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New types of general single/multiple integral inequalities

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Abstract

By introducing some concepts such as multiple integral inner product (MIIP) and multiple integral inner product space (MIIPS), new series of single/multiple integral inequalities are developed in a systematic way, which produce more accurate bounds on the cross terms from the direct Lyapunov method than those in the literature. Some previous integral inequalities including both single and multiple integral inequalities can be regarded as special cases of the proposed inequalities. Accordingly, such integral inequalities are less conservative in comparison with the existing integral inequalities.

Keywords: Integral inequality; Integral inner product space; Time-delay systems; Stability analysis

1 Introduction

A time-delay control system whose future evolution depends not only on the current state but also on the past state of the system is a special class of infinite dimensional system [1, 2]. A large quantity of practical systems, such as engineering, physics, biology, and economics, can be modeled as time-delay systems [3, 4]. Compared with delay-free systems, the time-delay systems have complicated dynamic features and need in-depth research. From the perspective of both theoretical and practical points, the stability problem of time-delay systems is a fundamental and challenging issue. Time-delays frequently encounter and usually lead to instability of time-delay systems. Therefore, the stability problem of time-delay systems has raised concerns in various fields.

As we all know, Lyapunov theory is a potent weapon to discuss the stability or stabilization and other related control issues, such as synchronization control, dissipativity and passivity, convergence, track control, adaptive control and filter design, and so on. Integral inequality is indispensable and plays a crucial role in addressing the aforementioned issues of various time-delay systems when applying the direct Lyapunov method. More attention has been paid to integral inequalities and a large variety of integral inequalities have been developed during the past several decades. Jensen integral inequality (JII) was initially established to deal with the stability of time-delay systems [4]. However, the JII entails considerable conservatism. Later, Wirtinger integral inequality (WII) was developed based on Fourier theory, which includes the JII as a special case and is less conservative than the JII [5–7]. Subsequently, we presented an extended Wirtinger integral inequality (EWII) in Ref. [8], which is less conservative than the WII. Recently, by introducing the Legendre polynomials, a canonical Bessel–Legendre integral inequality (BLII) [9–12] was developed to address the stability problem for time-delay systems. It is pointed out in Ref. [13] that the BLII includes the WII, the AFII, and the EWII as its special cases. Very recently, Yang Z et al. [14] proposed a new Henry–Gronwall integral inequality to analyze the stability of delayed fractional-order neural networks systems.

Recently, Zamart C and Botmart T [15] proposed a novel integral inequality with an exponential function, which covers the WII, to investigate the finite-time boundedness for some delayed generalized neural networks. Nowadays, Halanay inequality (HI) becomes an efficient tool to discuss stability of systems with time-varying delays. However, the ordinary HI has some limitations. In order to handle the stability issue of some fractional neural networks, a novel fractional non-autonomous HI has been developed in the literature [16]. Utilizing a vector-valued function to replace the scalar function in the ordinary HI, Mazenc et al. [17] established two vector-form Halanay inequalities, which extend the HI and provides relaxed versions of the HI.

Although various classes integral inequalities, such as the above inequalities and others, have been presented in the literature, on the one hand, these inequalities are scattered and unsystematic. Such inequalities are still conservative on the other hand. It is essential that we should establish a theoretical framework for single and multiple integral inequalities with less conservatism.

Inspired by the discussion above, we initiate the present work. The main contributions of this paper are summarized as follows. Firstly, a new type of multiple integral inner product (MIIP) and a new multiple integral inner product space (MIIPS) are creatively introduced. Secondly, with the help of such concepts, general single/multiple integral inequalities are to be developed in a systematic way, which generalize and outperform previous integral inequalities, including single and multiple integral inequalities.

Notations Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with Euclidean norm $\|\cdot\|$, $\mathbb{R}^{m\times n}$ is the set of all $m\times n$ real matrices. $P>0$ ($P\geq 0$) means that P is a symmetric and positive definite matrix (positive semi-definite matrix). The notations $\mathbb{S}_n$ and $\mathbb{S}_n^+$ represent the set of symmetric and symmetric positive definite matrices of $\mathbb{R}^{n\times n}$, respectively. The superscript "T" denotes the transpose of a matrix. The notation $\binom{k}{l}$ refers to the binomial coefficients given by $\frac{k!}{(k-l)!l!}$. $\mathbb{N}$ and $\mathbb{N}^+$ denote the sets of nonnegative and positive integers, respectively. Unless explicitly stated, in what follows, matrices are assumed to have compatible dimensions with context.

2 Preliminaries

In this section, we will introduce some definitions and preliminaries necessary to derive main results.

