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On the qualitative evaluation of the variable-order coupled boundary value problems with a fractional delay

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Abstract

The logical progression from the constant order differential equations is the field of variable-order differential equations. Such equations can frequently give a more succinct description of problems in the real world. In light of this, we therefore take into account a class of coupled boundary value problems under variable-order differentiation. By utilizing the fixed-point techniques of Banach and Schauder, we investigate the existence and uniqueness of solutions to the proposed problem. Also, sufficient results are documented for the necessary needs. Furthermore, some stability results based on the ideas of Ulam, Hyers, and Rassias are elaborated upon. Ultimately, appropriate examples and in-depth analysis are presented to support our results.

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1 Introduction

Since the noninteger-order derivative of a function produces its complete spectrum, including, in a particular case, the related integer-order counterpart, the work addressing noninteger-order derivatives and integrals has attracted significant interest from researchers in various fields of technology and science over the last decades. Several important applications arose in a variety of fields, including rheology, distribution theory, financial mathematics, fluid dynamics, viscoelasticity, etc. For instance, the author in [1] used bioengineering to apply fractional calculus concepts. See [2] and [3] for a chaos neuron concept utilizing a fractional calculus for fractional physics and dynamics, respectively. Similarly, see [4] for the delay fractional-order model of HIV/AIDS and malaria. We respectively refer to [5] and [6] for the fundamental ideas, theory, and applications of fractional calculus. In addition, a lot of academics have thought about the field dealing with differential equations of noninteger-order because there is more freedom and a more thorough dynamics is produced when applying these equations in mathematical models of real-world systems. Regarding applications in viscoelasticity, physics, and dynamics, reputable results can be found in [7–9], and [10], respectively. As a result, scholars have developed a number of concepts, including qualitative theory, stability analysis, and numerical interpretation. For instance, some fundamental theories of the aforementioned subjects can be found in [11], while fundamental ideas are referred to in [12].

It is astounding how many different engineering disciplines use boundary value problems (BVPs). As a result, the qualitative elements of the aforementioned field have been thoroughly researched for arbitrary-order differential equations. Using a fixed-point (FP) methodology, the necessary qualitative consequences for a number of BVPs of fractional calculus have been examined. To ensure the existence and uniqueness of various initial and boundary value problems utilizing FP theory, we refer to [13–19]. These are some noteworthy results in this regard. Additionally, in [20–22], several authors extended the FP technique to analyze multipoint, nonlocal, and initial value problems for the existence of solutions. The ability of delay-type equations of integer, as well as fractional, order to study the relationship between a current state of a phenomenon and its past is crucial.

Delay problems can be classified into three categories: discrete, proportional, and continuous delay equations. As a result, the field of fractional-order delay differential equations (FODDEs) has received a lot of attention recently. FODDEs are crucial in the simulation of numerous biological and physical events and processes. FODDEs have many applications in a variety of subjects, including electrodynamics, quantum mechanics, cell growth, the dynamics of both linear and nonlinear systems, and astronomy. Several types of fractional pantograph differential equations for numerical analysis were studied in [23–25], using polynomials, wavelet methods, and other tools.

Fixed fractional-order derivatives have received a great deal of attention up to this point. In this regard, a number of differential operators were presented whose properties may be examined regarding the computational algorithms and the existence of uniqueness results for delay problems. We refer to [26–29] for further information. Regarding the decomposition method for treating FODDEs, see [30], and for numerical analysis of the aforementioned issues using various numerical techniques, such as wavelet, operational matrices, etc., see [31, 32]. We cite [33, 34] for a qualitative theory using the FP theory and degree method, respectively. Additionally, in the following papers [35, 36], researchers have looked into various problems related to qualitative analysis.

In 1993, another type of differential operator that treats the order as a continuous function gained popularity. Samko and his coauthor presented the aforementioned concept in [37]. The operator is more adaptable and has a greater degree of freedom when order is chosen as a variable. Furthermore, many problems still exist whose dynamics cannot be adequately investigated by means of conventional fractional-order operators. As a result, during the past 20 years, academics have used variable-order differential operators more and more to derive the existence and uniqueness (EU), stability, and numerical results. A theoretical analysis has been performed by the authors of [38] for a few situations with changing orders. Results about the extremal solutions, as well as for stability analysis and existence theory, have been developed in [39–41]. Also, the work [42] provides some helpful applications to real situations in the aforementioned field.

Typically, stability theory is required for dynamical problems. Lyapunov and Mittag-Leffler stabilities, as well as exponential kinds of stabilities, have been successfully established for common classical fractional calculus problems. Ulam–Hyers (HU) stability has recently received proper attention. For instance, [43] has proved the aforementioned stability for a class of Hilfer FODEs. Additionally, the stability of the tumor-immune FODDEs system has been extracted in [44]. The aforementioned stability has also been derived in [45] for a coupled system of fractional-order deferential equations (FODEs). Several stability and existence results using the FP approach have been examined in [46]. In [47, 48], the existence theory and stability analysis of several FODDEs have been studied. The stability indicated above has also been deduced for a family of linear FODDEs in [49].

