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Hermite–Hadamard type inequalities for multiplicatively harmonic convex functions

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Abstract

In this work, the notion of a multiplicative harmonic convex function is examined, and Hermite–Hadamard inequalities for this class of functions are established. Many inequalities of Hermite–Hadamard type are also taken into account for the product and quotient of multiplicative harmonic convex functions. In addition, new multiplicative integral-based inequalities are found for the quotient and product of multiplicative harmonic convex and harmonic convex functions. In addition, we provide certain upper limits for such classes of functions. The obtained results have been verified by providing examples with included graphs. The findings of this study may encourage more research in several scientific areas.

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Keywords: Multiplicative calculus; Multiplicatively harmonic convex functions; Hermite–Hadamard inequality

1 Introduction

In the 17th century, Newton and Leibniz independently discovered differential and integral calculus. Since then, differentiation and integration have been considerable significant in analysis and calculus. These are operations on numbers that allow for infinitesimal addition and subtraction.

Grossman and Katz defined a new type of derivative and integral between 1967 and 1970, which replaced the roles of addition and subtraction with multiplication and division. They called the newly established calculus "multiplicative calculus." Multiplicative calculus, also known as non-Newtonian calculus, is only applicable to positive functions. Because of its restricted area of application, it is not as well known as the calculus of Leibniz and Newton. One of the initial studies of multiplicative calculus was performed in the 1970s [14]. The classical calculus introduced by Newton and Leibniz in the 17th century was modified in this study. Since the area of application of multiplicative calculus is quite limited, it is not as popular as the calculus of Newton and Leibniz. However, a number of intriguing results have been obtained in various fields. For example, in [5] Bashirov et al. gave a fundamental theorem of multiplicative calculus. In [6], Bashirov and Riza introduced complex multiplicative calculus. In [11] and [17], some properties of stochastic multiplicative calculus have been studied. For some applications and other aspects of this discipline, see [2, 4, 25, 28] and the references cited therein.

Recall that the concept of multiplicative integral called * integral is represented by $\int_{\varrho_1}^{\varrho_2}(\Upsilon(\varkappa_1))^{d\varkappa_1}$, while the ordinary integral is represented by $\int_{\varrho_1}^{\varrho_2}(\Upsilon(\varkappa_1))\,d\varkappa_1$. This is due to the fact that in the definition of a classical Riemann integral of Υ on $[\varrho_1,\varrho_2]$, the sum of the products of terms is used, while in the definition of the multiplicative integral of Υ on $[\varrho_1,\varrho_2]$, the products of terms raised to certain powers are used.

The following connection between Riemann integral and * integral exists [5].

Proposition 1.1 *If Υ is Riemann integrable on $[\varrho_1,\varrho_2]$, then Υ is * integrable on $[\varrho_1,\varrho_2]$ and*

$$\int_{\varrho_1}^{\varrho_2}\big(\Upsilon(\varkappa_1)\big)^{d\varkappa_1} = e^{\int_{\varrho_1}^{\varrho_2}\ln(\Upsilon(\varkappa_1))\,d\varkappa_1}.$$

In [5], Bashirov et al. show the following results and notations for the * integral.

Definition 1.1 If Υ is positive and Riemann integrable on $[\varrho_1,\varrho_2]$, then Υ is * integrable on $[\varrho_1,\varrho_2]$ and

(1) $\int_{\varrho_1}^{\varrho_2}((\Upsilon(\varkappa_1))^p)^{d\varkappa_1} = \int_{\varrho_1}^{\varrho_2}((\Upsilon(\varkappa_1))^{d\varkappa_1})^p$,
(2) $\int_{\varrho_1}^{\varrho_2}(\Upsilon(\varkappa_1)\omega(\varkappa_1))^{d\varkappa_1} = \int_{\varrho_1}^{\varrho_2}(\Upsilon(\varkappa_1))^{d\varkappa_1}.\int_{\varrho_1}^{\varrho_2}(\omega(\varkappa_1))^{d\varkappa_1}$,
(3) $\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\omega(\varkappa_1)})^{d\varkappa_1} = \frac{\int_{\varrho_1}^{\varrho_2}(\Upsilon(\varkappa_1))^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\omega(\varkappa_1))^{d\varkappa_1}}$,
(4) $\int_{\varrho_1}^{\varrho_2}(\Upsilon(\varkappa_1))^{d\varkappa_1} = \int_{\varrho_1}^{\mu}(\Upsilon(\varkappa_1))^{d\varkappa_1}.\int_{\mu}^{\varrho_2}(\Upsilon(\varkappa_1))^{d\varkappa_1}$, $\varrho_1 \le \mu \le \varrho_2$,
(5) $\int_{\varrho_1}^{\varrho_1}(\Upsilon(\varkappa_1))^{d\varkappa_1} = 1$ and $\int_{\varrho_1}^{\varrho_2}(\Upsilon(\varkappa_1))^{d\varkappa_1} = (\int_{\varrho_2}^{\varrho_1}(\Upsilon(\varkappa_1))^{d\varkappa_1})^{-1}$.

Convexity, on the other hand, is important in a variety of fields, including mathematical finance, economics, engineering, management sciences, and optimization theory. By employing convexity, several extensions and generalizations of integral inequalities were discovered along with their useful applications (see [1, 8, 18, 27]).

2 Preliminaries

Definition 2.1 A non-empty set $\aleph$ is said to be convex if for every $\varrho_1,\varrho_2 \in \aleph$ we have

$$\varrho_1 + \kappa(\varrho_2 - \varrho_1) \in \aleph, \quad \forall\kappa \in [0,1].$$

Definition 2.2 The function $\Upsilon : [\varrho_1,\varrho_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex in the classical sense if the following inequality holds:

$$\Upsilon\big(\kappa\varkappa_1 + (1-\kappa)\varkappa_2\big) \le \kappa\Upsilon(\varkappa_1) + (1-\kappa)\Upsilon(\varkappa_2)$$

for all $\varkappa_1,\varkappa_2 \in [\varrho_1,\varrho_2]$ and $\kappa \in [0,1]$.

The function Υ is said to be concave if $-\Upsilon$ is convex.

The Hermite–Hadamard inequality is one of the most well-known inequalities involving the integral mean of a convex function. This double inequality is stated as follows (see [13, 15, 26]).

Definition 2.3 Let $\Upsilon : I = [\varrho_1, \varrho_2] \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function. Then

$$\Upsilon\left(\frac{\varrho_1 + \varrho_2}{2}\right) \le \frac{1}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \Upsilon(\varkappa_1)\, d\varkappa_1 \le \frac{\Upsilon(\varrho_1) + \Upsilon(\varrho_2)}{2}. \tag{2.1}$$

Both inequalities hold in the reversed direction if Υ is concave.

