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The Stenger conjectures and the A-stability of collocation Runge-Kutta methods

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Abstract

Stenger conjectures are claims about the location of the eigenvalues of matrices whose elements are certain integrals involving basic Lagrange interpolating polynomials supported on the zeros of orthogonal polynomials. In this paper, we show the validity of the extended Stenger conjecture for families of classical orthogonal polynomials. We also show the validity of the restricted Strenger conjecture for a family of Jacobi and generalized Laguerre orthogonal polynomials. A connection with the A-stability of the collocation Runge-Kutta methods is investigated.

Mathematics Subject Classification: Zeros of orthogonal polynomials; Matrix eigenvalues; Collocation Runge-Kutta; A-stability

1 Introduction

Given a non-negative weight function ω over an interval $[a,b]$, $-\infty \leq a < b \leq +\infty$, denote by π_k, $k \geq 0$, the orthogonal polynomials with respect to the scalar product

$$\langle f,g\rangle_\omega = \int_a^b f(x)g(x)\omega(x)\,dx.$$

Let the matrices $U_n = [u_{ij}^{(n)}]$ and $V_n = [v_{ij}^{(n)}]$ be defined by

$$u_{ij}^{(n)} = \int_a^{x_i} \ell_j^{(n)}(x)\omega(x)\,dx, \qquad v_{ij}^{(n)} = \int_{x_i}^b \ell_j^{(n)}(x)\omega(x)\,dx, \quad i,j = 1,2,\ldots,n, \tag{1.1}$$

where x_i, $i = 1,\ldots,n$, are the zeros of the orthogonal polynomial π_n and $\ell_1^{(n)},\ldots,\ell_n^{(n)}$, are the associated basic Lagrange interpolating polynomials of degree $(n-1)$, i.e.,

$$\ell_k^{(n)}(x) = \prod_{1\leq j\leq n, j\neq k} \frac{x-x_j}{x_k-x_j}, \quad k = 1,2,\ldots,n.$$

The following conjecture is stated in [1] and [2].

Extended Stenger Conjecture *Given an almost everywhere positive weight function ω on $[a,b]$ such that $-\infty \leq a < b \leq +\infty$, or equivalently, the associated orthogonal polynomials*

π_n, the eigenvalues of each of the matrices U_n and V_n defined in (1.1) lie in the open right half of the complex plane.

The validity of the conjecture is established for Sinc interpolation in [3]. The conjecture is shown to hold for Legendre polynomials in [4] and its equivalence to the A-stability of the n-stage Gauss–Runge–Kutta method is proved. Moreover, in [4] the conjecture is shown to hold for U_n in the case of Jacobi orthogonal polynomials with parameters $(1,0)$. In [4], the authors provided a counter-example to the conjecture for some piecewise constant weight functions. In this paper, we show the validity of the conjecture for a specific family of weight functions, which enables us in particular to prove the conjecture for a family of Jacobi, generalized Laguerre orthogonal polynomials and Jacobi–Koonwinder orthogonal polynomials.

In the restricted Stenger conjecture, the matrices U_n (with a finite) and V_n (with b finite) are defined by

$$u_{ij}^{(n)} = \int_a^{x_i} \ell_j^{(n)}(x)\,dx, \qquad v_{ij}^{(n)} = \int_{x_i}^{b} \ell_j^{(n)}(x)\,dx, \quad i,j = 1,2,\dots,n, \tag{1.2}$$

and the conjecture states

Restricted Stenger Conjecture *Given an almost everywhere positive weight function ω on $[a,b]$ or equivalently the associated orthogonal polynomials π_n, the eigenvalues of each of the matrices U_n and V_n defined in (1.2) lie on the open right half of the complex plane.*

It has been shown in [4] that the restricted conjecture is wrong in general. For instance, they showed, at least numerically, that the real part of some eigenvalues of the matrix U_n with $n = 5$ and x_i, $i = 1,2,\dots,5$, the zeros of Gegenbauer polynomial $C_5^{(10)}$, are negative. However, several conjectures on the range of validity of the restricted Stenger conjecture for Jacobi and generalized Laguerre orthogonal polynomials were advanced in [4]. In this paper, we show the validity of the restricted Stenger conjecture for a family of classical orthogonal polynomials.

2 Extended Stenger conjecture for a family of weight functions

In the following, we give a general result on the validity of the extended Stenger conjecture for a specific family of weight functions. The theorem and its proof were inspired by the methodology introduced in [4].

Theorem 2.1 *Let ω be weight function on $[a,b]$, $-\infty \le a < b < +\infty$, supposed to be positive on (a,b). Assume that, for any complex polynomial P of degree at most $(n-1)$, there exist a complex polynomial Q of degree at most n and a real differentiable strictly increasing positive function Φ on $(a,b]$ meeting the following three requirements:*

$$\int_a^x P(t)\omega(t)\,dt = Q(x)\Phi(x) \quad \textit{for all } x \in [a,b], \tag{2.1}$$

$$\lim_{x\to a} \Phi(x)\big|Q(x)\big|^2 = 0, \qquad \Phi'|Q|^2 \in L^1\big([a,b]\big). \tag{2.2}$$

Then the extended Stenger conjecture associated with ω is satisfied for the matrix U_n defined in (1.1).

Proof Let λ be an eigenvalue of the matrix U_n. Thus, there exists a complex vector

$$(y_1, y_2, \dots, y_n) \neq (0, 0, \dots, 0) \tag{2.3}$$

such that

$$\int_a^{x_i} \sum_{k=1}^{n} y_k \ell_k^{(n)}(t)\omega(t)\,dt = \lambda y_i, \quad i = 1, \dots, n.$$

Denote by $\mathbf{y}$ the unique complex polynomial of degree at most $(n-1)$ such that $\mathbf{y}(x_i) = y_i$. We have

$$\int_a^{x_i} \mathbf{y}(t)\omega(t)\,dt - \lambda \mathbf{y}(x_i) = 0, \quad i = 1, 2, \dots, n. \tag{2.4}$$