Definition 1 A type of multiple integrals defined by

$$\langle p_k,p_l\rangle_m \triangleq \int_a^b ds_1\int_{s_1}^b ds_2\cdots\int_{s_{m-1}}^b p_k(s_m)p_l(s_m)\,ds_m,$$

$$[p_k,p_l]_m \triangleq \int_a^b ds_1\int_a^{s_1} ds_2\cdots\int_a^{s_{m-1}} p_k(s_m)p_l(s_m)\,ds_m,$$

between polynomial functions $p_k(u)$, $p_l(u)$, $k,l \in \mathbb{N}$ is said to be a multiple integral inner product (MIIP). Correspondingly, the set consisting of the polynomial functions defined such inner product is said to be a multiple integral inner product space (MIIPS). In particular, when $m = 1$, the MIIP and MIIPS become ordinary integral inner product and ordinary integral inner product space, respectively.

Definition 2 Two polynomial functions $p_k(u)$, $p_l(u)$, $k,l \in \mathbb{N}$ are said to be orthogonal if they satisfy

$$\langle p_k, p_l\rangle_m = \begin{cases} 0, & k \neq l, \\ r_k, & k = l, \end{cases}$$

or

$$[p_k, p_l]_m = \begin{cases} 0, & k \neq l, \\ \widetilde{r}_k, & k = l, \end{cases}$$

where $r_k, \widetilde{r}_k$ are nonzero numbers. In reality, $r_k, \widetilde{r}_k$ can be expressed as two types of norms, i.e., $r_k = \|p_k(\cdot)\|_1^2, \widetilde{r}_k = \|p_k(\cdot)\|_2^2$, where $\|\cdot\|_1$, $\|\cdot\|_2$ represent norms derived by the aforementioned inner products $\langle\cdot,\cdot\rangle_m$, $[\cdot,\cdot]_m$, respectively.

Definition 3 The Legendre polynomial function for $k \in \mathbb{N}$ over an interval $[a,b]$ is defined by

$$L_k(u) = (-1)^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{k+1}{i} \left(\frac{u-a}{b-a}\right)^i,$$

or

$$\widetilde{L}_k(u) = (-1)^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{k+1}{i} \left(\frac{b-u}{b-a}\right)^i.$$

The set of the Legendre polynomials $\{L_k(\cdot) \text{ or } \widetilde{L}_k(\cdot), k \in \mathbb{N}\}$ forms an orthogonal sequence with respect to the inner product described in Definition 1 when $m = 1$.

Thus, the Legendre polynomials $\{L_k(\cdot) \text{ or } \widetilde{L}_k(\cdot), k \in \mathbb{N}\}$ have the following properties:

1) Orthogonality:

$$\forall k,l \in \mathbb{N}, \quad \int_a^b L_k(u)L_l(u)\,du = \begin{cases} 0, & k \neq l, \\ \frac{b-a}{2k+1}, & k = l, \end{cases}$$

$$\forall k,l \in \mathbb{N}, \quad \int_a^b \widetilde{L}_k(u)\widetilde{L}_l(u)\,du = \begin{cases} 0, & k \neq l, \\ \frac{b-a}{2k+1}, & k = l. \end{cases}$$

2) Boundary conditions:

$$L_k(a) = (-1)^k, \qquad L_k(b) = 1, \quad k \in \mathbb{N},$$

$$\widetilde{L}_k(b) = (-1)^k, \qquad \widetilde{L}_k(a) = 1, \quad k \in \mathbb{N}.$$

3) Differentiation:

$$\frac{d}{du}L_k(u) = \begin{cases} 0, & k = 0, \\ \sum_{i=0}^{k-1} \frac{2i+1}{b-a}[1 - (-1)^{k+i}]L_i(u), & k \geq 1, \end{cases}$$

$$\frac{d}{du}\widetilde{L}_k(u) = \begin{cases} 0, & k = 0, \\ \sum_{i=0}^{k-1} \frac{2i+1}{b-a}[-1 + (-1)^{k+i}]\widetilde{L}_i(u), & k \geq 1. \end{cases}$$

Proof Proof of these properties can be found in Ref. [18]. □

3 Main results

In this section, we will develop new series of single/multiple integral inequalities and their corollaries.

Theorem 1 *For a constant matrix $M \in \mathbb{S}_n^+$, two scalars a, b satisfying $a < b$, a vector-valued function $\omega(\cdot) : [a, b] \to \mathbb{R}^n$ and $m \in \mathbb{N}^+$, there exists a polynomial function $p_k(s)$ with degree of k, $k = 0, 1, 2, \dots, N$ satisfying orthogonality, i.e., there existing a polynomial function $p_l(s)$ with degree of l, $l = 0, 1, 2, \dots, N$ such that*

$$\langle p_k, p_l \rangle_m = \begin{cases} 0, & k \neq l, \\ r_k, & k = l, \end{cases}$$

or

$$[p_k, p_l]_m = \begin{cases} 0, & k \neq l, \\ \widetilde{r}_k, & k = l, \end{cases}$$