The Hermite–Hadamard inequality can be considered as an improvement of the concept of convexity. Many researchers have looked into this inequality in great detail since it was discovered independently by Hermite (1883) and Hadamard (1896). Especially, in the previous two decades, many researchers have focused on obtaining novel boundaries for the left and right sides of the Hermite–Hadamard inequality. For some findings that enhance, improve, and expand the inequalities in (2.1), please refer to the monographs [3, 9, 12, 20, 22–24].

Now it is time to present basic definitions and results. Note that I is an interval.

Definition 2.4 [26] A function $\Upsilon : I \to (0, \infty)$ is said to be log- or multiplicatively convex on set $\aleph$ if

$$\Upsilon\big((1-\kappa)\varkappa_1 + \kappa\varkappa_2\big) \le \big[\Upsilon(\varkappa_1)\big]^{1-\kappa}\big[\Upsilon(\varkappa_2)\big]^{\kappa}, \quad \forall \kappa \in [0,1].$$

Definition 2.5 [16] Let $I \subseteq \mathbb{R} \setminus \{0\}$ be an interval. A function $\Upsilon : I \to \mathbb{R}$ is said to be a harmonically convex function if

$$\Upsilon\left(\frac{\varkappa_1\varkappa_2}{\kappa\varkappa_1 + (1-\kappa)\varkappa_2}\right) \le (1-\kappa)\Upsilon(\varkappa_1) + \kappa\Upsilon(\varkappa_2)$$

for all $\varkappa_1, \varkappa_2 \in I$, $\kappa \in [0,1]$.

Definition 2.6 [19] Let $I \subseteq \mathbb{R} \setminus \{0\}$ be an interval. A function $\Upsilon : I \to \mathbb{R}$ is said to be a multiplicative harmonic convex function or a log harmonic convex function if

$$\Upsilon\left(\frac{\varkappa_1\varkappa_2}{\kappa\varkappa_1 + (1-\kappa)\varkappa_2}\right) \le \big(\Upsilon(\varkappa_1)\big)^{(1-\kappa)}\big(\Upsilon(\varkappa_2)\big)^{\kappa} \tag{2.2}$$

for all $\varkappa_1, \varkappa_2 \in I$, $\kappa \in [0,1]$.

In [16], Iscan proved the Hermite–Hadamard inequality for harmonically convex functions as follows.

Definition 2.7 Let $\Upsilon : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and let $\varrho_1, \varrho_2 \in I$ with $\varrho_1 < \varrho_2$. If $\Upsilon \in L[\varrho_1, \varrho_2]$, then the following inequalities hold:

$$\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1 + \varrho_2}\right) \le \frac{\varrho_1\varrho_2}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\, d\varkappa_1 \le \frac{\Upsilon(\varrho_1) + \Upsilon(\varrho_2)}{2}. \tag{2.3}$$

Some recent results for Hermite–Hadamard inequalities for harmonically convex functions can be seen in [7, 10, 21].

3 Hermite–Hadamard integral inequalities

In this section, we obtain some Hermite–Hadamard type integral inequalities in the setting of multiplicative calculus for multiplicative harmonic and harmonic convex functions.

Throughout the paper, for the sake of simplicity, we use the following notations for special means of two non-negative numbers ϱ_1, ϱ_2 ($\varrho_1 < \varrho_2$):

(1) arithmetic mean:

$$A(\varrho_1,\varrho_2) = \frac{\varrho_1 + \varrho_2}{2}, \quad \varrho_1,\varrho_2 \in \mathbb{R}^+;$$

(2) geometric mean:

$$G(\varrho_1,\varrho_2) = \sqrt{\varrho_1\varrho_2}, \quad \varrho_1,\varrho_2 \in \mathbb{R}^+;$$

(3) logarithmic mean:

$$L(\varrho_1,\varrho_2) = \frac{\varrho_2 - \varrho_1}{\log(\varrho_2) - \log(\varrho_1)}, \quad \varrho_1 \neq \varrho_2, \varrho_1,\varrho_2 \neq 0, \varrho_1,\varrho_2 \in \mathbb{R}^+;$$

(4) generalized log-mean:

$$L_n(\varrho_1,\varrho_2) = \left[\frac{\varrho_2^{n+1} - \varrho_1^{n+1}}{(n+1)(\varrho_2 - \varrho_1)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\backslash\{-1,0\}, \varrho_1,\varrho_2 \in \mathbb{R}^+;$$

(5) identric mean:

$$I = I(\varrho_1,\varrho_2) = \begin{cases} \varrho_1, & \text{if } \varrho_1 = \varrho_2, \\ \frac{1}{e}\big(\frac{\varrho_2^{\varrho_2}}{\varrho_1^{\varrho_1}}\big)^{\frac{1}{\varrho_2-\varrho_1}}, & \text{if } \varrho_1 \neq \varrho_2, \varrho_1,\varrho_2 > 0. \end{cases}$$

Now we give our first main result.

Theorem 3.1 *Let* Υ *be a positive multiplicative harmonic convex function on interval* $[\varrho_1,\varrho_2]$. *Then*

$$\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1 + \varrho_2}\right) \leq \left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \leq G\big(\Upsilon(\varrho_1),\Upsilon(\varrho_2)\big). \tag{3.1}$$

The above inequality is called the Hermite–Hadamard integral inequality for multiplicative harmonic convex functions.

Proof Let Υ be a positive multiplicative harmonic convex function. For $\kappa = \frac{1}{2}$ in (2.2), we have

$$\Upsilon\left(\frac{2\varkappa_1\varkappa_2}{\varkappa_1 + \varkappa_2}\right) \leq \big(\Upsilon(\varkappa_1),\Upsilon(\varkappa_2)\big)^{\frac{1}{2}}. \tag{3.2}$$

This implies that

$$\begin{aligned}
&\ln\left(\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\right)\\
&\leq \ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\left(\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)+\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)\right).
\end{aligned}$$

Integrating with respect to κ on $[0,1]$, we have

$$\begin{aligned}
&\ln\left(\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\right)\\
&\leq \frac{1}{2}\int_0^1 \ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)d\kappa+\frac{1}{2}\int_0^1 \ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)d\kappa\\
&= \frac{1}{2}\left(\left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1+\left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)\\
&= \left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1.
\end{aligned}$$

Thus, we have

$$\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\leq\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})}. \tag{3.3}$$

