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Orthogonal neutrosophic 2-metric spaces

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Abstract

In this study, we introduce the notion of an orthogonal neutrosophic 2-metric space and prove the common fixed-point theorem on an orthogonal neutrosophic 2-metric space. From the obtained results, we give an example to support our results.

Mathematics Subject Classification: 47H10; 54H25

Keywords: Fuzzy 2-metric spaces; Neutrosophic 2-metric space; Orthogonal neutrosophic 2-metric space; Common fixed point

1 Introduction

Nowadays, a fuzzy concept has become the subject of several research works. Finding the fuzzy equivalents of the classical set theory is one of the advancements made to the basic theory of fuzzy sets provided by Zadeh [1]. Following that, the use of a fuzzy metric space in applied sciences including fixed-point theory, image and signal processing, medical imaging, and decision making occurred. The concept of intuitionistic fuzzy metric spaces was first proposed by Park [2]. The domains of population dynamics [3] computer programming [4], chaos control [5], nonlinear dynamical system [6], and medicine [7] are only a few examples of the scientific and technological fields that have utilized it. Gahler [8] presented a study on a 2-metric space. Schweizer and Sklar [9] explored the statistical metric spaces. The concept of intuitionistic fuzzy sets was presented by Atanassov [10] and Çoker [11] and the concept of intuitionistic fuzzy topological was discussed in [12]. In [13] the authors introduced the concepts of intuitionistic fuzzy 2-normed spaces and in [14] intuitionistic fuzzy 2-metric spaces.

Bera and Mahapatra [15] established the neutrosophic soft linear space. The neutrosophic normed linear space was established by Bera and Mahapatra [16]. The concept of an orthogonal neutrosophic metric space was introduced by Ishtiaq et al. [17] who proved several fixed-point results in the context of an orthogonal neutrosophic metric space. The contraction mapping was used to prove common fixed-point results in the context of a neutrosophic metric space established by Jeyaraman and Sowndrarajan [18]. Several fixed-point results in weak and rational $(\alpha - \psi)$-contractions in an ordered 2-metric space were established by Fathollahi et al. [19]. Many authors like Salama and Alblowi [20] worked on neutrosophic topological spaces and Al-omeri et al. [21] worked on a neutrosophic cone metric space, etc. Mursaleen and Lohani [22] introduced the idea of an intuitionistic 2-normed space and an intuitionistic 2-metric space. Ali Asghar and Aftab Hussain [23] established the basic properties of N2MSs and demonstrated some fixed-point findings. Umar Ishtiaq [24] introduced the notion of ONMSs and investigated some fixed-point results. The idea of orthogonality has several applications in mathematics. The notion of orthogonality in a metric space was established by Eshagi Gordji, Ramezani, De la Sen and Cho [25] and also expanded the findings in the setting of a metric space with new orthogonality and proved fixed-point theorems.

The main objectives of this study are as follows:

(i) To introduce the concept of an orthogonal neutrosophic 2-metric space (ON2MS).
(ii) To prove common fixed-point results on the orthogonal neutrosophic 2-metric space.
(iii) To enhance the literature of an intuitionistic fuzzy 2-metric space and a neutrosophic metric space.
(iv) To prove the uniqueness of the solution of integral equations.

Now, we provide some basic definitions to help to understand the main section.

2 Preliminaries

Here, "con-t-nm" means continuous triangular-norm, "con-t-conm" means continuous-triangular-conorm, "NMS" means neutrosophic metric space, "N2MS" means neutrosophic 2-metric space, "ON2MS" means orthogonal neutrosophic 2-metric space. Some basic definitions are given below:

Definition 2.1 [26] Let $*: [0,1] \times [0,1] \to [0,1]$ be a con-t-nm on a binary operation, then:

(I) $*$ is associative and commutative;
(II) $*$ is continuous;
(III) $\mu * 1 = \mu$ for all $\mu \in [0,1]$;
(IV) $\mu * \alpha \leq \eta * \gamma$, when $\mu \leq \eta$ and $\alpha \leq \gamma$ for all $\mu, \alpha, \eta, \gamma \in [0,1]$.

Definition 2.2 Let $+: [0,1] \times [0,1] \to [0,1]$ be a con-t-conm on a binary operation, then it satisfies (I), (II), (IV), and

(III) $\mu + 0 = \mu$ for all $\mu \in [0,1]$.

Definition 2.3 Let Φ be the universe. A neutrosophic set (NS) $\mathcal{A}$ in Φ is characterized by a truth membership function $\mathcal{Q}_{\mathcal{A}}$, an indeterminacy membership function $\mathcal{F}_{\mathcal{A}}$, and a falsity membership function $\mathcal{G}_{\mathcal{A}}$, where $\mathcal{Q}_{\mathcal{A}}$, $\mathcal{F}_{\mathcal{A}}$, and $\mathcal{G}_{\mathcal{A}}$ are real standard elements of $[0,1]$. This can be written as:

$$\mathcal{A} = \{\langle \nu, (\mathcal{Q}_{\mathcal{A}}(\nu), \mathcal{F}_{\mathcal{A}}(\nu), \mathcal{G}_{\mathcal{A}}(\nu))\rangle : \nu \in \Phi, \mathcal{Q}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}}, \mathcal{G}_{\mathcal{A}} \in]^{-}0, 1^{+}[\}.$$

There is no restriction on the sum of $\mathcal{Q}_{\mathcal{A}}(\nu)$, $\mathcal{F}_{\mathcal{A}}(\nu)$, and $\mathcal{G}_{\mathcal{A}}(\nu)$ and so $0^{-} \leq \mathcal{Q}_{\mathcal{A}}(\nu) + \mathcal{F}_{\mathcal{A}}(\nu) + \mathcal{G}_{\mathcal{A}}(\nu) \leq 3^{+}$.

Definition 2.4 [27] Let $\Phi \neq \emptyset$. A 6-tuple $(\Phi, \mathcal{Q}, \mathcal{F}, \mathcal{G}, *, +)$, where $*$ is a con-t-nm, $+$ is a con-t-conm, $\mathcal{Q}$, $\mathcal{F}$, and $\mathcal{G}$ are neutrosophic sets on $\Phi \times \Phi \times (0,\infty)$. If $(\Phi, \mathcal{Q}, \mathcal{F}, \mathcal{G}, *, +)$, satisfies the conditions below for all $\nu, \varrho, \mathfrak{z} \in \Phi$, and $\wp, \mathfrak{s} > 0$:

(N1) $\mathcal{Q}(\nu, \varrho, \wp) + \mathcal{F}(\nu, \varrho, \wp) + \mathcal{G}(\nu, \varrho, \wp) \leq 3$;
(N2) $0 \leq \mathcal{Q}(\nu, \varrho, \wp) \leq 1$;

(N3) $\mathcal{Q}(\nu,\varrho,\wp) = 1$ if and only if $\nu = \varrho$;
(N4) $\mathcal{Q}(\nu,\varrho,\wp) = \mathcal{Q}(\varrho,\nu,\wp)$;
(N5) $\mathcal{Q}(\nu,\varsigma,\wp+\mathfrak{s}) \geq \mathcal{Q}(\nu,\varrho,\wp) * \mathcal{Q}(\varrho,\varsigma,\mathfrak{s})$;
(N6) $\mathcal{Q}(\nu,\varrho,\cdot)\colon [0,\infty) \to [0,1]$ is continuous;
(N7) $\lim_{\wp\to\infty} \mathcal{Q}(\nu,\varrho,\wp) = 1$;
(N8) $0 \leq \mathcal{F}(\nu,\varrho,\wp) \leq 1$;
(N9) $\mathcal{F}(\nu,\varrho,\wp) = 0$ if and only if $\nu = \varrho$;
(N10) $\mathcal{F}(\nu,\varrho,\wp) = \mathcal{F}(\varrho,\nu,\wp)$;
(N11) $\mathcal{F}(\nu,\varsigma,\wp+\mathfrak{s}) \leq \mathcal{F}(\nu,\varrho,\wp) + \mathcal{F}(\varrho,\varsigma,\mathfrak{s})$;
(N12) $\mathcal{F}(\nu,\varrho,\cdot)\colon [0,\infty) \to [0,1]$ is continuous;
(N13) $\lim_{\wp\to\infty} \mathcal{F}(\nu,\varrho,\wp) = 0$;
(N14) $0 \leq \mathcal{G}(\nu,\varrho,\wp) \leq 1$;
(N15) $\mathcal{F}(\nu,\varrho,\wp) = 0$ if and only if $\nu = \varrho$;
(N16) $\mathcal{G}(\nu,\varrho,\wp) = \mathcal{G}(\varrho,\nu,\wp)$;
(N17) $\mathcal{G}(\nu,\varsigma,\wp+\mathfrak{s}) \leq \mathcal{G}(\nu,\varrho,\wp) + \mathcal{G}(\varrho,\varsigma,\mathfrak{s})$;
(N18) $\mathcal{G}(\nu,\varrho,\cdot)\colon [0,\infty) \to [0,1]$ is continuous;
(N19) $\lim_{\wp\to\infty} \mathcal{G}(\nu,\varrho,\wp) = 0$;
(N20) if $\wp \leq 0$, then $\mathcal{Q}(\nu,\varrho,\wp) = 0$, $\mathcal{F}(\nu,\varrho,\wp) = 1$, $\mathcal{G}(\nu,\varrho,\wp) = 1$.

Then, $(\mathcal{Q},\mathcal{F},\mathcal{G})$ is a neutrosophic metric and $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+)$ is a NMS.

Definition 2.5 [28] The 5-tuple $(\Phi,\mathcal{Q},\mathcal{F},*,+)$ is called an intuitionistic fuzzy 2-metric space if Φ is any nonvoid set, $*$ is a con-t-nm, $+$ is a con-t-conm, and $\mathcal{Q}$, $\mathcal{F}$ are fuzzy sets on $\Phi\times\Phi\times\Phi\times(0,\infty)$, then it satisfies for all $\nu,\varrho,\varsigma,\mathfrak{w}\in\Phi$, and $\mathfrak{s},\wp>0$:

(a) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) + \mathcal{F}(\nu,\varrho,\varsigma,\wp) \leq 1$;
(b) Let ν, ϱ of Φ, there exists an element ς of Φ such that $0 \leq \mathcal{Q}(\nu,\varrho,\varsigma,\wp) \leq 1$;
(c) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) = 1$ if at least two of ν, ϱ, ς are equal;
(d) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) = \mathcal{Q}(\nu,\varsigma,\varrho,\wp) = \mathcal{Q}(\varrho,\varsigma,\nu,\wp)$ for all ν, ϱ, ς in Φ;
(e) $\mathcal{Q}(\nu,\varrho,\mathfrak{w},\wp) * \mathcal{Q}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) * \mathcal{Q}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \leq \mathcal{Q}(\nu,\varrho,\varsigma,\wp+\mathfrak{s}+\mathfrak{h})$ for all $\nu,\varrho,\varsigma,\mathfrak{w}\in\Phi$;
(f) $\mathcal{Q}(\nu,\varrho,\varsigma,\cdot)\colon (0,\infty) \to (0,1]$ is continuous;
(g) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) < 1$;
(h) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) = 0$ if at least two of ν, ϱ, ς are equal;
(i) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) = \mathcal{F}(\nu,\varsigma,\varrho,\wp) = \mathcal{F}(\varrho,\varsigma,\nu,\wp)$ for all ν, ϱ, ς in Φ;
(j) $\mathcal{F}(\nu,\varrho,\mathfrak{w},\wp) + \mathcal{F}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) + \mathcal{F}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \geq \mathcal{F}(\nu,\varrho,\varsigma,\wp+\mathfrak{s}+\mathfrak{h})$;
(k) $\mathcal{F}(\nu,\varrho,\varsigma,\cdot)\colon (0,\infty) \to (0,1]$ is continuous.