where $r_k, \widetilde{r}_k$ are nonzero real numbers. Then the following inequalities hold

$$\int_a^b ds_1 \int_{s_1}^b ds_2 \cdots \int_{s_{m-1}}^b \omega^T(s_m)M\omega(s_m)\, ds_m \geq \sum_{k=0}^N H_m^{-1}(k,k)\Omega_{m,k}^T M \Omega_{m,k}, \tag{1}$$

$$\int_a^b ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{m-1}} \omega^t(s_m)M\omega(s_m)\, ds_m \geq \sum_{k=0}^N \widetilde{H}_m^{-1}(k,k)\widetilde{\Omega}_{m,k}^T M \widetilde{\Omega}_{m,k}, \tag{2}$$

where

$$H_m(k,k) = \int_a^b ds_1 \int_{s_1}^b ds_2 \cdots \int_{s_{m-1}}^b p_k(s_m)p_k(s_m)\, ds_m,$$

$$\widetilde{H}_m(k,k) = \int_a^b ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{m-1}} p_k(s_m)p_k(s_m)\, ds_m,$$

$$\Omega_{m,k} = \int_a^b ds_1 \int_{s_1}^b ds_2 \cdots \int_{s_{m-1}}^b p_k(s_m)\omega(s_m)\, ds_m,$$

$$\widetilde{\Omega}_{m,k} = \int_a^b ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{m-1}} p_k(s_m)\omega(s_m)\, ds_m,$$
$$a \le s_i \le b, i = 1, 2, \ldots, m.$$

Proof We first prove Ineq. (1).

Since $M \in \mathbb{S}_n^+$, there exists a real orthogonal matrix Q such that

$$M = Q \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} Q^T,$$

where $\lambda_i > 0$, $i = 1, 2, \ldots, n$ are eigenvalues of M.

Define

$$\bar{M} = Q \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{bmatrix} Q^T.$$

Apparently, one has $\bar{M}^T = \bar{M}$ and $\bar{M}^2 = M$.

Setting

$$W(s) = \begin{bmatrix} \bar{M}\omega(s) & p_0(s)\bar{M} & p_1(s)\bar{M} & \cdots & p_N(s)\bar{M} \end{bmatrix},$$

then one has $W^T(s)W(s) \ge 0$. Namely,

$$\begin{bmatrix} \omega^T(s)M\omega(s) & p_0(s)\omega^T(s)M & p_1(s)\omega^T(s)M & \cdots & p_N(s)\omega^T(s)M \\ * & p_0^2(s)M & p_0(s)p_1(s)M & \cdots & p_0(s)p_N(s)M \\ * & * & p_1^2(s)M & \cdots & p_1(s)p_N(s)M \\ * & * & \vdots & \ddots & \vdots \\ * & * & * & \cdots & p_N^2(s)M \end{bmatrix} \ge 0.$$

Integrating both sides of the above inequality with respect to s for m times and observing that the definitions of $H_m(k,k)$, $\Omega_{m,k}$, we reach

$$\begin{bmatrix} (1,1) & \Omega_{m,0}^T M & \Omega_{m,1}^T M & \cdots & \Omega_{m,N}^* M \\ * & H_m(0,0)M & 0 & \cdots & 0 \\ * & * & H_m(1,1)M & \cdots & 0 \\ * & * & \vdots & \ddots & \vdots \\ * & * & * & \cdots & H_m(N,N)M \end{bmatrix} \ge 0,$$

where $(1,1) = \int_a^b ds_1 \int_{s_1}^b ds_2 \cdots \int_{s_{m-1}}^b \omega^T(s_m)M\omega(s_m)\, ds_m$.

Utilizing Schur complement for the above matrix inequality, we arrive at

$$\int_a^b ds_1 \int_{s_1}^b ds_2 \cdots \int_{s_{m-1}}^b \omega^T(s_m)M\omega(s_m)\,ds_m - \sum_{k=0}^N H_m^{-1}(k,k)\Omega_{m,k}^T M\Omega_{m,k} \geq 0.$$

Ineq. (1) thus holds.

By analogy, Ineq. (2) can also be proved. This completes the proof of Theorem 1. □

Remark 1 It is easy to see that the proposed multiple integral Inequalities (1) and (2) can produce more and more accurate bounds of the derivative of the chosen Lyapunov functional as N increases. Theoretically, when $N \to +\infty$, the conservatism of Ineq. (1), Ineq. (2) will asymptotically vanish, for more detail, see [13].

Retrieving Ineq. (1) and Ineq. (2) with $\omega(u) = \dot{x}(t)$ in Theorem 1, we immediately obtain the following corollary.