According to condition (2.1), there exists a complex polynomial Q of degree at most n such that

$$\int_a^{x} \mathbf{y}(t)\omega(t)\,dt = Q(x)\Phi(x), \tag{2.5}$$

where Φ is a real differentiable strictly increasing positive function on $(a, b]$ and thus is non-negative on $[a, b]$. Let ω_i, $i = 1, 2, \dots, n$, be the weights of the Gaussian quadrature with respect to the weight function ω. Denote by $\bar{\mathbf{y}}$ the conjugate of $\mathbf{y}$. From (2.4), we have

$$\sum_{i=1}^{n} \frac{\omega_i}{\Phi(x_i)} \bar{\mathbf{y}}(x_i) \int_a^{x_i} \mathbf{y}(t)\omega(t)\,dt = \lambda \sum_{i=1}^{n} \frac{\omega_i}{\Phi(x_i)} \left|\mathbf{y}(x_i)\right|^2.$$

Thus, according to (2.5), we have

$$\sum_{i=1}^{n} \omega_i \bar{\mathbf{y}}(x_i) Q(x_i) = \lambda \sum_{i=1}^{n} \frac{\omega_i}{\Phi(x_i)} \left|\mathbf{y}(x_i)\right|^2.$$

Applying Gaussian quadrature to the polynomial $\bar{\mathbf{y}}Q$ of degree at most $(2n-1)$, we obtain

$$\int_a^{b} \bar{\mathbf{y}}(x) Q(x)\omega(x)\,dx = \lambda \sum_{i=1}^{n} \frac{\omega_i}{\Phi(x_i)} \left|\mathbf{y}(x_i)\right|^2,$$

or equivalently

$$\int_a^{b} \frac{\bar{\mathbf{y}}(x)\omega(x)}{\Phi(x)} \left[\int_a^{x} \mathbf{y}(t)\omega(t)\,dt\right] dx = \lambda \sum_{i=1}^{n} \frac{\omega_i}{\Phi(x_i)} \left|\mathbf{y}(x_i)\right|^2. \tag{2.6}$$

The real part of the left-hand side of (2.6) is given by

$$\frac{1}{2} \int_a^{b} \frac{1}{\Phi(x)} \left(\bar{\mathbf{y}}(x)\omega(x) \left[\int_a^{x} \mathbf{y}(t)\omega(t)\,dt\right] + \mathbf{y}(x)\omega(x) \left[\int_a^{x} \bar{\mathbf{y}}(t)\omega(t)\,dt\right] \right) dx.$$

Moreover, using the product derivative formula, we note that

$$\bar{\mathbf{y}}(x)\omega(x)\left[\int_a^x \mathbf{y}(t)\omega(t)\,dt\right] + \mathbf{y}(x)\omega(x)\left[\int_a^x \bar{\mathbf{y}}(t)\omega(t)\,dt\right] = \frac{d}{dx}\left|\int_a^x \mathbf{y}(t)\omega(t)\,dt\right|^2.$$

Thus, the real part of the left-hand side of (2.6) is given by

$$\frac{1}{2}\int_a^b \frac{1}{\Phi(x)}\frac{d}{dx}\left|\int_a^x \mathbf{y}(t)\omega(t)\,dt\right|^2 dx = \frac{1}{2}\int_a^b \frac{1}{\Phi(x)}\frac{d}{dx}\left(\left|\Phi(x)Q(x)\right|^2\right)dx, \tag{2.7}$$

while the real part of the right-hand side of (2.6) is equal to

$$\mathrm{Re}(\lambda)\sum_{i=1}^n \frac{\omega_i}{\Phi(x_i)}\left|\mathbf{y}(x_i)\right|^2.$$

Thus, from (2.6), to prove that the real part of λ is non-negative, it is sufficient to prove the non-negativity of (2.7). Since the function $\Phi'|Q|^2 \in L^1([a,b])$, integration by parts shows that (2.7) is equal to

$$\frac{1}{2}\Phi(b)\left|Q(b)\right|^2 - \frac{1}{2}\lim_{x\to a}\Phi(x)\left|Q(x)\right|^2 + \frac{1}{2}\int_a^b \Phi'(x)\left|Q(x)\right|^2 dx. \tag{2.8}$$

Moreover, taking into consideration that $\lim_{x\to a}\Phi(x)|Q(x)|^2 = 0$ and Φ is a strictly increasing positive function on $(a,b]$ shows that the quantity (2.8) is non-negative and thereby $\mathrm{Re}(\lambda) \geq 0$. Let us assume that $\mathrm{Re}(\lambda) = 0$. Then necessarily (2.7) vanishes and hence

$$\int_a^b \Phi'(x)\left|Q(x)\right|^2 dx = 0.$$

Therefore, $\Phi' Q \equiv 0$ on $(a,b]$. On account of the assumption that Φ is strictly increasing on $(a,b]$, this automatically implies that Q is identically zero on $(a,b]$. Hence, from (2.5) and the positivity of ω on (a,b), we can conclude that $\mathbf{y}$ is identically zero, which contradicts (2.3). □

Remark 2.1 Note that one can replace the condition $\lim_{x\to a}\Phi(x)|Q(x)|^2 = 0$ in Theorem 2.1 by the less restrictive conditions of the non-negativity of (2.8) or the non-negativity of $\frac{1}{2}\Phi(b)|Q(b)|^2 - \frac{1}{2}\lim_{x\to a}\Phi(x)|Q(x)|^2$.

Applying similar arguments, the matrices V_n leads to the following theorem.

Theorem 2.2 *Let ω be weight function on $[a,b]$, $-\infty < a < b \leq +\infty$, supposed to be positive on (a,b). Assume that, for any complex polynomial P of degree at most $(n-1)$, there exist a complex polynomial Q of degree at most n and a real differentiable strictly decreasing positive function Φ on $(a,b]$ meeting the following three requirements*:

$$\int_x^b P(t)\omega(t)\,dt = Q(x)\Phi(x) \quad \textit{for all } x \in [a,b],$$

$$\lim_{x\to b}\Phi(x)\left|Q(x)\right|^2 = 0, \qquad \Phi'|Q|^2 \in L^1\big([a,b]\big).$$

Then the extended Stenger conjecture associated with ω is satisfied for the matrix V_n defined in (1.1).

