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Reich–Krasnoselskii-type fixed point results with applications in integral equations

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Abstract

In this paper, motivated by Reich contraction and tool of measure of noncompactness, some generalizations of Reich, Kannan, Darbo, Sadovskii, and Krasnoselskii type fixed point results are presented by considering a pair of maps A, B on a nonempty closed subset M of a Banach space X into X. The existence of a solution to the equation $Ax + Bx = x$, where A is k-set contractive and B is a generalized Reich contraction, is established. As applications, it is established that the main result of this paper can be applied to learn conditions under which a solution of a nonlinear integral equation exists. Further we explain this phenomenon with the help of a practical example to approximate such solutions by using fixed point techniques. The graphs of exact and approximate solutions are also given to attract readers for further research activities.

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1 Introduction

Fixed point theory is one of the most developed and applicable branches of nonlinear mathematical analysis. It based on the well-known principle that every operator equation can be transformed into a fixed point problem, and vice versa.

In essence, the fixed point theory has two branches, namely the metric fixed point theory and topological fixed point theory. Topological fixed point theory is the most important branch of nonlinear analysis. It has a strong and useful history of more than a century. In 1912 Brouwer proved his well-known fixed point theorem. Later in 1922, Banach proved the most versatile result known as Banach contraction principle [38]. The most applicable generalization of Brouwer's theorem was presented in 1930 by Schauder using compact operators. Banach and Schauder's results remain most celebrated results in fixed point theory. The following two results of Brouwer and Banach were established in 1912 and 1922, respectively, and are given as follows.

Theorem BR (*Brouwer*, [31]) *Every compact convex nonempty subset B of $\mathbb{R}^n$ has a fixed-point property.*

Theorem BC (*Banach*, [31]) *Any contraction mapping of a complete nonempty metric space* Ω_1 *into* Ω_1 *has a unique fixed point in* Ω_1.

Both theorems are crucial in the existence theory of differential and integral equations. In the literature, there are many generalizations of these results. The most famous generalization of Theorem BR is Schauder's fixed point theorem.

Theorem SH (*Schauder's second theorem*, [31]) *Let* Ω_1 *be a nonempty convex subset of a normed space* L *and* S *be a continuous operator of* Ω_1 *into a compact set* $B \subseteq \Omega_1$. *Then* S *has a fixed point*.

A variety of generalizations of these results can be seen in [8, 13, 15, 18, 20, 21, 33, 35], and in the references therein.

Writing the physical problems into mathematical form produces mathematical equations like differential, integral, linear, and nonlinear equations. All these equations can be solved by fixed point techniques. Before solving them, the existence theory for the fixed points of operators plays very a important role. A number of results regarding differential and integral equations in connection with their existence theory can be seen in [1, 2, 6, 9, 11, 17, 23, 24, 28, 32].

In 1958, while studying the existence theory of neutral and delayed differential equations, it was observed that the solution might be expressed as a sum of compact and contractive operators. Working on this idea, Krasnoselskii proved his fixed point results for the sum of compact and contractive operators. The importance and applications of such theorems for the existence of solutions to the equation $x = Ax + Bx$ can be seen in [10, 12, 14, 25–27, 30, 36, 37].

The Krasnoselskii fixed point theorem can be stated as:

Theorem 1 ([31]) *Consider a Banach space* X *and let* M *be its nonempty convex closed subset. Suppose* A *and* B *map* M *into* X *such that*

(1) $y, \varkappa \in M$ *implies* $Ay + B\varkappa \in M$;

(2) B *is a contraction mapping*;

(3) A *is compact and continuous*.

Then there is $x \in M$ *such that* $Ax + Bx = x$.

This Krasnoselskii's theorem is a generalization of Schauder's fixed point theorem and Banach contraction principle, as we can see by taking $B = O$ and $A = O$, respectively.

To weaken the compactness condition used in Schauder's fixed point theorem, Darbo and Sadovskii [38, p. 500] generalized Schauder's fixed point theorem by introducing k-set contractive and condensing operators (noncompact operators) in the following way.

Theorem 2 (*Darbo*) *Assume that*

(*i*) *the mapping* $T : M \subseteq X \to M$ *is* k-*set contractive*,

(*ii*) M *is a nonempty, convex, bounded, and closed subset of a Banach space* X.

Then there exists $p \in M$ *such that* $p = Tp$.

Remark 3 Since every compact operator is k-set contractive with $k = 0$, Darbo's theorem is a generalization of Schauder's fixed point theorem.

Theorem 4 (*Sadovskii*) *Assume that*

(i) $T : M \subseteq X \to M$ *is a condensing operator,*

(ii) M *is a nonempty, convex, bounded, and closed subset of a Banach space* X.

Then T *has a fixed point.*

Remark 5 Since every k-set contractive operator with $0 \leq k < 1$ is condensing, Sadovskii's theorem is an extension of Darbo's theorem. For more about the above discussed results, one can see the related monographs and articles [3, 4, 16, 34].

Since Krasnoselskii combined the results of Banach and Schauder, in the above results Darbo and Sadovskii generalized Schauder's theorem. The next result was established by Reich [29], which is a generalization of Kannan fixed point theorem and Banach contraction principle.

Theorem 6 ([29]) *Consider a complete metric space* X *with metric* d *on it and let* $T : X \to X$ *be a mapping with the following property*:

$$d(Tq, Tp) \leq a_1 d(q, Tq) + a_2 d(p, Tp) + a_3 d(q, p), \quad q, p \in X,$$

where a_1, a_2, a_3 *are nonnegative and satisfy* $a_1 + a_2 + a_3 < 1$. *Then* T *has a unique fixed point.*

Remark 7 Letting $a_1 = a_2 = 0$ gives Banach fixed point theorem, and taking $a_1 = a_2$, $a_3 = 0$ gives Kannan fixed point theorem [19].