Corollary 1 *For a constant matrix $M \in \mathbb{S}_n^+$, a vector-valued function $x(\cdot) : [a,b] \to \mathbb{R}^n$, orthogonal polynomials $p_k(s)$ and $m \in \mathbb{N}^+$ such that the following integrations are well-defined, then*

$$\int_a^b ds_1 \int_{m_1}^b ds_2 \cdots \int_{s_{m-1}}^b \dot{x}^T(s_m)M\dot{x}(s_m)\,ds_m \geq \sum_{k=0}^N H_m^{-1}(k,k)\Theta_{m,k}^T M\Theta_{m,k}, \quad (3)$$

$$\int_a^b ds_1 \int_a^{s_s} ds_2 \cdots \int_a^{s_{m-1}} \dot{x}^T(s_m)M\dot{x}(s_m)\,ds_m \geq \sum_{k=0}^N \widetilde{H}_m^{-1}(k,k)\widetilde{\Theta}_{m,k}^t M\widetilde{\Theta}_{m,k}, \quad (4)$$

where

$$\Theta_{m,k} = \int_a^b ds_1 \int_{s_1}^b ds_2 \cdots \int_{s_{m-1}}^b p_k(s_m)\dot{x}(s_m)\,ds_m,$$
$$\widetilde{\Theta}_{m,k} = \int_a^b ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{m-1}} p_k(s_m)\dot{x}(s_m)\,ds_m,$$
$$a \leq s_i \leq b, i = 1,2,\ldots,m.$$

In Theorem 1 and Corollary 1, when $m = 1$, Ineq. (1)–(4) are called single-integral inequalities and when $m \geq 2$ Ineq. (1)–(4) are called multiple-integral inequalities.

Remark 2 Although Ineqs. (1)–(4) have been developed, they cannot be successfully applied to discuss relevant practical problems that are still unknown but have a polynomial function of degree and orthogonality. In other words, Theorem 1 and Corollary 1 are only results of existence. These inequalities cannot be applied to specific problems until the polynomial function $p_k(\cdot)$ is determined.

Next, we will determine $p_k(\cdot)$.

In what follows, we consider the following two cases:

a) In the case of $m = 1$.

For brevity in form, we firstly give result for the case when $m = 1$ in Ineq. (1) and Ineq. (2), that is

Corollary 2 *For a constant matrix $M \in \mathbb{S}_n^+$, and a vector-valued function $\omega(\cdot) : [a,b] \to \mathbb{R}^n$, there exists a polynomial function $p_k(u)$ with degree of k on the interval $[a,b]$, $k \in \mathbb{N}$ and satisfying*

$$\int_a^b p_k(u)p_l(u)\,du = \begin{cases} 0, & k \neq l, \\ r_k \neq 0, & k = l. \end{cases}$$

Then the following inequality holds:

$$\int_a^b \omega^T(u)M\omega(u)\,du \geq \sum_{k=0}^{N} \frac{1}{r_k}\Omega_k^T M \Omega_k,$$

where $N \in \mathbb{N}$, $\Omega_k = \int_a^b p_k(u)\omega(u)\,du$.

The subsequent effort is to determine the $p_k(\cdot)$. Observe that the aforementioned Legendre polynomial $L_k(\cdot)$, $\widetilde{L}_k(\cdot)$ satisfies orthogonality and other properties, numerous references, for example [9–12], all chose $p_k(\cdot) = L_k(\cdot)$ or $p_k(\cdot) = \widetilde{L}_k(\cdot)$ and obtained the BLI. In addition, other polynomials can be chosen to act as $L_k(\cdot)$ or $\widetilde{L}_k(\cdot)$, such as those polynomials proposed in Ref. [19], which were constructed by the odd and even properties at the central point of the integral range $[a,b]$. That is

$$\begin{aligned} &p_0(u) = 1, \qquad p_1(u) = u - \frac{a+b}{2}, \qquad p_2(u) = \left(u - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}, \\ &p_3(u) = \left(u - \frac{a+b}{2}\right)^3 - \frac{3(b-a)^2}{20}\left(u - \frac{a+b}{2}\right), \qquad \ldots. \end{aligned}$$

Then, based on Theorem 1, we derive the following inequality when $m = 1$, $N = 3$:

