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On stability of almost surjective functional equations of uniformly convex Banach spaces

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Abstract

Let Y be a uniformly convex space with power type p, and let $(G,+)$ be an abelian group, $\delta,\varepsilon \geq 0$, $0<r<1$. We first show a stability result for approximate isometries from an arbitrary Banach space into Y. This is a generalization of Dolinar' results for (δ,r)-isometries of Hilbert spaces and L_p $(1<p<\infty)$ spaces. As a result, we prove that if a standard mapping $F:G\to Y$ satisfies $d(u,F(G))\leq \delta\|u\|^r$ for every $u\in Y$ and

$$\bigl|\|F(x)-F(y)\|-\|F(x-y)\|\bigr|\leq\varepsilon,\quad x,y\in G,$$

then there is an additive operator $A:G\to Y$ such that

$$\|F(x)-Ax\|=o\bigl(\|F(x)\|\bigr)\quad \text{as } \|F(x)\|\to\infty.$$

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1 Introduction

Let $(G,+)$ be a group and let Y be a Banach space. Sikorska [18] firstly investigated the approximation stability by an additive function for near-surjective mappings $F:G\to Y$ satisfying

$$\bigl|\|F(x)-F(y)\|-\|F(x-y)\|\bigr|\leq\varepsilon, \tag{1.1}$$

and showed that the near-surjectivity assumption is essential. In this paper, we continue to study asymptotical stability for the above functional equations with a nonsurjectivity condition, where Y is a uniformly convex Banach space with power type p. Let us first review the historical development of stability for equation (1.1).

In the fall of 1940, Ulam [20] raised the following question:

Question 1.1 (Ulam [20]) Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist $\delta>0$ such that if a function $h:G_1\to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, Hyers [9] answered the question of Ulam for the case where G_1 and G_2 are both Banach spaces.

Theorem 1.2 (Hyers [9]) *Let X, Y be Banach spaces, and let $F : X \to Y$ be a mapping with*

$$\|F(x+y) - F(x) - F(y)\| \leq \varepsilon, \quad x, y \in X, \tag{1.2}$$

for some $\varepsilon \geq 0$. Then there is an additive mapping $A : X \to Y$ such that

$$\|F(x) - Ax\| \leq \varepsilon, \quad x \in X.$$

The stability result above for (1.2) is called the Hyers–Ulam stability. In 1978, Rassias [17] obtained the following exciting result, which weakened the condition for the bound of the norm of $F(x+y) - F(x) - F(y)$.

Theorem 1.3 (Rassias [17]) *Let X, Y be Banach spaces, and let $F : X \to Y$ be a mapping satisfying*

$$\|F(x+y) - F(x) - F(y)\| \leq \varepsilon\big(\|x\|^p + \|y\|^p\big), \quad x, y \in X, \tag{1.3}$$

for some $\varepsilon \geq 0$ and $0 \leq p < 1$. Then there is an additive mapping $A : X \to Y$ such that

$$\|F(x) - Ax\| \leq \frac{2\varepsilon}{2 - 2^p}, \quad x \in X.$$

The stability phenomenon established by Rassias is called the Hyers–Ulam–Rassias stability. For some recent work on Hyers–Ulam–Rassias stability and related topics, one may refer to [1, 10, 13, 15].

In 2003, by using a stability result of ε-isometries which was established by Omladič and Šemrl [14], Tabor [19] firstly got the perturbation stability result for the Fischer–Muszély functional equation

$$\big|\|F(x+y)\| - \|F(x) + F(y)\|\big| \leq \varepsilon, \tag{1.4}$$

where $F : G \to Y$ is surjective. Further, Sikorska [18] showed the stability of the functional equation for (1.1).

Theorem 1.4 (Sikorska [18]) *Let $(G, +)$ be an abelian group, Y be a Banach space, $\delta, \varepsilon \geq 0$. If $F : G \to Y$ is a δ-surjective mapping satisfying*

$$\big|\|F(x-y)\| - \|F(x) - F(y)\|\big| \leq \varepsilon, \quad x, y \in G,$$

then

$$\|F(x+y) - F(x) - F(y)\| \leq 5\varepsilon + 5\delta, \quad x, y \in G.$$

Since 2013, Dong generalized stability of near-surjective equations (1.1) and (1.4), where the error is unbounded (see [5–8]). However, the near-surjectivity assumption also cannot be omitted.

This paper is organized as follows. In Sect. 2, we show that if a mapping $f : X \to Y$ with $f(0) = 0$ satisfies

$$\big| \| f(x) - f(y) \| - \|x - y\| \big| \le \delta \big(\|x\|^r + \|y\|^r \big) + \varepsilon,$$

where X is a Banach space, Y is a uniformly convex space with power type p, and $\delta, \varepsilon \ge 0$, $0 < r < 1$, then there are two nonnegative constants $K(\delta, r, p)$, $\widetilde{K}(\varepsilon)$ with $\lim_{\delta \to 0} K(\delta, r, p) = 0$, $\lim_{\varepsilon \to 0} \widetilde{K}(\varepsilon) = 0$ and a linear isometry $U : X \to Y$ such that

$$\| f(x) - Ux \| \le K(\delta, r, p) \max\big\{ \|x\|^r, \|x\|^{1-(1-r)/p} \big\} + \widetilde{K}(\varepsilon) \max\big\{ 1, \|x\|^{1-1/p} \big\}, \quad x \in X.$$

This is a generalization of Dolinar' results [4] for (δ, r)-isometries of Hilbert spaces and L_p $(1 < p < \infty)$ spaces. In Sect. 3, by using the stability result above for perturbation isometries, we obtain that if $F : G \to Y$ satisfies (1.1) and $d(u, F(G)) \le \delta \|u\|^r$ for every $u \in Y$, then there is an additive operator $A : G \to Y$ such that

$$\| F(x) - Ax \| = o\big(\| F(x) \| \big) \quad \text{as } \| F(x) \| \to \infty.$$

As a result, we also give a stability result for the mappings which preserve the equality of distance.