To prove the second inequality in (3.1), we have

$$\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})}\\
&= \left(e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})d\varkappa_1}\right)^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})}\\
&= e^{((\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})d\varkappa_1)}\\
&= e^{(\int_0^1\ln(\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))d\kappa)}\\
&\leq e^{(\int_0^1\ln((\Upsilon(\varrho_2))^{(1-\kappa)}(\Upsilon(\varrho_1))^{\kappa})d\kappa)}\\
&= G\big(\Upsilon(\varrho_1),\Upsilon(\varrho_2)\big).
\end{aligned} \tag{3.4}$$

Hence, the second inequality is proved. By adding both inequalities we get (3.1). □

Example 1 Let the function Υ be defined as $\Upsilon(\varkappa)=e^{\varkappa}$. Then the function Υ is multiplicative harmonic convex on $[\varrho_1,\varrho_2]$. We have

$$\begin{aligned}
&\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)=e^{\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}},\\
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}=\left(e^{(\frac{\varrho_2^2-\varrho_1^2}{2}-(2\varrho_2\ln\varrho_2-2\varrho_2)+(2\varrho_1\ln\varrho_1-2\varrho_1))}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}},
\end{aligned}$$

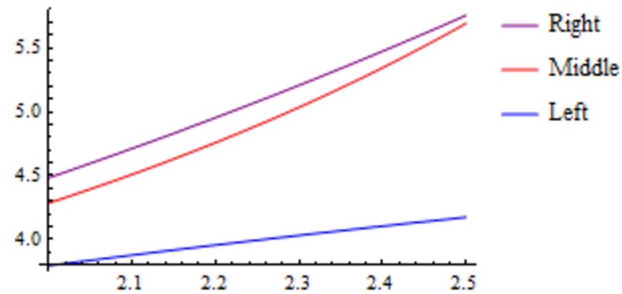


Figure 1 Graphical visualization of Theorem 3.1

$$G\big(\Upsilon(\varrho_1),\Upsilon(\varrho_2)\big)=e^{\frac{\varrho_1+\varrho_2}{2}}.$$

The graph of the inequalities of Example 1 is depicted in Fig. 1 for $\varrho_1 = 1$ and $\varrho_2 \in [2, 2.5]$, which demonstrates the validity of Theorem 3.1.

Theorem 3.2 *Let Υ and ω be two positive multiplicative harmonic convex functions on interval $[\varrho_1,\varrho_2]$. Then*

$$\begin{aligned}&\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\omega\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\\&\quad\le\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\int_{\varrho_1}^{\varrho_2}\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\\&\quad\le G\big(\Upsilon(\varrho_1),\Upsilon(\varrho_2)\big)G\big(\omega(\varrho_1),\omega(\varrho_2)\big).\end{aligned}\tag{3.5}$$

Proof We have

$$\begin{aligned}&\ln\left(\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\omega\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\right)\\&\quad=\ln\left(\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\right)+\ln\left(\omega\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\right)\\&\quad\le\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)^{\frac{1}{2}}\\&\qquad+\ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)^{\frac{1}{2}}\\&\quad=\frac{1}{2}\left(\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)+\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)\right)\\&\qquad+\frac{1}{2}\left(\ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)+\ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)\right).\end{aligned}\tag{3.6}$$

Integrating with respect to κ on $[0,1]$, we have

$$\begin{aligned}&\ln\left(\Upsilon\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\omega\left(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\right)\right)\\&\quad\le\frac{1}{2}\int_0^1\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)d\kappa+\frac{1}{2}\int_0^1\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)d\kappa\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^1 \ln\biggl(\omega\biggl(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\biggr)\biggr)\,d\kappa+\frac{1}{2}\int_0^1 \ln\biggl(\omega\biggl(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\biggr)\biggr)\,d\kappa\\
&=\frac{1}{2}\biggl(\biggl(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\biggr)\biggl(\int_{\varrho_1}^{\varrho_2}\ln\biggl(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\biggr)\,d\varkappa_1\biggr)+\biggl(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\biggr)\biggl(\int_{\varrho_1}^{\varrho_2}\ln\biggl(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\biggr)\,d\varkappa_1\biggr)\biggr)\\
&+\frac{1}{2}\biggl(\biggl(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\biggr)\biggl(\int_{\varrho_1}^{\varrho_2}\ln\biggl(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\biggr)\,d\varkappa_1\biggr)+\biggl(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\biggr)\biggl(\int_{\varrho_1}^{\varrho_2}\ln\biggl(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\biggr)\,d\varkappa_1\biggr)\biggr)\\
&=\biggl(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\biggr)\biggl(\int_{\varrho_1}^{\varrho_2}\ln\biggl(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\biggr)\,d\varkappa_1\biggr)+\biggl(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\biggr)\biggl(\int_{\varrho_1}^{\varrho_2}\ln\biggl(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\biggr)\,d\varkappa_1\biggr). \quad (3.7)
\end{aligned}
$$

Thus, we have

$$
\Upsilon\biggl(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\biggr)\omega\biggl(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\biggr)\le\biggl(\int_{\varrho_1}^{\varrho_2}\biggl(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\biggr)^{d\varkappa_1}\int_{\varrho_1}^{\varrho_2}\biggl(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\biggr)^{d\varkappa_1}\biggr)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \quad (3.8)
$$

and the first inequality is proved.

To prove the second inequality in (3.5), we have

$$
\begin{aligned}
&\biggl(\biggl(\int_{\varrho_1}^{\varrho_2}\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\,d\varkappa_1\biggr)\biggl(\int_{\varrho_1}^{\varrho_2}\frac{\omega(\varkappa_1)}{\varkappa_1^2}\,d\varkappa_1\biggr)\biggr)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\\
&=\bigl(e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1+\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}\bigr)^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})}\\
&=e^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})(\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1+\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1)}\\
&=\bigl(e^{\int_0^1(\ln(\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))+\ln(\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})))\,d\kappa}\bigr)\\
&\le\bigl(e^{\int_0^1(\ln((\Upsilon(\varrho_2))^{(1-\kappa)}(\Upsilon(\varrho_1))^{\kappa})\,d\kappa+\int_0^1\ln((\omega(\varrho_2))^{(1-\kappa)}(\Upsilon(\varrho_1))^{\kappa})\,d\kappa)}\bigr)\\
&=G\bigl(\Upsilon(\varrho_1),\Upsilon(\varrho_2)\bigr)G\bigl(\omega(\varrho_1),\omega(\varrho_2)\bigr). \quad (3.9)
\end{aligned}
$$

Hence, the second inequality is proved. □

Theorem 3.3 *Let Υ and ω be two positive multiplicative harmonic convex functions on interval $[\varrho_1,\varrho_2]$. Then*

$$
\frac{\Upsilon(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})}{\omega(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})}\le\biggl(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\biggr)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\le\frac{G(\Upsilon(\varrho_1),\Upsilon(\varrho_2))}{G(\omega(\varrho_1),\omega(\varrho_2))}. \quad (3.10)
$$