To combine the generalized Reich contraction and k-set contractive mappings in the form of operator equation $x = Ax + Bx$ and generalize Reich, Kannan, Darbo, Sadovskii, and Krasnoselskii type fixed point results, we need the following definitions.

Definition 8 ([5, 7]) Let X be a Banach space and $B(X)$ be the collection of all bounded subsets of X. A mapping μ of $B(X)$ into $[0, +\infty)$ is called a measure of noncompactness if the following conditions hold for all $E, F \in B(X)$:

(1) $\mu(F) = \mu(\overline{F})$;

(2) $\mu(F \cup E) = \max\{\mu(F), \mu(E)\}$;

(3) $\mu(F) = 0 \Leftrightarrow F$ is precompact.

The following conditions can also be deduced:

(4) $\mu(E + F) \leq \mu(E) + \mu(F)$;

(5) $E \subseteq F$ implies $\mu(E) \leq \mu(F)$.

Definition 9 ([37]) Let X be a Banach space and $T : M \subseteq X \longrightarrow X$ be a mapping. Then T is called k-set contractive if the following conditions hold for any bounded subset E of M:

(1) T is continuous and bounded;

(2) $\mu(T(E)) \leq k\mu(E)$.

Also T is strictly k-set contractive if

(1) T is k-set contractive;

(2) $\mu(T(E)) < k\mu(E)$ with $\mu(E) \neq 0$.

Finally, T is condensing if T is strictly 1-set contractive.

Now we define generalized Kannan contractions in Banach spaces.

Definition 10 Consider a Banach space X and let M be its nonempty subset. Suppose A and B map M into X. Then B is called a generalized Kannan contraction if

$$\|By - By'\| \leq \alpha\big(\|Ax - (I - B)y\| + \|Ax - (I - B)y'\|\big),$$

with $\alpha < \frac{1}{2}$ and all $x, y, y' \in M$.

Remark 11 If $A = O$ (the zero operator), we obtain a Kannan contraction.

To investigate the novelty of the above definition, the following example is important.

Example 12 Let $B : \mathbb{R} \to \mathbb{R}$ be the mapping defined by

$$Bx = \begin{cases} \frac{x}{3} & \text{if } 0 \leq x < 1, \\ \frac{1}{6} & \text{if } x = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Then with usual metric induced from the usual norm $|\cdot|$ on $\mathbb{R}$, we have

$$\frac{1}{9} = d\left(B(0), B\left(\frac{1}{3}\right)\right) = \frac{1}{2}\left[d\big(0, B(0)\big) + d\left(\frac{1}{3}, B\left(\frac{1}{3}\right)\right)\right] = \frac{1}{9}$$

so the condition of a Kannan mapping fails, but if we define $Ax = \frac{2}{3}$, then the condition of a generalized Kannan contraction holds.

The above example and definition motivated us to define the following.

Definition 13 Consider a Banach space X and let M be its nonempty subset. Suppose A and B map M into X. Then B is called a generalized Reich contraction if

$$\|By - By'\| \leq a_1\|Ax - (I - B)y\| + a_2\|Ax - (I - B)y'\| + a_3\|y - y'\|,$$

for nonnegative numbers a_1, a_2, and a_3 with $a_1 + a_2 + a_3 < 1$ and all $y, y', x \in M$.

Remark 14 Taking $A = O$ (the zero operator), we obtain a Reich contraction.

Lemma 15 ([37]) *Let M be a subset of a Banach space X and A be a Lipschitz mapping of M into X such that*

$$\|Ax - Ay\| \leq k\|x - y\|$$

for $x, y \in M$. Then $\mu(A(E)) \leq k\mu(E)$ holds for every bounded subset E of M.

2 Main results

The following theorem shows that a unique solution exists for the operator equation $x = Ax + Bx$ if A is a contraction and B is a generalized Reich contraction.

Theorem 16 *Let X be a Banach space and M be its nonempty closed subset. Consider the mappings $A : M \to X$ and $B : M \to X$ such that*

(1) $\|By - By'\| \le a_1\|Av - (I - B)y\| + a_2\|Av - (I - B)y'\| + a_3\|y - y'\|$ with $a_1 + a_2 + a_3 < 1$ and all $y, y', v \in M$;

(2) $\|Ax - Ax'\| \le k\|x - x'\|$ with $k < \frac{1-a_3}{1+a_2}$;

(3) $Ap + Bq \in M$ for all $p, q \in M$.

Then there exists a unique $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Proof Fix $x \in M$ and define $H : M \to M$ by $H(y) = Ax + By$. Using (1), and since condition (1) holds for all $v \in M$, we have

$$\begin{aligned}
\|H(y) - H(y')\| &= \|By - By'\| \\
&\le a_1\|Ax - (I - B)y\| + a_2\|Ax - (I - B)y'\| + a_3\|y - y'\| \\
&= a_1\|y - H(y)\| + a_2\|y' - H(y')\| + a_3\|y - y'\|.
\end{aligned}$$

This shows that H is a Reich contraction. Hence by [29, Theorem 3], there is a unique $Gx \in M$ that such that $Gx = Ax + B(Gx)$. Now

$$\begin{aligned}
\|Gx - Gx'\| &= \|Ax + B(Gx) - (Ax' + B(Gx'))\| \\
&= \|(B(Gx) - B(Gx')) - (Ax' - Ax)\| \\
&\le a_1\|Ax - (I - B)Gx\| + a_2\|Ax - (I - B)Gx'\| \\
&\quad + a_3\|Gx - Gx'\| + \|Ax' - Ax\| \\
&= a_2\|Ax - (I - B)Gx'\| + a_3\|Gx - Gx'\| + \|Ax' - Ax\| \\
&= (a_2 + 1)\|Ax - Ax'\| + a_3\|Gx - Gx'\|.
\end{aligned}$$

Thus

$$\|Gx - Gx'\| \le \left(\frac{a_2 + 1}{1 - a_3}\right)\|Ax - Ax'\| \le k\left(\frac{a_2 + 1}{1 - a_3}\right)\|x - x'\|.$$

Thus $G : M \to M$ is a contraction. Using Banach contraction principle, there is an $x \in M$ such that $Gx = x$. Since for this $x \in M$ there is a $Gx \in M$ such that $Gx = Ax + B(Gx)$, this implies that $x = Gx = Ax + B(Gx) = Ax + Bx$.