In this paper, the letters X, Y are used to denote real Banach spaces, X^*, Y^* are their dual spaces. We also denote that G is an abelian group. For a real Banach space X, we denote by S_X and B_X the unit sphere and the closed unit ball of X, respectively.

2 Asymptotical stability of nonsurjective (δ, r, ε)-isometries

The main results of this section are inspired by [2, Theorem 2.5].

Definition 2.1 Let $f : X \to Y$ be a mapping, $\delta, \varepsilon \ge 0$, $0 < r < 1$. Then f is called a (δ, r, ε)-isometry if

$$\big| \| f(x) - f(y) \| - \|x - y\| \big| \le \delta \big(\|x\|^r + \|y\|^r \big) + \varepsilon, \quad x, y \in X. \tag{2.1}$$

We say that f is standard if $f(0) = 0$.

Given a nonzero $x \in X$, we define $g : \mathbb{R} \to Y$ as follows:

$$g(t) = \frac{f(tx)}{\|x\|}, \quad t \in \mathbb{R}.$$

Then for each $s, t \in \mathbb{R}$,

$$\begin{aligned} \big| \| g(t) - g(s) \| - |s - t| \big| &\le \frac{\big| \| f(tx) - f(sx) \| - |s - t| \, \|x\| \big|}{\|x\|} \\ &\le \delta \big(|s|^r + |t|^r \big) \|x\|^{r-1} + \varepsilon \|x\|^{-1}. \end{aligned} \tag{2.2}$$

Lemma 2.2 *Suppose that* $g : \mathbb{R} \to Y$ *is defined as above. Then for every* $n \in \mathbb{N}$, *there exists* $\varphi_n \in S_{Y^*}$ *such that*

$$\left|\langle \varphi_n, g(t)\rangle - t\right| \le 3\delta n^r \|x\|^{r-1} + 2\varepsilon \|x\|^{-1}, \quad t \in [0, n].$$

Proof By Hahn–Banach theorem (see [12, p. 75–76]), for each $n \in \mathbb{N}$, there exists $\varphi_n \in S_{Y^*}$ so that

$$\langle \varphi_n, g(n)\rangle = \|g(n)\| = \frac{f(nx)}{\|x\|}.$$

Then

$$n - \delta n^r \|x\|^{r-1} - \varepsilon \|x\|^{-1} \le \langle \varphi_n, g(n)\rangle \le n + \delta n^r \|x\|^{r-1} + \varepsilon \|x\|^{-1}.$$

Given $n \in \mathbb{N}$, on the one hand,

$$\langle \varphi_n, g(t)\rangle \le \|g(t)\| \le t + \delta t^r \|x\|^{r-1} + \varepsilon \|x\|^{-1} \le t + \delta n^r \|x\|^{r-1} + \varepsilon \|x\|^{-1}, \quad t \in [0, n].$$

On the other hand,

$$\langle \varphi_n, g(n)\rangle - \langle \varphi_n, g(t)\rangle \le \|g(n) - g(t)\| \le n - t + 2\delta n^r \|x\|^{r-1} + \varepsilon \|x\|^{-1}, \quad t \in [0, n].$$

Thus,

$$\begin{aligned}\langle \varphi_n, g(t)\rangle &= \langle \varphi_n, g(n)\rangle - \left(\langle \varphi_n, g(n)\rangle - \langle \varphi_n, g(t)\rangle\right) \\ &\ge n - \delta n^r \|x\|^{r-1} - \varepsilon \|x\|^{-1} - \left(n - t + 2\delta n^r \|x\|^{r-1} + \varepsilon \|x\|^{-1}\right) \\ &\ge t - 3\delta n^r \|x\|^{r-1} - 2\varepsilon \|x\|^{-1}, \quad t \in [0, n].\end{aligned}$$

Therefore,

$$\left|\langle \varphi_n, g(t)\rangle - t\right| \le 3\delta n^r \|x\|^{r-1} + 2\varepsilon \|x\|^{-1}, \quad t \in [0, n]. \qquad \square$$

The modulus of convexity of a Banach space Y is the function $\delta_Y : [0, 2] \to [0, 1]$ defined by

$$\delta_Y(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in B_Y, \|x - y\| \ge \epsilon\right\}.$$

Definition 2.3 ([16]) A Banach space Y is said to be uniformly convex if $\delta_Y(\epsilon) > 0$ for all $0 < \epsilon \le 2$. If $p \ge 1$, we say that a uniformly convex Banach space Y has power type p if there is a constant $C > 0$ so that $\delta_Y(\epsilon) \ge C\epsilon^p$ for all $0 < \epsilon \le 2$.

Remark 2.4 Pisier [16] showed that every uniformly convex Banach spaces can be renormed to admit power type p for some $2 \le p < +\infty$.

Theorem 2.5 *Suppose that Y is a uniformly convex space with power type p, and that $f : X \to Y$ is a standard (δ, r, ε)-isometry. Then there are constants $K(\delta, r, p) \geq 0$, $\widetilde{K}(\varepsilon) \geq 0$ with $\lim_{\delta\to 0} K(\delta, r, p) = 0$, $\lim_{\varepsilon\to 0} \widetilde{K}(\varepsilon) = 0$ and a linear isometry $U : X \to Y$ such that*

$$\begin{aligned}\|f(x) - Ux\| &\leq K(\delta, r, p)\max\{\|x\|^r, \|x\|^{1-(1-r)/p}\} \\ &\quad + \widetilde{K}(\varepsilon)\max\{1, \|x\|^{1-1/p}\}, \quad x \in X.\end{aligned}$$

Proof Given $x \in X \setminus \{0\}$, $t \in \mathbb{R}$, $n \in \mathbb{N}$, let $g(t) = \frac{f(tx)}{\|x\|}$, $y_n = \frac{f(2^n x)}{\|2^n x\|}$, $z_n = \frac{f(2^n x)}{\|f(2^n x)\|}$. Then $z_n \in S_Y$ and

$$\|y_n - z_n\| = \|f(2^n x)\| \left|\frac{1}{\|2^n x\|} - \frac{1}{\|f(2^n x)\|}\right| \leq \frac{\delta\|2^n x\|^r + \varepsilon}{\|2^n x\|} = \delta\|2^n x\|^{r-1} + \varepsilon\|2^n x\|^{-1}. \quad (2.3)$$