Theorem 2 *For a constant matrix $M \in \mathbb{S}_n^+$, two scalars a, b satisfying $a < b$ and a vector-valued function $\omega(\cdot) : [a,b] \to \mathbb{R}^n$ such that the integrations below are well-defined, then the following inequality holds*:

$$(b-a)\int_a^b \omega^T(u)M\omega(u)\,du \geq \Omega_0^T M\Omega_0 + 3\Omega_1^T M\Omega_1 + 5\Omega_2^T M\Omega_2 + 7\Omega_3^T M\Omega_3, \tag{5}$$

where

$$\begin{aligned} \Omega_0 &= \int_a^b \omega(s)\,ds, \\ \Omega_1 &= \int_a^b \omega(s)\,ds - \frac{2}{b-a}\int_a^b ds\int_a^s \omega(r)\,dr, \\ \Omega_2 &= \int_a^b \omega(s)\,ds - \frac{6}{b-a}\int_a^b ds\int_a^s \omega(r)\,dr + \frac{12}{(b-a)^2}\int_a^b ds\int_a^s du\int_a^u \omega(r)\,dr, \\ \Omega_3 &= \int_a^b \omega(s)\,ds - \frac{12}{b-a}\int_a^b ds\int_a^s \omega(r)\,dr + \frac{60}{(b-a)^2}\int_a^b ds\int_a^s du\int_a^u \omega(r)\,dr \\ &\quad - \frac{120}{(b-a)^3}\int_a^b ds_1\int_a^{s_1} ds_2\int_a^{s_2} ds_3\int_a^{s_3} \omega(s_4)\,ds_4. \end{aligned}$$

Remark 3 Obviously, the aforementioned JII, WII, EWII can be regarded as special cases of Inq. (5) when $N = 0, 1, 2$, respectively. Hence Inq. (5) generalizes them and has is conservative than them.

Replacing $\omega(\cdot)$ by $\dot{x}(\cdot)$ in Theorem 2, one immediately obtains the following corollary.

Corollary 3 *For a given $M \in \mathbb{S}_n^+$, the following inequality holds for all continuously differentiable functions $\dot{x}(\cdot)$ in $[a,b] \to \mathbb{R}^n$:*

$$(b-a)\int_a^b \dot{x}^T(s)M\dot{x}(s)\,ds \geq \Theta_0^T M\Theta_0 + 3\Theta_1^T M\Theta_1 + 5\Theta_2^T M\Theta_2 + 7\Theta_3^T M\Theta_3, \tag{6}$$

where

$$\begin{aligned}
\Theta_0 &= x(b) - x(a),\\
\Theta_1 &= x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)\,ds,\\
\Theta_2 &= x(b) - x(a) - \frac{6}{b-a}\int_a^b x(b)\,ds + \frac{12}{(b-a)^2}\int_a^b ds\int_a^s x(r)\,dr,\\
\Theta_3 &= x(b) + x(a) - \frac{12}{b-a}\int_a^b x(s)\,ds + \frac{60}{(b-a)^2}\int_a^b ds\int_a^s x(r)\,dr\\
&\quad - \frac{120}{(b-a)^3}\int_a^b ds\int_a^s du\int_a^u x(r)\,dr.
\end{aligned}$$

Next, we choose $p_k(\cdot) = L_k(\cdot)$ or $p_k(\cdot) = \widetilde{L}_k(\cdot) \triangleq \widetilde{p}_k(\cdot)$ in Corollary 2 and derive a type of single integral inequalities on the basis of Corollary 2. It should be pointed out that the approach we take is different from those in the literature, although we also choose $p_k(\cdot) = L_k(\cdot)$, like in Refs. [9–12]. Correspondingly, we formulate the single integral inequalities distinct in form from those in the literature mentioned above, and the proposed single integral inequalities are more practical to use. For more details, see Theorem 3.

Theorem 3 *Suppose a constant matrix $M \in \mathbb{S}_n^+$, two scalars a, b satisfying $a < b$ and a vector-valued function $\omega(\cdot) : [a,b] \to \mathbb{R}^n$, then the following inequality holds for any $N \in \mathbb{N}$:*

$$(b-a)\int_a^b \omega^T(u)M\omega(u)\,du \geq \sum_{k=0}^{N}(2k+1)\Xi_k^T M\Xi_k, \tag{7}$$

$$(b-a)\int_a^b \omega^T(u)M\omega(u)\,du \geq \sum_{k=0}^{N}(2k+1)\widetilde{\Xi}_k^T M\widetilde{\Xi}_k, \tag{8}$$

where

$$\begin{aligned}
\Xi_k &= p_k(a)\Omega_0 + \frac{1}{b-a}\bigl\{\bigl[1-(-1)^k\bigr]\Pi_{0,0} + 3\bigl[1-(-1)^{k+1}\bigr]\Pi_{1,0} + \cdots + 2(2k-1)\Pi_{k-1,0}\bigr\},\\
\widetilde{\Xi}_b &= \widetilde{p}_b(b)\widetilde{\Omega}_0\\
&\quad - \frac{1}{b-a}\bigl\{\bigl[-1+(-1)^k\bigr]\widetilde{\Pi}_{0,0} + 3\bigl[-1+(-1)^{k+1}\bigr]\widetilde{\Pi}_{1,0} + \cdots - 2(2k-1)\widetilde{\Pi}_{k-1,0}\bigr\},
\end{aligned}$$