For uniqueness, suppose $s, s' \in M$ are such that $s = As + Bs$ and $s' = As' + Bs'$. Then

$$\begin{aligned}
\|s - s'\| &= \|(As + Bs) - (As' + Bs')\| \\
&= \|(Bs - Bs') - (As' - As)\| \\
&\le \|Bs - Bs'\| + \|As - As'\| \\
&\le a_1\|As - (I - B)s\| + a_2\|As - (I - B)s'\| + a_3\|s - s'\| + k\|s - s'\|
\end{aligned}$$

$$
\begin{aligned}
&= a_2\|As - As'\| + a_3\|s - s'\| + k\|s - s'\| \\
&\leq a_2k\|s - s'\| + a_3\|s - s'\| + k\|s - s'\| \\
&= (a_2k + a_3 + k)\|s - s'\|.
\end{aligned}
$$

This means that

$$
\big(1 - (a_2k + a_3 + k)\big)\|s - s'\| \leq 0,
$$

showing that $s = s'$. □

Remark 17 If we substitute $A = O$ in conditions (1), (2), and (3) of Theorem 16, we obtain the well-known theorem of Reich (see [29, Theorem 3]).

Corollary 18 *Let X be a Banach space and M be a nonempty closed subset of X. Consider mappings $A : M \to X$ and $B : M \to X$ such that*

(1) $\|By - By'\| \leq \alpha(\|Ax - (I - B)y\| + \|Ax - (I - B)y'\|)$ *with* $\alpha < \frac{1}{2}$ *and for all* $x, y, y' \in M$;

(2) $\|Ax - Ax'\| \leq k\|x - x'\|$ *with* $k < \frac{1}{1+\alpha}$;

(3) $Ap + Bq \in M$ *for all* $p, q \in M$.

Then there exists a unique $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Remark 19 Letting $A = O$ in conditions (1), (2), and (3), we obtain Kannan fixed point theorem (see [19, p. 406]). The above corollary is a combined form of contraction mapping and generalized Kannan contraction mapping.

Corollary 20 *Let X be a Banach space and M be a nonempty closed subset of X. Consider mappings $A : M \to X$ and $B : M \to X$ such that*

(1) $\|By - By'\| \leq \alpha\|Av - (I - B)y\|$ *with* $\alpha < 1$ *and for all* $v, y, y' \in M$;

(2) $\|Ax - Ax'\| \leq k\|x - x'\|$ *with* $k < 1$;

(3) $Ap + Bq \in M$ *for all* $p, q \in M$.

Then there is a unique $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Corollary 21 *Let X be a Banach space and M be a nonempty closed subset of X. Consider mappings $A : M \to X$ and $B : M \to X$ such that*

(1) $\|By - By'\| \leq c\|y - y'\|$ *with* $c < 1$ *and for all* $y, y' \in M$;

(2) $\|Ax - Ax'\| \leq k\|x - x'\|$ *with* $k < 1 - c$;

(3) $Ap + Bq \in M$ *for all* $p, q \in M$.

Then there is a unique $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

In the next result, with the help of Sadovskii fixed point theorem and Reich contraction theorem, we find that a solution of the operator equation $x = Ax + Bx$ exists, where A is a strictly k-set contractive mapping and B is a generalized Reich contraction.

Theorem 22 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $A : M \to X$ and $B : M \to X$ such that*

(1) $\|By - By'\| \leq a_1\|Ax - (I - B)y\| + a_2\|Ax - (I - B)y'\| + a_3\|y - y'\|$ *for* $a_1 + a_2 + a_3 < 1$ *and for all* $x, y', y \in M$;

(2) *A is strictly* $(\frac{1-a_3}{1+a_2})$*-set contractive mapping*;
(3) $Ax + By \in M$.
Then there exists $\varkappa \in M$ *such that* $A\varkappa + B\varkappa = \varkappa$.

Proof Fix $Ax \in A(M)$ and define $H : M \to M$ by $H(y) = Ax + By$. Using (1),

$$\begin{aligned}\|H(y) - H(y')\| &= \|By - By'\| \\ &\le a_1\|Ax - (I - B)y\| + a_2\|Ax - (I - B)y'\| + a_3\|y - y'\| \\ &= a_1\|y - H(y)\| + a_2\|y' - H(y')\| + a_3\|y - y'\|.\end{aligned}$$

This shows that H is a Reich contraction. Hence by [29, Theorem 3], there is a unique $G(Ax) \in M$ such that $G(Ax) = Ax + B(G(Ax))$. Now

$$\begin{aligned}\|G(Ax) - G(Ax')\| &= \|Ax + B(G(Ax)) - (Ax' + B(G(Ax')))\| \\ &= \|(B(G(Ax)) - B(G(Ax'))) - (Ax' - Ax)\| \\ &\le (a_1\|Ax - (I - B)G(Ax)\| + a_2\|Ax - (I - B)G(Ax')\| \\ &\quad + a_3\|G(Ax) - G(Ax')\| + \|Ax' - Ax\|) \\ &= a_2\|Ax - (I - B)G(Ax')\| \\ &\quad + a_3\|G(Ax) - G(Ax')\| + \|Ax' - Ax\| \\ &= (a_2 + 1)\|Ax - Ax'\| + a_3\|G(Ax) - G(Ax')\|.\end{aligned}$$

Thus

$$\|(G \circ A)x - (G \circ A)x'\| \le \left(\frac{a_2 + 1}{1 - a_3}\right)\|Ax - Ax'\|.$$

This shows that $G \circ A$ is a continuous function of M into X.