From Lemma 2.2, there exists $\varphi_{2^n} \in S_{Y^*}$ so that

$$|\langle\varphi_{2^n}, g(t)\rangle - t| \leq 3\delta 2^{nr}\|x\|^{r-1} + 2\varepsilon\|x\|^{-1}, \quad t \in [0, 2^n].$$

This implies that

$$\begin{aligned}&\left|\left\langle\varphi_{2^n}, \frac{g(2^n)}{2^n}\right\rangle - 1\right| \leq \frac{3\delta 2^{nr}\|x\|^{r-1} + 2\varepsilon\|x\|^{-1}}{2^n}, \quad \text{and} \\ &\left|\left\langle\varphi_{2^n}, \frac{g(2^{n-1})}{2^{n-1}}\right\rangle - 1\right| \leq \frac{3\delta 2^{nr}\|x\|^{r-1} + 2\varepsilon\|x\|^{-1}}{2^{n-1}}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\|y_{n-1} + y_n\|}{2} &\geq \left\langle\varphi_{2^n}, \frac{y_{n-1} + y_n}{2}\right\rangle \\ &= \frac{1}{2}\left\langle\varphi_{2^n}, \frac{g(2^{n-1})}{2^{n-1}} + \frac{g(2^n)}{2^n}\right\rangle \\ &\geq 1 - \frac{1}{2}\left(3\delta 2^{nr}\|x\|^{r-1} + 2\varepsilon\|x\|^{-1}\right)\left(\frac{1}{2^n} + \frac{1}{2^{n-1}}\right) \\ &\geq 1 - \frac{9}{2}\left(\delta\|2^n x\|^{r-1} + \varepsilon\|2^n x\|^{-1}\right).\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}\frac{\|z_{n-1} + z_n\|}{2} &= \frac{\|y_{n-1} + y_n\|}{2} - \left(\frac{\|y_{n-1} + y_n\|}{2} - \frac{\|z_{n-1} + z_n\|}{2}\right) \\ &\geq \frac{\|y_{n-1} + y_n\|}{2} - \frac{1}{2}\left(\|y_{n-1} - z_{n-1}\| + \|y_n - z_n\|\right) \\ &\geq 1 - \frac{9}{2}\left(\delta\|2^n x\|^{r-1} + \varepsilon\|2^n x\|^{-1}\right) \\ &\quad - \frac{1}{2}\left(\delta\|2^{n-1} x\|^{r-1} + \varepsilon\|2^{n-1} x\|^{-1} + \delta\|2^n x\|^{r-1} + \varepsilon\|2^n x\|^{-1}\right) \\ &= 1 - C_1(\delta, r)\|2^n x\|^{r-1} - \widetilde{C}_1(\varepsilon)\|2^n x\|^{-1},\end{aligned}$$

where $C_1(\delta,r) = (\frac{2^{1-r}+10}{2})\delta$, $\widetilde{C_1}(\varepsilon) = 6\varepsilon$. Since Y is uniformly convex with power type p, there is a constant $C > 0$ such that $\delta_Y(\epsilon) \geq C\epsilon^p$ for all $0 < \epsilon \leq 2$. Then

$$\begin{aligned} C\|z_{n-1} - z_n\|^p &\leq \delta_Y\big(\|z_{n-1} - z_n\|\big) \leq 1 - \frac{\|z_{n-1} + z_n\|}{2} \\ &\leq C_1(\delta,r)\big\|2^n x\big\|^{r-1} + \widetilde{C_1}(\varepsilon)\big\|2^n x\big\|^{-1}. \end{aligned}$$

It follows that $\|z_{n-1} - z_n\| \leq (\frac{C_1(\delta,r)}{C})^{1/p}\|2^n x\|^{(r-1)/p} + (\frac{\widetilde{C_1}(\varepsilon)}{C})^{1/p}\|2^n x\|^{-1/p}$. Again by (2.3),

$$\begin{aligned} \|y_{n-1} - y_n\| &\leq \|y_{n-1} - z_{n-1}\| + \|z_{n-1} - z_n\| + \|z_n - y_n\| \\ &\leq \delta\big\|2^{n-1}x\big\|^{r-1} + \varepsilon\big\|2^{n-1}x\big\|^{-1} + \left(\frac{C_1(\delta,r)}{C}\right)^{1/p}\big\|2^n x\big\|^{(r-1)/p} \\ &\quad + \left(\frac{\widetilde{C_1}(\varepsilon)}{C}\right)^{1/p}\big\|2^n x\big\|^{-1/p} + \delta\big\|2^n x\big\|^{r-1} + \varepsilon\big\|2^n x\big\|^{-1} \\ &= \big(2^{1-r} + 1\big)\delta\big\|2^n x\big\|^{r-1} + 3\varepsilon\big\|2^n x\big\|^{-1} + \left(\frac{C_1(\delta,r)}{C}\right)^{1/p}\big\|2^n x\big\|^{(r-1)/p} \\ &\quad + \left(\frac{\widetilde{C_1}(\varepsilon)}{C}\right)^{1/p}\big\|2^n x\big\|^{-1/p}. \end{aligned}$$

Because of $0 < \frac{1-r}{p} \leq 1 - r < 1$, then $(2^n)^{r-1} \leq (2^n)^{(r-1)/p}$ and $2^{-n} \leq 2^{-n/p}$. Thus

$$\begin{aligned} \|y_{n-1} - y_n\| &\leq 2^{n(r-1)/p}\left(\left(\frac{C_1(\delta,r)}{C}\right)^{1/p}\|x\|^{(r-1)/p} + \big(2^{1-r} + 1\big)\delta\|x\|^{r-1}\right) \\ &\quad + 2^{-n/p}\left(\left(\frac{\widetilde{C_1}(\varepsilon)}{C}\right)^{1/p}\|x\|^{-1/p} + 3\varepsilon\|x\|^{-1}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \left\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}}\right\| &= \|y_{n-1} - y_n\| \cdot \|x\| \\ &\leq 2^{n(r-1)/p}\left(\left(\frac{C_1(\delta,r)}{C}\right)^{1/p}\|x\|^{1-(1-r)/p} + \big(2^{1-r} + 1\big)\delta\|x\|^{r}\right) \\ &\quad + 2^{-n/p}\left(\left(\frac{\widetilde{C_1}(\varepsilon)}{C}\right)^{1/p}\|x\|^{1-1/p} + 3\varepsilon\right). \end{aligned}$$