Definition 2.6 The 6-tuple $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+)$ is said to be a N2MS if Φ is any nonempty set, $*$ is a con-t-nm, $+$ is a con-t-conm, and $\mathcal{Q}$, $\mathcal{F}$, $\mathcal{G}$ are neutrosophic sets on $\Phi\times\Phi\times\Phi\times(0,\infty)$, then it satisfies for all $\nu,\varrho,\varsigma,\mathfrak{w}\in\Phi$, and $\mathfrak{s},\wp>0$;

(N2MS1) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) + \mathcal{F}(\nu,\varrho,\varsigma,\wp) + \mathcal{G}(\nu,\varrho,\varsigma,\wp) \leq 3$;
(N2MS2) Let ν, ϱ of Φ, there exists an element ς of Φ such that $0 \leq \mathcal{Q}(\nu,\varrho,\varsigma,\wp) \leq 1$;
(N2MS3) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) = 1$ if at least two of ν, ϱ, ς are equal;
(N2MS4) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) = \mathcal{Q}(\nu,\varsigma,\varrho,\wp) = \mathcal{Q}(\varrho,\varsigma,\nu,\wp)$;
(N2MS5) $\mathcal{Q}(\nu,\varrho,\mathfrak{w},\wp) * \mathcal{Q}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) * \mathcal{Q}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \leq \mathcal{Q}(\nu,\varrho,\varsigma,\wp+\mathfrak{s}+\mathfrak{h})$;
(N2MS6) $\mathcal{Q}(\nu,\varrho,\varsigma,\cdot)\colon (0,\infty) \to (0,1]$ is continuous for all $\nu,\varrho,\varsigma\in\Phi$ such that $\nu\perp\varrho\perp\varsigma$;

(N2MS7) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) \le 1$;
(N2MS8) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) = 0$ if at least two of ν, ϱ, ς are equal;
(N2MS9) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) = \mathcal{F}(\nu,\varsigma,\varrho,\wp) = \mathcal{F}(\varrho,\varsigma,\nu,\wp)$;
(N2MS10) $\mathcal{F}(\nu,\varrho,\mathfrak{w},\wp) + \mathcal{F}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) + \mathcal{F}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \ge \mathcal{F}(\nu,\varrho,\varsigma,\wp + \mathfrak{s} + \mathfrak{h})$;
(N2MS11) $\mathcal{F}(\nu,\varrho,\varsigma,\cdot): (0,\infty) \to (0,1]$ is continuous;
(N2MS12) $\mathcal{G}(\nu,\varrho,\varsigma,\wp) \le 1$;
(N2MS13) $\mathcal{G}(\nu,\varrho,\varsigma,\wp) = 0$ if at least two of ν, ϱ, ς are equal;
(N2MS14) $\mathcal{G}(\nu,\varrho,\varsigma,\wp) = \mathcal{G}(\nu,\varsigma,\varrho,\wp) = \mathcal{G}(\varrho,\varsigma,\nu,\wp)$;
(N2MS15) $\mathcal{G}(\nu,\varrho,\mathfrak{w},\wp) + \mathcal{G}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) + \mathcal{G}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \ge \mathcal{G}(\nu,\varrho,\varsigma,\wp + \mathfrak{s} + \mathfrak{h})$;
(N2MS16) $\mathcal{G}(\nu,\varrho,\varsigma,\cdot): (0,\infty) \to (0,1]$ is continuous.

Here, the functions $\mathcal{Q}(\nu,\varrho,\varsigma,\wp)$, $\mathcal{F}(\nu,\varrho,\varsigma,\wp)$, and $\mathcal{G}(\nu,\varrho,\varsigma,\wp)$ denotes the degree of nearness, the degree of nonnearness, and the degree of naturalness between ν, ϱ, and ς with respect to $\wp$, respectively.

Now, we define the notion of ON2MS

Definition 2.7 The 6-tuple $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is said to be a ON2MS if Φ is any nonempty set, $*$ is a con-t-nm, $+$ is a con-t-conm, and $\mathcal{Q}$, $\mathcal{F}$, $\mathcal{G}$ are neutrosophic sets on $\Phi \times \Phi \times \Phi \times (0,\infty)$, then it satisfies for all $\nu,\varrho,\varsigma,\mathfrak{w} \in \Phi$ and $\mathfrak{s},\wp > 0$;

(ON2MS1) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) + \mathcal{F}(\nu,\varrho,\varsigma,\wp) + \mathcal{G}(\nu,\varrho,\varsigma,\wp) \le 3$ for all $\nu,\varrho,\varsigma \in \Phi$, $\wp > 0$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS2) Let ν, ϱ of Φ, there exists an element ς of Φ such that $0 \le \mathcal{Q}(\nu,\varrho,\varsigma,\wp) \le 1$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS3) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) = 1$ if at least two of ν, ϱ, ς are equal such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS4) $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) = \mathcal{Q}(\nu,\varsigma,\varrho,\wp) = \mathcal{Q}(\varrho,\varsigma,\nu,\wp)$ for all ν, ϱ, ς in Φ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS5) $\mathcal{Q}(\nu,\varrho,\mathfrak{w},\wp) * \mathcal{Q}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) * \mathcal{Q}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \le \mathcal{Q}(\nu,\varrho,\varsigma,\wp + \mathfrak{s} + \mathfrak{h})$ for all $\nu,\varrho,\varsigma,\mathfrak{w} \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS6) $\mathcal{Q}(\nu,\varrho,\varsigma,\cdot): (0,\infty) \to (0,1]$ is continuous for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS7) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) \le 1$, for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS8) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) = 0$ if at least two of ν, ϱ, ς are equal for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS9) $\mathcal{F}(\nu,\varrho,\varsigma,\wp) = \mathcal{F}(\nu,\varsigma,\varrho,\wp) = \mathcal{F}(\varrho,\varsigma,\nu,\wp)$ for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS10) $\mathcal{F}(\nu,\varrho,\mathfrak{w},\wp) + \mathcal{F}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) + \mathcal{F}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \ge \mathcal{F}(\nu,\varrho,\varsigma,\wp + \mathfrak{s} + \mathfrak{h})$, for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS11) $\mathcal{F}(\nu,\varrho,\varsigma,\cdot): (0,\infty) \to (0,1]$ is continuous for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS12) $\mathcal{G}(\nu,\varrho,\varsigma,\wp) \le 1$, for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS13) $\mathcal{G}(\nu,\varrho,\varsigma,\wp) = 0$ if at least two of ν, ϱ, ς are equal for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS14) $\mathcal{G}(\nu,\varrho,\varsigma,\wp) = \mathcal{G}(\nu,\varsigma,\varrho,\wp) = \mathcal{G}(\varrho,\varsigma,\nu,\wp)$ for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;
(ON2MS15) $\mathcal{G}(\nu,\varrho,\mathfrak{w},\wp) + \mathcal{G}(\nu,\mathfrak{w},\varsigma,\mathfrak{s}) + \mathcal{G}(\mathfrak{w},\varrho,\varsigma,\mathfrak{h}) \ge \mathcal{G}(\nu,\varrho,\varsigma,\wp + \mathfrak{s} + \mathfrak{h})$, for all $\nu,\varrho,\varsigma \in \Phi$ such that $\nu \perp \varrho \perp \varsigma$;

(ON2MS16) $\mathcal{G}(\nu,\varrho,\varsigma,\cdot)\colon (0,\infty)\to(0,1]$ is continuous for all $\nu,\varrho,\varsigma\in\Phi$ such that $\nu\perp\varrho\perp\varsigma$.

Definition 2.8 Suppose $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is a ON2MS. Suppose $\mathfrak{h}\in(0,1)$, $\wp>0$ and $\nu\in\Phi$. The set $\mathbb{B}(\nu,\mathfrak{h},\wp)=\{\varrho\in\Phi\colon \mathcal{Q}(\nu,\varrho,\varsigma,\wp)>1-\mathfrak{h},\mathcal{F}(\nu,\varrho,\varsigma,\wp)<\mathfrak{h}$ and $\mathcal{G}(\nu,\varrho,\varsigma,\wp)<\mathfrak{h}$ for all $\varsigma\in\Phi\}$ is called the open ball with center ν and radius $\mathfrak{h}$ with respect to $\wp$.

Definition 2.9 Suppose $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is a ON2MS. Then, an open set of $\mathcal{U}\subset\Phi$ of its points is the center of a open ball contained in $\mathcal{U}$. The open set in a N2MS $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is represented by $\mathbb{U}$.

Definition 2.10 Assume $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is a ON2MS. A sequence $(\nu_{\mathfrak{n}})$ in Φ is a Cauchy one if for each $\epsilon>0$ and each $\wp>0$, there exist $\mathfrak{n}^*\in\mathbb{N}$ such that $\mathcal{Q}(\nu_{\mathfrak{n}},\nu_{\mathfrak{m}},\mathfrak{h},\wp)>1-\mathfrak{h}$, $\mathcal{F}(\nu_{\mathfrak{n}},\nu_{\mathfrak{m}},\mathfrak{h},\wp)<\mathfrak{h}$ and $\mathcal{G}(\nu_{\mathfrak{n}},\nu_{\mathfrak{m}},\mathfrak{h},\wp)<\mathfrak{h}$ for all $\mathfrak{n},\mathfrak{m}\geq\mathfrak{n}^*$ for all $\mathfrak{h}\in\Phi$.

Definition 2.11 Suppose $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is a ON2MS. A sequence $\nu=(\nu_\iota)$ is convergent to $\mathfrak{l}\in\Phi$, with respect to the ON2MS if, for every $\epsilon>0$ and $\wp>0$, there exist $\iota_0\in\mathbb{N}$ such that $\mathcal{Q}(\nu_\iota,\mathfrak{l},\mathfrak{h},\wp)>1-\epsilon$, $\mathcal{F}(\nu_\iota,\mathfrak{l},\mathfrak{h},\wp)<\epsilon$, and $\mathcal{G}(\nu_\iota,\mathfrak{l},\mathfrak{h},\wp)\epsilon$ for all $\iota\geq\iota_0$ and for all $\mathfrak{h}\in\Phi$. In this case, we write $(\mathcal{Q},\mathcal{F},\mathcal{G})_2-\lim\nu=\mathfrak{l}$ (or) $\nu_\iota\xrightarrow{(\mathcal{Q},\mathcal{F},\mathcal{G})_2}\mathfrak{l}$ as $\iota\to\infty$.

Definition 2.12 Let $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ be a ON2MS. Define $\tau_{(\mathcal{Q},\mathcal{F},\mathcal{G})_2}=\Upsilon\subset\Phi\colon$ for each $\nu\in\Phi$, there exist $\wp>0$ and $\mathfrak{h}\in(0,1)$ such that $\mathbb{B}(\nu,\mathfrak{h},\wp)\subset\Phi\}$. Then, $\tau_{(\mathcal{Q},\mathcal{F},\mathcal{G})_2}$ is a topology on $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$.

Definition 2.13 Let $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ be a ON2MS. If each Cauchy sequence converges with respect to $\varsigma(\mathcal{Q},\mathcal{F},\mathcal{G})_2$ then it is called complete.

