_t^d\|_{L^2}\|\partial_\beta v^d\|_{L^2} + \|\Delta\rho\|_{L^4}\|\nabla v_t^d\|_{L^4}\|\partial_\beta v^d\|_{L^2} \\ &\lesssim + \|\partial_\beta\rho\|_{L^2}\|v_t^d\|_{L^\infty}\|\partial_\beta v^d\|_{L^2} + \varepsilon_0\|v_t^d\|_{H^2}^2 + C(\varepsilon_0)\|v^d\|_{H^3}^2 \\ &\lesssim C(\varepsilon_0)\big(\|v_t^a\|_{H^3}^2 + \|\sigma^a\|_{H^5}^2 + \|\sigma^a\|_{H^3}^2\|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^4 + 1\big)\mathcal{E}^2 \\ &\quad + \varepsilon_0\|v^d\|_{H^4}^2 + \|S_N^a\|_{H^2}^2. \end{aligned} \tag{3.56}$$

Arguing in a way similar to that used above, it is easy to verify that

$$\begin{aligned} \sum_{j=3}^{11} I_j \lesssim\ & C(\varepsilon_0)\big(\|\sigma^a\|_{H^4}^2\|v^a\|_{H^4}^2\mathcal{E}^2 + \|v^a\|_{H^4}^2\mathcal{E}^2 + \|v^a\|_{H^4}^2\mathcal{E}^4\big) \\ &+ \varepsilon_0\|v^d\|_{H^4}^2 + \big(\|v_t^a\|_{H^4}^2 + \|\sigma^a\|_{H^5}^2 + 1\big)\mathcal{E}^2 + \mathcal{E}^4 + \|S_N^a\|_{H^3}^2 \\ \lesssim\ & C(\varepsilon_0)\big(\|\sigma^a\|_{H^4}^2\|v^a\|_{H^4}^2 + \|v^a\|_{H^4}^2 + \|v_t^a\|_{H^4}^2 + \|\sigma^a\|_{H^5}^2 + 1\big)\mathcal{E}^2 \\ &+ \varepsilon_0\|v^d\|_{H^4}^2 + \|S_N^a\|_{H^3}^2. \end{aligned} \tag{3.57}$$

To estimate the term $I_{12}$ in (3.45), we shall further rewrite $I_{12}$ as follows by using the divergence-free condition and integration by parts.

$$\begin{aligned} &-\kappa \int \partial_\beta\big(\nabla\rho\Delta\sigma^d\big)\cdot\partial_\beta v^d\,\mathrm{d}x \\ &\quad = -\kappa\int \partial_\beta\nabla\sigma^d\cdot\nabla\big(\nabla\rho\cdot\partial_\beta v^d\big)\,\mathrm{d}x \\ &\qquad - \kappa\int\Bigg(\sum_{\substack{\mu+\nu=\beta\\|\mu|\geq 1}}\binom{\mu}{\nu}\partial_\mu\nabla\rho\cdot\partial_\nu\Delta\sigma^d\Bigg)\cdot\partial_\beta v^d\,\mathrm{d}x. \end{aligned} \tag{3.58}$$

Therefore, using (3.36), (3.37), and Hölder's and Cauchy–Schwarz's inequalities, we derive from (3.58) that

$$I_{12} \lesssim C(\varepsilon_0)\big(\|\sigma^d\|_{H^4}^2 + \|v^d\|_{H^3}^2\big) + \varepsilon_0\|v^d\|_{H^4}^2 \lesssim \mathcal{E}^2 + \varepsilon_0\|v^d\|_{H^4}^2. \tag{3.59}$$