Put $C_2(\delta,r,p) = (\frac{C_1(\delta,r)}{C})^{1/p} + (2^{1-r} + 1)\delta$, $\widetilde{C_2}(\varepsilon) = (\frac{\widetilde{C_1}(\varepsilon)}{C})^{1/p} + 3\varepsilon$. Then $\lim_{\delta\to 0} C_2(\delta,r,p) = 0$, $\lim_{\varepsilon\to 0}\widetilde{C_2}(\varepsilon) = 0$, and

$$\begin{aligned} \left\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}}\right\| &\leq C_2(\delta,r,p)2^{n(r-1)/p}\max\big\{\|x\|^r, \|x\|^{1-(1-r)/p}\big\} \\ &\quad + \widetilde{C_2}(\varepsilon)2^{-n/p}\max\big\{1, \|x\|^{1-1/p}\big\}. \end{aligned}$$

Let $n, m \in \mathbb{N}$ with $n > m$. We have

$$\begin{aligned}\left\| \frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \right\| &\le C_2(\delta, r, p) \sum_{k=m+1}^{n} 2^{k(r-1)/p} \max\left\{ \|x\|^r, \|x\|^{1-(1-r)/p} \right\} \\ &\quad + \widetilde{C}_2(\varepsilon) \sum_{k=m+1}^{n} 2^{-k/p} \max\left\{ 1, \|x\|^{1-1/p} \right\} \\ &\le \frac{2^{(m+1)(r-1)/p}}{1 - 2^{(r-1)/p}} C_2(\delta, r, p) \max\left\{ \|x\|^r, \|x\|^{1-(1-r)/p} \right\}. \\ &\quad + \frac{2^{-(m+1)/p}}{1 - 2^{-1/p}} \widetilde{C}_2(\varepsilon) \max\left\{ 1, \|x\|^{1-1/p} \right\}. \end{aligned} \tag{2.4}$$

It follows from $\frac{r-1}{p} < 0$ that $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence. We define $U : X \to Y$ by

$$Ux = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

From $0 < r < 1$ and since f is a (δ, r, ε)-isometry, we obtain that

$$\begin{aligned}\big| \|Ux - Uy\| - \|x - y\| \big| &= \lim_{n\to\infty} \left| \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right\| - \|x - y\| \right| \\ &\le \lim_{n\to\infty} \frac{\delta \cdot 2^{nr}(\|x\|^r + \|y\|^r) + \varepsilon}{2^n} \\ &= \lim_{n\to\infty} \frac{\delta(\|x\|^r + \|y\|^r) + \varepsilon}{2^{n(1-r)}} \\ &= 0 \quad \text{for each } x, y \in X. \end{aligned}$$

Since Y is strictly convex, U is a linear isometry. Taking $m = 0$ in (2.4) and letting $n \to \infty$, we obtain that

$$\|f(x) - Ux\| \le K(\delta, r, p) \max\left\{ \|x\|^r, \|x\|^{1-(1-r)/p} \right\} + \widetilde{K}(\varepsilon) \max\left\{ 1, \|x\|^{1-1/p} \right\}, \quad x \in X.$$

Here $K(\delta, r, p) = \frac{2^{(r-1)/p}}{1-2^{(r-1)/p}} C_2(\delta, r, p)$, $\widetilde{K}(\varepsilon) = \frac{2^{-1/p}}{1-2^{-1/p}} \widetilde{C}_2(\varepsilon)$. Clearly, $\lim_{\delta\to 0} K(\delta, r, p) = 0$ and $\lim_{\varepsilon\to 0} \widetilde{K}(\varepsilon) = 0$. □

Note that Hilbert spaces have power type 2 and L_p spaces have power type p if $p > 2$; and 2 if $1 < p \le 2$ (see [3, Theorem 1, p. 69]). Then, by Theorem 2.5, we have the following corollaries which were obtained by Dolinar.

Corollary 2.6 ([4, Proposition 2]) *Let Y be a Hilbert space, and let $f : X \to Y$ be a standard mapping satisfying*

$$\big| \|f(x) - f(y)\| - \|x - y\| \big| \le \delta \|x - y\|^r, \quad x, y \in X, \tag{2.5}$$

for some $\delta \ge 0$ and $0 < r < 1$. Then there exist a linear isometry $U : X \to Y$ and a constant $K(\delta, r) \ge 0$ such that $\lim_{\delta\to 0} K(\delta, r) = 0$ and

$$\|f(x) - Ux\| \le K(\delta, r) \max\left\{ \|x\|^r, \|x\|^{(1+r)/2} \right\}, \quad x \in X.$$

Corollary 2.7 ([4, Proposition 3]) *Let $Y = L_p$ $(1 < p < \infty)$, and let $f : X \to Y$ be a standard mapping satisfying*

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \le \delta\|x - y\|^r, \quad x, y \in X, \tag{2.6}$$

for some $\delta \ge 0$ and $0 < r < 1$. Then there exist a linear isometry $U : X \to Y$ and a constant $K(\delta, r, p) \ge 0$ such that $\lim_{\delta\to 0} K(\delta, r, p) = 0$ and

$$\|f(x) - Ux\| \le K(\delta, r, p) \max\{\|x\|^r, \|x\|^{1-(1-r)/p}\}, \quad x \in X.$$

3 (δ, r)-Surjective functional equations

Let $(G, +)$ be an abelian group, $0 \in G$ be a unit element, and Y be a Banach space. We say that a mapping $F : G \to Y$ is (δ, r)-surjective if $d(u, F(G)) \le \delta\|u\|^r$ for every $u \in Y$; F is said to be standard if $F(0) = 0$.