Theorem 2.1 *Every open ball $\mathbb{B}(\nu,\mathfrak{h},\wp)$ in ON2MS is an open set.*

Proof Consider $\mathbb{B}(\nu,\mathfrak{h},\wp)$ to be an open ball with center ν and radius $\mathfrak{h}$. Assume $\varrho\in\mathbb{B}(\nu,\mathfrak{h},\wp)$. Therefore, $\mathcal{Q}(\nu,\varrho,\varsigma,\wp)>1-\mathfrak{h}$, $\mathcal{F}(\nu,\varrho,\varsigma,\wp)<\mathfrak{h}$, and $\mathcal{G}(\nu,\varrho,\varsigma,\wp)<\mathfrak{h}$ for each $\varsigma\in\Xi$. There exists $\frac{\wp}{3}\in(0,\wp)$ such that $\mathcal{Q}(\nu,\varrho,\mathfrak{p},\frac{\wp}{3})>1-\mathfrak{h}$, $\mathcal{F}(\nu,\varrho,\mathfrak{p},\frac{\wp}{3})<\mathfrak{h}$, and $\mathcal{G}(\nu,\varrho,\mathfrak{p},\frac{\wp}{3})<\mathfrak{h}$, due to $\mathcal{Q}(\nu,\varrho,\varsigma,\wp)>1-\mathfrak{h}$. If we take $\mathfrak{h}_0=\mathcal{Q}(\nu,\varrho,\mathfrak{p},\frac{\wp}{3})$, then for $\mathfrak{h}_0>1-\mathfrak{h}$, $\epsilon\in(0,1)$ will exist such that $\mathfrak{h}_0>1-\epsilon>1-\mathfrak{h}$. Given $\mathfrak{h}_0$ and ϵ such that $\mathfrak{h}_0>1-\epsilon$, then $\{\mathfrak{h}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^6\in(0,1)$ such that $\mathfrak{h}_0*\mathfrak{h}_1*\mathfrak{h}_2>1-\epsilon$, $(1-\mathfrak{h}_0)+(1-\mathfrak{h}_3)+(1-\mathfrak{h}_4)\leq\epsilon$, and $(1-\mathfrak{h}_0)+(1-\mathfrak{h}_5)+(1-\mathfrak{h}_6)\leq\epsilon$. Choose $\mathfrak{h}_7=\max\{\mathfrak{h}_{\mathfrak{i}}\}_{\mathfrak{i}=1}^6$. Consider $\mathbb{B}(\varrho,1-\mathfrak{h}_7,\frac{\wp}{3})$. To show that $\mathbb{B}(\varrho,1-\mathfrak{h}_7,\frac{\wp}{3})\subset\mathbb{B}(\nu,\mathfrak{h},\wp)$, consider $\mathfrak{v}\in\mathbb{B}(\varrho,1-\mathfrak{h}_7,\frac{\wp}{3})$, then $\mathcal{Q}(\nu,\mathfrak{p},\varsigma,\frac{\wp}{3})>\mathfrak{h}_7$, $\mathcal{F}(\nu,\mathfrak{p},\varsigma,\frac{\wp}{3})<\mathfrak{h}_7$, and $\mathcal{Q}(\nu,\mathfrak{p},\varsigma,\frac{\wp}{3})<\mathfrak{h}_7$ and $\mathcal{F}(\mathfrak{p},\varrho,\varsigma,\frac{\wp}{3})>\mathfrak{h}_7$, $\mathcal{F}(\mathfrak{p},\varrho,\varsigma,\frac{\wp}{3})<\mathfrak{h}_7$, and $\mathcal{G}(\mathfrak{p},\varrho,\varsigma,\frac{\wp}{3})<\mathfrak{h}_7$. Then,

$$
\begin{aligned}
\mathcal{Q}(\nu,\varrho,\varsigma,\wp)&\geq\mathcal{Q}\left(\nu,\varrho,\mathfrak{p},\frac{\wp}{3}\right)*\mathcal{Q}\left(\nu,\mathfrak{p},\varsigma,\frac{\wp}{3}\right)*\mathcal{Q}\left(\mathfrak{p},\varrho,\varsigma,\frac{\wp}{3}\right)\\
&\geq\mathfrak{h}_0*\mathfrak{h}_7*\mathfrak{h}_7\geq\mathfrak{h}_0*\mathfrak{h}_1*\mathfrak{h}_2\geq1-\epsilon>1-\mathfrak{h},\\
\mathcal{F}(\nu,\varrho,\varsigma,\wp)&\geq\mathcal{F}\left(\nu,\varrho,\mathfrak{p},\frac{\wp}{3}\right)*\mathcal{F}\left(\nu,\mathfrak{p},\varsigma,\frac{\wp}{3}\right)*\mathcal{F}\left(\mathfrak{p},\varrho,\varsigma,\frac{\wp}{3}\right)\\
&\geq(1-\mathfrak{h}_0)+(1-\mathfrak{h}_7)+(1-\mathfrak{h}_7)\\
&\geq(1-\mathfrak{h}_0)+(1-\mathfrak{h}_1)+(1-\mathfrak{h}_2)\leq\epsilon<\mathfrak{h},
\end{aligned}
$$

$$\mathcal{G}(\nu,\varrho,\varsigma,\wp) \geq \mathcal{Q}\left(\nu,\varrho,\mathfrak{p},\frac{\wp}{3}\right) * \mathcal{Q}\left(\nu,\mathfrak{p},\varsigma,\frac{\wp}{3}\right) * \mathcal{Q}\left(\mathfrak{p},\varrho,\varsigma,\frac{\wp}{3}\right)$$
$$\leq (1-\mathfrak{h}_0) + (1-\mathfrak{h}_7) + (1-\mathfrak{h}_7)$$
$$\leq (1-\mathfrak{h}_0) + (1-\mathfrak{h}_1) + (1-\mathfrak{h}_2) \leq \epsilon < \mathfrak{h}.$$

We obtain $\mathfrak{v} \in \mathbb{B}(\nu,\mathfrak{h},\wp)$ and $\mathbb{B}(\varrho, 1-\mathfrak{h}_7, \frac{\wp}{3}) \subset \mathbb{B}(\nu,\mathfrak{h},\wp)$. □

Theorem 2.2 *Every ON2MS is Hausdorff.*

Proof Let $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+)$ be a N2MS. Let ν and ϱ be points in Φ. Then, $0 < \mathcal{Q}(\nu,\varrho,\varsigma,\wp) < 1$, $0 < \mathcal{F}(\nu,\varrho,\varsigma,\wp) < 1$, and $0 < \mathcal{G}(\nu,\varrho,\varsigma,\wp) < 1$ for every $\varsigma \in \Phi$. Put $\mathfrak{h}_1 = \mathcal{Q}(\nu,\varrho,\varsigma_1,\wp)$, $1-\mathfrak{h}_2 = \mathcal{F}(\nu,\varrho,\varsigma_1,\wp)$, and $1-\mathfrak{h}_3 = \mathcal{G}(\nu,\varrho,\varsigma_1,\wp)$, $\mathfrak{h}_4 = \mathcal{Q}(\nu,\varrho,\mathfrak{p},\frac{\wp}{3})$, $1-\mathfrak{h}_5 = \mathcal{F}(\nu,\varrho,\mathfrak{p},\frac{\wp}{3})$, $1-\mathfrak{h}_6 = \mathcal{G}(\nu,\varrho,\mathfrak{p},\frac{\wp}{3})$ and $\mathfrak{h} = \max\{\mathfrak{h}_1, 1-\mathfrak{h}_2, 1-\mathfrak{h}_3, \mathfrak{h}_4, 1-\mathfrak{h}_5, 1-\mathfrak{h}_6\}$. For each $\mathfrak{h}_0 \in (\mathfrak{h},1)$ there exist $\mathfrak{h}_7$ and $\mathfrak{h}_8$ such that $\mathfrak{h}_4 * \mathfrak{h}_7 * \mathfrak{h}_7 \geq \mathfrak{h}_0$ and $(1-\mathfrak{h}_5) * (1-\mathfrak{h}_8) * (1-\mathfrak{h}_8) \leq 1-\mathfrak{h}_0$. Put $\mathfrak{h}_9 = \max\{\mathfrak{h}_7,\mathfrak{h}_8\}$ and consider the open balls $\mathbb{B}(\nu, 1-\mathfrak{h}_9, \frac{\wp}{3})$ and $\mathbb{B}(\varrho, 1-\mathfrak{h}_9, \frac{\wp}{3})$. Then, clearly

$$\varrho\mathbb{B}\left(\nu, 1-\mathfrak{h}_9, \frac{\wp}{3}\right) \cap \mathbb{B}\left(\varrho, 1-\mathfrak{h}_9, \frac{\wp}{3}\right) = \emptyset.$$

If there is $\mathfrak{p} \in \mathbb{B}(\nu, 1-\mathfrak{h}_9, \frac{\wp}{3}) \cap \mathbb{B}(\varrho, 1-\mathfrak{h}_9, \frac{\wp}{3}) = \emptyset$, then

$$\mathfrak{h}_1 = \mathcal{Q}(\nu,\varrho,\varsigma_1,\wp) \geq \mathcal{Q}\left(\nu,\mathfrak{p},\varsigma_1,\frac{\wp}{3}\right) * \mathcal{Q}\left(\mathfrak{p},\varrho,\varsigma_1,\frac{\wp}{3}\right) * \mathcal{Q}\left(\nu,\varrho,\mathfrak{p},\frac{\wp}{3}\right)$$
$$\geq \mathfrak{h}_4 * \mathfrak{h}_9 * \mathfrak{h}_9 \geq \mathfrak{h}_4 * \mathfrak{h}_7 * \mathfrak{h}_7 \geq \mathfrak{h}_0 > \mathfrak{h}_1$$

and similarly, $1-\mathfrak{h}_2 < 1-\mathfrak{h}_2$, is its contrary. Hence, $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+)$ is Hausdorff. □

3 Main results

Lemma 1 *If $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is a N2MS. Then, $\mathcal{Q}(\nu,\varrho,\varsigma,\wp)$ is nondecreasing, $\mathcal{F}(\nu,\varrho,\varsigma,\wp)$ is nonincreasing, and $\mathcal{G}(\nu,\varrho,\varsigma,\wp)$ is nonincreasing for all $\nu,\varrho,\varsigma \in \Phi$.*

Proof Let $\mathfrak{s},\wp > 0$ be any points such that $\wp > \mathfrak{s} \cdot \wp = \mathfrak{s} + \frac{\wp-\mathfrak{s}}{2} + \frac{\wp-\mathfrak{s}}{2}$. Hence, we have

$$\mathcal{Y}(\nu,\varrho,\varsigma,\wp) = \mathcal{Y}\left(\nu,\varrho,\varsigma,\mathfrak{s} + \frac{\wp-\mathfrak{s}}{2} + \frac{\wp-\mathfrak{s}}{2}\right),$$
$$\leq \mathcal{Y}(\nu,\varrho,\varsigma,\mathfrak{s}) + \mathcal{Y}\left(\nu,\varsigma,\varsigma,\frac{\wp-\mathfrak{s}}{2}\right) + \mathcal{Y}\left(\varsigma,\varrho,\varsigma,\frac{\wp-\mathfrak{s}}{2}\right) = \mathcal{Y}(\nu,\varrho,\varsigma,\mathfrak{s})$$

and

$$\mathcal{G}(\nu,\varrho,\varsigma,\wp) = \psi\left(\nu,\varrho,\varsigma,\mathfrak{s} + \frac{\wp-\mathfrak{s}}{2} + \frac{\wp-\mathfrak{s}}{2}\right),$$
$$\leq \mathcal{G}(\nu,\varrho,\varsigma,\mathfrak{s}) + \psi\left(\nu,\varsigma,\varsigma,\frac{\wp-\mathfrak{s}}{2}\right) + \mathcal{G}\left(\varsigma,\varrho,\varsigma,\frac{\wp-\mathfrak{s}}{2}\right) = \mathcal{G}(\nu,\varrho,\varsigma,\mathfrak{s}).$$

Similarly, $\mathcal{Q}(\nu,\varrho,\varsigma,\wp) > \mathcal{Q}(\nu,\varrho,\varsigma,\mathfrak{s})$. □

From Lemma 1, let $(\Phi, \mathcal{Q}, \mathcal{F}, \mathcal{G}, *, +, \perp)$ be a ON2MS with the following conditions:

$$\lim_{\wp\to\infty} \mathcal{Q}(\nu, \varrho, \varsigma, \wp) = 1, \qquad \lim_{\wp\to\infty} \mathcal{F}(\nu, \varrho, \varsigma, \wp) = 0 \quad \text{and} \quad \lim_{\wp\to\infty} \mathcal{G}(\nu, \varrho, \varsigma, \wp) = 0.$$