From (2), the above inequality, and Lemma 15, we deduce

$$\mu(G \circ A(N)) = \mu(G(A(N))) \le \left(\frac{a_2 + 1}{1 - a_3}\right)\mu(A(N)) < \mu(N)).$$

Using Sadovskii fixed point theorem, there is $x \in M$ such that $(G \circ A)x = x$. Also for $Ax \in A(M)$ there is a unique $G(Ax)$ such that $G(Ax) = Ax + B(G(Ax))$, therefore $x = G(Ax) = Ax + B(G(Ax)) = Ax + Bx$. □

Remark 23 If we take $A = O$, we obtain Reich contraction theorem which is a generalization of Banach contraction principle and Kannan contraction theorem (see [22, p. 400]). Also taking $B = O$ with $b = c = 0$, we get Sadovskii fixed point theorem which is a generalization Schauder's fixed point theorem.

Corollary 24 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $A : M \to X$ and $B : M \to X$ such that*
(1) *B is a contraction mapping for* $c < 1$;
(2) *A is a strictly* $(1 - c)$*-set contractive mapping*;

(3) $Ax + By \in M$.

Then there is $\varkappa \in M$ *such that* $A\varkappa + B\varkappa = \varkappa$.

Proof Letting $a_2 = 0 = a_1$ and $a_3 = c$ in Theorem 22, we obtain the above corollary. □

Remark 25 Since every compact operator is a strictly $(1 - c)$-set contractive mapping, Corollary 24 and Theorem 22 are generalizations of Krasnoselskii's fixed point theorem.

Remark 26 Theorem 22 is a generalization of Theorem 2.11 in [37].

Corollary 27 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings* $A : M \to X$ *and* $B : M \to X$ *such that*

(1) $\|By - By'\| \leq \alpha(\|Ax - (I - B)y\| + b\|Ax - (I - B)y'\|)$ *for* $\alpha < \frac{1}{2}$ *and for all* $x, y, y' \in M$;

(2) *A is a strictly* $(\frac{1}{1+\alpha})$*-set contractive mapping*;

(3) $Ax + By \in M$.

Then there exists $\varkappa \in M$ *such that* $A\varkappa + B\varkappa = \varkappa$.

Corollary 28 *Consider a Banach space X and let M be a subset of X such that M is closed non-empty convex and bounded. Consider the mappings* $A : M \longrightarrow X$ *and* $B : M \longrightarrow X$ *such that*

(1) $\|By - By'\| \leq b\|Ax - (I - B)y'\|$ *for* $b < 1$, *all* $x\ y, y' \in M$;

(2) *A is condensing mapping of M into X*;

(3) $Ax + By \in M$.

Then there exists $\varkappa \in M$ *such that* $A\varkappa + B\varkappa = \varkappa$.

Suppose A and B are two operators of M into X. By using a measure of noncompactness, it makes sense to impose a condition on the operators A and B such that

$$\mu\big(A(E) + B(F)\big) < \mu(E) \quad \text{for all } E \subseteq F \text{ with } \mu(E) \neq 0.$$

For example, if A and B are compact operators, then

$$\mu\big(A(E) + B(F)\big) \leq \mu(A(E) + \mu\big(B(F)\big) = 0 < \mu(E) \quad \text{for all } E \subseteq F \text{ with } \mu(F) \neq 0.$$

Also, if A is a condensing operator and $B(F)$ lies in a compact subset of X, then

$$\mu\big(A(E) + B(F)\big) \leq \mu(A(E) + \mu\big(B(F)\big) = \mu(A(E) < \mu(E)$$

for all $E \subseteq F$ with $\mu(E) \neq 0$.

Theorem 29 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings* $B : M \to X$ *and* $A : M \to X$ *such that*

(1) $\|By - By'\| \leq a_1\|Ax - (I - B)y\| + a_2\|Ax - (I - B)y'\| + a_3\|y - y'\|$ *for* $a_1 + a_2 + a_3 < 1$ *and for all* $x, y, y' \in M$;

(2) $\mu(A(E) + B(M)) < \mu(E)$ *for all* $E \subseteq M$ *with* $\mu(E) \neq 0$;

(3) $Ax + By \in M$ *for all* $x, y \in M$, *where* $A : M \to X$ *is a continuous mapping.*

Then there is $x \in M$ *such that* $x = Ax + Bx$.