Theorem 3.1 *Suppose that Y is a uniformly convex space with power type p, and that $F : G \to Y$ is a (δ, r)-surjective mapping with*

$$\big|\|F(x) - F(y)\| - \|F(x - y)\|\big| \le \varepsilon,$$

where $\delta, \varepsilon \ge 0$ and $0 < r < 1$. Then there are constants $K(\delta, r, p)$ and $\widetilde{K}(2\varepsilon)$ such that

$$\begin{aligned}\|F(x + y) - F(x) - F(y)\| \le{}& K(\delta, r, p) \max\{\|F(y)\|^r, \|F(y)\|^{1-(1-r)/p}\}\\ &+ \widetilde{K}(2\varepsilon) \max\{1, \|F(y)\|^{1-1/p}\} + \varepsilon, \quad x, y \in G.\end{aligned}$$

Proof Given $x \in G$, a set-valued mapping $\Phi_x : Y \to 2^Y$ is defined by

$$\Phi_x(u) = \{F(a_u + x) - F(x) : a_u \in F^{-1}(B(u, \delta\|u\|^r))\}, \quad u \in Y.$$

When F is (δ, r)-surjective, this entails that $\Phi_x(u) \neq \emptyset$ for every $u \in Y$. Fixing $u, v \in Y$, for each $z_u \in \Phi_x(u)$, $z_v \in \Phi_x(v)$, there exist $a_u \in F^{-1}(B(u, \delta\|u\|^r))$ and $a_v \in F^{-1}(B(v, \delta\|v\|^r))$ so that $z_u = F(a_u + x) - F(x)$ and $z_v = F(a_v + x) - F(x)$. Then

$$\begin{aligned}\big|\|z_u - z_v\| - \|u - v\|\big| &\le \big|\|F(a_u + x) - F(a_v + x)\| - \|F(a_u - a_v)\|\big|\\ &\quad + \big|\|F(a_u - a_v)\| - \|F(a_u) - F(a_v)\|\big|\\ &\quad + \|F(a_u) - u\| + \|F(a_v) - v\|\\ &\le \delta(\|u\|^r + \|v\|^r) + 2\varepsilon.\end{aligned} \tag{3.1}$$

In particular, if $u = v$ (i.e., $z_u, z_v \in \Phi_x(u)$), we have

$$\|z_u - z_v\| \le 2\delta\|u\|^r + 2\varepsilon. \tag{3.2}$$

Let $g_x : Y \to Y$ be an arbitrary selection of Φ_x. It follows from (3.1) that

$$\big|\|g_x(u) - g_x(v)\| - \|u - v\|\big| \le \delta(\|u\|^r + \|v\|^r) + 2\varepsilon. \tag{3.3}$$

Then, by Theorem 2.5, there exist a linear isometry $U_{g_x} : Y \to Y$ and two constants $K(\delta, r, p)$, $\widetilde{K}(2\varepsilon)$ such that

$$\begin{aligned} \|g_x(u) - g_x(0) - U_{g_x}(u)\| \le K(\delta, r, p) \max\{\|u\|^r, \|u\|^{1-(1-r)/p}\} \\ + \widetilde{K}(2\varepsilon) \max\{1, \|u\|^{1-1/p}\}, \quad u \in Y. \end{aligned} \tag{3.4}$$

Since $g_x(0) \in \Phi_x(0)$, we can find $a \in F^{-1}(0)$ so that $g_x(0) = F(a + x) - F(x)$. Thus,

$$\|g_x(0)\| \le \big| \|F(a + x) - F(x)\| - \|F(a)\| \big| + \|F(a)\| \le \varepsilon.$$

Therefore,

$$\begin{aligned} \|g_x(u) - U_{g_x}(u)\| \le K(\delta, r, p) \max\{\|u\|^r, \|u\|^{1-(1-r)/p}\} \\ + \widetilde{K}(2\varepsilon) \max\{1, \|u\|^{1-1/p}\} + \varepsilon, \quad u \in Y. \end{aligned} \tag{3.5}$$

Assume that $h_x : Y \to Y$ is another selection of Φ_x. Combining (3.2) and (3.5), we obtain that

$$\begin{aligned} \|U_{g_x}(u) - U_{h_x}(u)\| &\le \|U_{g_x}(u) - g_x(u)\| + \|g_x(u) - h_x(u)\| + \|h_x(u) - U_{h_x}(u)\| \\ &\le 2K(\delta, r, p) \max\{\|u\|^r, \|u\|^{1-(1-r)/p}\} + 2\widetilde{K}(2\varepsilon) \max\{1, \|u\|^{1-1/p}\} \\ &\quad + 2\delta\|u\|^r + 4\varepsilon, \quad u \in Y. \end{aligned} \tag{3.6}$$

Note that if $0 < r < 1$, $p \ge 1$, and U_{g_x}, U_{h_x} are two linear isometries, then for each $u \in Y$,

$$\begin{aligned} \|U_{g_x}(u) - U_{h_x}(u)\| &= \frac{\|U_{g_x}(nu) - U_{h_x}(nu)\|}{n} \\ &\le \frac{2K(\delta, r, p) \max\{\|nu\|^r, \|nu\|^{1-(1-r)/p}\}}{n} \\ &\quad + \frac{2\widetilde{K}(2\varepsilon) \max\{1, \|nu\|^{1-1/p}\} + 2\delta\|u\|^r + 4\varepsilon}{n} \\ &\to 0, \quad n \to \infty. \end{aligned}$$

This implies that $U_{g_x} = U_{h_x}$. We denote U_{g_x} by U_x.