with

$$\begin{aligned}
&\varXi_k(\alpha) \triangleq \int_\alpha^b p_k(s)\omega(s)\,ds, \qquad \varXi_k \triangleq \varXi_k(a) = \int_a^b p_k(s)\omega(s)\,ds,\\
&\varOmega_0(s) \triangleq \int_s^b \omega(u)\,du, \qquad \varOmega_0 \triangleq \varOmega_0(a) = \int_a^b \omega(u)\,du,\\
&\varPi_{0,0} = \int_a^b p_0(s)\varOmega_0(s)\,ds = \int_a^b ds \int_s^b \omega(u)\,du,\\
&\varPi_{1,0} = \int_a^b p_1(s)\varOmega_0(s)\,ds = \int_a^b p_1(s)\int_s^b \omega(u)\,du\,ds, \qquad \ldots,\\
&\varPi_{k-1,0} = \int_a^b p_{k-1}(s)\varOmega_0(s)\,ds = \int_a^b p_{k-1}(s)\int_s^b \omega(u)\,du\,ds;\\
&\widetilde{\varOmega}_0(s) = \int_a^s \omega(u)\,du, \qquad \widetilde{\varOmega}_0 \triangleq \widetilde{\varOmega}_0(b) = \int_a^b \omega(u)\,du,\\
&\widetilde{\varXi}_k(\beta) \triangleq \int_a^\beta \widetilde{p}_k(s)\omega(s)\,ds, \qquad \widetilde{\varXi}_k \triangleq \widetilde{\varXi}_k(b) = \int_a^b \widetilde{p}_k(s)\omega(s)\,ds,\\
&\widetilde{\varPi}_{0,0} = \int_a^b \widetilde{p}_0(s)\widetilde{\varOmega}_0(s)\,ds = \int_a^b ds \int_a^s \omega(u)\,du,\\
&\widetilde{\varPi}_{1,0} = \int_a^b \widetilde{p}_1(s)\widetilde{\varOmega}_0(s)\,ds = \int_a^b \widetilde{p}_1(s)\int_a^s \omega(u)\,du\,ds, \qquad \ldots,\\
&\widetilde{\varPi}_{k-1,0} = \int_a^b \widetilde{p}_{k-1}(s)\widetilde{\varOmega}_0(s)\,ds = \int_a^b \widetilde{p}_{k-1}(s)\int_a^s \omega(u)\,du\,ds.
\end{aligned}$$

b) In the case of $m \geq 2$.

In the case when $m \geq 2$, one cannot directly choose $p_k(\cdot) = L_k(\cdot)$ or $p_k(\cdot) = \widetilde{L}_k(\cdot)$ like a) due to the fact that $L_k(\cdot)$ or $\widetilde{L}_k(\cdot), k \in \mathbb{N}$ is not orthogonal sequence with respect to the inner product described in Definition 1 when $m \geq 2$. It is thus crucial to find some orthogonal polynomial sequence $\{p_k(\cdot), k \in \mathbb{N}\}$ in the MIIPS when $m \geq 2$.

A simple approach to seek such $\{p_k(\cdot), k \in \mathbb{N}\}$ is systematically computed by Schmidt orthogonal algorithm with a basis of polynomial functions of degree k, $k = 0, 1, 2, \ldots, N$ with respect to s in the MIIPS when $m \geq 2$ such as $p_0(s) = 1, p_1(s) = s - a, p_2(s) = (s - a)^2, \ldots, p_N(s) = (s - a)^N$.

For simplicity, the following notations are introduced and defined as:

$$\begin{aligned}
&\varPi_{0^2} \triangleq \varPi_{0,0} = \int_a^b p_0(s)\varOmega_0(s)\,ds = \int_a^b ds \int_s^b \omega(u)\,du,\\
&\varPi_{0^{k+1}}(\alpha) \triangleq \varPi_{0,0^k}(\alpha) = \int_\alpha^b p_0(s)\varPi_{0^k}(s)\,ds,\\
&\varPi_{0^{k+1}} \triangleq \varPi_{0^{k+1}}(a) = \int_a^b p_0(s)\varPi_{0^k}(s)\,ds,\\
&\widetilde{\varPi}_{0^2} \triangleq \widetilde{\varPi}_{0,0} = \int_a^b p_0(s)\widetilde{\varOmega}_0(s)\,ds = \int_a^b ds \int_a^s \omega(u)\,du,\\
&\widetilde{\varPi}_{0^{k+1}}(\beta) \triangleq \widetilde{\varPi}_{0,0^k}(\beta) = \int_a^\beta p_0(s)\widetilde{\varPi}_{0^k}(s)\,ds,
\end{aligned}$$

$$\widetilde{\Pi}_{0^{k+1}} \triangleq \widetilde{\Pi}_{0^{k+1}}(b) = \int_a^b p_0(s)\widetilde{\Pi}_{0^k}(s)\,ds.$$