3 Application to Jacobi polynomials

In the following, we shall use Theorem 2.1 to prove the validity of the extended Stenger conjecture for a family of Jacobi orthogonal polynomials. Jacobi polynomials $P_n^{(\alpha,\beta)}$ are orthogonal polynomials on $[-1,1]$ with respect to the weight function

$$\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1. \tag{3.1}$$

They admit the following explicit expressions

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k.$$

Lemma 3.1 *Let $\alpha > -1$ be a real number and P a polynomial of degree m. Then,*

$$\int_{-1}^{x} P(t)(1+t)^{\alpha}\,dt = R(x)(1+x)^{\alpha+1},$$

where R is the polynomial

$$R(x) = \sum_{k=0}^{m} \frac{(-1)^k P^{(k)}(x)(1+x)^k}{(\alpha+1)(\alpha+2)\dots(\alpha+k+1)}.$$

Proof We proceed by induction of the degree of the polynomial P. For constant polynomials, the claim is trivial. Let P be a polynomial of degree n. Integration by parts leads to

$$\int_{-1}^{x} P(t)(1+t)^{\alpha}\,dt = \frac{(1+x)^{\alpha+1}P(x)}{\alpha+1} - \frac{1}{\alpha+1}\int_{-1}^{x} P'(t)(1+t)^{\alpha+1}\,dt.$$

We conclude the proof using the induction hypothesis. □

Corollary 3.1 *The extended Stenger conjecture holds true in each of the following case:*

- *(a) for U_n in the case of Jacobi polynomials $P_n^{(m,\alpha)}$ for all $n \geq 1$, $\alpha > -1$ and $m = 0,1$.*
- *(b) for V_n in the case of Jacobi polynomials $P_n^{(\alpha,m)}$ for all $n \geq 1$, $\alpha > -1$ and $m = 0,1$.*

Proof (a) We shall consider the cases $m = 0,1$ separately.

• *Case $m = 0$*: According to Lemma 3.1, the weight function $\omega^{(0,\alpha)}(t) = (1+t)^{\alpha}$ satisfies the conditions of Theorem 2.1, with $\Phi(x) = (1+x)^{\alpha+1}$.

• *Case $m = 1$*: According to Lemma 3.1, for any polynomial P of degree at most $n-1$, we have

$$\int_{-1}^{x} P(t)(1-t)(1+t)^{\alpha}\,dt = Q(x)(1+x)^{\alpha+1},$$

where Q is the polynomial of degree at most n given by

$$Q(x) = \sum_{k=0}^{n} \frac{(-1)^k (P(x)(1-x))^{(k)}(1+x)^k}{(\alpha+1)(\alpha+2)\dots(\alpha+k+1)}.$$

Thus, the weight function $\omega^{(1,\alpha)}(t) = (1-t)(1+t)^\alpha$ satisfies the conditions of Theorem 2.1, with $\Phi(x) = (1+x)^{\alpha+1}$. The proof of (b) is a direct consequence of the fact that the eigenvalues of V_n for $P_n^{(\alpha,\beta)}$ coincide with the eigenvalues of U_n for $P_n^{(\beta,\alpha)}$ (see [4]). □

4 Application to generalized Laguerre polynomials

In the following, we shall use Theorem 2.2 to prove the validity of the extended Stenger conjecture for a family of generalized Laguerre orthogonal polynomials.

Generalized Laguerre polynomials $L_n^{(\alpha)}$ are orthogonal polynomials on $[0,+\infty)$ with respect to the weight function

$$\omega^{(\alpha)}(x) = x^\alpha e^{-x}, \quad \alpha > -1. \tag{4.1}$$

They admit explicit expressions as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}.$$

Lemma 4.1 *Let P be a polynomial of degree m. Then,*

$$\int_x^{+\infty} P(t)e^{-t}\,dt = R(x)e^{-x},$$

where R is the polynomial

$$R(x) = \sum_{k=0}^{m} P^{(k)}(x).$$

Proof We proceed by induction of the degree of the polynomial P. For constant polynomials, the claim is trivial. Let P be a polynomial of degree m. Integration by parts leads to

$$\int_x^{+\infty} P(t)e^{-t}\,dt = P(x)e^{-x} + \int_x^{+\infty} P'(t)e^{-t}\,dt.$$

We conclude the proof using the induction hypothesis. □

Corollary 4.1 *The extended Stenger conjecture holds true for V_n in the case of generalized Laguerre polynomials $L_n^{(m)}$ for all $n \geq 1$ and $m = 0, 1$.*

Proof We shall consider the cases $m = 0$ and 1 separately.

- *Case $m = 0$*: According to Lemma 4.1, the weight function $\omega^{(0)}(t) = e^{-t}$ satisfies the conditions of Theorem 2.2 with $\Phi(x) = e^{-x}$.
- *Case $m = 1$*: According to Lemma 3.1, for any polynomial of degree $n-1$, we have

$$\int_x^{+\infty} P(t)te^{-t}\,dt = Q(x)e^{-x},$$

where Q is the polynomial of degree at most n

$$Q(x) = \sum_{k=0}^{n} \frac{d^k(xP(x))}{dx^k}.$$

Thus, the weight function $\omega^{(1)}(t) = te^{-t}$ satisfies the conditions of Theorem 2.2 with $\Phi(x) = e^{-x}$. □

Remark 4.1 Similar arguments as in the proof of Theorem 2.1 can be applied to show that the extended Stenger conjecture is valid for U_n for the Legendre weight $\omega \equiv 1$ and for V_n for the Laguerre weight $\omega = e^{-x}$ if, in the definition of the matrices U_n and V_n, we take the real numbers $x_1, x_2, \dots, x_n$ as the nodes of a $(2n, n)$ positive quadrature. Similarly, the extended Stenger conjecture can be shown to hold when we take $x_1, x_2, \dots, x_n$ as the zeros of the othogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\omega(x)\,dx + r_0 f(a)g(a), \quad r_0 > 0,$$

for U_n for Jacobi weights $\omega^{(m,\beta)}$ with $m = 0$ or 1 and $\beta > -1$ (Jacobi–Koornwinder polynomials [5]) and for V_n for the Laguerre weights $x^m e^{-x}$ with $m = 0$ or 1 (Laguerre–Koornwinder polynomials [6]) when we replace $r_0 f(a)g(a)$ by $r_0 f(b)g(b)$ in the Sobolev inner product.

5 Restricted Stenger conjecture

For the rest of the paper, we shall study the restricted Stenger conjecture. It has been shown in [4] that the conjecture is wrong in general. For instance, they showed, at least numerically, that the real part of some eigenvalues of the matrix U_n with $n = 5$ and $x_i, i = 1, 2, \dots, 5$, the zeros of Gegenbauer polynomial $C_5^{(10)}$, are negative. The validity of the restricted Stenger conjecture for Legendre polynomial is shown in [4] and its equivalence with the A-stability of the n-stage Gauss–Runge–Kutta method is established. Several conjectures on the range of validity of the restricted Stenger conjecture for Jacobi and generalized Laguerre orthogonal polynomials were advanced in [4].