In what follows, we shall prove that

$$U_x = id_Y.$$

Firstly, we show

$$U_{x_1} = U_{x_2}, \quad x_1, x_2 \in G.$$

Given $x, y \in G$, let $u = F(y)$, and then $y \in F^{-1}(u) \subseteq F^{-1}(B(u, \delta\|u\|^r))$ and $F(y + x) - F(x)$ is a value of a selection at u of Φ_x. By (3.5), we have

$$\begin{aligned} \|F(y + x) - F(x) - U_x(u)\| \le K(\delta, r, p) \max\{\|u\|^r, \|u\|^{1-(1-r)/p}\} \\ + \widetilde{K}(2\varepsilon) \max\{1, \|u\|^{1-1/p}\} + \varepsilon. \end{aligned} \tag{3.7}$$

For every $x_1, x_2 \in G$,

$$\begin{aligned}\|U_{x_1}(u) - U_{x_2}(u)\| &\le \|F(y+x_1) - F(x_1) - U_{x_1}(u)\| + \|F(y+x_2) - F(x_2) - U_{x_2}(u)\| \\ &\quad + \|F(y+x_1) - F(y+x_2)\| + \|F(x_1) - F(x_2)\| \\ &\le 2K(\delta, r, p)\max\{\|u\|^r, \|u\|^{1-(1-r)/p} + 2\widetilde{K}(2\varepsilon)\max\{1, \|u\|^{1-1/p}\} \\ &\quad + 3\varepsilon + \|F(x_1 - x_2)\| + \|F(x_1) - F(x_2)\|.\end{aligned}$$

Combining $0 < r < 1$ and the linearity of U_{x_1}, U_{x_2}, we obtain that $U_{x_1}(u) = U_{x_2}(u)$ for all $u \in F(G)$. Since F is (δ, r)-surjective, for each $w \in Y$, $n \in \mathbb{N}$, there exists $z_n \in G$ such that $\|nw - F(z_n)\| \le \delta\|nw\|^r$. Then

$$\begin{aligned}\|U_{x_1}(w) - U_{x_2}(w)\| &= \frac{\|U_{x_1}(nw) - U_{x_2}(nw)\|}{n} \\ &\le \frac{\|U_{x_1}(nw) - U_{x_1}(F(z_n))\|}{n} + \frac{\|U_{x_2}(nw) - U_{x_2}(F(z_n))\|}{n} \\ &\quad + \frac{\|U_{x_1}(F(z_n)) - U_{x_2}(F(z_n))\|}{n} \\ &\le 2\delta\frac{\|nw\|^r}{n} \to 0, \quad n \to \infty.\end{aligned}$$

This implies that $x_1, x_2 \in G$, $U_{x_1} = U_{x_2}$. Putting $x = 0$ in (3.7), we obtain that

$$\begin{aligned}\|u - U_0(u)\| &\le K(\delta, r, p)\max\{\|u\|^r, \|u\|^{1-(1-r)/p}\} \\ &\quad + \widetilde{K}(2\varepsilon)\max\{1, \|u\|^{1-1/p}\} + \varepsilon, \quad u \in F(G).\end{aligned} \tag{3.8}$$

In the following, we prove

$$U_0 = id_Y.$$

Again since F is (δ, r)-surjective, for every $w \in Y$, $n \in \mathbb{N}$, there exists $z_n \in G$ such that $\|nw - F(z_n)\| \le \delta\|nw\|^r$. This entails that $\lim_{n\to\infty} \frac{F(z_n)}{n} = w$. Then by (3.8), we obtain that for each $w \in Y$,

$$\begin{aligned}\|w - U_0(w)\| &= \frac{\|nw - U_0(nw)\|}{n} \\ &\le \frac{\|nw - F(z_n)\|}{n} + \frac{\|F(z_n) - U_0(F(z_n))\|}{n} + \frac{\|U_0(F(z_n)) - U_0(nw)\|}{n} \\ &\le 2\delta\frac{\|nw\|^r}{n} + \frac{K(\delta, r, p)\max\{\|F(z_n)\|^r, \|F(z_n)\|^{1-(1-r)/p}\}}{n} \\ &\quad + \frac{\widetilde{K}(2\varepsilon)\max\{1, \|F(z_n)\|^{1-1/p}\} + \varepsilon}{n} \\ &\to 0, \quad n \to \infty.\end{aligned}$$

Thus

$$U_0 = id_Y.$$

Therefore, from (3.7), we obtain that

$$\begin{aligned}\|F(x+y)-F(x)-F(y)\| &\le K(\delta,r,p)\max\{\|F(y)\|^r,\|F(y)\|^{1-(1-r)/p}\}\\ &\quad+\widetilde{K}(2\varepsilon)\max\{1,\|F(y)\|^{1-1/p}\}+\varepsilon,\quad x,y\in G.\end{aligned}$$

□

Theorem 3.2 *Suppose that Y is a uniformly convex space with power type p, and that $F:G\to Y$ is a (δ,r)-surjective mapping with*

$$\big|\|F(x)-F(y)\|-\|F(x-y)\|\big|\le\varepsilon, \tag{3.9}$$

where $\delta,\varepsilon\ge 0$ and $0<r<1$. Then there is an additive mapping $A:G\to Y$ such that

$$\|F(x)-Ax\|=o(\|F(x)\|),\quad \|F(x)\|\to\infty.$$

Proof By Theorem 3.1,

$$\begin{aligned}\|F(x+y)-F(x)-F(y)\| &\le K(\delta,r,p)\max\{\|F(y)\|^r,\|F(y)\|^{1-(1-r)/p}\}\\ &\quad+\widetilde{K}(2\varepsilon)\max\{1,\|F(y)\|^{1-1/p}\}+\varepsilon,\quad x,y\in G.\end{aligned}$$

If we replace y for x in inequality above, then

$$\begin{aligned}\|F(2x)-2F(x)\| &\le K(\delta,r,p)\max\{\|F(x)\|^r,\|F(x)\|^{1-(1-r)/p}\}\\ &\quad+\widetilde{K}(2\varepsilon)\max\{1,\|F(x)\|^{1-1/p}\}+\varepsilon.\end{aligned}$$

By substituting $2^n x$ for x, and dividing by 2^{n+1} in the inequality above, we observe that

$$\begin{aligned}\left\|\frac{F(2^{n+1}x)}{2^{n+1}}-\frac{F(2^n x)}{2^n}\right\| &\le \frac{K(\delta,r,p)\max\{\|F(2^n x)\|^r,\|F(2^n x)\|^{1-(1-r)/p}\}}{2^{n+1}}\\ &\quad+\frac{\widetilde{K}(2\varepsilon)\max\{1,\|F(2^n x)\|^{1-1/p}\}+\varepsilon}{2^{n+1}}.\end{aligned} \tag{3.10}$$