Continuing on the same path, here we argue that coupled fractional-order differential delay equations (CFODDEs, for short) have not received adequate scrutiny. To close this gap, the following BVPs of mixed-type delays CFODEs are taken into consideration:

$$\begin{cases} {}^{C}D_{0+}^{\rho(s)}w(s) = Q(s, w(s-\kappa), w(\sigma s), v(\sigma s)), & \sigma \in (0,1), \rho(s) \in (0,1], \\ {}^{C}D_{0+}^{\theta(s)}v(s) = Q(s, v(s-\kappa), v(\sigma s), w(\sigma s)), & \sigma \in (0,1), \theta(s) \in (0,1], \\ w(0) = \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(w(\eta), v(\eta))\,d\eta + w_0, & \xi \in (0,1], \\ v(0) = \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(v(\eta), w(\eta))\,d\eta + v_0, & \lambda \in (0,1], \end{cases} \tag{1.1}$$

where $Q : K \times \mathbb{R}^3 \to \mathbb{R}$ and $h : K \times K \to \mathbb{R}$, $K = [0, S]$, while ${}^{C}D_{0+}^{\rho(s)}$, ${}^{C}D_{0+}^{\theta(s)}$ refer to the Caputo derivatives of variable orders $\rho(s)$ and $\theta(s)$, respectively. We note that [50, 51] recently reported some updated results on the difficulties involving variable order. We will adhere to the same steps that are described in these publications. Moreover, EU and stability results for the aforementioned variable FODDEs containing mixed-type delay terms are developed using the conventional fixed point approach and nonlinear functional analysis. Further, many types of Ulam–Hyers stability, including generalized Hyers–Ulam (GHU), Hyers–Ulam–Rassias (UHR), and generalized Rassias–Hyers–Ulam (GUHR) are investigated. We also support our analysis with appropriate test-case examples. Under boundary circumstances, the corresponding stability was examined for fixed fractional-order problems [52–54]. We will construct our study using FP theory [55].

2 Basic concepts

We need the following axillary results:

Definition 2.1 ([37]) Let $L(K)$ be the space of all integrable functions on K. In the Riemann–Liouville perspective, the variable-order fractional integral of $w \in L(K)$ is defined as

$$I_{+0}^{\rho(s)}w(s) = \frac{1}{\Gamma(\rho(s))}\int_0^s (s-\eta)^{\rho(\eta)-1}w(\eta)\,d\eta,$$

where $\rho : K \to (0, 1]$ is a continuous function.

Definition 2.2 ([37]) According to Caputo, the variable-order fractional derivative is defined as

$${}^{C}D_0^{\rho(s)}w(s) = \frac{1}{\Gamma(u-\rho(s))}\int_0^s (s-\eta)^{u-\rho(s)-1}w^{(u)}(\eta)\,d\eta, \quad u \in \mathbb{N}.$$

Lemma 2.3 ([6]) *Let $w \in L(K)$. For a fractional order $\rho \in (0, 1]$, the following relation is true*:

$$I_{+0}^{\rho}\,{}^{C}D_{+0}^{\rho}w(s) = w(s) - w(0).$$

Theorem 2.4 ([55]) *Assume that $\mho$ is a Banach space and $Z \neq \emptyset$ is a closed, convex subset of $\mho$. If $\Omega : Z \to Z$ is a continuous function so that $\Omega(Z)$ is a relatively compact subset of $\mho$, then Ω has at least one FP in Z.*

3 Main consequences

In this part, we discuss the EU for the considered problem (1.1). Assume that $u \in \{1, 2, 3, \dots\}$, and consider a partition of K as follows:

$$\bigl\{K_1 = [0, s_1], K_2 = (s_1, s_2], K_3 = (s_2, s_3], \dots, K_u = (s_{u-1}, s_u]\bigr\}.$$

Let $\rho, \theta : K \to (0, 1]$ be a piecewise functions such that

$$\rho(s) = \sum_{j=1}^{u} \rho_j(s) q_j(s) = \begin{cases} \rho_1, & \text{if } s \in K_1, \\ \rho_2, & \text{if } s \in K_2, \\ \vdots \\ \rho_u, & \text{if } s \in K_u, \end{cases}$$

and

$$\theta(s) = \sum_{j=1}^{u} \theta_j(s) q_j(s) = \begin{cases} \theta_1, & \text{if } s \in K_1, \\ \theta_2, & \text{if } s \in K_2, \\ \vdots \\ \theta_u, & \text{if } s \in K_u, \end{cases}$$

where $\rho_j, \theta_j \in (0, 1]$ are constants and q_j represents the indicator function of $K_j = (s_{j-1}, s_j]$ with $j = 1, 2, \dots, u$ such that $s_0 = 0$, $s_u = S$, and

$$q_j(s) = \begin{cases} 1, & \text{for } s \in K_j, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $\mho_j = C[K_j, \mathbb{R}]$ is a Banach space endowed with the norm $\|w\| = \max_{s \in K} |w(s)|$. Consequently, the left side of the problem under discussion can be expressed as (at $u = 1$)

$$\begin{cases} {}^{C}D_{0+}^{\rho(s)} w(s) = \int_0^{s_1} \frac{(s-\eta)^{-\rho_1}}{\Gamma(1-\rho_1)} w^1(\eta)\, d\eta + \cdots + \int_{s_{u-1}}^{s} \frac{(s-\eta)^{-\rho_j}}{\Gamma(1-\rho_j)} w^1(\eta)\, d\eta, \\ {}^{C}D_{0+}^{\theta(s)} \nu(s) = \int_0^{s_1} \frac{(s-\eta)^{-\theta_1}}{\Gamma(1-\theta_1)} \nu^1(\eta)\, d\eta + \cdots + \int_{s_{u-1}}^{s} \frac{(s-\eta)^{-\theta_j}}{\Gamma(1-\theta_j)} \nu^1(\eta)\, d\eta. \end{cases} \tag{3.1}$$