In the following, we use the techniques introduced in [4] to prove the validity of the restricted Stenger conjecture for monotonous differentiable weight functions.

Theorem 5.1 *For differentiable non-increasing (resp. non-decreasing) weight functions ω over $[a, b]$, all the eigenvalues of the matrices U_n (resp. V_n) defined in (1.2) have positive real part.*

Proof Let λ be an eigenvalue of the matrix U_n. Thus there exists a non-zero complex vector $y = (y_1, y_2, \dots, y_n)$ such that

$$\int_a^{x_i} \sum_{k=1}^{n} y_k \ell_k^{(n)}(x)\,dx = \lambda y_i, \quad i = 1, \dots, n.$$

Denote by $\mathbf{y}$ the unique complex polynomial of degree at most $(n-1)$ such that $\mathbf{y}(x_i) = y_i$. Thus,

$$\int_a^{x_i} \mathbf{y}(x)\,dx - \lambda \mathbf{y}(x_i) = 0, \quad i = 1, 2, \dots, n.$$

The polynomial

$$Q(x) = \int_a^{x} \mathbf{y}(t)\,dt - \lambda \mathbf{y}(x)$$

is of degree at most n such that $Q(x_i) = 0$ for $i = 1, 2, \dots, n$. Therefore, there exists a complex constant K such that

$$\int_a^x \mathbf{y}(t)\,dt - \lambda \mathbf{y}(x) = K\pi_n(x). \tag{5.1}$$

Multiplying (5.1) by $\omega(x)\bar{\mathbf{y}}(x)$ and integrating over $[a,b]$, we obtain

$$\int_a^b \omega(x)\bar{\mathbf{y}}(x)\left[\int_a^x \mathbf{y}(t)\,dt\right]dx = \lambda \int_a^b \omega(x)\left|\mathbf{y}(x)\right|^2 dx \tag{5.2}$$

The real part of the left side of (5.2) is given by

$$\frac{1}{2}\int_a^b \omega(x)\frac{d}{dx}\left|\int_a^x \mathbf{y}(t)\,dt\right|^2 dx, \tag{5.3}$$

while the real part of right side of (5.2) is given by

$$\mathrm{Re}(\lambda)\int_a^b \omega(x)\left|\mathbf{y}(x)\right|^2 dx.$$

Thus, to show the non-negativity of $\mathrm{Re}(\lambda)$, we should prove the non-negativity of (5.3). Integration by parts shows that (5.3) is equal to

$$\frac{1}{2}\omega(b)\left|\int_a^b \mathbf{y}(t)\,dt\right|^2 - \frac{1}{2}\int_a^b \omega'(x)\left|\int_a^x \mathbf{y}(t)\,dt\right|^2 dx. \tag{5.4}$$

Since the non-negative differentiable weight function ω is a non-increasing function on $[a,b]$, the expression (5.4) is non-negative and therefore the real part of λ is non-negative. To prove that $\mathrm{Re}(\lambda) > 0$, we should prove that (5.4) is strictly positive. We first note that (5.4) is equal to zero only when the weight function is a constant function and that

$$\int_a^b \mathbf{y}(t)\,dt = 0. \tag{5.5}$$

In this case, one can use the methodology in [4] (page 5) to conclude that $\mathbf{y}$ is orthogonal to all polynomials of degree at most $(n-1)$ and in particular $\int_a^b \mathbf{y}^2(t)\,dt = 0$ and thus $\mathbf{y} \equiv \mathbf{0}$. This contradiction shows that $\mathrm{Re}(\lambda) > 0$. Similar arguments can be used to prove the theorem for the matrices V_n. □

As an application of Theorem 5.1, we give instances for which the restricted Stenger conjecture holds for Jacobi and generalized Laguerre orthogonal polynomials.

The following is a direct consequence of Theorem 5.1.

Corollary 5.1 *The restricted Stenger conjecture holds true in each of the following cases*:

- (*a*) *for* U_n *in the case of Jacobi polynomials* $P_n^{(\alpha,\beta)}$ *for all* $n \geq 1$, $\alpha \geq 0$ *and* $-1 < \beta \leq 0$.
- (*b*) *for* V_n *in the case of Jacobi polynomials* $P_n^{(\alpha,\beta)}$ *for all* $n \geq 1$, $\beta \geq 0$ *and* $-1 < \alpha \leq 0$.
- *for* U_n *in the case of generalized Laguerre polynomials* $L_n^{(\alpha)}$ *for all* $n \geq 1$ *and* $-1 < \alpha \leq 0$.

Proof The statement (a) is a direct consequence of the fact that for $\alpha \geq 0$ and $-1 < \beta \leq 0$ the Jacobi weight function (3.1) is a non-increasing functionon $[-1, 1]$ since

$$\frac{d}{dx}\omega^{\alpha,\beta}(x) = -(1-x)^{\alpha-1}(1+x)^{\beta-1}\big((\alpha+\beta)x + (\alpha-\beta)\big) \leq 0, \quad \text{for any } x \in [-1,1].$$

Statement (b) is a consequence of the fact proved in [4] that $U_n^{(\alpha,\beta)} = V_n^{(\beta,\alpha)}$, i.e.; the matrice U_n for the Jacobi polynomials with parameters (α, β) coincide with the matrix V_n for Jacobi polynomials with parameters (β, α). To prove (c), we simply remark that, for $-1 < \alpha \leq 0$, the generalized Laguerre weight function (4.1) is non-increasing on $[0, +\infty[$, i.e.,

$$\frac{d}{dx}\omega^{(\alpha)}(x) = x^{\alpha-1}e^{-x}(\alpha - x) \leq 0, \quad \text{for any } x \in [0, +\infty[.$$

This concludes the proof of the corollary. □

6 Restricted Stenger conjecture for Jacobi orthogonal polynomials

In this section, we show that the eigenvalues of the restricted matrices U_n and V_n defined in (1.2) coincide with the zeros of certain polynomials with coefficients expressed in terms of the value of the successive derivatives of the orthogonal polynomials at the endpoints of the interval. This allows us to restate the restricted Stenger conjecture for Jacobi orthogonal polynomials as a result already proved in [7] and permits us to improve the results of Corollary 5.1. To state the results in their full generality, we shall assume that in the definition of the restricted matrices U_n and V_n, the real numbers $x_1, x_2, \ldots, x_n$ are arbitrary distinct real numbers in the interval $[a, b]$.