Next, we study the relationship between $\|F(2^n x)\|$ and $2^n\|F(x)\|$. Let $x=0$ and $y=x$ in (3.9), then

$$\big|\|F(-x)\|-\|F(x)\|\big|\le\varepsilon. \tag{3.11}$$

Letting $y=-x$ in (3.9), we have

$$\big|\|F(2x)\|-\|F(x)-F(-x)\|\big|\le\varepsilon. \tag{3.12}$$

Then

$$\|F(2x)\|\le\|F(x)-F(-x)\|+\varepsilon\le\|F(-x)\|+\|F(x)\|+\varepsilon\le 2(\|F(x)\|+\varepsilon).$$

By mathematical induction, we obtain that

$$\|F(2^n x)\|\le 2^n\|F(x)\|+(2^{n+1}-2)\varepsilon\le 2^n\|F(x)\|+2^{n+1}\varepsilon.$$

Since $0 < r < 1$, we have

$$\|F(2^n x)\|^r \le 2^{nr}\|F(x)\|^r + 2^{nr}(2\varepsilon)^r. \tag{3.13}$$

Combing (3.10) and (3.13), we observe that

$$\begin{aligned}\left\|\frac{F(2^{n+1}x)}{2^{n+1}} - \frac{F(2^n x)}{2^n}\right\| \le \frac{K(\delta,r,p)}{2}\max\bigl\{&2^{n(r-1)}\|F(x)\|^r + 2^{n(r-1)}(2\varepsilon)^r,\\ &2^{n(r-1)/p}\|F(x)\|^{1-(1-r)/p} + 2^{n(r-1)/p}(2\varepsilon)^{1-(1-r)/p}\bigr\}\\ &+ \frac{\widetilde{K}(2\varepsilon)}{2}\max\bigl\{2^{-n}, 2^{-n/p}\|F(x)\|^{1-1/p} + 2^{-n/p}(2\varepsilon)^{1-1/p}\bigr\} + \varepsilon/2^{n+1}.\end{aligned} \tag{3.14}$$

Note that $0 < r < 1$, $p \ge 1$, and then for $n > m$,

$$\sum_{k=m+1}^{n} 2^{k(r-1)}, \sum_{k=m+1}^{n} 2^{k(r-1)/p}, \sum_{k=m+1}^{n} 2^{-k}, \sum_{k=m+1}^{n} 2^{-k/p} \to 0, \quad \text{as } m \to \infty. \tag{3.15}$$

It follows that $\{\frac{F(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in G$ and thus there is a limit function

$$Ax = \lim_{n\to\infty}\frac{F(2^n x)}{2^n}, \quad x \in G.$$

Then by (3.14) and (3.15), there exist constants $L_1, L_2, L_3, L_4 > 0$ such that

$$\begin{aligned}\|F(x) - Ax\| &= \sum_{n=0}^{\infty}\left\|\frac{F(2^{n+1}x)}{2^{n+1}} - \frac{F(2^n x)}{2^n}\right\|\\ &\le L_1\|F(x)\|^r + L_2\|f(x)\|^{1-(1-r)/p} + L_3\|F(x)\|^{1-1/p} + L_4.\end{aligned} \tag{3.16}$$

Clearly,

$$\|F(x) - Ax\| = o(\|F(x)\|), \quad \|F(x)\| \to \infty.$$

Now, we only need to prove that A is additive. For every $x, y \in G$,

$$\begin{aligned}&\left\|\frac{F(2^n(x+y))}{2^n} - \frac{F(2^n x)}{2^n} - \frac{F(2^n y)}{2^n}\right\|\\ &\quad\le \frac{K(\delta,r,p)\max\{\|F(2^n y)\|^r, \|F(2^n y)\|^{1-(1-r)/p}\}}{2^n}\\ &\qquad+ \frac{\widetilde{K}(2\varepsilon)\max\{1, \|F(2^n y)\|^{1-1/p}\} + \varepsilon}{2^n}.\end{aligned} \tag{3.17}$$

Note that $\frac{\|F(2^n y)\|^r}{2^n} = (\frac{\|F(2^n y)\|}{2^n})^r \cdot 2^{(r-1)n} \to 0$ as $n \to \infty$. Similarly, we also have $\frac{\|F(2^n y)\|^{1-(1-r)/p}}{2^n}$, $\frac{\|F(2^n y)\|^{1-1/p}}{2^n} \to 0$ as $n \to \infty$. By letting $n \to \infty$ in (3.17), we obtain $\|A(x+y) - Ax - Ay\| = 0$, i.e., A is additive. □

By Theorem 3.2, we obtain the following corollaries.

Corollary 3.3 *Suppose that Y is a Hilbert space, and that $F : G \to Y$ is a (δ, r)-surjective mapping with*

$$\big| \|F(x) - F(y)\| - \|F(x-y)\| \big| \le \varepsilon,$$

where $\delta, \varepsilon \ge 0$ and $0 < r < 1$. Then there is an additive mapping $A : G \to Y$ such that

$$\|F(x) - Ax\| = o\big(\|F(x)\|\big), \quad \|F(x)\| \to \infty.$$

Corollary 3.4 *Suppose that $Y = L_p$ $(1 < p < \infty)$, and that $F : G \to Y$ is a (δ, r)-surjective mapping with*

$$\big| \|F(x) - F(y)\| - \|F(x-y)\| \big| \le \varepsilon,$$

where $\delta, \varepsilon \ge 0$ and $0 < r < 1$. Then there is an additive mapping $A : G \to Y$ such that

$$\|F(x) - Ax\| = o\big(\|F(x)\|\big), \quad \|F(x)\| \to \infty.$$

As an application of Theorem 3.2, we show the following stability result for maps which preserve equality of distance.