In light of (3.1), we can formulate our hypothetical problem as

$$\begin{cases} Q(s, w(s-\kappa), w(\sigma s), \nu(\sigma s)) \\ \quad = \int_0^{s_1} \frac{(s-\eta)^{-\rho_1}}{\Gamma(1-\rho_1)} (w^1(\eta))\, d\eta + \cdots + \int_{s_{u-1}}^{s} \frac{(s-\eta)^{-\rho_j}}{\Gamma(1-\rho_j)} (\nu^1(\eta))\, d\eta, \\ Q(s, \nu(s-\kappa), \nu(\sigma s), w(\sigma s)) \\ \quad = \int_0^{s_1} \frac{(s-\eta)^{-\theta_1}}{\Gamma(1-\theta_1)} (\nu^1(\eta))\, d\eta + \cdots + \int_{s_{u-1}}^{s} \frac{(s-\eta)^{-\theta_j}}{\Gamma(1-\theta_j)} (\nu^1(\eta))\, d\eta. \end{cases}$$

Now, assume that $w, \nu \in C([0,S],\mathbb{R})$ are such that we need to deal with

$$\begin{cases} {}^{C}D^{\rho_j}_{s_{j-1}} w(s) = Q(s, w(s-\kappa), w(\sigma s), \nu(\sigma s)), \quad s \in K_j, \\ {}^{C}D^{\theta_j}_{s_{j-1}} \nu(s) = Q(s, \nu(s-\kappa), \nu(\sigma s), w(\sigma s)), \quad s \in K_j, \\ w(s_{j-1}) = \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(w(\eta), \nu(\eta))\, d\eta + w_0, \quad \xi \in (0,1], \\ \nu(s_{j-1}) = \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(\nu(\eta), w(\eta))\, d\eta + \nu_0, \quad \lambda \in (0,1]. \end{cases}$$

Lemma 3.1 *If $p, q \in L(K_j)$, then the problem*

$$\begin{cases} D^{\rho_j}_{0+} w(s) = p(s), \\ D^{\rho_j}_{0+} \nu(s) = q(s), \\ w(s_{j-1}) = \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(\eta, w(\eta), \nu(\eta))\, d\eta + w_0, \\ \nu(s_{j-1}) = \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(\eta, \nu(\eta), w(\eta))\, d\eta + \nu_0, \end{cases} \tag{3.2}$$

has the solution below

$$\begin{cases} w(s) = w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(\eta, w(\eta), \nu(\eta))\, d\eta + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} p(\eta)\, d\eta, \\ \nu(s) = \nu_0 + \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(\eta, \nu(\eta), w(\eta))\, d\eta + \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\theta_j-1} q(\eta)\, d\eta. \end{cases}$$

Proof Utilizing Lemma 2.3, after applying the integrals $I^{\rho_j}_{s_{j-1}}$ and $I^{\theta_j}_{s_{j-1}}$ to Problem (3.2), we have

$$\begin{cases} w(s) = A + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} p(\eta)\, d\eta, \\ \nu(s) = B + \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\theta_j-1} q(\eta)\, d\eta. \end{cases} \tag{3.3}$$

Letting $s \to 0$ in (3.3) and applying the initial conditions, one has

$$\begin{cases} A = w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(\eta, w(\eta), \nu(\eta))\, d\eta, \\ B = \nu_0 + \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(\eta, \nu(\eta), w(\eta))\, d\eta. \end{cases}$$

Hence, we obtain that

$$\begin{cases} w(s) = w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(\eta, w(\eta), \nu(\eta))\, d\eta + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} p(\eta)\, d\eta, \\ \nu(s) = \nu_0 + \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(\eta, \nu(\eta), w(\eta))\, d\eta + \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\theta_j-1} q(\eta)\, d\eta. \end{cases}$$

□

Corollary 3.2 *According to Lemma 3.1, the considered system (1.1) has a solution described as*

$$\begin{cases} w(s) = w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(\eta, w(\eta), \nu(\eta))\, d\eta \\ \qquad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} Q(\eta, w(\eta-\kappa), w(\sigma\eta), \nu(\sigma\eta))\, d\eta, \\ \nu(s) = \nu_0 + \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(\eta, \nu(\eta), w(\eta))\, d\eta \\ \qquad + \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\theta_j-1} Q(\eta, \nu(\eta-\kappa), \nu(\sigma\eta), w(\sigma\eta))\, d\eta. \end{cases}$$

The following hypotheses are very important to obtain the EU of the system (1.1):

(H_1) For $\vartheta, \varpi, \varkappa, \widetilde{\vartheta}, \widetilde{\varpi}, \widetilde{\varkappa} \in \mathbb{R}$, there is $L_Q > 0$ such that

$$\left|Q(s,\vartheta,\varpi,\varkappa) - Q(s,\widetilde{\vartheta},\widetilde{\varpi},\widetilde{\varkappa})\right| \le L_Q\left(|\vartheta - \widetilde{\vartheta}| + |\varpi - \widetilde{\varpi}| + |\varkappa - \widetilde{\varkappa}|\right).$$

(H_2) For $\varpi, \varkappa, \widetilde{\varpi}, \widetilde{\varkappa} \in \mathbb{R}$, there is $L_h > 0$ such that

$$\left|h(s,\varpi,\varkappa) - h(s,\widetilde{\varpi},\widetilde{\varkappa})\right| \le L_h\left(|\varpi - \widetilde{\varpi}| + |\varkappa - \widetilde{\varkappa}|\right).$$

Theorem 3.3 *If the hypotheses* (H_1) *and* (H_2) *hold, then the considered problem* (1.1) *possesses a unique solution, provided that the following condition is true*:

$$\left(\frac{2L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{3L_Q S^{\rho_j}}{\Gamma(\rho_j+1)}\right) < 1 \quad \text{and} \quad \left(\frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{3L_Q S^{\theta_j}}{\Gamma(\theta_j+1)}\right) < 1. \tag{3.4}$$

Proof Define an operator $\Omega : \mho_j \times \mho_j \to \mho_j$ by

$$\begin{aligned}\Omega(w,v)(s) = w_0 &+ \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\big(\eta, w(\eta), v(\eta)\big)\,d\eta \\ &+ \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} Q\big(\eta, w(\eta-\kappa), w(\sigma\eta), v(\sigma\eta)\big)\,d\eta.\end{aligned}$$

It is clear that finding a solution to the problem (1.1) is equivalent to finding the coupled FP of the operator Ω. For this, we need to show that Ω is a condensing operator.