Lemma 2 *Let $(\Phi, \mathcal{Q}, \mathcal{F}, \mathcal{G}, *, +, \perp)$ be a ON2MS. If there exists $\ell \in (0,1)$ such that $\mathcal{Q}(\nu, \varrho, \varsigma, \ell\wp + 0) \geq \mathcal{Q}(\nu, \varrho, \varsigma, \wp)$, $\mathcal{F}(\nu, \varrho, \varsigma, \ell\wp + 0) \leq \mathcal{F}(\nu, \varrho, \varsigma, \wp)$, and $\mathcal{G}(\nu, \varrho, \varsigma, \ell\wp + 0) \leq \mathcal{G}(\nu, \varrho, \varsigma, \wp)$ for all $\nu, \varrho, \varsigma \in \Phi$ with $\varsigma \neq \nu$, $\varsigma \neq \varrho$, and $\wp > 0$, then $\nu = \varrho$.*

Proof Since

$$\mathcal{Q}(\nu, \varrho, \varsigma, \wp) \geq \mathcal{Q}(\nu, \varrho, \varsigma, \ell\wp + 0) \geq \mathcal{Q}(\nu, \varrho, \varsigma, \wp),$$
$$\mathcal{F}(\nu, \varrho, \varsigma, \wp) \leq \mathcal{F}(\nu, \varrho, \varsigma, \ell\wp + 0) \leq \mathcal{F}(\nu, \varrho, \varsigma, \wp)$$

and

$$\mathcal{G}(\nu, \varrho, \varsigma, \wp) \leq \mathcal{G}(\nu, \varrho, \varsigma, \ell\wp + 0) \leq \mathcal{G}(\nu, \varrho, \varsigma, \wp),$$

for all $\wp > 0$, $\mathcal{Q}(\nu, \varrho, \varsigma, \cdot)$, $\mathcal{F}(\nu, \varrho, \varsigma, \cdot)$, and $\mathcal{G}(\nu, \varrho, \varsigma, \cdot)$ are constant. Since $\lim_{\wp\to\infty} \mathcal{Q}(\nu, \varrho, \varsigma, \wp) = 1$, $\lim_{\wp\to\infty} \mathcal{F}(\nu, \varrho, \varsigma, \wp) = 0$ and $\lim_{\wp\to\infty} \mathcal{G}(\nu, \varrho, \varsigma, \wp) = 0$, then $\mathcal{Q}(\nu, \varrho, \varsigma, \wp) = 1$, $\mathcal{F}(\nu, \varrho, \varsigma, \wp) = 0$ and $\mathcal{G}(\nu, \varrho, \varsigma, \wp) = 0$. Consequently, for all $\wp > 0$. Hence, $\nu = 0$ because $\varsigma \neq \nu$, $\varsigma \neq \varrho$. □

Lemma 3 *Let $(\Phi, \mathcal{Q}, \mathcal{F}, \mathcal{G}, *, +, \perp)$ be a ON2MS and let $\lim_{\wp\to\infty} \nu_{\mathfrak{n}} = \nu$, $\lim_{\wp\to\infty} \varrho_{\mathfrak{n}} = \varrho$. Then, it satisfies for all $\tau \in \Phi$ and $\wp \geq 0$:*

(1)

$$\lim_{\mathfrak{n}\to\infty} \inf \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \geq \mathcal{Q}(\nu, \varrho, \tau, \wp),$$
$$\lim_{\mathfrak{n}\to\infty} \sup \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \leq \mathcal{F}(\nu, \varrho, \tau, \wp)$$

and

$$\lim_{\mathfrak{n}\to\infty} \sup \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \leq \mathcal{G}(\nu, \varrho, \tau, \wp).$$

(2)

$$\mathcal{Q}(\nu, \varrho, \tau, \wp) \geq \lim_{\mathfrak{n}\to\infty} \sup \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp),$$
$$\mathcal{F}(\nu, \varrho, \tau, \wp + 0) \leq \lim_{\mathfrak{n}\to\infty} \inf \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp)$$

and

$$\mathcal{G}(\nu, \varrho, \tau, \wp + 0) \leq \lim_{\mathfrak{n}\to\infty} \inf \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp).$$

Proof For all $\tau \in \Phi$ and $\wp \geq 0$, we have

$$\begin{aligned}\mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) &\geq \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \nu, \wp_1) * \mathcal{Q}(\nu_{\mathfrak{n}}, \nu, \tau, \wp_2) * \mathcal{Q}(\nu, \varrho_{\mathfrak{n}}, \tau, \wp), \quad \wp_1 + \wp_2 = 0 \\ &\geq \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \nu, \wp_1) * \mathcal{Q}(\nu_{\mathfrak{n}}, \nu, \tau, \wp_2) * \mathcal{Q}(\nu, \varrho_{\mathfrak{n}}, \varrho, \wp_3) \\ &\quad * \mathcal{Q}(\nu, \varrho, \tau, \wp_4) * \mathcal{Q}(\varrho, \varrho_{\mathfrak{n}}, \tau, \wp), \quad \wp_3 + \wp_4 = 0,\end{aligned}$$

which implies $\lim_{\mathfrak{n}\to\infty} \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \geq 1 * 1 * 1 * \mathcal{Q}(\nu, \varrho, \tau, \wp) * 1 = \mathcal{Q}(\nu, \varrho, \tau, \wp)$, also

$$\begin{aligned}\mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) &\leq \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \nu, \wp_1) + \mathcal{F}(\nu_{\mathfrak{n}}, \nu, \tau, \wp_2) + \mathcal{F}(\nu, \varrho_{\mathfrak{n}}, \tau, \wp), \quad \wp_1 + \wp_2 = 0 \\ &\leq \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \nu, \wp_1) + \mathcal{F}(\nu_{\mathfrak{n}}, \nu, \tau, \wp_2) + \mathcal{F}(\nu, \varrho_{\mathfrak{n}}, \varrho, \wp_3) \\ &\quad + \mathcal{F}(\nu, \varrho, \tau, \wp_4) + \mathcal{F}(\varrho, \varrho_{\mathfrak{n}}, \tau, \wp), \quad \wp_3 + \wp_4 = 0,\end{aligned}$$

which implies

$$\lim_{\mathfrak{n}\to\infty} \sup \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \leq 0 + 0 + 0 + \mathcal{F}(\nu, \varrho, \tau, \wp) + 0 = \mathcal{F}(\nu, \varrho, \tau, \wp)$$

and

$$\begin{aligned}\mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) &\leq \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \nu, \wp_1) + \mathcal{G}(\nu_{\mathfrak{n}}, \nu, \tau, \wp_2) + \mathcal{G}(\nu, \varrho_{\mathfrak{n}}, \tau, \wp), \quad \wp_1 + \wp_2 = 0 \\ &\leq \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \nu, \wp_1) + \mathcal{G}(\nu_{\mathfrak{n}}, \nu, \tau, \wp_2) + \mathcal{G}(\nu, \varrho_{\mathfrak{n}}, \varrho, \wp_3) \\ &\quad + \mathcal{G}(\nu, \varrho, \tau, \wp_4) + \mathcal{G}(\varrho, \varrho_{\mathfrak{n}}, \tau, \wp_4), \quad \wp_3 + \wp_4 = 0,\end{aligned}$$

which implies $\lim_{\mathfrak{n}\to\infty} \sup \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \leq 0 + 0 + 0 + \mathcal{G}(\nu, \varrho, \tau, \wp) + 0 = \mathcal{G}(\nu, \varrho, \tau, \wp)$.

Let $\epsilon > 0$ be given. For all $\tau \in \nu$ and $\wp > 0$, we have

$$\begin{aligned}\mathcal{Q}(\nu, \varrho, \tau, \wp + 2\epsilon) &\geq \mathcal{Q}\left(\nu, \varrho, \nu_{\mathfrak{n}}, \frac{\epsilon}{2}\right) * \mathcal{Q}\left(\nu, \nu_{\mathfrak{n}}, \tau, \frac{\epsilon}{2}\right) * \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho, \tau, \wp + \epsilon) \\ &\geq \mathcal{Q}\left(\nu, \varrho, \nu_{\mathfrak{n}}, \frac{\epsilon}{2}\right) * \mathcal{Q}\left(\nu, \nu_{\mathfrak{n}}, \tau, \frac{\epsilon}{2}\right) * \mathcal{Q}\left(\nu_{\mathfrak{n}}, \varrho, \varrho_{\mathfrak{n}}, \frac{\epsilon}{2}\right) \\ &\quad * \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) * \mathcal{Q}\left(\varrho_{\mathfrak{n}}, \varrho, \tau, \frac{\epsilon}{2}\right).\end{aligned}$$

Consequently,

$$\mathcal{Q}(\nu, \varrho, \tau, \wp + 2\epsilon) \geq \lim_{\mathfrak{n}\to\infty} \sup \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp).$$

Letting $\epsilon \to 0$, we have

$$\mathcal{Q}(\nu, \varrho, \tau, \wp + 0) \geq \lim_{\mathfrak{n}\to\infty} \sup \mathcal{Q}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp).$$

Also, we have

$$\begin{aligned}\mathcal{F}(\nu, \varrho, \tau, \wp + 2\epsilon) &\leq \mathcal{F}\left(\nu, \varrho, \nu_{\mathfrak{n}}, \frac{\epsilon}{2}\right) \diamond \mathcal{F}\left(\nu, \nu_{\mathfrak{n}}, \tau, \frac{\epsilon}{2}\right) \diamond \mathcal{F}(\nu_{\mathfrak{n}}, \varrho, \tau, \wp + \epsilon) \\ &\geq \mathcal{F}\left(\nu, \varrho, \nu_{\mathfrak{n}}, \frac{\epsilon}{2}\right) \diamond \mathcal{F}\left(\nu, \nu_{\mathfrak{n}}, \tau, \frac{\epsilon}{2}\right) \diamond \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \\ &\quad \diamond \mathcal{F}\left(\varrho_{\mathfrak{n}}, \varrho, \tau, \frac{\epsilon}{2}\right) \diamond \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \diamond \mathcal{F}\left(\varrho_{\mathfrak{n}}, \varrho, \tau, \frac{\epsilon}{2}\right).\end{aligned}$$

Consequently,

$$\mathcal{F}(\nu, \varrho, \tau, \wp + 2\epsilon) \leq \lim_{\mathfrak{n}\to\infty} \inf \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp).$$

Letting $\epsilon \to 0$, we have

$$\mathcal{F}(\nu, \varrho, \tau, \wp + 0) \leq \lim_{\mathfrak{n}\to\infty} \inf \mathcal{F}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp)$$

and

$$\begin{aligned}\mathcal{G}(\nu, \varrho, \tau, \wp + 2\epsilon) &\leq \mathcal{G}\left(\nu, \varrho, \nu_{\mathfrak{n}}, \frac{\epsilon}{2}\right) \diamond \mathcal{G}\left(\nu, \nu_{\mathfrak{n}}, \tau, \frac{\epsilon}{2}\right) \diamond \mathcal{G}(\nu_{\mathfrak{n}}, \varrho, \tau, \wp + \epsilon) \\ &\geq \mathcal{G}\left(\nu, \varrho, \nu_{\mathfrak{n}}, \frac{\epsilon}{2}\right) \diamond \mathcal{G}\left(\nu, \nu_{\mathfrak{n}}, \tau, \frac{\epsilon}{2}\right) \diamond \mathcal{G}\left(\nu_{\mathfrak{n}}, \varrho, \varrho_{\mathfrak{n}}, \frac{\epsilon}{2}\right) \diamond \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \\ &\quad \diamond \mathcal{G}\left(\varrho_{\mathfrak{n}}, \varrho, \tau, \frac{\epsilon}{2}\right) \diamond \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp) \diamond \mathcal{G}\left(\varrho_{\mathfrak{n}}, \varrho, \tau, \frac{\epsilon}{2}\right).\end{aligned}$$

Consequently,

$$\mathcal{G}(\nu, \varrho, \tau, \wp + 2\epsilon) \leq \lim_{\mathfrak{n}\to\infty} \inf \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp).$$

Letting $\epsilon \to 0$, we have

$$\mathcal{G}(\nu, \varrho, \tau, \wp + 0) \leq \lim_{\mathfrak{n}\to\infty} \inf \mathcal{G}(\nu_{\mathfrak{n}}, \varrho_{\mathfrak{n}}, \tau, \wp). \qquad \square$$