Definition 3.5 ([21]) Let X, Y be Banach spaces, we say that a map $T : X \to Y$ preserves the equality of distance if

$$\|x-y\| = \|u-v\| \quad \Rightarrow \quad \big\|T(x) - T(y)\big\| = \big\|T(u) - T(v)\big\|, \quad x, y, u, v \in X.$$

Lemma 3.6 ([21]) *Let X, Y be Banach spaces with $\dim X \ge 2$, and let $T : X \to Y$ be a mapping which preserves the equality of distance. If for each $\eta > 0$ there exist $x, y \in X$ with $x \ne y$ so that $\|T(x) - T(y)\| < \eta$, then T is uniformly continuous.*

Theorem 3.7 *Let X be a Banach space with $\dim X \ge 2$, Y be a uniformly convex space with power type p, and let $f : X \to Y$ be a (δ, r)-surjective standard mapping, where $\delta, \varepsilon \ge 0$, $0 < r < 1$. If*

$$\|x-y\| = \|u-v\| \quad \Rightarrow \quad \big| \|f(x) - f(y)\| - \|f(u) - f(v)\| \big| \le \varepsilon, \quad x, y, u, v \in X, \tag{3.18}$$

then there exist a constant $\alpha > 0$ and a linear isometry $U : X \to Y$ such that

$$\|f(x) - \alpha Ux\| = o\big(\|f(x)\|\big), \quad \|f(x)\| \to \infty.$$

Proof Substituting $u = x - y$ and $v = 0$ in (3.18), we have

$$\big| \|f(x) - f(y)\| - \|f(x-y)\| \big| \le \varepsilon.$$

By Theorem 3.2, we can find an additive mapping $A : X \to Y$ such that

$$\|f(x) - Ax\| = o\big(\|f(x)\|\big), \quad \|f(x)\| \to \infty.$$

For every $x, y, u, v \in X$ with $\|x - y\| = \|u - v\|$, it follows from (3.18) that

$$\left| \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right\| - \left\| \frac{f(2^n u)}{2^n} - \frac{f(2^n v)}{2^n} \right\| \right| \le \frac{\varepsilon}{2^n}.$$

Letting $n \to \infty$ in the inequality above, $\|Ax - Ay\| = \|Au - Av\|$, i.e., A preserves equality of distance. By the additivity of the mapping A, for each $\eta > 0$ there exist $x, y \in X$ with $x \neq y$ such that $\|Ax - Ay\| < \eta$. According to Lemma 3.6, A is uniformly continuous and then A is linear.

In the following, we show that there exist $\alpha > 0$ and a linear isometry $U : X \to Y$ so that $A = \alpha U$.

For each $x \in S_X$, put $\alpha = \|Ax\|$. Then

$$\|Ay\| = \|y\| \cdot \left\| A\left(\frac{y}{\|y\|}\right) \right\| = \alpha \|y\| \quad \text{for all } y \in X \setminus \{0\}.$$

This entails $\alpha > 0$. We define a mapping $U : X \to Y$ by $U = \alpha^{-1} A$. Clearly, U is a linear isometry and

$$\|f(x) - \alpha Ux\| = o(\|f(x)\|), \quad \|f(x)\| \to \infty. \qquad \square$$

The following example shows that the conditions of (δ, r)-surjectivity and uniform convexity in Theorem 3.7 cannot be removed.

Example 3.8 Let $X = \ell_\infty^2$ and $Y = \ell_\infty^3$, where ℓ_∞^n denotes the vector space $\mathbb{R}^n$, endowed with the supremum norm $\|\cdot\|$ defined for $x = (x_1, x_2, \ldots, x_n) \in \ell_\infty^n$ by $\|(x_1, x_2, \ldots, x_n)\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$. Then Y is not uniformly convex. We define $f : \ell_\infty^2 \to \ell_\infty^3$ as follows:

$$f(s,t) = (s, t, \max\{|s|, |t|\}), \quad (s,t) \in \ell_\infty^2.$$

Clearly, f is a nonlinear isometry and (3.18) holds. It is also easy to show that f does not satisfy the (δ, r)-surjectivity condition, where $\delta \geq 0$, $0 < r < 1$.

Next, we shall prove that there is no linear isometry $U : \ell_\infty^2 \to \ell_\infty^3$ and $\alpha > 0$ such that

$$\|f(s,t) - \alpha U(s,t)\| = o(\|f(s,t)\|), \quad \|f(s,t)\| \to \infty.$$

Otherwise, if there were a linear isometry $U : \ell_\infty^2 \to \ell_\infty^3$ and a constant $\alpha > 0$ such that the above formula holds, then

$$\lim_{\|f(s,t)\| \to \infty} \frac{\|f(s,t) - \alpha U(s,t)\|}{\|f(s,t)\|} = \lim_{\|(s,t)\| \to \infty} \frac{\|f(s,t) - \alpha U(s,t)\|}{\|(s,t)\|} = 0.$$

It follows that for every $(s,t) \in \ell_\infty^2 \setminus \{0\}$,

$$\alpha U(s,t) = \lim_{n \to \infty} \frac{f(n(s,t))}{n} = (s, t, \max\{|s|, |t|\}) = f(s,t).$$

This is a contradiction to the linearity of U.

4 Conclusion

In this article, we mainly studied the stability of functional equation (1.1) with a nonsurjectivity condition. Firstly, we gave an asymptotical stability result of perturbed isometries of uniformly convex spaces. Next, we showed that if a standard mapping $F : G \to Y$ satisfies (1.1) and $d(u, F(G)) \leq \delta \|u\|^r$ for every $u \in Y$, where G is an abelian group and Y is a uniformly convex space with power type p, then there is an additive operator $A : G \to Y$ such that $\|F(x) - Ax\| = o(\|F(x)\|)$ as $\|F(x)\| \to \infty$. As an application, we gave a stability result for the mappings which preserve the equality of distance. For more results in modular spaces and Orlicz-binomial spaces, one may refer to [11, 22].

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Author contributions

All authors contributed to the study conception. Material preparation and analysis were performed by Yuqi Sun and Wen Zhang. We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work. All authors wrote the main manuscript text and reviewed the manuscript.

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