Proof We have

$$
\begin{aligned}
&\ln\biggl(\frac{\Upsilon(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})}{\omega(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})}\biggr)\\
&=\ln\biggl(\Upsilon\biggl(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\biggr)\biggr)-\ln\biggl(\omega\biggl(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2}\biggr)\biggr)\\
&\le\ln\biggl(\Upsilon\biggl(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\biggr)\Upsilon\biggl(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\biggr)\biggr)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&- \ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\left(\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)+\ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)\right)\\
&\quad - \frac{1}{2}\left(\ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)+\ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)\right).
\end{aligned}
\tag{3.11}
$$

Integrating the above inequality with respect to κ on $[0,1]$, we have

$$
\begin{aligned}
&\ln\left(\frac{\Upsilon(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})}{\omega(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})}\right)\\
&\leq \frac{1}{2}\int_0^1 \ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)d\kappa + \frac{1}{2}\int_0^1 \ln\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)d\kappa\\
&\quad - \frac{1}{2}\int_0^1 \ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)d\kappa - \frac{1}{2}\int_0^1 \ln\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right)d\kappa\\
&= \frac{1}{2}\left(\left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)+\left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)\right)\\
&\quad - \frac{1}{2}\left(\left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)+\left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)\right)\\
&= \left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)-\left(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)\left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)\\
&= \left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})}-\left(\int_{\varrho_1}^{\varrho_2}\ln\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)d\varkappa_1\right)^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})}.
\end{aligned}
$$

Hence, we have

$$
\frac{\Upsilon(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})}{\omega(\frac{2\varrho_1\varrho_2}{\varrho_1+\varrho_2})} \leq \left(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}
\tag{3.12}
$$

and the first inequality is proved.

To prove the second inequality in (3.10), we have

$$
\begin{aligned}
&\left(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\\
&= \left(e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1-\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}\right)^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})}\\
&= e^{(\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1})(\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1-\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1)}\\
&= e^{(\int_0^1(\ln(\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))-\ln(\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})))\,d\kappa)}\\
&\leq e^{(\int_0^1(\ln((\Upsilon(\varrho_2))^{(1-\kappa)}(\Upsilon(\varrho_1))^{\kappa})\,d\kappa-\int_0^1\ln((\omega(\varrho_2))^{(1-\kappa)}(\Upsilon(\varrho_1))^{\kappa})\,d\kappa))}\\
&= \frac{G(\Upsilon(\varrho_1),\Upsilon(\varrho_2))}{G(\omega(\varrho_1),\omega(\varrho_2))}.
\end{aligned}
\tag{3.13}
$$

Hence, the second inequality is proved. □

Theorem 3.4 *Let Υ and ω be harmonic convex and multiplicative harmonic convex functions, respectively. Then we have*

$$\left(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \leq \frac{(\frac{(\Upsilon(\varrho_2))^{\Upsilon(\varrho_2)}}{(\Upsilon(\varrho_1))^{\Upsilon(\varrho_1)}})^{\frac{1}{\Upsilon(\varrho_2)-\Upsilon(\varrho_1)}}}{G(\omega(\varrho_1),\omega(\varrho_2))e}. \tag{3.14}$$

Proof Note that

$$\begin{aligned}
&\left(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= \left(\frac{e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}}{e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= \left(e^{(\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1-\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1)}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= e^{(\int_0^1\ln(\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))\,d\kappa-\int_0^1\ln(\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))\,d\kappa)} \\
&\leq \frac{(\frac{(\Upsilon(\varrho_2))^{\Upsilon(\varrho_2)}}{(\Upsilon(\varrho_1))^{\Upsilon(\varrho_1)}})^{\frac{1}{\Upsilon(\varrho_2)-\Upsilon(\varrho_1)}}}{G(\omega(\varrho_1),\omega(\varrho_2))e}.
\end{aligned}$$

□

Example 2 Let the functions Υ, ω be defined as $\Upsilon(\varkappa) = \varkappa$ and $\omega(\varkappa) = e^{\varkappa}$. Then Υ and ω are harmonic convex and multiplicative harmonic convex functions, respectively,

$$\left(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} = \left(\frac{e^{(-\varrho_2\ln\varrho_2+\varrho_2+\varrho_1\ln\varrho_1-\varrho_1)}}{e^{(\frac{\varrho_2^2-\varrho_1^2}{2}-(2\varrho_2\ln\varrho_2-2\varrho_2)+(2\varrho_1\ln\varrho_1-2\varrho_1))}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}$$

and

$$\frac{(\frac{(\Upsilon(\varrho_2))^{\Upsilon(\varrho_2)}}{(\Upsilon(\varrho_1))^{\Upsilon(\varrho_1)}})^{\frac{1}{\Upsilon(\varrho_2)-\Upsilon(\varrho_1)}}}{G(\omega(\varrho_1),\omega(\varrho_2))e} = \frac{(\frac{\varrho_2^{\varrho_2}}{\varrho_1^{\varrho_1}})^{\frac{1}{\varrho_2-\varrho_1}}}{e^{\frac{\varrho_1+\varrho_2+2}{2}}}.$$

The graph of the inequalities of Example 2 is depicted in Fig. 2 for $\varrho_1 = 1$ and $\varrho_2 \in [2, 3]$, which demonstrates the validity of Theorem 3.4.

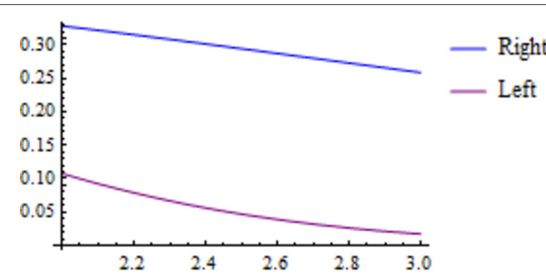


Figure 2 Graphical visualization of Theorem 3.4

Theorem 3.5 *Let Υ and ω be multiplicative harmonic convex and harmonic convex functions, respectively. Then we have*

$$\left(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \leq \frac{G(\Upsilon(\varrho_1),\Upsilon(\varrho_2))e}{(\frac{(\omega(\varrho_2))^{\omega(\varrho_2)}}{(\omega(\varrho_1))^{\omega(\varrho_1)}})^{\frac{1}{\omega(\varrho_2)-\omega(\varrho_1)}}}. \tag{3.15}$$