Let $w, v, \widetilde{w}, \widetilde{v} \in \mho_j$. Using $(S - s_{j-1})^{\rho_j} \le S^{\rho}$, we have

$$\begin{aligned}&\left\|\Omega(w,v) - \Omega(\widetilde{w},\widetilde{v})\right\| \\ &= \max_{s\in K}\left|\left\{w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\big(\eta, w(\eta), v(\eta)\big)\,d\eta \right.\right. \\ &\quad \left. + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} Q\big(\eta, w(\eta-\kappa), w(\sigma\eta), v(\sigma\eta)\big)\,d\eta\right\} \\ &\quad - \left\{w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\big(\eta, \widetilde{w}(\eta), \widetilde{v}(\eta)\big)\,d\eta \right. \\ &\quad \left.\left. + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} Q\big(\eta, \widetilde{w}(\eta-\kappa), \widetilde{w}(\sigma\eta), \widetilde{v}(\sigma\eta)\big)\,d\eta\right\}\right| \\ &\le \max_{s\in K}\left\{\int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} \left|h\big(\eta, w(\eta), v(\eta)\big) - h\big(\eta, \widetilde{w}(\eta), \widetilde{v}(\eta)\big)\right|\,d\eta \right. \\ &\quad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} \left|Q\big(\eta, w(\eta-\kappa), w(\sigma\eta), v(\sigma\eta)\big)\right. \\ &\quad \left.\left. - Q\big(\eta, \widetilde{w}(\eta-\kappa), \widetilde{w}(\sigma\eta), \widetilde{v}(\sigma\eta)\big)\right|\,d\eta\right\}.\end{aligned}$$

Applying (H_1) and (H_2), one has

$$\begin{aligned}
&\left\|\Omega(w,v)-\Omega(\widetilde{w},\widetilde{v})\right\| \\
&\leq \frac{L_h}{\Gamma(\xi)}\int_0^S (S-\eta)^{\xi-1}\,d\eta \times \left(\|w-\widetilde{w}\|+\|v-\widetilde{v}\|\right) \\
&\quad + \frac{1}{\Gamma(\rho_j)}\int_{s_{j-1}}^S (S-\eta)^{\rho_j-1}\,d\eta \times \left(2\|w-\widetilde{w}\|+\|v-\widetilde{v}\|\right) \\
&\leq \frac{L_hS^{\xi}}{\Gamma(\xi+1)}\left(\|w-\widetilde{w}\|+\|v-\widetilde{v}\|\right)+\frac{L_Q(S-s_{j-1})^{\rho_j}}{\Gamma(\rho_j+1)}\left(2\|w-\widetilde{w}\|+\|v-\widetilde{v}\|\right) \qquad (3.5)\\
&\leq \frac{L_hS^{\xi}}{\Gamma(\xi+1)}\left(\|w-\widetilde{w}\|+\|v-\widetilde{v}\|\right)+\frac{L_QS^{\rho_j}}{\Gamma(\rho_j+1)}\left(2\|w-\widetilde{w}\|+\|v-\widetilde{v}\|\right) \\
&= \left(\frac{L_hS^{\xi}}{\Gamma(\xi+1)}+\frac{2L_QS^{\rho_j}}{\Gamma(\rho_j+1)}\right)\|w-\widetilde{w}\|+\left(\frac{L_hS^{\xi}}{\Gamma(\xi+1)}+\frac{L_QS^{\rho_j}}{\Gamma(\rho_j+1)}\right)\|v-\widetilde{v}\|. \\
&= \left(\frac{2L_hS^{\xi}}{\Gamma(\xi+1)}+\frac{3L_QS^{\rho_j}}{\Gamma(\rho_j+1)}\right)\left\|(w,v)-(\widetilde{w},\widetilde{v})\right\|.
\end{aligned}$$

Similarly, using $(S-s_{j-1})^{\theta_j}\leq S^{\theta}$, we get

$$\left\|\Omega(v,w)-\Omega(\widetilde{v},\widetilde{w})\right\|\leq\left(\frac{2L_hS^{\lambda}}{\Gamma(\lambda+1)}+\frac{3L_QS^{\theta_j}}{\Gamma(\theta_j+1)}\right)\left\|(v,w)-(\widetilde{v},\widetilde{w})\right\|. \qquad (3.6)$$

It follows from (3.4), (3.5), and (3.6) that Ω is a condensing operator, and this finishes the proof. □

Now, we prove that the considered problem (1.1) has at leas one solution on bounded sets. For this, we suggest the following assumptions.