Lemma 4 *Let $(\Phi, \mathcal{Q}, \mathcal{F}, \mathcal{G}, *, +, \perp)$ be a N2MS. Let Υ and Λ be a continuous self-map on Φ and $[\Upsilon, \Lambda]$ are compatible. Let $\nu_{\mathfrak{n}}$ be a sequence in Φ such that $\Upsilon\nu_{\mathfrak{n}} \to \omega$ and $\Lambda\nu_{\mathfrak{n}} \to \omega$. Then, $\Upsilon\Lambda\nu_{\mathfrak{n}} \to \Lambda\omega$.*

Proof Since Υ, Λ are compatible maps, $\Upsilon\Lambda\nu_{\mathfrak{n}} \to \Upsilon\omega$, $\Lambda\Upsilon\nu_{\mathfrak{n}} \to \Lambda\omega$ and so, $\mathcal{Q}(\Upsilon\Lambda\nu_{\mathfrak{n}}, \Upsilon\omega, \tau, \frac{\wp}{3}) \to 1$, $\mathcal{F}(\Lambda\Upsilon\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \frac{\wp}{3}) \to 0$ and $\mathcal{G}(\Lambda\Upsilon\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \frac{\wp}{3}) \to 0$ for all $\tau \in \Phi$ and $\wp > 0$,

$$\begin{aligned}\mathcal{Q}(\Upsilon\Lambda\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \wp) &\geq \mathcal{Q}\left(\Upsilon\Lambda\nu_{\mathfrak{n}}, \Lambda\omega, \Lambda\Upsilon\nu_{\mathfrak{n}}, \frac{\wp}{3}\right) * \mathcal{Q}\left(\Upsilon\Lambda\nu_{\mathfrak{n}}, \Lambda\Upsilon\nu_{\mathfrak{n}}, \tau, \frac{\wp}{3}\right) \\ &\quad * \mathcal{Q}\left(\Lambda\Upsilon\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \frac{\wp}{3}\right) \\ &\geq \mathcal{Q}\left(\Lambda\Upsilon\nu_{\mathfrak{n}}, \Lambda\omega, \Upsilon\Lambda\nu_{\mathfrak{n}}, \frac{\wp}{3}\right) * \mathcal{Q}\left(\Lambda\Upsilon\nu_{\mathfrak{n}}, \omega\Lambda\nu_{\mathfrak{n}}, \tau, \frac{\wp}{3}\right) \\ &\quad * \mathcal{Q}\left(\Lambda\Upsilon\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \frac{\wp}{3}\right) \to 1.\end{aligned}$$

Also, we have

$$\begin{aligned}\mathcal{F}(\Upsilon\Lambda\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \wp) &\leq \mathcal{F}\left(\Upsilon\Lambda\nu_{\mathfrak{n}}, \Lambda\omega, \Lambda\Upsilon\nu_{\mathfrak{n}}, \frac{\wp}{3}\right) + d\mathcal{F}\left(\Upsilon\Lambda\nu_{\mathfrak{n}}, \Lambda\Upsilon\nu_{\mathfrak{n}}, \tau, \frac{\wp}{3}\right) \\ &\quad + \mathcal{F}\left(\Lambda\Upsilon\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \frac{\wp}{3}\right) \\ &\leq \mathcal{F}\left(\Lambda\Upsilon\nu_{\mathfrak{n}}, \Lambda\omega, \Upsilon\Lambda\nu_{\mathfrak{n}}, \frac{\wp}{3}\right) + \mathcal{F}\left(\Lambda\Upsilon\nu_{\mathfrak{n}}, \omega\Lambda\nu_{\mathfrak{n}}, \tau, \frac{\wp}{3}\right) \\ &\quad + \mathcal{F}\left(\Lambda\Omega\nu_{\mathfrak{n}}, \Lambda\omega, \tau, \frac{\wp}{3}\right) \to 0.\end{aligned}$$

For all $\tau \in \Xi$ and $\wp > 0$, and

$$\begin{aligned}\mathcal{G}(\Upsilon\Lambda\nu_{\mathfrak{n}},\Lambda\omega,\tau,\wp) &\leq \mathcal{G}\left(\Upsilon\Lambda\nu_{\mathfrak{n}},\Lambda\omega,\Lambda\Upsilon\nu_{\mathfrak{n}},\frac{\wp}{3}\right)+\mathcal{G}\left(\Upsilon\Lambda\nu_{\mathfrak{n}},\Lambda\Upsilon\nu_{\mathfrak{n}},\tau,\frac{\wp}{3}\right)\\&\quad+\mathcal{G}\left(\Lambda\Upsilon\nu_{\mathfrak{n}},\Lambda\omega,\tau,\frac{\wp}{3}\right)\\&\leq \mathcal{G}\left(\Lambda\Upsilon\nu_{\mathfrak{n}},\Lambda\omega,\Upsilon\Lambda\nu_{\mathfrak{n}},\frac{\wp}{3}\right)+\mathcal{G}\left(\Lambda\Upsilon\nu_{\mathfrak{n}},\omega\Lambda\nu_{\mathfrak{n}},\tau,\frac{\wp}{3}\right)\\&\quad+\mathcal{G}\left(\Lambda\Upsilon\nu_{\mathfrak{n}},\Lambda\omega,\tau,\frac{\wp}{3}\right)\to 0.\end{aligned}$$

For all $\tau \in \Phi$ and $\wp > 0$. Hence, $\Upsilon\Lambda\nu_{\mathfrak{n}} \to \Lambda\omega$. □

Theorem 3.1 *Let $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ be an orthogonal complete neutrosophic 2-metric space with "$*$" as con-t-nm and "$+$" as con-t-conm. Let Θ and Γ be continuous self-mappings on Φ. Then, Θ and Γ have a unique common fixed point in Φ if and only if there exist two self-mappings Υ, Λ of Φ satisfying*:

(1) $\Upsilon\Phi \subset \Gamma\Phi$, $\Lambda\Phi \subset \Theta\Phi$;
(2) *The pair $\{\Upsilon,\Theta\}$ and $\{\Lambda,\Gamma\}$ are compatible*;
(3) *Υ, Λ, Θ, Γ be $\perp$-preserving*;
(4) *There exists $\ell \in (0,1)$ such that for every $\nu,\varrho,\varsigma \in \Phi$ and $\wp > 0$,*

$$\begin{aligned}\mathcal{Q}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) &\geq \min\bigl\{\mathcal{Q}(\Theta\nu,\Gamma\nu,\varsigma,\wp),\mathcal{Q}(\Upsilon\nu,\Theta\nu,\varsigma,\wp),\\&\qquad\mathcal{Q}(\Lambda\varrho,\Gamma\varrho,\varsigma,\wp),\mathcal{Q}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp)\bigr\},\\\mathcal{F}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) &\leq \max\bigl\{\mathcal{F}(\Theta\nu,\Gamma\nu,\varsigma,\wp),\mathcal{F}(\Upsilon\nu,\Theta\nu,\varsigma,\wp),\\&\qquad\mathcal{F}(\Lambda\varrho,\Gamma\varrho,\varsigma,\wp),\mathcal{F}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp)\bigr\},\\\mathcal{G}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) &\leq \max\bigl\{\mathcal{F}(\Theta\nu,\Gamma\nu,\varsigma,\wp),\mathcal{F}(\Upsilon\nu,\Theta\nu,\varsigma,\wp),\\&\qquad\mathcal{F}(\Lambda\varrho,\Gamma\varrho,\varsigma,\wp),\mathcal{F}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp)\bigr\}.\end{aligned}$$

Then Υ, Λ, Θ, and Γ have a unique common fixed point in Φ.

Proof Suppose Θ and Γ have a unique common fixed point, say $\mathfrak{r} \in \Phi$. Define $\Upsilon: \Phi \to \Phi$ by $\Upsilon\nu = \mathfrak{r}$ for all $\nu \in \Phi$ and $\Lambda: \Phi \to \Phi$ by $\Lambda\nu = \mathfrak{r}$ for all $\nu \in \Phi$. Then, it satisfies(1)–(4)

Conversely, if there exist two self-mappings Υ, Λ of ν this satisfies (1)–(4). From (1) if two sequences are $\nu_{\mathfrak{n}}$ and $\varrho_{\mathfrak{n}}$ of Φ such that $\varrho_{2\mathfrak{n}-1} = \Gamma\nu_{2\mathfrak{n}-1}$ and $\nu_{2\mathfrak{n}-1} = \Theta\nu_{2\mathfrak{n}} = \Lambda\nu_{2\mathfrak{n}-1}$ for $\mathfrak{n} = 1,2,3$. Putting $\nu = \nu_{2\mathfrak{n}}$ and $\nu = \nu_{2\mathfrak{n}+1}$ in condition (4), for all $\varsigma \in \Phi$ and $\wp > 0$.

Since $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is an orthogonal complete neutrosophic 2-metric space there exists $\nu_0 \in \Phi$, such that

$$\nu_0 \perp \varrho, \quad \text{for all } \varrho \in \Phi,$$

$$\text{i.e., } \nu_0 \perp \Upsilon\nu_0 \quad \text{take}$$

$$\nu_{\mathfrak{n}} = \Upsilon^{\mathfrak{n}}\nu_0 = \Upsilon\nu_{\mathfrak{n}-1}, \quad \text{for all } \mathfrak{n} \in \mathcal{F}.$$

Since Υ is $\perp$-preserving, $\{\nu_{\mathfrak{n}}\}$ is an O-sequence. Now, since Υ is an $\perp$-contraction, we can obtain

$$\begin{aligned}
\mathcal{Q}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+2},\varsigma,\ell\wp) &= \mathcal{Q}(\Upsilon\nu_{2\mathfrak{n}},\Lambda\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)\\
&\geq \min\bigl\{\mathcal{Q}(\Theta\nu_{2\mathfrak{n}},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{Q}(\Upsilon\nu_{2\mathfrak{n}},\Theta\nu_{2\mathfrak{n}},\varsigma,\wp),\\
&\quad \mathcal{Q}(\Lambda\varrho_{2\mathfrak{n}+1},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{Q}(\Upsilon\nu_{2\mathfrak{n}},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp)\bigr\}\\
&\geq \min\bigl\{\mathcal{Q}(\varrho\nu_{2\mathfrak{n}},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp),\mathcal{Q}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)\bigr\},\\
\mathcal{F}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+2},\varsigma,\ell\wp) &= \mathcal{F}(\Upsilon\nu_{2\mathfrak{n}},\Lambda\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)\\
&\leq \max\bigl\{\mathcal{F}(\Theta\nu_{2\mathfrak{n}},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{F}(\Upsilon\nu_{2\mathfrak{n}},\Theta\nu_{2\mathfrak{n}},\varsigma,\wp),\\
&\quad \mathcal{F}(\Lambda\varrho_{2\mathfrak{n}+1},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{F}(\Upsilon\nu_{2\mathfrak{n}},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp)\bigr\}\\
&\leq \max\bigl\{\mathcal{F}(\varrho\nu_{2\mathfrak{n}},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp),\mathcal{F}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)\bigr\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+2},\varsigma,\ell\wp) &= \mathcal{G}(\Upsilon\nu_{2\mathfrak{n}},\Lambda\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)\\
&\leq \max\bigl\{\mathcal{G}(\Theta\nu_{2\mathfrak{n}},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{G}(\Upsilon\nu_{2\mathfrak{n}},\Theta\nu_{2\mathfrak{n}},\varsigma,\wp),\\
&\quad \mathcal{G}(\Lambda\varrho_{2\mathfrak{n}+1},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{G}(\Upsilon\nu_{2\mathfrak{n}},\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp)\bigr\}\\
&\leq \max\bigl\{\mathcal{G}(\varrho\nu_{2\mathfrak{n}},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp),\mathcal{G}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)\bigr\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathcal{Q}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+2},\varsigma,\ell\wp) \geq \mathcal{Q}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp),\\
&\mathcal{F}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+2},\varsigma,\ell\wp) \leq \mathcal{F}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)
\end{aligned}$$

and

$$\mathcal{G}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+2},\varsigma,\ell\wp) \leq \mathcal{G}(\varrho\nu_{2\mathfrak{n}+1},\varrho\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp).$$

By using Lemma 1 and letting $\nu = \nu_{2\mathfrak{n}+1}$ and $\varrho = \nu_{2\mathfrak{n}+1}$ in condition (4), we have

$$\begin{aligned}
&\mathcal{Q}(\varrho_{2\mathfrak{n}+2},\varrho_{2\mathfrak{n}+3},\varsigma,\ell\wp) \geq \mathcal{Q}(\varrho_{2\mathfrak{n}+1},\varrho_{2\mathfrak{n}+1},\varsigma,\wp),\\
&\mathcal{F}(\varrho_{2\mathfrak{n}+2},\varrho_{2\mathfrak{n}+3},\varsigma,\wp) \leq \mathcal{F}(\varrho_{2\mathfrak{n}+1},\varrho_{2\mathfrak{n}+1},\varsigma,\wp)
\end{aligned}$$

and

$$\mathcal{G}(\varrho_{2\mathfrak{n}+2},\varrho_{2\mathfrak{n}+3},\varsigma,\wp) \leq \mathcal{G}(\varrho_{2\mathfrak{n}+1},\varrho_{2\mathfrak{n}+1},\varsigma,\wp),$$

for all $\varsigma \in \Phi$ and $\wp > 0$.