Theorem 4 *For a constant matrix $M \in \mathbb{S}_n^+$, a vector-valued function $\omega(\cdot) : [a,b] \to \mathbb{R}^n$ such that the integrations below are well-defined, the following double-integral inequalities hold*

$$\begin{aligned}
&\frac{(b-a)^2}{2}\int_a^b ds \int_s^b \omega^T(u)M\omega(u)\,du \\
&\quad\geq \Omega_{2,0}^T M\Omega_{2,0} + 8\Omega_{2,1}^T M\Omega_{2,1} + 27\Omega_{2,2}^T M\Omega_{2,2} + 64\Omega_{2,3}^T M\Omega_{2,3} + \cdots,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\frac{(b-a)^2}{2}\int_a^b ds \int_a^s \omega^T(u)M\omega(u)\,du \\
&\quad\geq \widetilde{\Omega}_{2,0}^T M\widetilde{\Omega}_{2,0} + 8\widetilde{\Omega}_{2,1}^T M\widetilde{\Omega}_{2,1} + 27\widetilde{\Omega}_{2,2}^T M\widetilde{\Omega}_{2,2} + 64\widetilde{\Omega}_{2,3}^T M\widetilde{\Omega}_{2,3} + \cdots,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&\Omega_{2,0} = \Pi_{0^2}, \qquad \Omega_{2,1} = -\Pi_{0^2} + \frac{3}{b-a}\Pi_{0^3}, \\
&\Omega_{2,2} = \Pi_{0^2} - \frac{8}{b-a}\Pi_{0^3} + \frac{20}{(b-a)^2}\Pi_{0^4}, \\
&\Omega_{2,3} = -\Pi_{0^2} + \frac{15}{b-a}\Pi_{0^3} - \frac{90}{(b-a)^2}\Pi_{0^4} + \frac{210}{(b-a)^3}\Pi_{0^5}, \\
&\widetilde{\Omega}_{2,0} = \widetilde{\Pi}_{0^2}, \qquad \widetilde{\Omega}_{2,1} = \widetilde{\Pi}_{0^2} - \frac{3}{b-a}\widetilde{\Pi}_{0^3}, \\
&\widetilde{\Omega}_{2,2} = \widetilde{\Pi}_{0^2} - \frac{8}{b-a}\widetilde{\Pi}_{0^3} + \frac{20}{(b-a)^2}\widetilde{\Pi}_{0^4}, \\
&\widetilde{\Omega}_{2,3} = \widetilde{\Pi}_{0^2} - \frac{15}{b-a}\widetilde{\Pi}_{0^3} + \frac{90}{(b-a)^2}\widetilde{\Pi}_{0^4} - \frac{210}{(b-a)^3}\widetilde{\Pi}_{0^5}.
\end{aligned}$$

Proof We first prove Ineq. (9).

Choosing $p_{2,0}(s) = p_0(s) = 1$, according to the Schmidt orthogonal algorithm in the MI-IPS with $m = 2$, one has

$$p_{2,k}(s) = p_k(s) - \sum_{i=0}^{k} \frac{\langle p_k(s), p_{2,i}(s)\rangle_2}{\langle p_{2,i}(s), p_{2,i}(s)\rangle_2} p_{2,i}(s), \quad k = 1, 2, \ldots, N.$$

Computing directly yields

$$\begin{aligned}
&p_{2,1}(s) = \frac{2}{3}(b-a)\left(-1 + \frac{3}{2}\frac{s-a}{b-a}\right), \\
&p_{2,2}(s) = \frac{3}{10}(b-a)^2\left[1 - 4\frac{s-a}{b-a} + \frac{10}{3}\frac{(s-a)^2}{(b-a)^2}\right], \\
&p_{2,3}(s) = \frac{4}{35}(b-a)^3\left[-1 + \frac{15}{2}\frac{s-a}{b-a} - 15\frac{(s-a)^2}{(b-a)^2} + \frac{35}{4}\frac{(s-a)^3}{(b-a)^3}\right].
\end{aligned}$$

In addition, after easy computing, we get

$$H_2(0,0) = \frac{(b-a)^2}{2}, \qquad H_2(1,1) = \frac{(b-a)^4}{36},$$

$$H_2(2,2)=\frac{(b-a)^6}{600},\qquad H_2(3,3)=\frac{(b-a)^8}{9800}.$$

In the light of Theorem 1, Ineq. (9) can be obtained. Following the same procedure as that derived Ineq. (9), one can prove Ineq. (10). This completes the proof of Theorem 4. □