(H_3) For $M_Q>0$, we have

$$\left|Q(s,\vartheta,\varpi,\varkappa)\right|\leq M_Q\left(|\vartheta|+|\varpi|+|\varkappa|\right),\quad \text{for } \vartheta,\varpi,\varkappa\in\mathbb{R}.$$

(H_4) If $M_h>0$, then

$$\left|h(s,\vartheta,\varpi)\right|\leq M_h\left(|\vartheta|+|\varpi|\right),\quad \text{for } \vartheta,\varpi\in\mathbb{R}.$$

Theorem 3.4 *Based on the conditions* (H_1)–(H_4), *the suggested problem* (1.1) *has at least one solution in the bounded set* $Z\times Z=\{(w,v)\in\mho_j\times\mho_j:\|w\|\leq\ell$ *and* $\|v\|\leq\ell\}$, *with* $\beta=\frac{2L_hS^{\xi}}{\Gamma(\xi+1)}+\frac{3L_QS^{\rho_j}}{\Gamma(\rho_j+1)}$ *and* $\beta^*=\frac{2L_hS^{\lambda}}{\Gamma(\lambda+1)}+\frac{3L_QS^{\theta_j}}{\Gamma(\theta_j+1)}$.

Proof We split the proof into the following steps:

(I) Show that $\Omega:Z\times Z\to Z$ is bounded. Assuming that $(w,v)\in Z\times Z$, we have

$$\begin{aligned}
\left\|\Omega(w,v)\right\| = \max_{s\in K}\Bigg|&w_0+\int_0^S\frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)}h\big(\eta,w(\eta),v(\eta)\big)\,d\eta \\
&+\frac{1}{\Gamma(\rho_j)}\int_{s_{j-1}}^s(s-\eta)^{\rho_j-1}Q\big(\eta,w(\eta-\kappa),w(\sigma\eta),v(\sigma\eta)\big)\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\le w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} \big|h\big(\eta, w(\eta), v(\eta)\big)\big| d\eta \\
&\quad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} \big|Q\big(\eta, w(\eta-\kappa), w(\sigma\eta), v(\sigma\eta)\big)\big| \, d\eta \\
&\le w_0 + \left(\frac{2L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{3L_Q S^{\rho_j}}{\Gamma(\rho_j+1)}\right)\ell \le \ell.
\end{aligned}$$

Analogously, one can write

$$\big\|\Omega(v,w)\big\| \le v_0 + \left(\frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{3L_Q S^{\theta_j}}{\Gamma(\theta_j+1)}\right)\ell \le \ell.$$

Hence, $(\Omega(w,v), \Omega(v,w)) \in Z \times Z$, therefore Ω maps a bounded set into a bounded set in $\mho_j$.

(II) Prove that Ω is continuous. Assuming that sequences (w_u, v_u) converge to (w,v) in $Z \times Z$, for each $s \in K$, we get

$$\begin{aligned}
&\big\|\Omega(w_u,v_u) - \Omega(w,v)\big\| \\
&\quad = \max_{s\in K_j} \bigg|\bigg\{ w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\big(\eta, w_u(\eta), v_u(\eta)\big)\, d\eta \\
&\qquad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} Q\big(\eta, w_u(\eta-\kappa), w_u(\sigma\eta), v_u(\sigma\eta)\big)\, d\eta \bigg\} \\
&\qquad - \bigg\{ w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\big(\eta, w(\eta), v(\eta)\big)\, d\eta \\
&\qquad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} Q\big(\eta, w(\eta-\kappa), w(\sigma\eta), v(\sigma\eta)\big)\, d\eta \bigg\}\bigg| \\
&\quad \le \max_{s\in K_j} \bigg\{ \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} \big|h\big(\eta, w_u(\eta), v_u(\eta)\big) - h\big(\eta, w(\eta), v(\eta)\big)\big|\, d\eta \\
&\qquad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1} \big|Q\big(\eta, w_u(\eta-\kappa), w_u(\sigma\eta), v_u(\sigma\eta)\big) \\
&\qquad - Q\big(\eta, w(\eta-\kappa), w(\sigma\eta), v(\sigma\eta)\big)\big|\, d\eta \bigg\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\big\|\Omega(w_u,v_u) - \Omega(w,v)\big\| \\
&\quad \le \max_{s\in K_j} \bigg\{ M_h \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)}\, d\eta \times \big(|w_u - w| + |v_u - v|\big) \\
&\qquad + \frac{M_Q}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j-1}\, d\eta \times \big(2|w_u - w| + |v_u - v|\big) \bigg\} \\
&\quad \le \max_{s\in K_j} \bigg\{ M_h \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)}\, d\eta + \frac{2M_Q}{\Gamma(\rho_j)} (s-\eta)^{\rho_j-1}\, d\eta \bigg\} \big(\|w_u - w\| + \|v_u - v\|\big).
\end{aligned}$$

Similarly, one can write

$$\begin{aligned}
&\bigl\| \Omega(v_u, w_u) - \Omega(v, w) \bigr\| \\
&\quad \le \max_{s\in K_j} \biggl\{ M_h \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)}\, d\eta + \frac{2M_Q}{\Gamma(\theta_j)} (s-\eta)^{\theta_j - 1}\, d\eta \biggr\} \bigl(\|w_u - w\| + \|v_u - v\| \bigr).
\end{aligned}$$

Since $w_u \to w$, $v \to v_u$ as $u \to \infty$, and Ω is bounded, we have $\|\Omega(w_u, v_u) - \Omega(w, v)\| \to 0$ and $\|\Omega(v_u, w_u) - \Omega(v, w)\| \to 0$ as $u \to 0$. This proves that Ω is continuous.