Proof Note that

$$\begin{aligned}
&\left(\frac{\int_{\varrho_1}^{\varrho_2}(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}{\int_{\varrho_1}^{\varrho_2}(\frac{\omega(\varkappa_1)}{\varkappa_1^2})^{d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= \left(\frac{e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}}{e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= \left(e^{(\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1-\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1)}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= e^{(\int_0^1\ln(\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))\,d\kappa-\int_0^1\ln(\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))\,d\kappa)} \\
&\leq e^{(\ln(G(\Upsilon(\varrho_1),\Upsilon(\varrho_2)))-\ln(\frac{(\omega(\varrho_2))^{\omega(\varrho_2)}}{(\omega(\varrho_1))^{\omega(\varrho_1)}})^{\frac{1}{\omega(\varrho_2)-\omega(\varrho_1)}}+1)} \\
&= \frac{G(\Upsilon(\varrho_1),\Upsilon(\varrho_2))e}{(\frac{(\omega(\varrho_2))^{\omega(\varrho_2)}}{(\omega(\varrho_1))^{\omega(\varrho_1)}})^{\frac{1}{\omega(\varrho_2)-\omega(\varrho_1)}}}.
\end{aligned}$$

□

Theorem 3.6 *Let Υ and ω be harmonic convex and multiplicative harmonic convex functions, respectively. Then we have*

$$\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\int_{\varrho_1}^{\varrho_2}\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&\quad\leq \frac{(\frac{(\Upsilon(\varrho_2))^{\Upsilon(\varrho_2)}}{(\Upsilon(\varrho_1))^{\Upsilon(\varrho_1)}})^{\frac{1}{\Upsilon(\varrho_2)-\Upsilon(\varrho_1)}}G(\omega(\varrho_1),\omega(\varrho_2))}{e}.
\end{aligned} \tag{3.16}$$

Proof Note that

$$\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\int_{\varrho_1}^{\varrho_2}\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= \left(e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}.e^{\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= \left(e^{(\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1+\int_{\varrho_1}^{\varrho_2}\ln(\frac{\omega(\varkappa_1)}{\varkappa_1^2})\,d\varkappa_1)}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= e^{(\int_0^1\ln(\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))+\int_0^1\ln(\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}))\,d\kappa)} \\
&\leq \frac{(\frac{(\Upsilon(\varrho_2))^{\Upsilon(\varrho_2)}}{(\Upsilon(\varrho_1))^{\Upsilon(\varrho_1)}})^{\frac{1}{\Upsilon(\varrho_2)-\Upsilon(\varrho_1)}}G(\omega(\varrho_1),\omega(\varrho_2))}{e}.
\end{aligned}$$

□

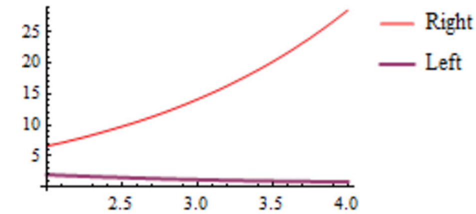


Figure 3 Graphical visualization of Theorem 3.6

Example 3 Under the assumptions of Example 2, we have

$$\begin{aligned}&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa)}{\varkappa_1^2}\right)^{d\varkappa_1}\int_{\varrho_1}^{\varrho_2}\left(\frac{\omega(\varkappa)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\\&\quad=\left(e^{(-\varrho_2\ln\varrho_2+\varrho_2+\varrho_1\ln\varrho_1-\varrho_1)}\left(e^{(\frac{\varrho_2^2-\varrho_1^2}{2}-(2\varrho_2\ln\varrho_2-2\varrho_2)+(2\varrho_1\ln\varrho_1-2\varrho_1))}\right)\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\end{aligned}$$

and

$$\frac{\left(\frac{(\Upsilon(\varrho_2))^{\Upsilon(\varrho_2)}}{(\Upsilon(\varrho_1))^{\Upsilon(\varrho_1)}}\right)^{\frac{1}{\Upsilon(\varrho_2)-\Upsilon(\varrho_1)}}G(\omega(\varrho_1),\omega(\varrho_2))}{e}=\left(\frac{\varrho_2^{\varrho_2}}{\varrho_1^{\varrho_1}}\right)^{\frac{1}{\varrho_2-\varrho_1}}e^{\frac{\varrho_1+\varrho_2-2}{2}}.$$

The graph of the inequalities of Example 3 is depicted in Fig. 3 for $\varrho_1 = 1$ and $\varrho_2 \in [2,4]$, which demonstrates the validity of Theorem 3.6.

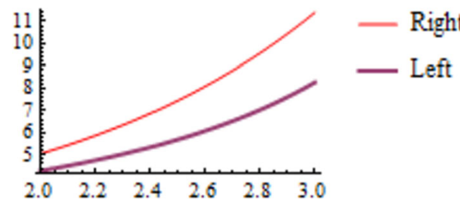
Theorem 3.7 *Let $\Upsilon : I \to \mathbb{R}$ be a positive multiplicative harmonic convex function where $\varrho_1, \varrho_2 \in I$ and $\varrho_1 < \varrho_2$. Then*

$$\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\le\frac{\Upsilon(\varrho_1)+\Upsilon(\varrho_2)}{2}. \tag{3.17}$$

Proof Let Υ be a positive multiplicative harmonic convex function. Then we have

$$\begin{aligned}&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}\\&\quad=e^{(\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})d\varkappa_1)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}}\\&\quad=e^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}(\int_{\varrho_1}^{\varrho_2}\ln(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2})d\varkappa_1)}\\&\quad=e^{\int_0^1\ln\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})d\kappa}\\&\quad\le\int_0^1 e^{\ln\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})d\kappa}\\&\quad=\int_0^1\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)d\kappa\\&\quad\le\int_0^1\left[\left(\Upsilon(\varrho_1)\right)^{\kappa}\left(\Upsilon(\varrho_2)\right)^{1-\kappa}\right]d\kappa\\&\quad=\int_0^1\Upsilon(\varrho_2)\left(\frac{\Upsilon(\varrho_1)}{\Upsilon(\varrho_2)}\right)^{\kappa}d\kappa\end{aligned}$$

Figure 4 Graphical visualization of Theorem 3.7



$$= \frac{\Upsilon(\varrho_1) - \Upsilon(\varrho_2)}{\log \Upsilon(\varrho_1) - \log \Upsilon(\varrho_2)}$$
$$\leq \frac{\Upsilon(\varrho_1) + \Upsilon(\varrho_2)}{2}. \tag{3.18}$$

Hence, we get the required result. □

Example 4 Under the assumption of Example 1, we have

$$\left(\int_{\varrho_1}^{\varrho_2} \left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} = \left(e^{(\frac{\varrho_2^2-\varrho_1^2}{2}-(2\varrho_2 \ln \varrho_2 - 2\varrho_2)+(2\varrho_1 \ln \varrho_1 - 2\varrho_1))}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}$$

and

$$\frac{\Upsilon(\varrho_1) + \Upsilon(\varrho_2)}{2} = \frac{e^{\varrho_1} + e^{\varrho_2}}{2}.$$

The graph of the inequalities of Example 4 is depicted in Fig. 4 for $\varrho_1 = 1$ and $\varrho_2 \in [2,3]$, which demonstrates the validity of Theorem 3.7.