Proof Fix $Ax \in A(M)$ and define $H : M \to M$ by $H(y) = Ax + By$. Using (1),

$$\begin{aligned}\|H(y) - H(y')\| &= \|By - By'\| \\ &\leq a_1\|Ax - (I-B)y\| + a_2\|Ax - (I-B)y'\| + a_3\|y - y'\| \\ &= a_1\|y - H(y)\| + a_2\|y' - H(y')\| + a_3\|y - y'\|.\end{aligned}$$

Thus by [29, Theorem 3], there is a unique $G(Ax) \in M$ that such that $G(Ax) = Ax + B(G(Ax))$. Now

$$\begin{aligned}\|G(Ax) - G(Ax')\| &= \|Ax + B(G(Ax)) - (Ax' + B(G(Ax')))\| \\ &= \|(B(G(Ax)) - B(G(Ax'))) - (Ax' - Ax)\| \\ &\leq (a_1\|Ax - (I-B)G(Ax)\| + a_2\|Ax - (I-B)G(Ax')\| \\ &\quad + a_3\|G(Ax) - G(Ax')\| + \|Ax' - Ax\|) \\ &= a_2\|Ax - (I-B)G(Ax')\| \\ &\quad + a_3\|G(Ax) - G(Ax')\| + \|Ax' - Ax\| \\ &= (a_2 + 1)\|Ax - Ax'\| + a_3\|G(Ax) - G(Ax')\|.\end{aligned}$$

Thus

$$\|(G \circ A)x - (G \circ A)x'\| \leq \left(\frac{a_2 + 1}{1 - a_3}\right)\|Ax - Ax'\|.$$

This shows that $G \circ A$ is a continuous function of M into M. Using (2),

$$\mu((G \circ A)E)) = \mu(G(A(E))) = \mu(A(E) + B(G(A(E)))) \leq \mu(A(E) + B(M)) < \mu(E).$$

Using Sadovskii fixed point theorem, there is $x \in M$ such that $(G \circ A)x = x$. Also for $Ax \in A(M)$, there is a unique $G(Ax)$ such that $G(Ax) = Ax + B(G(Ax))$, therefore $x = Ax + Bx$. □

Corollary 30 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $B : M \to X$ and $A : M \to X$ such that $B(M)$ lies in a compact subset of X and*

(1) $\|By - By'\| \leq a_1\|Ax - (I-B)y\| + a_2\|Ax - (I-B)y'\| + a_3\|y - y'\|$ *for* $a_1 + a_2 + a_3 < 1$ *and for all* $y', y, x \in M$;

(2) *A is a condensing mapping*;

(3) *$Aq + Bp \in M$ for all $q, p \in M$.*

Then there exists $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Proof Since A is a condensing mapping and $B(M)$ lies in a compact subset of X,

$$\mu(A(E) + B(M)) \leq \mu(A(E)) + \mu(B(M)) = \mu(A(E)) < \mu(E).$$

Hence, by Theorem 29, the proof is complete. □

Corollary 31 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $B : M \to X$ and $A : M \to X$ such that*

(1) $Ax + By \in M$;

(2) *B is a contraction mapping and $B(M)$ lies in a compact subset of X;*

(3) *A is a condensing mapping.*

Then there exists $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Proof Putting $a_1 = a_2 = 0$ in Corollary 30, we obtain the required result. □

Remark 32 Corollary 31 is a variant of Krasnoselskii fixed point theorem and Theorem 2.6 in [37].

Corollary 33 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $B : M \to X$ and $A : M \to X$ such that*

(1) *A is a k-set contractive mapping for $0 \leq k < 1$;*

(2) *B is a contraction mapping and $B(M)$ lies in a compact subset of X;*

(3) $Ax + By \in M$.

Then there exists $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Proof Since every k-set contractive mapping for $0 \leq k < 1$ is condensing, by Corollary 31, we have the required result. □

Remark 34 If $B = O$ (the zero operator) in Corollary 33, we obtain Darbo fixed point theorem.

Corollary 35 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $B : M \to X$ and $A : M \to X$ such that*

(1) $Ax + By \in M$;

(2) *B is a contraction mapping and $B(M)$ lies in a compact subset of X;*

(3) *A is a compact operator.*

Then there exists $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Proof As every compact operator is k-set contractive for $0 \leq k < 1$, by Corollary 33, we have a fixed point for the operator equation $A\varkappa + B\varkappa = \varkappa$. □

Remark 36 If $B = O$ (the zero operator), we obtain Schauder's fixed point theorem (see [38, p. 56]).

Theorem 37 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $B : M \to X$ and $A : M \to X$ such that*

(1) *$\|By - By'\| \leq q\|Ax - (y - By')\|$ for all $x, y, y' \in M$ with $q < \frac{1}{2}$;*

(2) *$\mu(A(E) + B(M)) < \mu(E)$ for all $E \subseteq M$ with $\mu(E) \neq 0$.*

Then there is $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

Proof Fix $x \in M$ and define a mapping $H : M \to M$ by $H(y) = Ax + By$. Now (1) implies that

$$\begin{aligned}\|H(y) - H(y')\| &= \|By - By'\| \\ &\leq q\|y - (Ax + By')\| \\ &= q\|y - H(y')\|.\end{aligned}$$

Thus by [22, p. 394], there is a unique $Gx \in M$ that such that $Gx = Ax + B(Gx)$, where $G : M \to M$ is a function. Now

$$\begin{aligned}\|Gx - Gx'\| &= \|Ax + B(Gx) - (Ax' + B(Gx'))\| \\ &= \|(B(Gx) - B(Gx')) - (Ax' - Ax)\| \\ &\leq \|B(Gx) - B(Gx')\| + \|Ax' - Ax)\| \\ &\leq q\|Ax' - (Gx - B(Gx'))\| + \|Ax' - Ax\| \\ &= q\|Gx - (Ax' + B(Gx'))\| + \|Ax - Ax'\| \\ &= q\|Gx - Gx'\| + \|Ax - Ax'\|.\end{aligned}$$

Thus

$$\|Gx - Gx'\| \leq \frac{1}{1-q}\|Ax - Ax'\|.$$

This shows that G is continuous.