In general, we obtain that for all $\varsigma \in \Phi$ and $\wp > 0$ and $\mathfrak{n} = 1, 2, 3, \ldots$.

$$\begin{aligned}
&\mathcal{Q}(\varrho_{\mathfrak{n}},\varrho_{\mathfrak{n}+1},\varsigma,\ell\wp) \geq \mathcal{Q}(\varrho_{\mathfrak{n}-1},\varrho_{\mathfrak{n}},\varsigma,\wp),\\
&\mathcal{F}(\varrho_{\mathfrak{n}},\varrho_{\mathfrak{n}+1},\varsigma,\ell\wp) \leq \mathcal{Q}(\varrho_{\mathfrak{n}-1},\varrho_{\mathfrak{n}},\varsigma,\wp)
\end{aligned}$$

and

$$\mathcal{G}(\varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \varsigma, \ell\wp) \leq \mathcal{G}(\varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \varsigma, \wp).$$

Thus, for all $\varsigma \in \Phi$ and $\wp > 0$, and $\mathfrak{n} = 1, 2, 3, \ldots$

$$\mathcal{Q}(\varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \varsigma, \ell\wp) \geq \mathcal{Q}\left(\varrho_0, \varrho_1, \varsigma, \frac{\wp}{\ell^{\mathfrak{n}}}\right), \tag{1}$$

$$\mathcal{F}(\varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \varsigma, \ell\wp) \leq \mathcal{F}\left(\varrho_0, \varrho_1, \varsigma, \frac{\wp}{\ell^{\mathfrak{n}}}\right), \tag{2}$$

$$\mathcal{G}(\varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \varsigma, \ell\wp) \leq \mathcal{G}\left(\varrho_0, \varrho_1, \varsigma, \frac{\wp}{\ell^{\mathfrak{n}}}\right). \tag{3}$$

To show that $\{\varrho_{\mathfrak{n}}\}$ is a Cauchy sequence in Φ, let $\mathfrak{m} > \mathfrak{n}$. Then, for all $\varsigma \in \Phi$ and $\wp > \nu$, we obtain

$$\begin{aligned}
\mathcal{Q}(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varsigma, \wp) &\geq \mathcal{Q}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) * \mathcal{Q}\left(\varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}}, \varsigma, \frac{\wp}{3}\right) * \mathcal{Q}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+1}, \varsigma, \frac{\wp}{3}\right) \\
&\geq \mathcal{Q}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) * \mathcal{Q}\left(\varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}}, \varsigma, \frac{\wp}{3}\right) * \mathcal{Q}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) \\
&\quad * \mathcal{Q}\left(\varrho_{\mathfrak{n}+2}, \varrho_{\mathfrak{n}+1}, \varsigma, \frac{\wp}{3^2}\right) * \mathcal{Q}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+2}, \varsigma, \frac{\wp}{3^2}\right) * \cdots \\
&\quad * \mathcal{Q}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{m}-1}, \varsigma, \frac{\wp}{3^{\mathfrak{m}-\mathfrak{n}}}\right), \\
\mathcal{F}(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varsigma, \wp) &\leq \mathcal{F}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) * \mathcal{F}\left(\varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}}, \varsigma, \frac{\wp}{3}\right) * \mathcal{F}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+1}, \varsigma, \frac{\wp}{3}\right) \\
&\leq \mathcal{F}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) * \mathcal{F}\left(\varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}}, \varsigma, \frac{\wp}{3}\right) * \mathcal{F}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) \\
&\quad * \mathcal{F}\left(\varrho_{\mathfrak{n}+2}, \varrho_{\mathfrak{n}+1}, \varsigma, \frac{\wp}{3^2}\right) * \mathcal{F}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+2}, \varsigma, \frac{\wp}{3^2}\right) * \cdots \\
&\quad * \mathcal{F}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{m}-1}, \varsigma, \frac{\wp}{3^{\mathfrak{m}-\mathfrak{n}}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varsigma, \wp) &\leq \mathcal{G}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) * \mathcal{G}\left(\varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}}, \varsigma, \frac{\wp}{3}\right) * \mathcal{G}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+1}, \varsigma, \frac{\wp}{3}\right) \\
&\leq \mathcal{G}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) * \mathcal{G}\left(\varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}}, \varsigma, \frac{\wp}{3}\right) * \mathcal{G}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+1}, \varrho_{\mathfrak{n}+1}, \frac{\wp}{3}\right) \\
&\quad * \mathcal{G}\left(\varrho_{\mathfrak{n}+2}, \varrho_{\mathfrak{n}+1}, \varsigma, \frac{\wp}{3^2}\right) * \mathcal{G}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}+2}, \varsigma, \frac{\wp}{3^2}\right) * \cdots \\
&\quad * \mathcal{G}\left(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{m}-1}, \varsigma, \frac{\wp}{3^{\mathfrak{m}-\mathfrak{n}}}\right).
\end{aligned}$$

Letting $\mathfrak{m}, \mathfrak{n} \to \infty$, we have

$$\lim_{\mathfrak{n}\to\infty} \mathcal{Q}(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varsigma, \wp) = 1, \qquad \lim_{\mathfrak{n}\to\infty} \mathcal{F}(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varsigma, \wp) = 0 \quad \text{and}$$
$$\lim_{\mathfrak{n}\to\infty} \mathcal{G}(\varrho_{\mathfrak{m}}, \varrho_{\mathfrak{n}}, \varsigma, \wp) = 0.$$

Thus, $\{\varrho_{\mathfrak{n}}\}$ is a Cauchy sequence in Φ. By completeness of Φ there exist $\mathfrak{r} \in \Phi$ such that

$$\lim_{\mathfrak{n}\to\infty} \varrho_{\mathfrak{n}} = \mathfrak{r}, \qquad \lim_{\mathfrak{n}\to\infty} \varrho_{2\mathfrak{n}-1} = \lim_{\mathfrak{n}\to\infty} \nu_{2\mathfrak{n}-1} = \lim_{\mathfrak{n}\to\infty} \Upsilon \nu_{2\mathfrak{n}-2} = \mathfrak{r}$$

and

$$\lim_{\mathfrak{n}\to\infty} \varrho_{2\mathfrak{n}} = \lim_{\mathfrak{n}\to\infty} \Theta \nu_{2\mathfrak{n}} = \lim_{\mathfrak{n}\to\infty} \Lambda \nu_{2\mathfrak{n}-1} = \mathfrak{r}.$$

From Lemma 4, we have

$$\Upsilon \Theta \nu_{2\mathfrak{n}+1} = \Theta \mathfrak{r} \quad \text{and} \quad \Lambda \Gamma \nu_{2\mathfrak{n}+1} = \Gamma \mathfrak{r}. \tag{4}$$

Meanwhile, for all $\varsigma \in \Phi$ with $\varsigma \neq \Theta \mathfrak{r}$ and $\varsigma \neq \Gamma \mathfrak{r}$ and $\wp > 0$, we have

$$\begin{aligned}
&\mathcal{Q}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Lambda\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \ell\wp) \\
&\quad \geq \min\bigl\{\mathcal{Q}(\Theta\Theta\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \wp), \mathcal{Q}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Theta\Theta\nu_{2\mathfrak{n}+1}, \varsigma, \wp), \\
&\qquad \mathcal{Q}(\Lambda\Gamma\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \ell\varsigma, \wp), \mathcal{Q}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \wp)\bigr\}, \\
&\mathcal{F}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Lambda\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \ell\wp) \\
&\quad \leq \max\bigl\{\mathcal{F}(\Theta\Theta\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \wp), \mathcal{F}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Theta\Theta\nu_{2\mathfrak{n}+1}, \varsigma, \wp), \\
&\qquad \mathcal{F}(\Lambda\Gamma\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \ell\varsigma, \wp), \mathcal{F}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \wp)\bigr\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{G}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Lambda\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \ell\wp) \\
&\quad \leq \max\bigl\{\mathcal{G}(\Theta\Theta\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \wp), \mathcal{G}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Theta\Theta\nu_{2\mathfrak{n}+1}, \varsigma, \wp), \\
&\qquad \mathcal{G}(\Lambda\Gamma\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \ell\varsigma, \wp), \mathcal{G}(\Upsilon\Theta\nu_{2\mathfrak{n}+1}, \Gamma\Gamma\nu_{2\mathfrak{n}+1}, \varsigma, \wp)\bigr\}.
\end{aligned}$$

Taking the limit as $\mathfrak{n} \to \infty$ and using (4), we have for all $\varsigma \in \Phi$ with $\varsigma \neq \Theta\mathfrak{r}$ and $\varsigma \neq \Gamma\mathfrak{r}$ and $\wp > 0$,

$$\begin{aligned}
&\mathcal{Q}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \ell\wp + 0) \\
&\quad \geq \min\bigl\{\mathcal{Q}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Theta\mathfrak{r}, \Theta\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Theta\mathfrak{r}, \Theta\mathfrak{r}, \varsigma, \wp)\bigr\} \\
&\quad = \mathcal{Q}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \\
&\mathcal{F}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \ell\wp + 0) \\
&\quad \leq \max\bigl\{\mathcal{F}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Theta\mathfrak{r}, \Theta\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Theta\mathfrak{r}, \Theta\mathfrak{r}, \varsigma, \wp)\bigr\} \\
&\quad = \mathcal{F}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{G}(\Theta\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\ell\wp+0)\\
&\quad\le\max\{\mathcal{G}(\Theta\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{G}(\Theta\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp),\mathcal{G}(\Gamma\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{G}(\Theta\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp)\}\\
&\quad=\mathcal{G}(\Theta\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp).
\end{aligned}$$

By Lemma 2, we have

$$\Theta\mathfrak{r}=\Gamma\mathfrak{r}. \tag{5}$$

From condition (4), we obtain for all $\varsigma\in\Phi$ with $\varsigma\neq\Upsilon\mathfrak{r}$, $\varsigma\neq\Gamma\mathfrak{r}$ and $\wp>0$,

$$\begin{aligned}
\mathcal{Q}(\Upsilon\mathfrak{r},\Lambda\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)&\ge\min\{\mathcal{Q}(\Theta\mathfrak{r},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{Q}(\Upsilon_{\mathfrak{r}},\Theta_{\mathfrak{r}},\varsigma,\wp)\\
&\qquad\mathcal{Q}(\Lambda\Gamma\nu_{2\mathfrak{n}+1},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{Q}(\Upsilon\mathfrak{r},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp)\},\\
\mathcal{F}(\Upsilon\mathfrak{r},\Lambda\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)&\le\max\{\mathcal{F}(\Theta\mathfrak{r},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{F}(\Upsilon_{\mathfrak{r}},\Theta_{\mathfrak{r}},\varsigma,\wp)\\
&\qquad\mathcal{F}(\Lambda\Gamma\nu_{2\mathfrak{n}+1},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{F}(\Upsilon\mathfrak{r},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp)\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\Upsilon\mathfrak{r},\Lambda\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\ell\wp)&\le\max\{\mathcal{G}(\Theta\mathfrak{r},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{G}(\Upsilon_{\mathfrak{r}},\Theta_{\mathfrak{r}},\varsigma,\wp)\\
&\qquad\mathcal{G}(\Lambda\Gamma\nu_{2\mathfrak{n}+1},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp),\mathcal{G}(\Upsilon\mathfrak{r},\Gamma\Gamma\nu_{2\mathfrak{n}+1},\varsigma,\wp)\}.
\end{aligned}$$