Proposition 6.1 *The eigenvalues of the restricted matrices U_n and V_n coincide with the zeros of the polynomials*

$$\Phi_n(z) = \sum_{k=0}^{n} \pi_n^{(k)}(a) z^k, \qquad \Psi_n(z) = \sum_{k=0}^{n} \pi_n^{(k)}(b)(-z)^k, \tag{6.1}$$

respectively, where $\pi_n(x) = (x - x_1)(x - x_2)\ldots(x - x_n)$ and $\pi_n^{(k)}$ refer to the kth derivative of the polynomial π_n.

Proof Let λ be an eigenvalue of the matrix U_n. Thus, from (5.1), there exists a non-zero complex vector $y = (y_1, y_2, \ldots, y_n)$ and a complex number C such that

$$\int_a^x \mathbf{y}(t)\, dt - \lambda \mathbf{y}(x) = C\pi_n(x), \tag{6.2}$$

where $\mathbf{y}$ the unique complex polynomial of degree $n - 1$ such that $\mathbf{y}(x_i) = y_i$. Writing the polynomial $\mathbf{y}$ and the polynomial π_n as

$$y(x) = \sum_{k=0}^{n-1} b_k (x-a)^k, \qquad \pi_n(x) = \sum_{k=0}^{n} \alpha_k (x-a)^k \quad \text{with } \alpha_k = \frac{\pi_n^{(k)}(a)}{k!},$$

and solving (6.2), we obtain the system

$$\frac{b_{k-1}}{k} - \lambda b_k = C\alpha_k, \quad k = 0, 1, \ldots, n,$$

with the convention that $b_{-1} = b_n = 0$. Solving for b_k leads to

$$b_{n-k} = C \sum_{j=0}^{k-1} \frac{(n-j)!}{(n-k)!} \lambda^{k-1-j} \alpha_{n-j}, \quad k = 1, \dots, n-1.$$

Moreover, using the fact that $b_0 = \lambda b_1 + C\alpha_1$, we obtain

$$b_0 = C \sum_{k=0}^{n-1} (n-k)! \lambda^{n-k-1} \alpha_{n-k}.$$

Inserting this equation into the relation $-\lambda b_0 = C\alpha_0$, we obtain

$$\sum_{k=0}^{n} k! \alpha_k \lambda^k = 0.$$

This proves that λ is a zero of the polynomial Φ_n. We shall now show that the eigenvalues of U_n are simple, so that the spectrum of U_n coincides with the zeros of Φ_n. Let $y = (y_1, y_2, \dots, y_n)$ and $z = (z_1, z_2, \dots, z_n)$ be two eigenvectors associated with the eigenvalue λ. Then, there exist two non-zero constants C_1, C_2 such that

$$\int_a^x \mathbf{y}(t)\,dt - \lambda \mathbf{y}(x) = C_1 \pi_n(x), \qquad \int_a^x \mathbf{z}(t)\,dt - \lambda \mathbf{z}(x) = C_2 \pi_n(x),$$

where $\mathbf{y}$ (resp. $\mathbf{z}$) is the unique complex polynomial of degree $(n-1)$ such that $\mathbf{y}(x_i) = y_i$ (resp. $\mathbf{z}(x_i) = z_i$). Thus, denoting $F(x) = (\mathbf{y}(x)/C_1) - (\mathbf{z}(x)/C_2)$, we have

$$\int_a^x F(t)\,dt - \lambda F(x) = 0 \quad \text{for all } x \in [a, b]. \tag{6.3}$$

Since F is a polynomial, this implies that $F \equiv 0$ and thus the vectors y and z are linearly dependent. The proof for V_n follows similar arguments. □

Proposition 6.2 *An eigenvector $(y_1, y_2, \dots, y_n)$ of the matrix U_n (resp. V_n) associated with the eigenvalue λ is given by $y_i = \mathbf{y}(x_i)$, where*

$$\mathbf{y}(x) = \sum_{k=1}^{n} \lambda^k \pi_n^{(k)}(x), \tag{6.4}$$

where λ is a zero of Φ_n (resp. Ψ_n).

Proof We should simply show that there is a constant C such that (6.2) holds. We have

$$\mathbf{y}(x) - \lambda \mathbf{y}'(x) = \sum_{k=1}^{n} \lambda^k \pi_n^{(k)}(x) - \sum_{k=1}^{n} \lambda^{k+1} \pi_n^{(k+1)}(x) = \lambda \pi_n'(x).$$

Integrating, we obtain

$$\int_a^x \mathbf{y}(t)\,dt - \lambda\big(\mathbf{y}(x) - \mathbf{y}(a)\big) = \lambda\big(\pi_n(x) - \pi_n(a)\big).$$

Since λ is a zero of Φ_n, we have $\mathbf{y}(a) = -\pi_n(a)$ and therefore

$$\int_a^x \mathbf{y}(t)\,dt - \lambda\mathbf{y}(x) = \lambda\pi_n(x). \tag{6.5}$$

This concludes the proof. □

In connection with the tau approximation for an eigenvalue problem, Csordas et al. [7] proved the Hurwitz stability of the polynomials

$$\Psi_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n}\left(\frac{d^k}{dx^k}P_n^{(\alpha,\beta)}(x)\right)_{x=1} z^k, \quad n \geq 2.$$

More precisely, they showed that if $-1 < \alpha \leq 1$ and $\beta > -1$, then the zeros of the polynomial $\Psi_n^{(\alpha,\beta)}$ lie in the half-plane $\operatorname{Re}(z) < 0$; and thus by virtue of Proposition 6.1, the eigenvalues of V_n for Jacobi polynomials lie in the half-plane $\operatorname{Re}(z) > 0$. Moreover, since for Jacobi polynomials, we have $U_n^{(\alpha,\beta)} = V_n^{(\beta,\alpha)}$, we conclude

Theorem 6.1 *The restricted Stenger conjecture holds true in each of the following cases*:

- *for U_n in the case of Jacobi polynomials $P_n^{(\alpha,\beta)}$ for all $n \geq 1$, $\alpha > -1$ and $-1 < \beta \leq 1$.*
- *for V_n in the case of Jacobi polynomials $P_n^{(\alpha,\beta)}$ for all $n \geq 1$, $\beta > -1$ and $-1 < \alpha \leq 1$.*