Using (2), we deduce that

$$\mu(G(E)) = \mu(A(E) + B(G(E))) \leq \mu(A(E) + B(M)) < \mu(E).$$

By Sadovskii fixed point theorem, there is $x \in M$ such that $Gx = x$. Also for $x \in M$, there is a unique Gx such that $Gx = Ax + B(Gx)$, therefore $x = Ax + Bx$. □

Corollary 38 *Consider a Banach space X and let M be a subset of X such that M is closed, nonempty, convex, and bounded. Consider mappings $B : M \to X$ and $A : M \to X$ such that $B(M)$ lies in a compact subset of X and*

(1) $\|By - By'\| \leq q\|Ax - (y - By')\|$ *for all* $x, y, y' \in M$ *with* $q < \frac{1}{2}$;

(2) *A is a condensing mapping.*

Then there is $\varkappa \in M$ such that $A\varkappa + B\varkappa = \varkappa$.

3 Application

Let $X = C[0,1]$, the Banach space of all continuous functions defined on $[0,1]$ with $\|\cdot\|_\infty$. Consider the following nonlinear integral equation;

$$y(t) = \frac{1}{(1-a)}\int_0^t g(s, y(s))\,ds + \left(\frac{c(y) - a}{1-a}\right)y(t), \quad t \in [0,1], \qquad (I)$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $c : [0,1] \to \mathbb{R}$ are continuous.

As

$$\frac{1}{(1-a)} = 1 + \frac{a}{(1-a)},$$

equation (I) can be written as

$$y(t) = \int_0^t g\big(s, y(s)\big)\,ds + \frac{a}{(1-a)}\int_0^t g\big(s, y(s)\big)\,ds + \left(\frac{c(y)-a}{1-a}\right)y(t).$$

We decompose the above integral equation into a sum of two operators

$$y(t) = Ay(t) + By(t),$$

where

$$Ay(t) = \int_0^t g\big(s, y(s)\big)\,ds$$

and

$$B\big(y(t)\big) = \frac{a}{(1-a)}\int_0^t g\big(s, y(s)\big)\,ds + \left(\frac{c(y)-a}{1-a}\right)y(t).$$

Let us assume the following conditions first:

($C1$) $|g(s,y(s)) - g(s,y'(s))| \le \lambda|y(s) - y'(s)|$,

($C2$) $c + 2a < 1$ and $\lambda + c \le 1$, where $\max|c(t)| = c$.

($C3$) $\max|g(s,0)| = \beta$.

First, we prove (3) of Theorem 16 and for this we define

$$S = \left\{ y \in X : \|y\| \le r, \text{ where } r \ge \frac{2\beta}{a(1-c)} \right\}$$

and consider for $y \in S$,

$$\begin{aligned}
\big|Ay(t) + By(t)\big| &= \left|\frac{1}{(1-a)}\int_0^t g\big(s,y(s)\big)\,ds + \left(\frac{c(y)-a}{1-a}\right)y(t)\right| \\
&\le \left|\frac{1}{(1-a)}\int_0^t g\big(s,y(s)\big)\,ds - \frac{1}{(1-a)}\int_0^t g(s,0)\,ds + \frac{1}{(1-a)}\int_0^t g(s,0)\,ds\right| \\
&\quad + \left|\left(\frac{c(y)-a}{1-a}\right)y(t)\right| \\
&\le \frac{1}{(1-a)}\left|\int_0^t \big\{g\big(s,y(s)\big) - g(s,0)\big\}\,ds\right| \\
&\quad + \left|\frac{1}{(1-a)}\int_0^t g(s,0)\,ds\right| + \left|\left(\frac{c(y)-a}{1-a}\right)y(t)\right| \\
&\le \frac{1}{1-a}\int_0^t \lambda\big|y(t)\big|\,ds + \frac{1}{(1-a)}\int_0^t \big|g(s,0)\big|\,ds + \left(\frac{c-a}{1-a}\right)\big|y(t)\big|.
\end{aligned}$$

Now taking the supremum and using $(C2)$ and $(C3)$, we have

$$\|Ay(t) + By(t)\| \leq \frac{1}{1-a}\beta + \left(\frac{\lambda + c - a}{1-a}\right)r \leq r,$$

proving (3).

We will show that the operators A and B satisfy (2) and (1) of our Theorem 16.

Claim: B satisfies (2) of Theorem 16.

For this, consider

$$\begin{aligned} B(y(t)) - B(y'(t)) &= \left\{\frac{a}{(1-a)}\int_0^t g(s, y(s))\,ds + \left(\frac{c(y) - a}{1-a}\right)y(t)\right\} \\ &\quad - \left\{\frac{a}{(1-a)}\int_0^t g(s, y'(s))\,ds + \left(\frac{c(y') - a}{1-a}\right)y'(t)\right\}, \end{aligned}$$

which gives

$$\begin{aligned} (1-a)\{B(y(t)) - B(y'(t))\} &= \left\{a\int_0^t g(s, y(s))\,ds + (c(y) - a)y(t)\right\} \\ &\quad - \left\{a\int_0^t g(s, y'(s))\,ds + (c(y) - a)y'(t)\right\}, \end{aligned}$$

further implying

$$\begin{aligned} \{B(y(t)) - B(y'(t))\} &= \left\{a\int_0^t g(s, y(s))\,ds + (c(y) - a)y(t) + aBy(t)\right\} \\ &\quad - \left\{a\int_0^t g(s, y'(s))\,ds + (c(y) - a)y'(t) + aBy'(t)\right\} \\ &= a\left\{\int_0^t g(s, y(s))\,ds - (I - B)y(t)\right\} + c(y)y(t) \\ &\quad - a\left\{\int_0^t g(s, y'(s))\,ds - (I - B)y'(t)\right\} - c(y')y'(t). \end{aligned}$$

Thus we have

$$\begin{aligned} |B(y(t)) - B(y'(t))| &= \left|a\left\{\int_0^t g(s, y(s))\,ds - (I - B)y(t)\right\} + c(y)y(t)\right. \\ &\quad \left. - a\left\{\int_0^t g(s, y'(s))\,ds - (I - B)y'(t)\right\} - c(y')y'(t)\right| \\ &\leq a|Ay(t) - (I - B)y(t)| + a|Ay'(t) - (I - B)y'(t)| \\ &\quad + |c(y)||y(t) - y'(t)|. \end{aligned}$$

Finally,

$$\|By - By'\| \leq a\|Ay - (I - B)y\| + a\|Ay' - (I - B)y'\| + c\|y - y'\|,$$

which shows that B satisfies (1).