(III) Claim that Ω is completely continuous. If $s_1, s_2 \in K$, $s_1 < s_2$, we get

$$\begin{aligned}
&\bigl| \Omega(w, v)(s_1) - \Omega(w, v)(s_2) \bigr| \\
&\quad = \Biggl| \biggl\{ w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\bigl(\eta, w(\eta), v(\eta)\bigr)\, d\eta \\
&\qquad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s_1} (s_1 - \eta)^{\rho_j - 1} Q\bigl(\eta, w(\eta - \kappa), w(\sigma\eta), v(\sigma\eta)\bigr)\, d\eta \biggr\} \\
&\qquad - \biggl\{ w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\bigl(\eta, w(\eta), v(\eta)\bigr)\, d\eta \\
&\qquad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s_2} (s_2 - \eta)^{\rho_j - 1} Q\bigl(\eta, w(\eta - \kappa), w(\sigma\eta), v(\sigma\eta)\bigr)\, d\eta \biggr\} \Biggr| \\
&\quad \le \frac{1}{\Gamma(\rho_j)} \biggl\{ \int_{s_{j-1}}^{s_1} (s_1 - \eta)^{\rho_j - 1}\, d\eta - \int_{s_{j-1}}^{s_2} (s_2 - \eta)^{\rho_j - 1}\, d\eta \biggr\} 2M_Q \ell.
\end{aligned}$$

By the same method, we have

$$\begin{aligned}
&\bigl| \Omega(v, w)(s_1) - \Omega(v, w)(s_2) \bigr| \\
&\quad \le \frac{1}{\Gamma(\theta_j)} \biggl\{ \int_{s_{j-1}}^{s} (s_1 - \eta)^{\theta_j - 1}\, d\eta - \int_{s_{j-1}}^{s_2} (s_2 - \eta)^{\theta_j - 1}\, d\eta \biggr\} 2M_Q \ell.
\end{aligned}$$

Since $s_1 \to s_2$, $|\Omega(w, v)(s_1) - \Omega(w, v)(s_2)| \to 0$ and $|\Omega(v, w)(s_1) - \Omega(v, w)(s_2)| \to 0$. From steps (I) and (II), Ω is bounded and continuous. Therefore, $\|\Omega(w, v)(s_1) - \Omega(w, v)(s_2)\| \to 0$ and $\|\Omega(v, w)(s_1) - \Omega(v, w)(s_2)\| \to 0$. Hence, Ω is completely continuous on $\mho_j$.

(IV) We shall show that the set $T = \{(w, v) \in \mho_j \times \mho_j : w = \alpha\Omega(w, v) \text{ and } v = \alpha^*\Omega(v, w)\}$ for some $\alpha, \alpha^* \in [0, 1]$ is bounded for a priori bounds. For any $w, v \in T$, we get

$$\begin{aligned}
\|w\| &= \max_{s_j\in K_j} \bigl| \alpha\Omega(w, v) \bigr| \\
&\le \max_{s\in K_j} \biggl| (w_0, v_0) + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\bigl(\eta, w(\eta), v(\eta)\bigr)\, d\eta \\
&\quad + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^{s} (s-\eta)^{\rho_j - 1} Q\bigl(\eta, w(\eta - \kappa), w(\sigma\eta), v(\sigma\eta)\bigr) \biggr| \\
&\le (w_0, v_0) + \biggl\{ \frac{2L_h S^{\xi}}{\Gamma(\xi + 1)} + \frac{3L_Q S^{\rho_j}}{\Gamma(\rho_j + 1)} \biggr\} \ell \le \ell.
\end{aligned} \tag{3.7}$$

Similarly, one has

$$\|v\| \leq (v_0, w_0) + \left\{ \frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{3L_Q S^{\theta_j}}{\Gamma(\theta_j+1)} \right\} \ell \leq \ell. \tag{3.8}$$

From (3.7) and (3.8), we see that

$$\ell \geq \frac{(w_0, v_0)}{1-\beta} \quad \text{and} \quad \ell \geq \frac{(v_0, w_0)}{1-\beta^*},$$

respectively, which leads to $\|w\| \leq \ell$ and $\|v\| \leq \ell$. Therefore, according to Theorem 2.4, the given problem (1.1) has at least one solution. □

4 Stability results

Here, we provide some precise predictions for the stability analysis of the presented problem (1.1). The following results are used to start this section:

Definition 4.1 The pair $(w(s), v(s))$ of the given system (1.1) is UH stable if, for every $\delta \geq 0$, there exists a constant $W_Q > 0$ such that for each solution $(w(s), v(s)) \in \mho_j \times \mho_j$ to the following inequalities:

$$\begin{cases} |{}^C D^{\rho_j}_{s_{j-1}} w(s) - Q(s, w(s-\kappa), w(\sigma s), v(\sigma s))| \leq \delta, \\ |{}^C D^{\theta_j}_{s_{j-1}} v(s) - Q(s, v(s-\kappa), v(\sigma s), w(\sigma s))| \leq \delta, \end{cases}$$

for all $s \in K_j$ and for a unique solution $(\widetilde{w}(s), \widetilde{v}(s)) \in \mho_j \times \mho_j$ of (1.1), one has

$$\left\| (w, \widetilde{w}) - (v, \widetilde{v}) \right\| \leq \delta W_Q.$$

Moreover, if there exists a function $\varphi : (0, \infty) \to (0, \infty)$ with $\varphi(0) = 0$ such that

$$\left\| (w, \widetilde{w}) - (v, \widetilde{v}) \right\| \leq W_Q \varphi(\delta),$$

then the solution is called GUH stable.

Remark 4.2 We say that the pair $(w(s), v(s)) \in \mho_j \times \mho_j$ is a solution to problem (1.1) if and only if there exist $U, V \in C(K_j)$, for every $s \in K_j$, such that

(i) $|U(s)| \leq \delta$ and $|V(s)| \leq \delta$;
(ii) ${}^C D^{\rho_j}_{s_{j-1}} w(s) - Q(s, w(s-\kappa), w(\sigma s), v(\sigma s)) - U(s) = 0$;
(iii) ${}^C D^{\theta_j}_{s_{j-1}} v(s) - Q(s, v(s-\kappa), v(\sigma s), w(\sigma s)) - V(s) = 0$.