In particular, the previous theorem shows that restricted Stenger conjecture holds for Chebyshev polynomials of the first and second kind T_n and S_n with

$$T_n(x) = P_n^{(-\frac{1}{2},-\frac{1}{2})}(x), \qquad S_n(x) = P_n^{(\frac{1}{2},\frac{1}{2})}(x).$$

Also, Theorem 6.1 shows that the restricted Stenger conjecture holds for Legendre polynomials $P_n^{(0,0)}$ and for Gegenbauer (or ultra-spherical) polynomials $C_n^{(\mu)}$ when $-\frac{1}{2} < \mu \leq \frac{3}{2}$, where

$$C_n^{(\mu)}(x) = P_n^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(x).$$

7 Restricted Stenger conjecture for generalized Laguerre polynomials

In the following, we show that the restricted Stenger conjecture for U_n holds for generalized Laguerre polynomials $L_n^{(\alpha)}$ when the parameter α is restricted to lie in the interval $(-1,1]$. An explicit expression of generalized Laguerre polynomials $L_n^{(\alpha)}$ is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\frac{\binom{n+\alpha}{n-k}}{k!}x^k.$$

Thus, the associated polynomials Φ_n according to Proposition 6.1 are given by

$$\Phi_n(z) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}z^k.$$

To study the restricted Stenger conjecture for Laguerre polynomials, we shall study instead the zeros of the degree n truncation of the binomial function $(1+z)^{n+\alpha}$, i.e.,

$$B_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{k} z^k = z^n \Phi_n(-1/z). \tag{7.1}$$

We shall need the following Eneström–Kakeya theorem [8].

Theorem 7.1 *Let $P_n(z) = \sum_{k=0}^{n} a_k z^k$ be any polynomial with $a_k > 0$ for all $0 \le k \le n$. Setting*

$$\gamma = \min_{0\le k<n} \frac{a_k}{a_{k+1}}, \qquad \mu = \max_{0\le k<n} \frac{a_k}{a_{k+1}},$$

then all the zeros of P_n are contained in the annulus

$$\gamma \le |z| \le \mu.$$

We introduce the polynomials

$$H_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{\alpha-1+k}{k} z^k. \tag{7.2}$$

The connection between the polynomials $B_n^{(\alpha)}$ and $H_n^{(\alpha)}$ is given by the following.

Lemma 7.1 *For any $z \neq -1$, we have*

$$B_n^{(\alpha)}(z) = (1+z)^n H_n^{(\alpha)}\left(\frac{z}{1+z}\right). \tag{7.3}$$

Proof We proceed by induction on $n \ge 1$. The claim is trivial for $n = 1$. Let us assume that (7.3) holds for n. We have

$$(1+z)^{n+1} H_{n+1}^{(\alpha)}\left(\frac{z}{z+1}\right) = (1+z)B_n^{(\alpha)}(z) + \binom{\alpha+n}{n+1} z^{n+1} = B_{n+1}^{(\alpha)}(z).$$

This concludes the proof. □

Proposition 7.1 *For $0 < \alpha \le 1$, all the zeros of the polynomial $B_n^{(\alpha)}$ lie in the half-plane* $\mathrm{Re}(z) < 0$.

Proof The condition $0 < \alpha \le 1$ guarantees that the coefficients of the polynomial $H_n^{(\alpha)}$ defined in (7.2) are positive. The successive ratios of the coefficients of $H_n^{(\alpha)}$ are

$$\xi_k = \frac{\binom{\alpha-1+k}{k}}{\binom{\alpha+k}{k+1}} = \frac{k+1}{\alpha+k}, \quad 0 \le k \le n-1,$$

which, for $0 < \alpha \le 1$, are non-increasing as k runs from 0 to $n-1$. Thus by the Eneström–Kakeya Theorem 7.1, the zeros of $H_n^{(\alpha)}$ are contained in the annulus

$$\xi_{n-1} = \frac{n}{n+\alpha-1} \le |z| \le \xi_0 = \frac{1}{\alpha}.$$

Since -1 is not a zero of $B_n^{(\alpha)}$ and in view of Lemma 7.1, the zeros of $B_n^{(\alpha)}$ satisfy

$$\frac{n}{n+\alpha-1} \leq \left|\frac{z}{1+z}\right| \leq \frac{1}{\alpha}. \tag{7.4}$$

Thus, we have

$$\alpha \leq \left|\frac{1}{z}+1\right| \leq \frac{n+\alpha-1}{n} \tag{7.5}$$

and for each zero z of B_n^{α}, $1/z$ lies in a ring with center $(-1,0)$ and radii smaller than 1. Thus, each zero z of B_n^{α} satisfies $\mathrm{Re}(z) < 0$. □

Propositions 6.1, 7.1, and (7.1) clearly prove the following.

Corollary 7.1 *The restricted Stenger conjecture holds for U_n for Laguerre polynomials $L_n^{(\alpha)}$ when* $0 < \alpha \leq 1$.

Remark 7.1 One can improve Corollary 7.1 by using the results in [6], where it is shown that for $n \geq 2$ and $-1 < \alpha \leq 1$, the polynomial

$$\sum_{k=0}^{n} \left(\frac{d^k}{dx^k} L_n^{(\alpha,M)}(x)\right)_{x=0} (-z)^k$$

is Hurwitz stable, where $L_n^{(\alpha,M)}$ is the Laguerre–Koornwinder polynomials defined by

$$L_n^{(\alpha,M)}(x) = \left(1 + \binom{n+\alpha}{n-1}\right) L_n^{(\alpha)}(x) + M \binom{n+\alpha}{n} \frac{d}{dx} L_n^{(\alpha)}(x), \quad M \geq 0.$$

Thus, in virtue of Proposition 6.1, the restricted Stenger conjecture for U_n holds for Laguerre–Koornwinder polynomials when $-1 < \alpha \leq 1$. The advantage of the proof of Corollary 7.1 we have provided is that it can be applied to other orthogonal polynomials using Eneström–Kakeya theorem or the many existing variants of it.

8 Restricted Stenger conjecture and blossoming

The notion of *polar form (or blossom)* (Ramshaw, 1989) associated with a polynomial P of degree n is defined as follows:

Definition 8.1 Let P be a complex polynomial of degree less or equal to n. There exists a unique multi-affine, symmetric function in n variables $p: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that for each z in $\mathbb{C}$ we have $p(z, z, \dots, z) = P(z)$. The function p is called the *polar form* or the *blossom* of the polynomial P.