Inserting all the above estimates into (3.45), we conclude that

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\partial_\beta v^d\|_{L^2}^2+\mu\|\partial_\beta\nabla v^d\|_{L^2}^2\\
&\lesssim\big(\|\sigma_t^a\|_{H^2}^2+\|\sigma^a\|_{H^5}^2+\|\sigma^a\|_{H^4}^2\|v^a\|_{H^4}^2+\|\sigma^d\|_{H^4}^2\\
&\quad+\|v^a\|_{H^4}^2+\|v^a\|_{H^4}^4+\|v_t^a\|_{H^4}^2+1\big)\mathcal{E}^2\\
&\quad+\|R_N^a\|_{H^4}^2+\|S_N^a\|_{H^3}^2+\varepsilon_0\|v^d\|_{H^4}^2.
\end{aligned}\tag{3.60}$$

Recalling Proposition 3.2 and the definition of $\rho_0$, we known the two norms $\|\cdot\|_s$ and $\|\cdot\|_{H^s}$ are equivalent. By applying a similar argument to the case $0\le|\beta|<3$, and then using (3.44), we arrive at, for sufficiently small $\varepsilon_0$.

$$\begin{aligned}
&\frac{d}{dt}\big(\|v^d\|_3^2+\|\sigma^d\|_{H^4}^2\big)\\
&\lesssim\big(\|\sigma_t^a\|_{H^2}^2+\|\sigma^a\|_{H^5}^2+\|\sigma^a\|_{H^4}^2\|v^a\|_{H^4}^2+\|\sigma^d\|_{H^4}^2+\|v^a\|_{H^4}^2+\|v^a\|_{H^4}^4\\
&\quad+\|v_t^a\|_{H^4}^2+1\big)\big(\|v^d\|_3^2+\|\sigma^d\|_{H^4}^2\big)+\|R_N^a\|_{H^4}^2+\|S_N^a\|_{H^4}^2.
\end{aligned}\tag{3.61}$$

This completes the proof. □

### 3.3 Existence of escape points

In this section, we are going to prove the Theorem 1.1. Let $(\sigma^a(t,x),v^a(t,x))$ be an approximate solution as in Lemma 3.1 with $N$ to be determined later. For any $\delta>0$, by Proposition 3.1, we known that exists a local-in time solution $(\sigma^\delta(t,x),v^\delta(t,x))$ with the initial data $(\sigma^a(0),v^a(0))$ to the full system (1.8). Obviously,

$$\big(\sigma^d(0),v^d(0)\big)=\big(\sigma^\delta(0)-\sigma^a(0),v^\delta(0)-v^a(0)\big)=0.$$

By (3.11) and Lemma 3.2, we have

$$\begin{aligned}
&\frac{d}{dt}\big(\|v^d\|_3^2+\|\sigma^d\|_{H^4}^2\big)\\
&\le C\big(\|\sigma_t^a\|_{H^2}^2+\|\sigma^a\|_{H^5}^2+\|\sigma^a\|_{H^4}^2\|v^a\|_{H^4}^2+\|\sigma^d\|_{H^4}^2+\|v^a\|_{H^4}^2+\|v^a\|_{H^4}^4\\
&\quad+\|v_t^a\|_{H^4}^2+1\big)\big(\|v^d\|_3^2+\|\sigma^d\|_{H^4}^2\big)+C\big(\|R_N^a\|_{H^4}^2+\|S_N^a\|_{H^4}^2\big)\\
&\le C\big(\|\sigma_t^a\|_{H^2}^2+\|\sigma^a\|_{H^5}^2+\|\sigma^a\|_{H^4}^2\|v^a\|_{H^4}^2+\|\sigma^d\|_{H^4}^2+\|v^a\|_{H^4}^2+\|v^a\|_{H^4}^4\\
&\quad+\|v_t^a\|_{H^4}^2+1\big)\big(\|v^d\|_3^2+\|\sigma^d\|_{H^4}^2\big)+C\delta^{2(N+1)}\exp\big(2(N+1)\lambda t\big).
\end{aligned}\tag{3.62}$$