Let $\mathfrak{n}\to\infty$, by condition (4), and Lemma 3, for all $\varsigma\in\Phi$

$$\begin{aligned}
\mathcal{Q}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\ell\wp+0)&\ge\min\{\mathcal{Q}(\Theta\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{Q}(\Upsilon\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp),\\
&\qquad\mathcal{Q}(\Gamma\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{Q}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)\}\\
&=\mathcal{Q}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\\
\mathcal{F}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\ell\wp+0)&\le\max\{\mathcal{F}(\Theta\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{F}(\Upsilon\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp),\\
&\qquad\mathcal{F}(\Gamma\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{F}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)\}\\
&=\mathcal{F}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\ell\wp+0)&\le\max\{\mathcal{G}(\Theta\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{G}(\Upsilon\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp),\\
&\qquad\mathcal{G}(\Gamma\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{G}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)\}\\
&=\mathcal{G}(\Upsilon\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp).
\end{aligned}$$

By Lemma 2, we have

$$\Upsilon\mathfrak{r}=\Gamma\mathfrak{r}. \tag{6}$$

For all $\varsigma \in \Phi$ with $\varsigma \neq \Upsilon\mathfrak{r}$ and $\varsigma \neq \Lambda\mathfrak{r}$ and $\wp > 0$, we have

$$\begin{aligned}
&\mathcal{Q}(\Upsilon\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \ell\wp) \\
&\quad \geq \min\bigl\{\mathcal{Q}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Lambda\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)\bigr\} \\
&\quad \geq \min\bigl\{\mathcal{Q}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Lambda\mathfrak{r}, \Upsilon\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)\bigr\} \\
&\quad = \mathcal{Q}(\Upsilon\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \wp), \\
&\mathcal{F}(\Upsilon\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \ell\wp) \\
&\quad \leq \max\bigl\{\mathcal{F}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Lambda\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)\bigr\} \\
&\quad \leq \max\bigl\{\mathcal{F}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Lambda\mathfrak{r}, \Upsilon\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)\bigr\} \\
&\quad = \mathcal{F}(\Upsilon\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \wp)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{G}(\Upsilon\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \ell\wp) \\
&\quad \leq \max\bigl\{\mathcal{G}(\Theta\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Lambda\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)\bigr\}, \\
&\quad \leq \max\bigl\{\mathcal{G}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Upsilon\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Lambda\mathfrak{r}, \Upsilon\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Gamma\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)\bigr\} \\
&\quad = \mathcal{G}(\Upsilon\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \wp).
\end{aligned}$$

By Lemma 2,

$$\Upsilon\mathfrak{r} = \Lambda\mathfrak{r}. \tag{7}$$

It follows that $\Upsilon\mathfrak{r} = \Lambda\mathfrak{r} = \Theta\mathfrak{r} = \Gamma\mathfrak{r}$. For all $\varsigma \in \Phi$ with $\varsigma \neq \Lambda\mathfrak{r}$ and $\varsigma \neq \mathfrak{r}$, and $\wp > 0$,

$$\begin{aligned}
\mathcal{Q}(\Upsilon\nu_{2\mathfrak{n}}, \Lambda\mathfrak{r}, \varsigma, \ell\wp) \geq{}& \min\bigl\{\mathcal{Q}(\Theta\nu_{2\mathfrak{n}}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Upsilon\nu_{2\mathfrak{n}}, \Theta\nu_{2\mathfrak{n}}, \varsigma, \wp) \\
&\mathcal{Q}(\Lambda\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\Upsilon\nu_{2\mathfrak{n}}, \Gamma_{\mathfrak{r}}, \varsigma, \wp)\bigr\}, \\
\mathcal{F}(\Upsilon\nu_{2\mathfrak{n}}, \Lambda\mathfrak{r}, \varsigma, \ell\wp) \leq{}& \max\bigl\{\mathcal{F}(\Theta\nu_{2\mathfrak{n}}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Upsilon\nu_{2\mathfrak{n}}, \Theta\nu_{2\mathfrak{n}}, \varsigma, \wp) \\
&\mathcal{F}(\Lambda\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{F}(\Upsilon\nu_{2\mathfrak{n}}, \Gamma_{\mathfrak{r}}, \varsigma, \wp)\bigr\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\Upsilon\nu_{2\mathfrak{n}}, \Lambda\mathfrak{r}, \varsigma, \ell\wp) \leq{}& \max\bigl\{\mathcal{G}(\Theta\nu_{2\mathfrak{n}}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Upsilon\nu_{2\mathfrak{n}}, \Theta\nu_{2\mathfrak{n}}, \varsigma, \wp) \\
&\mathcal{G}(\Lambda\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{G}(\Upsilon\nu_{2\mathfrak{n}}, \Gamma_{\mathfrak{r}}, \varsigma, \wp)\bigr\}.
\end{aligned}$$

Taking the limit as $\mathfrak{n} \to \infty$ and using (4) and Lemma 3, we have for all $\varsigma \in \Phi$ we $\varsigma \neq \Lambda\mathfrak{r}$, $\varsigma \neq \mathfrak{r}$ and $\wp > 0$

$$\begin{aligned}
\mathcal{Q}(\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \ell\wp + 0) \geq{}& \min\bigl\{\mathcal{Q}(\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\mathfrak{r}, \mathfrak{r}, \mathfrak{r}, \wp), \mathcal{Q}(\Lambda\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \wp), \mathcal{Q}(\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp)\bigr\} \\
\geq{}& \mathcal{Q}(\mathfrak{r}, \Gamma\mathfrak{r}, \varsigma, \wp) \geq \mathcal{Q}(\mathfrak{r}, \Lambda\mathfrak{r}, \varsigma, \wp),
\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\ell\wp+0) &\le \max\{\mathcal{F}(\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{F}(\mathfrak{r},\mathfrak{r},\mathfrak{r},\wp),\mathcal{F}(\Lambda\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\wp),\mathcal{F}(\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)\}\\ &\le \mathcal{F}(\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)\le \mathcal{F}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\wp)\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\ell\wp+0) &\le \max\{\mathcal{G}(\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp),\mathcal{G}(\mathfrak{r},\mathfrak{r},\mathfrak{r},\wp),\mathcal{G}(\Lambda\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\wp),\mathcal{G}(\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)\}\\ &\le \mathcal{G}(\mathfrak{r},\Gamma\mathfrak{r},\varsigma,\wp)\le \mathcal{G}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\wp).\end{aligned}$$

Hence, we have

$$\mathcal{Q}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\ell\wp)\ge \mathcal{Q}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\wp),\qquad \mathcal{F}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\ell\wp)\le \mathcal{F}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\wp)$$

and

$$\mathcal{G}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\ell\wp)\le \mathcal{G}(\mathfrak{r},\Lambda\mathfrak{r},\varsigma,\wp).$$

Therefore, $\Lambda\mathfrak{r}=\mathfrak{r}$. Thus, $\mathfrak{r}=\Upsilon\mathfrak{r}=\Lambda\mathfrak{r}=\Theta\mathfrak{r}=\Gamma\mathfrak{r}$. Hence, $\mathfrak{r}$ is a common fixed point of Υ, Λ, Θ, and Γ.

Let $\mathfrak{p}$ be another common fixed point of Υ, Λ, Θ, and Γ for all $\varsigma\in\Phi$ with $\varsigma\neq\mathfrak{r}$, $\varsigma\neq\mathfrak{p}$, and $\wp>0$, we have

$$\begin{aligned}&\nu_0\perp\nu^*,\\ &\nu_0\perp\varrho^*.\end{aligned}$$

Since, Γ is $\perp$-preserving, one writes

$$\begin{aligned}&\Upsilon^{\mathfrak{n}}\nu_0\perp\Upsilon^{\mathfrak{n}}\nu^*,\\ &\Upsilon^{\mathfrak{n}}\nu_0\perp\Upsilon^{\mathfrak{n}}\varrho^*.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{Q}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp) &= \mathcal{Q}(\Upsilon\mathfrak{r},\Lambda\mathfrak{p},\varsigma,\ell\wp)\\ &\ge \min\{\mathcal{Q}(\Theta\mathfrak{r},\Gamma\mathfrak{p},\varsigma,\wp),\mathcal{Q}(\Upsilon\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp),\mathcal{Q}(\Lambda\mathfrak{p},\Gamma\mathfrak{p},\varsigma,\wp),\\ &\quad \mathcal{Q}(\Upsilon\mathfrak{r},\Gamma\mathfrak{p},\varsigma,\wp)\}\\ &\ge \min\{\mathcal{Q}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\mathcal{Q}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\mathcal{Q}(\mathfrak{p},\mathfrak{p},\varsigma,\wp),\mathcal{Q}(\mathfrak{r},\mathfrak{p},\varsigma,\wp)\}\\ &\ge \mathcal{Q}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\\ \mathcal{F}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp) &= \mathcal{F}(\Upsilon\mathfrak{r},\Lambda\mathfrak{p},\varsigma,\ell\wp)\\ &\le \max\{\mathcal{F}(\Theta\mathfrak{r},\Gamma\mathfrak{p},\varsigma,\wp),\mathcal{F}(\Upsilon\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp),\mathcal{F}(\Lambda\mathfrak{p},\Gamma\mathfrak{p},\varsigma,\wp),\\ &\quad \mathcal{F}(\Upsilon\mathfrak{r},\Gamma\mathfrak{p},\varsigma,\wp)\}\\ &\le \max\{\mathcal{F}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\mathcal{F}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\mathcal{F}(\mathfrak{p},\mathfrak{p},\varsigma,\wp),\mathcal{F}(\mathfrak{r},\mathfrak{p},\varsigma,\wp)\}\\ &\le \mathcal{F}(\mathfrak{r},\mathfrak{p},\varsigma,\wp)\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp) &= \mathcal{G}(\Upsilon\mathfrak{r},\Lambda\mathfrak{p},\varsigma,\ell\wp)\\ &\leq \max\bigl\{\mathcal{G}(\Theta\mathfrak{r},\Gamma\mathfrak{p},\varsigma,\wp),\mathcal{G}(\Upsilon\mathfrak{r},\Theta\mathfrak{r},\varsigma,\wp),\mathcal{G}(\Lambda\mathfrak{p},\Gamma\mathfrak{p},\varsigma,\wp),\\ &\quad\mathcal{G}(\Upsilon\mathfrak{r},\Gamma\mathfrak{p},\varsigma,\wp)\bigr\}\\ &\leq \max\bigl\{\mathcal{G}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\mathcal{G}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\mathcal{G}(\mathfrak{p},\mathfrak{p},\varsigma,\wp),\mathcal{G}(\mathfrak{r},\mathfrak{p},\varsigma,\wp)\bigr\}\\ &\leq \mathcal{G}(\mathfrak{r},\mathfrak{p},\varsigma,\wp),\end{aligned}$$

which implies that

$$\begin{aligned}&\mathcal{Q}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp) \geq \mathcal{Q}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp),\\ &\mathcal{F}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp) \leq \mathcal{F}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp)\end{aligned}$$

and

$$\mathcal{G}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp) \leq \mathcal{G}(\mathfrak{r},\mathfrak{p},\varsigma,\ell\wp).$$

Hence, $\mathfrak{r} = \mathfrak{p}$. □

Example 3.2 Let $\Phi = [-1,2]$ and define a binary relation $\perp$ by $\nu \perp \varrho \perp \varsigma \iff \nu + \varrho + \varsigma \geq 0$. Define $\mathcal{Q}$, $\mathcal{F}$, $\mathcal{G}$ by,

$$\mathcal{Q}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) = \begin{cases} 1, & \text{if } \nu = \varrho = \varsigma,\\ \frac{\ell\wp}{\ell\wp+\min\{\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp\}}, & \text{if otherwise,}\end{cases}$$

$$\mathcal{F}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) = \begin{cases} 0, & \text{if } \nu = \varrho = \varsigma,\\ 1 - \frac{\ell\wp}{\ell\wp+\max\{\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp\}}, & \text{if otherwise,}\end{cases}$$

$$\mathcal{G}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) = \begin{cases} 0, & \text{if } \nu = \varrho = \varsigma,\\ \frac{\ell\wp+\max\{\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp\}}{\ell\wp}, & \text{if otherwise.}\end{cases}$$

With CTN $\mu * \alpha = \mu \cdot \alpha$ and CTCN $\mu + \alpha = \max\{\mu + \alpha\}$, $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+,\perp)$ is an O-complete N2MS. Also, observe that $\lim_{\mathfrak{n}\to\infty}\mathcal{Q}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) = 1$, $\lim_{\mathfrak{n}\to\infty}\mathcal{F}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) = 0$ and $\lim_{\mathfrak{n}\to\infty}\mathcal{G}(\Upsilon\nu,\Lambda\varrho,\varsigma,\ell\wp) = 0$ $\forall\nu,\varrho,\varsigma \in \Phi$.