Consider the following bilinear de Boor–Fix operator on the space of polynomials of degree less than or equal to n defined as follows: for two given polynomials P, Q of degree less than or equal to n, we define

$$[P, Q]_n = \sum_{k=0}^{n} \frac{(-1)^k}{n!} P^{(k)}(\tau) Q^{(n-k)}(\tau). \tag{8.1}$$

The expression $[P,Q]_n$ is independent of τ and for any complex numbers $u_1,u_2,\dots,u_n$ and any degree n polynomial P, we have

$$\big[P(z),(z-u_1)(z-u_2)\dots(z-u_n)\big]_n = p(u_1,u_2,\dots,u_n), \tag{8.2}$$

where p is the polar form of the polynomial P.

In the following, for generality, we assume that the $x_1,x_2,\dots,x_n$ in the definition of the resticted matrices U_n and V_n are arbitrary distinct real numbers in the interval $[a,b]$.

Theorem 8.1 *The eigenvalues of the restricted matrices U_n and V_n coincide with the zeros of the polynomials*

$$\Phi_n(z) = n!z^n f_n\left(\frac{a-x_1}{z},\frac{a-x_2}{z},\dots,\frac{a-x_n}{z}\right),$$

and

$$\Psi_n(z) = n!z^n f_n\left(\frac{x_1-b}{z},\frac{x_2-b}{z},\dots,\frac{x_n-b}{z}\right)$$

respectively, where f_n is the polar form of F_n; the degree n truncation of the exponential function, i.e.,

$$F_n(t) = \sum_{k=0}^{n}\frac{t^k}{k!}. \tag{8.3}$$

Proof Denote by R_n the polynomial

$$R_n(t) = \sum_{k=0}^{n}\frac{(a-t)^k}{k!}z^{n-k}.$$

We have $R_n^{(k)}(a) = (-1)^k z^{n-k}$. Thus, the polynomial Φ_n defined in (6.1) can be written as

$$\Phi_n(z) = \sum_{k=0}^{n}(-1)^k R_n^{(k)}(a)\pi_n^{(n-k)}(a) = n![R_n,\pi_n]_n.$$

Therefore, from (8.2), we have $\Phi_n(z) = n!r_n(x_1,x_2,\dots,x_n)$, where r_n is the blossom of the polynomial R_n. Moreover, from the definition of blossom, it is clear that

$$r_n(x_1,x_2,\dots,x_n) = z^n f_n\left(\frac{a-x_1}{z},\frac{a-x_2}{z},\dots,\frac{a-x_n}{z}\right).$$

This concludes the proof for U_n. Similar arguments provide a proof for V_n. □

We need the well-known coincidence theorem of Walsh and thus the need to define circular regions in the complex planes.

A *circular region* of the complex plane is defined as the image of either the closed or the open unit disc under a non-singular Moebius maps $\gamma(z)$ of the form

$$\gamma(z) = \frac{az+b}{cz+d},$$

where a, b, c, and d are complex numbers such that $ad - bc \neq 0$. Moebius maps are $1-1$ mapping of the extended plane into itself with the property of mapping every circle to either a circle or a line, and every line to either a circle or a line. A circular region is one of the following: an open disk, a closed disk, an open half-plane, a closed half-plane including ∞, the open exterior of a circle including ∞, or a closed exterior of a circle including ∞.

Theorem 8.2 (***Walsh coincidence theorem*** [9]). *Let f be a symmetric multi-affine function of n complex variables and total degree n. Let $u_1, u_2, \dots, u_n$ be n complex numbers which lie in a circular region $\mathcal{C}$. Then, there exists a ζ in $\mathcal{C}$ such that*

$$f(u_1, u_2, \dots, u_n) = f(\zeta, \zeta, \dots, \zeta).$$

From the previous theorem, we deduce the following.

Corollary 8.1 *The restricted Stenger conjecture holds for U_n and V_n, with $1 \leq n \leq 4$, for any weight function ω.*

Proof One can verify that the truncation polynomial of the exponential; F_n; for $1 \leq n \leq 4$ has all its zeros in the circular region $\mathcal{H}^- = \{z \in \mathbb{C} | \operatorname{Re}(z) < 0\}$. Let us assume that the polynomial Φ_n has a zero η with $\operatorname{Re}(\eta) \leq 0$. Then, we have

$$f_n\left(\frac{a - x_1}{\eta}, \frac{a - x_2}{\eta}, \dots, \frac{a - x_n}{\eta}\right) = 0.$$

Since the complex numbers $(a - x_i)/\eta$, $i = 1, 2, \dots, n$, belong to the circular region $\mathcal{H}^+ = \{z \in \mathbb{C} | \operatorname{Re}(z) \geq 0\}$, by Walsh coincidence theorem, there exists $\xi \in \mathcal{H}^+$ such that $f_n(\xi, \xi, \dots, \xi) = F_n(\xi) = 0$. This leads to a contradiction when $1 \leq n \leq 4$. Similar arguments can be applied to prove the theorem for V_n with $1 \leq n \leq 4$. □

Remark 8.1 In the proof of Corollary 8.1, the only used property of the zeros of the orthogonal polynomials is that they are real distinct and lie in the interval $[a, b]$. Thus, the conclusion of Corollary 8.1 remains true if we define the restricted matrices U_n and V_n with arbitrary distinct real numbers x_i, $i = 1, 2, \dots, n$, in the interval $]a, b[$ with $1 \leq n \leq 4$.

9 Restricted Stenger conjecture and collocation Runge–Kutta methods

Let $c_1, c_2, \dots, c_n$, be a set of n distinct collocation points in the interval $[0, 1]$. The collocation method of degree n based on the points c_i gives as a solution of the differential equation

$$y'(t) = f\big(t, y(t)\big), \qquad y(t_0) = y_0,$$

in each integration interval $[t_0, t_0 + h]$, as $y_1 = P(t_0 + h)$, where the collocation polynomial P is defined by

$$P(t_0) = y_0,$$
$$P'(t_0 + c_i h) = f\big(t_0 + c_i h, P(t_0 + c_i h)\big).$$

Given $b = (b_i)$, $A = (a_{i,j})$ $(i,j = 1,\dots,s)$ with b_i and $a_{i,j}$ real numbers and let $c_i = \sum_{j=1}^{s} a_{i,j}$. An s-stage Runge–Kutta method for solving the initial-value problem is given by

$$k_i = f\left(t_0 + c_i h, y_0 + h\sum_{j=1}^{s} a_{i,j}k_j\right), \quad i = 1,\dots,s,$$