Let

$$\begin{aligned}
T=\sup\Big\{t\ \Big|\ &\|v_t^a\|_{H^5}+\|\sigma_t^a\|_{H^5}+\|\sigma^a\|_{H^5}+\|v^a\|_{H^5}\le\frac{w}{2},\\
&\|\sigma^d\|_{H^4}+\|v^d\|_{H^3}\le\frac{w}{2}\Big\},
\end{aligned}\tag{3.63}$$

where $w$ is a small positive number that assures the local existence. If $T < \infty$, we have

$$\|v_t^a(T)\|_{H^5} + \|\sigma^a(T)\|_{H^5} + \|v^a(T)\|_{H^5} + \|\sigma_t^a(T)\|_{H^5} = \frac{w}{2}, \tag{3.64}$$

or

$$\|\sigma^d(T)\|_{H^4} + \|v^d(T)\|_{H^3} = \frac{w}{2}. \tag{3.65}$$

Since $(\sigma^d(0), v^d(0)) = (0,0)$, and $\|\sigma^a(0)\|_{H^5} = \|v^a(0)\|_{H^5} = O(\delta)$ by Lemma 3.1, $T$ is well defined for $\delta$ small enough.

Next we will prove that for $\theta$ small enough, $T^\delta \le T$ by contradiction, where $\theta = \delta e^{\lambda T^\delta}$ is defined by (3.2).

Let

$$\theta = \min\left\{\frac{w}{2C_{N,1}}, \frac{w}{2C_2}, \frac{C_3}{2C_{N,2}}, \frac{C_3}{4C_2}, 1\right\} > 0. \tag{3.66}$$

Suppose that $T^\delta > T$, then for $t \le T$. By the construction of approximate solutions in (3.3), we can deduce that

$$\begin{aligned}
&\|\sigma^a(t)\|_{H^5} + \|\sigma_t^a(t)\|_{H^5} + \|v^a(t)\|_{H^5} + \|v_t^a(t)\|_{H^5} \\
&\quad \le \sum_{j=1}^{N} C_j\delta^j \|\phi_j(t)\|_{H^5} + \sum_{j=1}^{N} C_j\delta^j \|\partial_t\phi_j(t)\|_{H^5} \\
&\qquad + \sum_{j=1}^{N} C_j\delta^j \|w_j(t)\|_{H^5} + \sum_{j=1}^{N} C_j\delta^j \|\partial_t w_j(t)\|_{H^5} \\
&\quad \le C_0 \sum_{j=1}^{N} \delta^j e^{j\lambda T} < C_0 \sum_{j=1}^{N} \delta^j e^{j\lambda T^\delta} \\
&\quad = C_0 \sum_{j=1}^{N} \theta^j \le C_{N,1}\theta \le \frac{w}{2}.
\end{aligned} \tag{3.67}$$

When $t = T$, we obtain

$$\|\sigma^a(T)\|_{H^5} + \|\sigma_t^a(T)\|_{H^5} + \|v^a(T)\|_{H^5} + \|v_t^a(T)\|_{H^5} < \frac{w}{2}. \tag{3.68}$$

Now we appeal to the definition of $T$ as in (3.63) and (3.62) to get, for $t \le T$,

$$\begin{aligned}
\frac{d}{dt}\left(\|\sigma^d(t)\|_{H^4}^2 + \|v^d(t)\|_3^2\right) &\le C\left(1 + \frac{w^2}{8}\right)\left(\|\sigma^d\|_{H^4}^2 + \|v^d\|_3^2\right) \\
&\quad + C\delta^{2(N+1)} e^{2(N+1)\lambda t}.
\end{aligned}$$

Then we choose $N > 0$ such that

$$C\left(1 + \frac{w^2}{8}\right) \le 2(N+1)\lambda.$$

Using Gronwall's inequality leads to

$$\left\|\sigma^d(t)\right\|_{H^4}^2 + \left\|v^d(t)\right\|_3^2 \le C\delta^{2(N+1)}e^{2(N+1)\lambda t}. \tag{3.69}$$