Define $\Upsilon,\Lambda,\Theta,\Gamma : \Phi \to \Phi$

$$\Upsilon\nu = \nu^2, \qquad \Lambda\nu = \nu, \qquad \Theta\nu = 4\nu^4 - 3, \qquad \Gamma\nu = 4\nu^2 - 3.$$

From this, we obtain

$$\begin{aligned}&\mathcal{Q}(\varrho_{\mathfrak{n}},\varrho_{\mathfrak{n}+1},\varsigma,\ell\wp) \geq \mathcal{Q}(\varrho_{\mathfrak{n}-1},\varrho_{\mathfrak{n}},\varsigma,\wp),\\ &\mathcal{F}(\varrho_{\mathfrak{n}},\varrho_{\mathfrak{n}+1},\varsigma,\ell\wp) \leq \mathcal{Q}(\varrho_{\mathfrak{n}-1},\varrho_{\mathfrak{n}},\varsigma,\wp),\\ &\mathcal{G}(\varrho_{\mathfrak{n}},\varrho_{\mathfrak{n}+1},\varsigma,\ell\wp) \geq \mathcal{Q}(\varrho_{\mathfrak{n}-1},\varrho_{\mathfrak{n}},\varsigma,\wp).\end{aligned}$$

This implies

$$\lim_{\mathfrak{n}\to\infty} \mathcal{Q}(\varrho_{\mathfrak{m}},\varrho_{\mathfrak{n}},\varsigma,\wp) = 1, \qquad \lim_{\mathfrak{n}\to\infty} \mathcal{F}(\varrho_{\mathfrak{m}},\varrho_{\mathfrak{n}},\varsigma,\wp) = 0 \quad \text{and}$$

$$\lim_{\mathfrak{n}\to\infty} \mathcal{G}(\varrho_{\mathfrak{m}},\varrho_{\mathfrak{n}},\varsigma,\wp) = 0 \quad \text{for all } \nu,\varrho,\varsigma \in \Phi.$$

All the conditions of the above theorem are satisfied and 1 is a common fixed point of Υ, Λ, Θ, and Γ.

4 Application

In this section, we given an application to the Fredholm integral equation as below:

Suppose $\mathcal{I} = \mathcal{C}([\rho,\pi],\mathbb{R})$ is the set of real-valued continuous functions defined on $[\rho,\pi]$. Consider the integral equation,

$$d(\varpi) = \mathfrak{f}(\varpi) + \delta \int_{\rho}^{\pi} \mathcal{U}_1(\varpi,\theta)\mathfrak{k}(\varpi)\,d\theta \quad \text{for all } \theta,\varpi \in [\rho,\pi], \tag{8}$$

$$d(\varpi) = \mathfrak{f}(\varpi) + \delta \int_{\rho}^{\pi} \mathcal{U}_1(\varpi,\theta)\mathfrak{k}(\varpi)\,d\theta \quad \text{for all } \theta,\varpi \in [\rho,\pi], \tag{9}$$

where $\delta > 0$, $\mathfrak{f}(\varpi)$ is a neutrosophic function of $\varpi : \varpi \in [\rho,\pi]$ and $\mathcal{U}_1,\mathcal{U}_2 : \mathcal{C}([\rho,\pi]\times\mathbb{R}) \to \mathbb{R}^+$. Define the binary relation $\perp$ on $\mathcal{X}$ by $\mathfrak{x} \perp \mathfrak{y} \perp \mathfrak{z}$ iff $\mathfrak{x}+\mathfrak{y}+\mathfrak{z} \geq 0$ and define $\mathcal{Q}$, $\mathcal{F}$ and $\mathcal{G}$ by

$$\mathcal{Q}(\Upsilon\nu, \Lambda\varrho, \varsigma, \ell\wp) = \min\left\{\frac{\ell\wp}{\ell\wp + \min\{\Upsilon\nu, \Lambda\varrho, \varsigma, \ell\wp\}}\right\} \quad \forall \nu,\varrho,\varsigma \in \Phi \text{ and } \wp > 0,$$

$$\mathcal{F}(\Upsilon\nu, \Lambda\varrho, \varsigma, \ell\wp) = \max\left\{1 - \frac{\ell\wp}{\ell\wp + \max\{\Upsilon\nu, \Lambda\varrho, \varsigma, \ell\wp\}}\right\} \quad \forall \nu,\varrho,\varsigma \in \Phi \text{ and } \wp > 0,$$

$$\mathcal{G}(\Upsilon\nu, \Lambda\varrho, \varsigma, \ell\wp) = \max\left\{\frac{\max\{\Upsilon\nu, \Lambda\varrho, \varsigma, \ell\wp\}}{\ell\wp}\right\} \quad \forall \nu,\varrho,\varsigma \in \Phi \text{ and } \wp > 0.$$

With con-t-nm and con-t-conm defined by $\rho * \pi = \rho \cdot \pi$ and $\rho + \pi = \max\{\rho,\pi\}$, then $(\Phi,\mathcal{Q},\mathcal{F},\mathcal{G},*,+)$ is a O-complete N2MS. Consider $\int_{\rho}^{\pi} d\theta \leq \ell\wp < 1$. Then, the neutrosophic integral equations (8) and (9) have a unique common solution.

Proof Define $\Upsilon, \Lambda : \Phi \to \Phi$ by

$$d(\varpi) = \mathfrak{f}(\varpi) + \delta \int_{\rho}^{\pi} \mathcal{U}_1(\varpi,\theta)\mathfrak{k}(\varpi)\,d\theta \quad \text{for all } \theta,\varpi \in [\rho,\pi], \tag{10}$$

$$d(\varpi) = \mathfrak{f}(\varpi) + \delta \int_{\rho}^{\pi} \mathcal{U}_1(\varpi,\theta)\mathfrak{k}(\varpi)\,d\theta \quad \text{for all } \theta,\varpi \in [\rho,\pi]. \tag{11}$$

The survival of a fixed of the operator $\mathcal{U}$ has come to the survival of solution of a neutrosophic integral equation,

$$\begin{aligned}
&\mathcal{Q}\big(\Upsilon\nu(\varpi), \Lambda\varrho(\varpi), \varsigma\varpi, \ell\wp\big) \\
&\quad = \sup_{\varpi\in[\rho,\pi]} \frac{\ell\wp}{\ell\wp + \min(\Upsilon\nu(\varpi), \Lambda\varrho(\varpi), \varsigma\varpi, \ell\wp)} \\
&\quad = \sup_{\varpi\in[\rho,\pi]} \ell\wp \Big/ \bigg(\ell\wp + \bigg|\mathfrak{f}(\varpi) + \delta \int_{\rho}^{\pi} \mathcal{U}_1(\varpi,\theta)\Upsilon\nu(\varpi) - \varsigma\varpi - \wp - \mathfrak{f}(\varpi)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \delta \int_{\rho}^{\pi} \mathcal{U}_2(\varpi,\theta)\Lambda\varrho(\varpi) - \varsigma\varpi - \wp\bigg|\bigg) \\
&\geq \sup_{\varpi\in[\rho,\pi]} \frac{\ell\wp}{\ell\wp + |\Upsilon\nu(\varpi) - \Lambda\varrho(\varpi) - \varsigma\varpi - \wp)|} \\
&\geq \mathcal{Q}(\Upsilon\nu, \Lambda\varrho, \varsigma, \wp),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}\big(\Upsilon\nu(\varpi), \Lambda\varrho(\varpi), \varsigma\varpi, \ell\wp\big) \\
&\quad = 1 - \sup_{\varpi\in[\rho,\pi]} \frac{\ell\wp}{\ell\wp + \max(\Upsilon\nu(\varpi), \Lambda\varrho(\varpi), \varsigma\varpi, \ell\wp)} \\
&\quad = 1 - \sup_{\varpi\in[\rho,\pi]} \ell\wp\bigg/\bigg(\ell\wp + \bigg|\mathfrak{f}(\varpi) + \delta\int_{\rho}^{\pi} \mathcal{U}_1(\varpi,\theta)\Upsilon\nu(\varpi) - \varsigma\varpi - \wp - \mathfrak{f}(\varpi) \\
&\qquad - \delta \int_{\rho}^{\pi} \mathcal{U}_2(\varpi,\theta)\Lambda\varrho(\varpi) - \varsigma\varpi - \wp\bigg|\bigg) \\
&\quad \leq 1 - \sup_{\varpi\in[\rho,\pi]} \frac{\ell\wp}{\ell\wp + |\Upsilon\nu(\varpi) - \Lambda\varrho(\varpi) - \varsigma\varpi - \wp)|} \\
&\quad \leq \mathcal{F}(\Upsilon\nu, \Lambda\varrho, \varsigma, \wp)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{G}\big(\Upsilon\nu(\varpi), \Lambda\varrho(\varpi), \varsigma\varpi, \ell\wp\big) \\
&\quad = \sup_{\varpi\in[\rho,\pi]} \frac{\ell\wp}{\ell\wp + \max(\Upsilon\nu(\varpi), \Lambda\varrho(\varpi), \varsigma\varpi, \ell\wp)} \\
&\quad = \sup_{\varpi\in[\rho,\pi]} \ell\wp\bigg/\bigg(\ell\wp + \bigg|\mathfrak{f}(\varpi) + \delta\int_{\rho}^{\pi} \mathcal{U}_1(\varpi,\theta)\Upsilon\nu(\varpi) - \varsigma\varpi - \wp - \mathfrak{f}(\varpi) \\
&\qquad - \delta \int_{\rho}^{\pi} \mathcal{U}_2(\varpi,\theta)\Lambda\varrho(\varpi) - \varsigma\varpi - \wp\bigg|\bigg) \\
&\quad \leq \sup_{\varpi\in[\rho,\pi]} \frac{\ell\wp}{\ell\wp + |\Upsilon\nu(\varpi) - \Lambda\varrho(\varpi) - \varsigma\varpi - \wp)|} \\
&\quad \leq \mathcal{F}(\Upsilon\nu, \Lambda\varrho, \varsigma, \wp).
\end{aligned}
$$

Hence, all the conditions of Theorem 3.1 are satisfied. Hence, Υ and Λ have a unique common solution. □

5 Conclusion

We introduced the notion of a neutrosophic metric space to an orthogonal neutrosophic 2-metric space that deals with greater ambiguity and uncertainty in engineering and research studies. Finally, we obtained the common fixed-point theorem in an orthogonal neutrosophic 2-metric space.

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