Claim: A satisfies (2).

For this, we write

$$\|Ay(t) - Ay'(t)\| \leq \left|\int_0^t \lambda \, ds\right| \|y - y'\| \leq \lambda \|y - y'\|,$$

where $\lambda < \frac{1-c}{1+a}$.

Hence all the conditions of Theorem 16 are satisfied, so we obtain $w \in S$ such that $w = Aw + Bw$, which is a solution of the integral equation (I).

We summarize all the above in the form of a theorem as follows.

Theorem 39 *If g satisfies ($C1$)–($C3$), then a solution to the integral equation (I) exists in S.*

In the next example, we apply our main Theorem 16 to obtain an approximate solution of a given nonlinear integral equation.

Example 40 Consider a special case of the integral equation

$$y(t) = \frac{1}{(1-a)} \int_0^t g(s, y(s)) \, ds + \left(\frac{ct - a}{1 - a}\right) y(t), \quad t \in [0,1], \tag{I}$$

namely

$$y(t) = \int_0^t \left(\frac{\sqrt{y(s)} y(t)}{\lambda e^{-s/2}} + \left(te^t\right)^{-1}\right) ds + \left(\frac{0.2 + ct - 0.2}{0.8}\right) y(t), \tag{J}$$

which has an exact solution $y(t) = e^{-t}$, $t \in [0,1]$, for $\lambda = \frac{8}{10c} = 4$ and $c = \frac{2}{10}$. Clearly, $2a + c < 1$ for $a = 0.2$, so using these values, ($C2$) is satisfied. From (I) and (J), one can see that

$$\left|g(t, y(t)) - g(t, y'(t))\right| \leq \frac{1}{\lambda e^{-1/2}} \left|y(t) - y'(t)\right|,$$

which implies that

$$\left\|g(t, y(t)) - g(t, y'(t))\right\| \leq \frac{\sqrt{e}}{4} \|y - y'\|$$

holds since $\frac{\sqrt{e}}{4} < 1$. Therefore all the conditions of the above theorem are satisfied. Now we consider the iterative procedure

$$y_{n+1}(t) = \int_0^t \frac{\sqrt{y_n(s)} y_n(t)}{\lambda e^{-s/2}} \, ds + \left(\frac{\{0.2 - ct\} - 0.2}{0.8}\right) y_n(t),$$

with an initial approximation $y_0(t) = 1 - t$.

The graphs of exact and approximate solutions and absolute error are given below after two iterations.

It can be seen that after two iterations the absolute error is not significant.

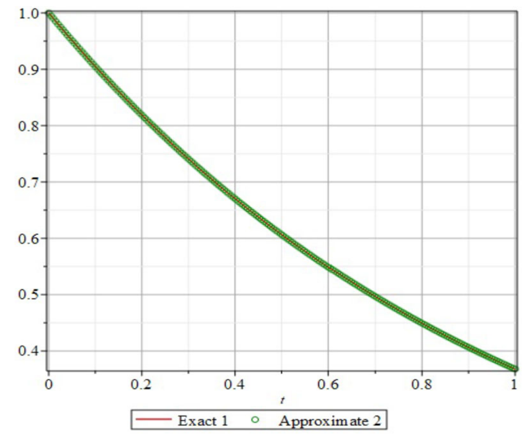

Figure 1 Show graphs of exact and approximate solutions

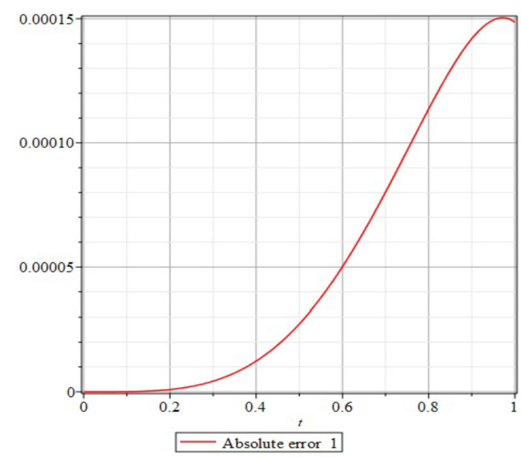

Figure 2 Shows the absolute error obtained from exact and approximate solutions

4 Conclusion

In this article, Kannan and Reich contractions are generalized and used to obtain different variants of Krasnoselskii's fixed point theorem for compact and noncompact operators. Examples and applications in the existence theory of integral equations are given to validate our presented work.