Hence

$$\begin{aligned}\left\|\sigma^d(t)\right\|_{H^4} + \left\|v^d(t)\right\|_{H^3} &\le C_2\delta^{N+1}e^{(N+1)\lambda t} < C_2\delta^{N+1}e^{(N+1)\lambda T^\delta} \\ &= C_2\theta^{N+1} < C_2\theta \le \frac{w}{2}.\end{aligned} \tag{3.70}$$

When $t = T$, we have

$$\left\|\sigma^d(T)\right\|_{H^4} + \left\|v^d(T)\right\|_{H^3} < \frac{w}{2}, \tag{3.71}$$

which, together with (3.68), leads to a contradiction to (3.64)–(3.65), thus $T^\delta \le T$.

Finally, utilizing (2.1) and Lemma 3.1, we obtain

$$\begin{aligned}\left\|v^a\left(T^\delta\right)\right\|_{L^2} &\ge \delta\left\|w_1\left(T^\delta\right)\right\|_{L^2} - \sum_{j=2}^{N}\delta^j\left\|w_j\left(T^\delta\right)\right\|_{L^2} \\ &\ge C_3\delta e^{\lambda T^\delta} - \sum_{j=2}^{N}C_j\delta^j e^{j\lambda T^\delta} \\ &= C_3\theta - \sum_{j=2}^{N}C_j\theta^j \\ &\ge C_3\theta - C_{N,2}\theta^2 \\ &\ge \frac{C_3}{2}\theta.\end{aligned}$$

Thus, when $t = T^\delta$, using (3.70), we deduce that,

$$\begin{aligned}\left\|v^\delta\left(T^\delta\right)\right\|_{L^2} &\ge \left\|v^a\left(T^\delta\right)\right\|_{L^2} - \left\|\left(v^\delta - v^a\right)\left(T^\delta\right)\right\|_{L^2} \\ &\ge \left\|v^a\left(T^\delta\right)\right\|_{L^2} - \left\|\left(v^\delta - v^a\right)\left(T^\delta\right)\right\|_{H^3} \\ &\ge \left\|v^a\left(T^\delta\right)\right\|_{L^2} - \left\|v^d\left(T^\delta\right)\right\|_{H^3} \\ &\ge \frac{C_3}{2}\theta - C_2\theta^{N+1} \\ &\ge \frac{C_3}{2}\theta - C_2\theta^2 \\ &\ge \frac{C_3}{4}\theta = \varepsilon_0 > 0.\end{aligned}$$

This completes the proof of Theorem 1.1 by defining $\varepsilon_0 = C_3\theta/4$.

## 4 Conclusion

This paper focuses on the Rayleigh–Taylor instability in the two-dimensional system of equations of inhomogeneous incompressible viscous fluids with capillarity effects in a horizontal periodic domain with infinite height. First, we use the modified variational method to construct (linear) unstable solutions for the linearized capillary Rayleigh–Taylor problem. Then, motivated by the Grenier's idea in [1], we further construct approximate solutions with higher-order growing modes to the capillary Rayleigh–Taylor problem and derive the error estimates between both the approximate solutions and nonlinear solutions of the capillary Rayleigh–Taylor problem. Finally, we prove the existence of escape points based on the bootstrap instability method of Hwang–Guo in [2], and thus obtain the nonlinear Rayleigh–Taylor instability result. Our instability result presents that the Rayleigh–Taylor instability can occur in the fluids with capillarity effects for any capillary coefficient $\kappa > 0$ if the critical capillary coefficient is infinite. In particular, it improves the previous Zhang's result in [3] with the smallness assumption of capillary coefficient.

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**Author contributions**
This work was carried out in collaboration between the three authors. Weiwei Wang designed the study and guided the research. Xuyan Zhang and Fangfang Tian performed the analysis and wrote the first draft of the manuscript. Xuyan Zhang, Fangfang Tian and Weiwei Wang managed the analysis of the study. The three authors read and approved the final manuscript.

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