Theorem 3.8 *Let $\Upsilon, \omega : I \to \mathbb{R}$ be two positive multiplicative harmonic convex functions where $\varrho_1, \varrho_2 \in I$ and $\varrho_1 < \varrho_2$. Then*

$$\left(\int_{\varrho_1}^{\varrho_2} \left(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \leq \frac{1}{4}\varphi(\varrho_1, \varrho_2), \tag{3.19}$$

where

$$\varphi(\varrho_1, \varrho_2) = \left(\Upsilon(\varrho_1)\right)^2 + \left(\Upsilon(\varrho_2)\right)^2 + \left(\omega(\varrho_1)\right)^2 + \left(\omega(\varrho_2)\right)^2.$$

Proof Let Υ, ω be two positive multiplicative harmonic convex functions. Then we have

$$\left(\int_{\varrho_1}^{\varrho_2} \left(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}$$
$$= e^{(\int_{\varrho_1}^{\varrho_2} \ln(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}) d\varkappa_1)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}}}$$
$$= e^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}(\int_{\varrho_1}^{\varrho_2} \ln(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}) d\varkappa_1)}$$
$$= e^{\int_0^1 \ln[\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})] d\kappa}$$
$$\leq \int_0^1 e^{\ln[\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})] d\kappa}$$

$$\begin{aligned}
&= \int_0^1 \left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2 + (1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2 + (1-\kappa)\varrho_1}\right)\right] d\kappa \\
&\leq \int_0^1 \left[\left(\Upsilon(\varrho_1)\right)^\kappa \left(\Upsilon(\varrho_2)\right)^{1-\kappa} \left(\omega(\varrho_1)\right)^\kappa \left(\omega(\varrho_2)\right)^{1-\kappa}\right] d\kappa \\
&= \Upsilon(\varrho_2)\omega(\varrho_2) \int_0^1 \left(\frac{\Upsilon(\varrho_1)\omega(\varrho_1)}{\Upsilon(\varrho_2)\omega(\varrho_2)}\right)^\kappa d\kappa \\
&= \frac{\Upsilon(\varrho_1)\omega(\varrho_1) - \Upsilon(\varrho_2)\omega(\varrho_2)}{\log(\Upsilon(\varrho_1)\omega(\varrho_1)) - \log(\Upsilon(\varrho_2)\omega(\varrho_2))} \\
&\leq \frac{\Upsilon(\varrho_1)\omega(\varrho_1) + \Upsilon(\varrho_2)\omega(\varrho_2)}{2} \\
&\leq \frac{1}{2}\int_0^1 \left[\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2 + (1-\kappa)\varrho_1}\right)\right)^2 + \left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2 + (1-\kappa)\varrho_1}\right)\right)^2\right] d\kappa \\
&\leq \frac{1}{2}\int_0^1 \left[\left(\left(\Upsilon(\varrho_1)\right)^\kappa \left(\Upsilon(\varrho_2)\right)^{1-\kappa}\right)^2 + \left(\left(\omega(\varrho_1)\right)^\kappa \left(\omega(\varrho_2)\right)^{1-\kappa}\right)^2\right] d\kappa \\
&= \frac{1}{4}\left[\frac{(\Upsilon(\varrho_1))^2 - (\Upsilon(\varrho_2))^2}{\log\Upsilon(\varrho_1) - \log\Upsilon(\varrho_2)}\right] + \frac{1}{4}\left[\frac{(\omega(\varrho_1))^2 - (\omega(\varrho_2))^2}{\log\omega(\varrho_1) - \log\omega(\varrho_2)}\right] \\
&\leq \frac{1}{4}\left(\Upsilon(\varrho_1)\right)^2 + \left(\Upsilon(\varrho_2)\right)^2 + \left(\omega(\varrho_1)\right)^2 + \left(\omega(\varrho_2)\right)^2. \qquad (3.20)
\end{aligned}$$

This completes the proof. □

Theorem 3.9 *Let $\Upsilon, \omega : I \to \mathbb{R}$ be two positive multiplicative harmonic convex functions where $\varrho_1, \varrho_2 \in I$ and $\varrho_1 < \varrho_2$. Then*

$$\left(\int_{\varrho_1}^{\varrho_2} \left(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \leq \frac{1}{8}\varphi(\varrho_1,\varrho_2) + \frac{1}{4}\Psi(\varrho_1,\varrho_2), \qquad (3.21)$$

where

$$\varphi(\varrho_1,\varrho_2) = \left(\Upsilon(\varrho_1)\right)^2 + \left(\Upsilon(\varrho_2)\right)^2 + \left(\omega(\varrho_1)\right)^2 + \left(\omega(\varrho_2)\right)^2$$

and

$$\Psi(\varrho_1,\varrho_2) = \Upsilon(\varrho_1)\omega(\varrho_1) + \Upsilon(\varrho_2)\omega(\varrho_2).$$

Proof Let Υ, ω be two positive multiplicative harmonic convex functions. Then using the inequality

$$\varrho_1\varrho_2 \leq \frac{1}{4}(\varrho_1 + \varrho_2)^2, \quad \forall \varrho_1, \varrho_2 \in \mathbb{R},$$