Definition 4.3 We say that the solution $(w(s), v(s))$ of the considered problem (1.1) is UHR stable for the continuous function $O \in \mho_j$, if there is a constant $W_Q > 0$ such that

$$\begin{cases} |{}^C D^{\rho_j}_{s_{j-1}} w(s) - Q(s, w(s-\kappa), w(\sigma s), v(\sigma s))| \leq O(s)\delta, \\ |{}^C D^{\theta_j}_{s_{j-1}} v(s) - Q(s, v(s-\kappa), v(\sigma s), w(\sigma s))| \leq O(s)\delta, \end{cases}$$

for all $s \in K_j$ and for a unique solution $(\widetilde{w}(s),\widetilde{v}(s)) \in \mho_j \times \mho_j$ of (1.1), we have

$$\left\|(w,\widetilde{w})-(v,\widetilde{v})\right\| \le \delta W_Q O(s).$$

Further, if there exists a function $\varphi:(0,\infty)\to(0,\infty)$ with $\varphi(0)=0$ such that

$$\left\|(w,\widetilde{w})-(v,\widetilde{v})\right\| \le W_Q O(s)\varphi(\delta),$$

then the solution is called GUHR stable.

Remark 4.4 For functions $U, V \in C(K_j)$, and for every $s \in K_j$, we have

(i) $|U(s)| \le \delta O(s)$ and $|V(s)| \le \delta O(s)$;

(ii) ${}^{C}D^{\rho_j}_{s_{j-1}} w(s) - Q(s, w(s-\kappa), w(\sigma s), v(\sigma s)) - U(s) = 0$;

(iii) ${}^{C}D^{\theta_j}_{s_{j-1}} v(s) - Q(s, v(s-\kappa), v(\sigma s), w(\sigma s)) - V(s) = 0$.

Lemma 4.5 *According to Remark 4.2 and Lemma 3.1, the solution of the perturbed system*

$$\begin{cases} D^{\rho_j}_{s_{j-1}} w(s) = Q(s, w(s-\kappa), w(\sigma s), v(\sigma s)) + U(s), & \rho_j \in (0,1], \\ D^{\theta_j}_{s_{j-1}} v(s) = Q(s, v(s-\kappa), v(\sigma s), w(\sigma s)) + V(s), & \theta_j \in (0,1], \\ w(0) = \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(w(\eta), v(\eta))\, d\eta + w_0, & \xi \in (0,1], \\ v(0) = \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h(v(\eta), w(\eta))\, d\eta + v_0, & \lambda \in (0,1], \end{cases} \tag{4.1}$$

satisfies the following inequality:

$$\left|w(s) - \Omega(w,v)(s)\right| \le \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)} \quad \textit{and} \quad \left|v(s) - \Omega(v,w)(s)\right| \le \frac{\delta S^{\theta_j}}{\Gamma(\theta_j+1)},$$

for all $s \in K$ with

$$\Omega(w,v)(s) = (w_0, v_0) + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\bigl(\eta, w(\eta), v(\eta)\bigr)\, d\eta + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} p(\eta)\, d\eta.$$

Proof Applying Lemma 3.1, problem (4.1) implies that

$$w(s) = w_0 + \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h\bigl(\eta, w(\eta), v(\eta)\bigr)\, d\eta + \frac{1}{\Gamma(\rho_j)} \int_0^s (s-\eta)^{\rho_j-1} U(\eta)\, d\eta + \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} U(\eta)\, d\eta,$$

and

$$v(s) = v_0 + \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h\bigl(\eta, v(\eta), w(\eta)\bigr)\, d\eta + \frac{1}{\Gamma(\theta_j)} \int_0^s (s-\eta)^{\theta_j-1} V(\eta)\, d\eta + \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\theta_j-1} V(\eta)\, d\eta,$$

which implies that

$$\left|w(s)-\Omega(w,v)(s)\right| \leq \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)} \quad \text{and} \quad \left|v(s)-\Omega(v,w)(s)\right| \leq \frac{\delta S^{\theta_j}}{\Gamma(\theta_j+1)}. \qquad \square$$

Theorem 4.6 *Assuming that* (H_1)*–*(H_4) *hold, the unique solution to the relevant problem* (1.1) *is UH and GUH stable, provided that* $1 \neq \frac{\Theta\Theta^*}{(1-\Theta)(1-\Theta^*)}$.

Proof Thanks to Lemma 4.5, if w^* and v^* are solutions to (1.1), then one has

$$\begin{aligned}
\left|w(s)-w^*(s)\right| &= \left|w(s)-\Omega\left(w^*,v^*\right)(s)\right| \\
&= \left|w(s)-\Omega(w,v)(s)+\Omega(w,v)(s)-\Omega\left(w^*,v^*\right)(s)\right| \\
&\leq \left|w(s)-\Omega(w,v)(s)\right| + \left|\Omega(w,v)(s)-\Omega\left(w^*,v^*\right)(s)\right| \\
&\leq \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)} + \left(\frac{L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{2L_Q S^{\rho_j}}{\Gamma(\rho_j+1)}\right)\left\|w-w^*\right\| \\
&\quad + \left(\frac{L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{L_Q S^{\rho_j}}{\Gamma(\rho_j+1)}\right)\left\|v-v^*\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left(1-\left(\frac{L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{2L_Q S^{\rho_j}}{\Gamma(\rho_j+1)}\right)\right)\left\|w-w^*\right\| \\
&\quad \leq \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)} + \left(\frac{L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{L_Q S^{\rho_j}}{\Gamma(\rho_j+1)}\right)\left\|v-v^*\right\|.
\end{aligned}$$