Remark 4 In our previous literature [20], by extreme value conditions of multiple variables function, some double integral inequalities were established, see [20, Remark 1] for details. These inequalities can be regarded as special cases of Lemma 4. If setting $\Omega_{2,2}=\Omega_{2,3}=0$ or $\widetilde{\Omega}_{2,2}=\widetilde{\Omega}_{2,3}=0$ in Theorem 4, one finds that Ineq. (9) and (10) reduce to those in Ref. [20]. Comparing with those in [20], one can find that Ineq. (9) and (10) can deliver more tight lower bounds of the terms $\int_a^b ds\int_s^b \omega^T(u)M\omega(u)\,du$ and $\int_a^b ds\int_a^u \omega^T(u)M\omega(u)\,du$. The resulting double integral inequalities have thus improved and Ineq. (9), (10) are less conservative. In addition, based on the statements in [20, Remark 2 and Remark 3], one finds that Ineq. (9), (10) remarkably generalize and enhance those double integral inequalities in Refs. [21, 22].

Substituting $\dot{x}(\cdot)$ for $\omega(\cdot)$ in Theorem 4, we readily obtain the following corollary.

Corollary 4 *For a given $M\in\mathbb{S}_n^+$, the following inequality holds for all continuously differentiable function $\dot{x}(\cdot)$ in $[a,b]\to\mathbb{R}^n$:*

$$\begin{aligned}
&\frac{(b-a)^2}{2}\int_a^b ds\int_s^b \dot{x}^T(u)M\dot{x}(u)\,du\\
&\quad\geq \Theta_{2,0}^T M\Theta_{2,0}+8\Theta_{2,1}^T M\Theta_{2,1}+27\Theta_{2,2}^T M\Theta_{2,2}+\cdots,\\
&\frac{(b-a)^2}{2}\int_a^b ds\int_a^s \dot{x}^T(u)M\dot{x}(u)\,du\\
&\quad\geq \widetilde{\Theta}_{2,0}^T M\widetilde{\Theta}_{2,0}+8\widetilde{\Theta}_{2,1}^T M\widetilde{\Theta}_{2,1}+27\widetilde{\Theta}_{2,2}^T M\widetilde{\Theta}_{2,2}+\cdots,
\end{aligned}$$

where

$$\begin{aligned}
\Theta_{2,0}&=(b-a)x(b)-\int_a^b x(s)\,ds,\\
\Theta_{2,1}&=\frac{b-a}{2}x(b)+\int_a^b x(s)\,ds-\frac{3}{b-a}\int_a^b ds\int_s^b x(u)\,du,\\
\Theta_{2,2}&=\frac{1}{3}(b-a)x(b)-\int_a^b x(s)\,ds+\frac{8}{b-a}\int_a^b ds\int_s^b x(u)\,du\\
&\quad-\frac{20}{(b-a)^2}\int_a^b ds\int_s^b du\int_u^b x(r)\,dr,\\
\widetilde{\Theta}_{2,0}&=-(b-a)x(a)+\int_a^b x(s)\,ds,\\
\widetilde{\Theta}_{2,1}&=\frac{b-a}{2}x(a)+\int_a^b x(s)\,ds-\frac{3}{b-a}\int_a^b ds\int_a^s x(u)\,du,\\
\widetilde{\Theta}_{2,2}&=-\frac{1}{3}(b-a)x(a)+\int_a^b x(s)\,ds-\frac{8}{b-a}\int_a^b ds\int_a^u x(u)\,du\\
&\quad+\frac{20}{(b-a)^2}\int_a^b ds\int_a^s du\int_a^u x(r)\,dr.
\end{aligned}$$

Remark 5 It should be pointed out that one can choose polynomial sequence else, for example, Legendre polynomials $\{L_k(\cdot), k \in \mathbb{N}\}$ or $\{\widetilde{L}_k(\cdot), k \in \mathbb{N}\}$, to replace $p_0(s) = 1, p_1(s) = s - a, p_2(s) = (s-a)^2, \dots, p_N(s) = (s-a)^N$, which are used to derive other double-integral inequalities similar to Theorem 4. Nevertheless, such other double-integral inequalities are more complex in form than those in Theorem 4.

Remark 6 By following the same procedure as that $m = 2$, some related multiple integral inequalities with $m \geq 3$ can be available without difficulty. Considering that they are rather complicated in form and they are seldom applied to analyze practical problems, we do not particularize here.

4 Conclusions

In this paper, we develop a class of single/multiple integral inequalities in a systemic way. Some integral inequalities in the literature are regarded as special cases of the proposed inequalities. In contrast, the proposed integral inequalities are less conservative. Extending the proposed single/multiple integral inequalities to the corresponding summation visions is our future research direction.

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The authors declare no competing interests.

Author contributions
LZ drafted the manuscript and gave the proofs of the theorems and the corollaries. HM modified the manuscript and polished the language of the paper. All authors read and approved the final manuscript.

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