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References

1. Abdeljawad, T., Agarwal, R.P., Karapınar, E., Kumari, P.S.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended *b*-metric space. Symmetry **11**(5), 686 (2019)
2. Ahmad, N., Mehmood, N., Akgül, A.: Applications of some new Krasnoselskii-type fixed-point results for generalized expansive and equiexpansive mappings. Adv. Cont. Discr. Mod. **2022**(1), 1–9 (2022)
3. Akhmerov, R.R., Kamenskii, M.I., Potapov, A.S., Rodkina, A.E., Sadovskii, B.N.: Measures of Noncompactness and Condensing Operators. Birkhäuser, Basel (1992)
4. Alotaibi, A., Mursaleen, M., Mohiuddine, S.A.: Application of measures of noncompactness to infinite system of linear equations in sequence spaces. Bull. Iran. Math. Soc. **41**(2), 519–527 (2015)
5. Ayerbe Toledano, J.M., Dominguez Benavides, T., López Acedo, G.: Measures of Noncompactness in Metric Fixed Point Theory. Oper. Theory Adv. Appl., vol. 99 (1997)
6. Bai, Z.: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal., Theory Methods Appl. **72**(2), 916–924 (2010)
7. Banaś, J., Jleli, M., Mursaleen, M., Samet, B., Vetro, C. (eds.): Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness. Springer, Singapore (2017)
8. Bharucha-Reid, A.T.: Fixed point theorems in probabilistic analysis. Bull. Am. Math. Soc. **82**(5), 641–657 (1976)
9. Bhaskar, T.G., Lakshmikantham, V.: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal., Theory Methods Appl. **65**(7), 1379–1393 (2006)
10. Burton, T.A.: A fixed-point theorem of Krasnoselskii. Appl. Math. Lett. **11**(1), 85–88 (1998)
11. Burton, T.A.: Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Courier Corporation (2014)
12. Burton, T.A., Purnaras, I.K.: A unification theory of Krasnoselskii for differential equations. Nonlinear Anal., Theory Methods Appl. **89**, 121–133 (2013)
13. Caristi, J.: Fixed point theorems for mappings satisfying inwardness conditions. Trans. Am. Math. Soc. **215**, 241–251 (1976)
14. Chen, Y.Z.: Krasnoselskii-type fixed point theorems using α-concave operators. J. Fixed Point Theory Appl. **22**, 52 (2020)
15. Ćirić, L.B.: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45**(2), 267–273 (1974)
16. Gabeleh, M., Malkowsky, E., Mursaleen, M., Rakočević, V.: A new survey of measures of noncompactness and their applications. Axioms **11**(6), 299 (2022)
17. Gul, R., Shah, K., Khan, Z.A., Jarad, F.: On a class of boundary value problems under ABC fractional derivative. Adv. Differ. Equ. **2021**(1), 437 (2021)
18. Jleli, M., Samet, B.: A generalized metric space and related fixed point theorems. Fixed Point Theory Appl. **2015**(1), 61 (2015)
19. Kannan, R.: Some results on fixed points—II. Am. Math. Mon. **76**(4), 405–408 (1969)
20. Karapınar, E.: Edelstein type fixed point theorems. Fixed Point Theory Appl. **2012**(1), 107 (2012)
21. Khamsi, M.A.: Remarks on Caristi's fixed point theorem. Nonlinear Anal., Theory Methods Appl. **71**(1–2), 227–231 (2009)
22. Malkowsky, E., Rakočević, V.:. Advanced Functional Analysis. CRC Press (2019)
23. Mehmood, N., Abbas, A., Abdeljawad, T., Akgül, A.: Existence results for ABC-fractional differential equations with non-separated and integral type of boundary conditions. Fractals **29**(05), 2140016 (2021)
24. Mehmood, N., Ahmad, N.: Existence results for fractional order boundary value problem with nonlocal non-separated type multi-point integral boundary conditions. AIMS Math. **5**, 385–398 (2020)
25. Park, S.: Generalizations of the Krasnoselskii fixed point theorem. Nonlinear Anal., Theory Methods Appl. **67**(12), 3401–3410 (2007)
26. Pourhadi, E., Saadati, R., Some, K.Z.: Krasnosel'skii-type fixed point theorems for Meir–Keeler-type mappings. Nonlinear Anal., Model. Control **25**(2), 257–265 (2020)
27. Przeradzki, B.: A generalization of Krasnosel'skii fixed point theorem for sums of compact and contractible maps with application. Cent. Eur. J. Math. **10**(6), 2012–2018 (2012)
28. Ran, A.C., Reurings, M.C.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **1**, 1435–1443 (2004)
29. Reich, S.: Some remarks concerning contraction mappings. Can. Math. Bull. **14**(1), 121–124 (1971)
30. Reich, S.: Fixed points of condensing functions. J. Math. Anal. Appl. **41**(2), 460–467 (1973)
31. Smart, D.R.:. Fixed Point Theorems. Cup Archive (1980)
32. Sun, S., Zhao, Y., Han, Z., Li, Y.: The existence of solutions for boundary value problem of fractional hybrid differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**(12), 4961–4967 (2012)
33. Suzuki, T.: A new type of fixed point theorem in metric spaces. Nonlinear Anal., Theory Methods Appl. **71**(11), 5313–5317 (2009)
34. Toledano, J.M., Benavides, T.D., Acedo, G.L.: Measures of Noncompactness in Metric Fixed Point Theory. Springer, Berlin (1997)
35. Valero, O.: On Banach fixed point theorems for partial metric spaces. Appl. Gen. Topol. **6**(2), 229–240 (2005)
36. Xiang, T., Georgiev, S.G.: Noncompact-type Krasnoselskii fixed-point theorems and their applications. Math. Methods Appl. Sci. **39**(4), 833–863 (2016)

37. Xiang, T., Yuan, R.: A class of expansive-type Krasnosel'skii fixed point theorems. Nonlinear Anal., Theory Methods Appl. **71**(7–8), 3229–3239 (2009)
38. Zeidler, E., Wadsack, P.R.: Nonlinear Functional Analysis and Its Applications: Fixed-Point Theorems. Springer, Berlin (1993). Transl. by Peter R. Wadsack.

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