we have

$$\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2} \left(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&= e^{\int_0^1 \ln[\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})]\, d\kappa}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right] d\kappa \\
&\leq \frac{1}{4}\int_0^1 \left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)+\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right]^2 d\kappa \\
&\leq \frac{1}{4}\int_0^1 \left[\Upsilon(\varrho_2)\left(\frac{\Upsilon(\varrho_1)}{\Upsilon(\varrho_2)}\right)^{\kappa}+\omega(\varrho_2)\left(\frac{\omega(\varrho_1)}{\omega(\varrho_2)}\right)^{\kappa}\right]^2 d\kappa \\
&= \frac{(\Upsilon(\varrho_2))^2}{4}\int_0^1 \left(\frac{\Upsilon(\varrho_1)}{\Upsilon(\varrho_2)}\right)^{2\kappa} d\kappa + \frac{(\omega(\varrho_2))^2}{4}\int_0^1 \left(\frac{\omega(\varrho_1)}{\omega(\varrho_2)}\right)^{2\kappa} d\kappa \\
&\quad + \frac{\Upsilon(\varrho_2)\omega(\varrho_2)}{2}\int_0^1 \left(\frac{\Upsilon(\varrho_1)\omega(\varrho_1)}{\omega(\varrho_2)\Upsilon(\varrho_2)}\right)^{\kappa} d\kappa \\
&= \frac{(\Upsilon(\varrho_2))^2}{8}\int_0^2 \left(\frac{\Upsilon(\varrho_1)}{\Upsilon(\varrho_2)}\right)^{\mathfrak{u}} d\mathfrak{u} + \frac{(\omega(\varrho_2))^2}{8}\int_0^2 \left(\frac{\omega(\varrho_1)}{\omega(\varrho_2)}\right)^{\mathfrak{u}} d\mathfrak{u} \\
&\quad + \frac{\Upsilon(\varrho_2)\omega(\varrho_2)}{2}\int_0^1 \left(\frac{\Upsilon(\varrho_1)\omega(\varrho_1)}{\omega(\varrho_2)\Upsilon(\varrho_2)}\right)^{\kappa} d\kappa \\
&= \frac{1}{4}\frac{\Upsilon(\varrho_1)+\Upsilon(\varrho_2)}{2}\frac{\Upsilon(\varrho_1)-\Upsilon(\varrho_2)}{\log\Upsilon(\varrho_1)-\log\Upsilon(\varrho_2)} + \frac{1}{4}\frac{\omega(\varrho_1)+\omega(\varrho_2)}{2}\frac{\omega(\varrho_1)-\omega(\varrho_2)}{\log\omega(\varrho_1)-\log\omega(\varrho_2)} \\
&\quad + \frac{1}{2}\frac{\Upsilon(\varrho_1)\omega(\varrho_1)-\Upsilon(\varrho_2)\omega(\varrho_2)}{\log(\Upsilon(\varrho_1)\omega(\varrho_1))-\log(\Upsilon(\varrho_2)\omega(\varrho_2))} \\
&\leq \frac{1}{8}\left[\left(\Upsilon(\varrho_1)\right)^2+\left(\Upsilon(\varrho_2)\right)^2+\left(\omega(\varrho_1)\right)^2+\left(\omega(\varrho_2)\right)^2\right] \\
&\quad + \frac{1}{4}\left[\Upsilon(\varrho_1)\omega(\varrho_1)+\Upsilon(\varrho_2)\omega(\varrho_2)\right]. \qquad (3.22)
\end{aligned}
$$

This completes the proof. □

Theorem 3.10 *Let $\Upsilon, \omega : I \to \mathbb{R}$ be positive multiplicative harmonic convex functions where $\varrho_1, \varrho_2 \in I$ and $\varrho_1 < \varrho_2$. Then*

$$
\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&\quad\leq \mathfrak{p}\frac{\Upsilon(\varrho_1)+\Upsilon(\varrho_2)}{2}\left[L_{\frac{1}{\mathfrak{p}}-1}\left(\Upsilon(\varrho_1),\Upsilon(\varrho_2)\right)\right]^{\frac{1-\mathfrak{p}}{\mathfrak{p}}} \\
&\qquad + \mathfrak{q}\frac{\omega(\varrho_1)+\omega(\varrho_2)}{2}\left[L_{\frac{1}{\mathfrak{q}}-1}\left(\omega(\varrho_1),\omega(\varrho_2)\right)\right]^{\frac{1-\mathfrak{q}}{\mathfrak{q}}}. \qquad (3.23)
\end{aligned}
$$

Proof Let Υ, ω be two positive multiplicative harmonic convex functions. Then using the Young's inequality

$$
\varrho_1\varrho_2 \leq \mathfrak{p}\varrho_1^{\frac{1}{\mathfrak{p}}} + \mathfrak{q}\varrho_2^{\frac{1}{\mathfrak{q}}}, \quad \mathfrak{p},\mathfrak{q} > 0, \mathfrak{p}+\mathfrak{q} = 1,
$$

we have

$$
\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}} \\
&\quad = e^{\int_0^1 \ln[\Upsilon(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})\omega(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1})]\, d\kappa}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^1 \left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right] d\kappa \\
&\leq \int_0^1 \left[\mathfrak{p}\left(\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)^{\frac{1}{\mathfrak{p}}} + \mathfrak{q}\left(\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\right)^{\frac{1}{\mathfrak{q}}}\right] d\kappa \\
&\leq \int_0^1 \left[\mathfrak{p}\left[\left(\Upsilon(\varrho_1)\right)^{\kappa}\left(\Upsilon(\varrho_2)\right)^{1-\kappa}\right]^{\frac{1}{\mathfrak{p}}} + \mathfrak{q}\left[\left(\omega(\varrho_1)\right)^{\kappa}\left(\omega(\varrho_2)\right)^{1-\kappa}\right]^{\frac{1}{\mathfrak{q}}}\right] d\kappa \\
&= \mathfrak{p}\left(\Upsilon(\varrho_2)\right)^{\frac{1}{\mathfrak{p}}} \int_0^1 \left(\frac{\Upsilon(\varrho_1)}{\Upsilon(\varrho_2)}\right)^{\frac{\kappa}{\mathfrak{p}}} d\kappa + \mathfrak{q}\left(\omega(\varrho_2)\right)^{\frac{1}{\mathfrak{q}}} \int_0^1 \left(\frac{\omega(\varrho_1)}{\omega(\varrho_2)}\right)^{\frac{\kappa}{\mathfrak{q}}} d\kappa \\
&= \mathfrak{p}^2\left(\Upsilon(\varrho_2)\right)^{\frac{1}{\mathfrak{p}}} \int_0^{\frac{1}{\mathfrak{p}}} \left(\frac{\Upsilon(\varrho_1)}{\Upsilon(\varrho_2)}\right)^{\mathfrak{u}} d\mathfrak{u} + \mathfrak{q}^2\left(\omega(\varrho_2)\right)^{\frac{1}{\mathfrak{q}}} \int_0^{\frac{1}{\mathfrak{q}}} \left(\frac{\omega(\varrho_1)}{\omega(\varrho_2)}\right)^{\mathfrak{v}} d\mathfrak{v} \\
&= \mathfrak{p}^2 \frac{(\Upsilon(\varrho_1))^{\frac{1}{\mathfrak{p}}} - (\Upsilon(\varrho_2))^{\frac{1}{\mathfrak{p}}}}{\log\Upsilon(\varrho_1) - \log\Upsilon(\varrho_2)} + \mathfrak{q}^2 \frac{(\omega(\varrho_1))^{\frac{1}{\mathfrak{q}}} - (\omega(\varrho_2))^{\frac{1}{\mathfrak{q}}}}{\log\omega(\varrho_1) - \log\omega(\varrho_2)} \\
&= \mathfrak{p}^2 \frac{(\Upsilon(\varrho_1))^{\frac{1}{\mathfrak{p}}} - (\Upsilon(\varrho_2))^{\frac{1}{\mathfrak{p}}}}{\Upsilon(\varrho_1) - \Upsilon(\varrho_2)} L\left(\Upsilon(\varrho_1), \Upsilon(\varrho_2)\right) \\
&\quad + \mathfrak{q}^2 \frac{(\omega(\varrho_1))^{\frac{1}{\mathfrak{q}}} - (\omega(\varrho_2))^{\frac{1}{\mathfrak{q}}}}{\omega(\varrho_1) - \omega(\varrho_2)} L\left(\omega(\varrho_1), \omega(\varrho_2)\right) \\
&\leq \mathfrak{p}\frac{\Upsilon(\varrho_1)+\Upsilon(\varrho_2)}{2}\left[L_{\frac{1}{\mathfrak{p}}-1}\left(\Upsilon(\varrho_1), \Upsilon(\varrho_2)\right)\right]^{\frac{1-\mathfrak{p}}{\mathfrak{p}}} \\
&\quad + \mathfrak{q}\frac{\omega(\varrho_1)+\omega(\varrho_2)}{2}\left[L_{\frac{1}{\mathfrak{q}}-1}\left(\omega(\varrho_1), \omega(\varrho_2)\right)\right]^{\frac{1-\mathfrak{q}}{\mathfrak{q}}}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.11 *Let $\Upsilon, \omega : I \to \mathbb{R}$ be increasing multiplicative harmonic convex functions where $\varrho_1, \varrho_2 \in I$ and $\varrho_1 < \varrho_2$. Then*