$$y_1 = y_0 + h\sum_{i=1}^{s} b_i k_i.$$

In [10], it is shown that a collocation method based on $c_1, c_2, \dots, c_n$ is equivalent to the Runge–Kutta method with matrix A and coefficient b given by

$$a_{ij} = \int_0^{c_i} \ell_j^n(\tau)\,d\tau \quad \text{and} \quad b_i = \int_0^1 \ell_i^n(\tau)\,d\tau, \quad i,j = 1,2,\dots,n, \tag{9.1}$$

where $\ell_i^n(\tau)$ are the elementary Lagrange interpolation polynomials based on c_i. Moreover, the order of the collocation method coincides with the order of the underlying quadrature formula. The stability function of the Runge-Kutta method is given by

$$R(z) = \frac{\det(I - zA + zeb^T)}{\det(I - zA)}; \quad e = (1,1,\dots,1)^T,$$

and the stability region of the method is defined by

$$S = \{z \in \mathbb{C}, |R(z)| \le 1\}.$$

A Runge–Kutta is said to be A-stable if its stability region satisfies

$$\mathcal{H}^- \subset S.$$

Theorem 9.1 *The stability function of the collocation Runge–Kutta method with data* (9.1) *is given by*

$$R(z) = \frac{f_n((1-c_1)z, (1-c_2)z, \dots, (1-c_n)z)}{f_n(-c_1 z, -c_2 z, \dots, -c_n z)}, \tag{9.2}$$

where f_n is the blossom of the degree n truncation of the exponential function F_n.

Proof One should note that the matrix A is nothing but the matrix U_n with $x_i = c_i$, $i = 1,2,\dots,n$ and $[a,b] = [0,1]$ and similarly the matrix $A - eb^T$ is the matrix $-V_n$. Thus, according to Theorem 8.1, we have

$$\det(I - zA) = (-1)^n z^n \det\left(U_n - \frac{1}{z}I\right) = \frac{z^n}{n!}\Phi_n\left(\frac{1}{z}\right) = f_n(-c_1 z, -c_2 z, \dots, -c_n z).$$

Similarly, we have

$$\det\bigl(I - zA + zeb^T\bigr) = f_n\bigl((1-c_1)z, (1-c_2)z, \dots, (1-c_n)z\bigr).$$

This concludes the proof. □

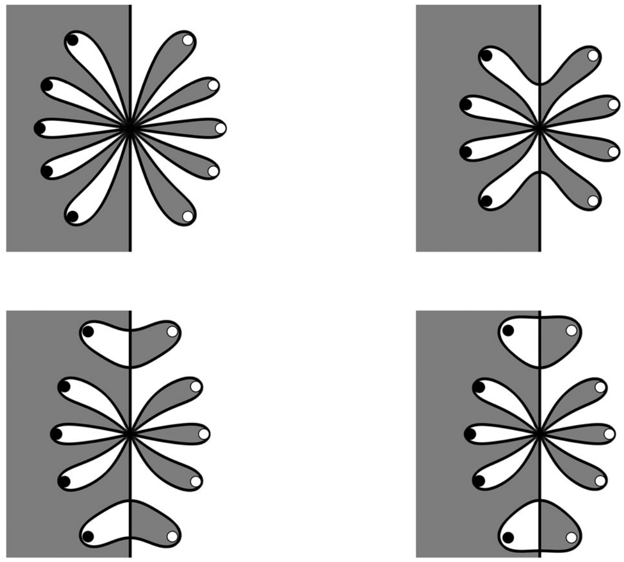

Figure 1 Order-star configuration of Legendre (top left), Chebyshev first kind (top right), Chebyshev second kind (bottom left), and Gegenbauer with $\mu = 1.4$ (bottom right) collocation Runge–Kutta methods for $n = 5$. The configurations show the *A*-stability of the methods

Corollary 9.1 *For symmetric weight functions ω over the interval $[0, 1]$, the A-stability of the collocation Runge–Kutta method with $c_1, c_2, \ldots, c_n$, as the zeros of the associated orthogonal polynomial π_n is equivalent to the validity of the restricted Stenger conjecture for ω.*

Proof If the method is A-stable, then necessarily the poles of R belongs to the complex right half-plane and therefore the restricted Stenger conjecture holds for ω. Let us now assume that the restricted Stenger conjecture holds for ω. Since the weight function is symmetric over the interval $[0, 1]$, the zeros $c_1, c_2, \ldots, c_n$, are symmetric with respect to the interval $[0, 1]$. Thus

$$f_n(-c_1 i\alpha, -c_2 i\alpha, \ldots, -c_n i\alpha) = \overline{f_n\bigl((1-c_1)i\alpha, (1-c_2)i\alpha, \ldots, (1-c_n)i\alpha\bigr)}$$

for any real number α. Therefore, $|R(z)| = 1$ for z in the imaginary axis. Moreover, since

$$\lim_{z\to\infty} R(z) = (-1)^n$$

and R has no poles in the complex left half-plane, we conclude that $\mathcal{H}^- \subset S$. This concludes the proof. □

From the previous corollary and Theorem 6.1, we conclude the following.

Corollary 9.2 *The collocation Runge-Kutta methods based on the zeros of the shifted Jacobi polynomials $\tilde{P}_n^{(\alpha,\alpha)}(x)$ with $-1 < \alpha \leq 1$ are A-stable. In particular, this is true for (see Fig. 1)*

- *Shifted Chebyshev polynomials* $\tilde{P}^{(-1/2,-1/2)}(x)$
- *Shifted Gegenbauer polynomials* $\tilde{P}^{(\mu-1/2,\mu-1/2)}(x)$; $-1/2 < \mu \leq 3/2$

Similarly, from Corollary 8.1, we conclude that any collocation Runge–Kutta method based on collocation points $c_1, c_2, \ldots, c_n$ symmetric with respect to the interval $[0,1]$, with $1 \leq n \leq 4$, is A-stable.

10 Conclusion

In this paper, we have shown the validity of the extended and restricted Stenger conjectures for a family of weight functions with specific properties. As applications, we proved the validity of the conjecture for a family of Jacobi and generalized Laguerre orthogonal polynomials. We related the restricted Stenger conjecture to the A-stability of collocation Runge-Kutta method when the collocation points are symmetric with respect to the interval $[0,1]$. This enabled us to prove the A-stability of a large family of collocation methods.

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