$$
\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2 \ln G(\omega(\varrho_1)),\omega(\varrho_2)}{\varrho_2-\varrho_1}} \left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2 \ln G(\Upsilon(\varrho_1)),\Upsilon(\varrho_2)}{\varrho_2-\varrho_1}} \\
&\quad \leq 2L\left[\Upsilon(\varrho_1)\omega(\varrho_2), \Upsilon(\varrho_2)\omega(\varrho_1)\right]. \qquad (3.24)
\end{aligned}
$$

Proof Let Υ, ω be two positive multiplicative harmonic convex functions. Then using the inequalities

$$
\begin{aligned}
\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right) &\leq \left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}, \\
\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right) &\leq \left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa},
\end{aligned}
$$

with $(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) \geq 0$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, and $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, we have

$$
\begin{aligned}
&\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa} \\
&\quad + \omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}
\end{aligned}
$$

$$
\begin{aligned}
&\le \Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\\
&\quad+\left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}\left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa}.
\end{aligned}
$$

Taking the logarithm and integrating the above inequality with respect to κ on $[0,1]$, we have

$$
\begin{aligned}
&\int_0^1 \ln\left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa}\right]d\kappa\\
&\quad+\int_0^1 \ln\left[\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}\right]d\kappa\\
&\le \int_0^1 \ln\left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right.\\
&\quad\left.+\left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}\left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa}\right]d\kappa.
\end{aligned}
$$

Since Υ and ω are increasing, we have

$$
\begin{aligned}
&\int_0^1 \ln\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)d\kappa\int_0^1 \ln\left[\left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa}\right]d\kappa\\
&\quad+\int_0^1 \ln\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)d\kappa\int_0^1 \ln\left[\left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}\right]d\kappa\\
&\le \int_0^1 \ln\left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right.\\
&\quad\left.+\left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}\left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa}\right]d\kappa,
\end{aligned}
$$

which means

$$
\begin{aligned}
&\ln G\left(\omega(\varrho_1),\omega(\varrho_2)\right)\int_0^1 \ln\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)d\kappa\\
&\quad+\ln G\left(\omega(\varrho_1),\omega(\varrho_2)\right)\int_0^1 \ln\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)d\kappa\\
&\le \int_0^1 \ln\left[\Upsilon\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_2+(1-\kappa)\varrho_1}\right)\omega\left(\frac{\varrho_1\varrho_2}{\kappa\varrho_1+(1-\kappa)\varrho_2}\right)\right.\\
&\quad\left.+\left(\Upsilon(\varrho_2)\right)^{(1-\kappa)}\left(\Upsilon(\varrho_1)\right)^{\kappa}\left(\omega(\varrho_1)\right)^{(1-\kappa)}\left(\omega(\varrho_2)\right)^{\kappa}\right]d\kappa.
\end{aligned}
$$

Now taking the exponential on both sides, we get the required result:

$$
\begin{aligned}
&\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\Upsilon(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2\ln G(\omega(\varrho_1)),\omega(\varrho_2)}{\varrho_2-\varrho_1}}\left(\int_{\varrho_1}^{\varrho_2}\left(\frac{\omega(\varkappa_1)}{\varkappa_1^2}\right)^{d\varkappa_1}\right)^{\frac{\varrho_1\varrho_2\ln G(\Upsilon(\varrho_1)),\Upsilon(\varrho_2)}{\varrho_2-\varrho_1}}\\
&\le 2L\left[\Upsilon(\varrho_1)\omega(\varrho_2),\Upsilon(\varrho_2)\omega(\varrho_1)\right]. \qquad (3.25)
\end{aligned}
$$

□

4 Concluding remarks

In this paper, we investigated multiplicative harmonic convex functions. We obtained a new version of the Hermite–Hadamard type integral inequality in the setting of multiplicative calculus for multiplicative harmonic convex and harmonic convex functions. We also derived several integral inequalities of Hermite–Hadamard type for the multiplication and division of harmonic convex and multiplicative harmonic convex functions in multiplicative calculus. New upper bounds for products of two multiplicative harmonic convex functions are also given. As a result, several new integral inequalities of Hermite–Hadamard type are established. Furthermore, by providing examples with included graphs, both obtained results have been verified and a better understanding of the subject has been achieved. In recent years, Hermite–Hadamard inequalities have grown into a significant tool for mathematical analysis, probability theory, optimization, and other fields of mathematics. Many studies have been dedicated to bringing a new dimension to the theory of inequalities. We believe that this class of functions will be deeply researched in this attractive and absorbing field of inequalities and also in different areas of pure and applied sciences. We also believe that our techniques and ideas will stimulate further research in this field.

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