Hence,

$$\left\|w-w^*\right\| - \frac{\Theta}{1-\Theta}\left\|v-v^*\right\| \leq \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)(1-\Theta)} = \delta\Delta. \tag{4.2}$$

Obviously, one has

$$\left\|v-v^*\right\| - \frac{\Theta^*}{1-\Theta^*}\left\|w-w^*\right\| \leq \frac{\delta S^{\theta_j}}{\Gamma(\theta_j+1)(1-\Theta^*)} = \delta\Delta^*, \tag{4.3}$$

where

$$\begin{aligned}
&\Theta = \left(\frac{L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{2L_Q S^{\rho_j}}{\Gamma(\rho_j+1)}\right), \qquad \Theta^* = \left(\frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{L_Q S^{\theta_j}}{\Gamma(\theta_j+1)}\right), \\
&\Delta = \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)(1-\Theta)}, \quad \text{and} \quad \Delta^* = \frac{\delta S^{\theta_j}}{\Gamma(\theta_j+1)(1-\Theta^*)},
\end{aligned}$$

respectively. The inequalities (4.2) and (4.3) can be written as

$$\begin{bmatrix} 1 & -\frac{\Theta}{1-\Theta} \\ -\frac{\Theta^*}{1-\Theta^*} & 1 \end{bmatrix} \begin{bmatrix} \|w-w^*\| \\ \|v-v^*\| \end{bmatrix} \leq \begin{bmatrix} \delta\Delta \\ \delta\Delta^* \end{bmatrix},$$

which leads to

$$\begin{bmatrix} \|w-w^*\| \\ \|v-v^*\| \end{bmatrix} \leq \begin{bmatrix} \frac{1}{E} & \frac{\Theta}{1-\Theta}\frac{1}{E} \\ \frac{\Theta^*}{1-\Theta^*}\frac{1}{E} & \frac{1}{E} \end{bmatrix} \begin{bmatrix} \delta\Delta \\ \delta\Delta^* \end{bmatrix}, \tag{4.4}$$

where $E = 1 - \frac{\Theta\Theta^*}{(1-\Theta)(1-\Theta^*)}$. Based on the system (4.4), one can write

$$\|w-w^*\| \leq \frac{\delta\Delta}{E} + \frac{\delta\Delta^*\Theta}{(1-\Theta)E}$$

and

$$\|v-v^*\| \leq \frac{\delta\Delta\Theta^*}{(1-\Theta^*)E} + \frac{\delta\Delta^*}{E}.$$

It follows that

$$\|w-w^*\| + \|v-v^*\| \leq \left(\frac{\Delta+\Delta^*}{E} + \frac{\Delta^*\Theta}{(1-\Theta)E} + \frac{\Delta\Theta^*}{(1-\Theta^*)E}\right)\delta.$$

Setting

$$W_Q = \left(\frac{\Delta+\Delta^*}{E} + \frac{\Delta^*\Theta}{(1-\Theta)E} + \frac{\Delta\Theta^*}{(1-\Theta^*)E}\right)$$

implies that

$$\|(w,w^*) - (v,v^*)\| \leq W_Q\delta.$$

Hence, the solution of (1.1) is UH stable. Moreover, for a function $\varphi(\delta) = \delta$, so that $\varphi(0) = 0$, we get

$$\|(w,w^*) - (v,v^*)\| \leq W_Q\varphi(\delta),$$

which guarantees the GUH stability (1.1). □

Lemma 4.7 *Under the conditions in Remark* 4.4, *for the solution of* (4.1), *the following is true*:

$$|w(s) - \Omega(w,v)(s)| \leq \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)}O(s) \quad \textit{and} \quad |v(s) - \Omega(v,w)(s)| \leq \frac{\delta S^{\theta_j}}{\Gamma(\theta_j+1)}.$$

Proof Using Remark 4.4, system (4.1) yields

$$\begin{aligned} w(s) = w_0 &+ \int_0^S \frac{(S-\eta)^{\xi-1}}{\Gamma(\xi)} h(\eta, w(\eta), v(\eta))\,d\eta \\ &+ \frac{1}{\Gamma(\rho_j)}\int_0^s (s-\eta)^{\rho_j-1}U(\eta)\,d\eta + \frac{1}{\Gamma(\rho_j)}\int_{s_{j-1}}^s (s-\eta)^{\rho_j-1}U(\eta)\,d\eta \end{aligned}$$

and

$$\nu(s) = \nu_0 + \int_0^S \frac{(S-\eta)^{\lambda-1}}{\Gamma(\lambda)} h\big(\eta, \nu(\eta), w(\eta)\big)\, d\eta$$
$$+ \frac{1}{\Gamma(\theta_j)} \int_0^s (s-\eta)^{\theta_j-1} V(\eta)\, d\eta + \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\theta_j-1} V(\eta)\, d\eta,$$

which both lead to

$$\begin{aligned}\big|w(s) - \Omega(w,\nu)(s)\big| &\le \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} \big|U(\eta)\big|\, d\eta \\ &\le \frac{1}{\Gamma(\rho_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} \delta O(s)\, d\eta \\ &\le \frac{\delta S^{\rho_j}}{\Gamma(\rho_j+1)} O(s)\end{aligned}$$

and

$$\begin{aligned}\big|\nu(s) - \Omega(\nu,w)(s)\big| &\le \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} \big|V(\eta)\big|\, d\eta \\ &\le \frac{1}{\Gamma(\theta_j)} \int_{s_{j-1}}^s (s-\eta)^{\rho_j-1} \delta V(s)\, d\eta \\ &\le \frac{\delta S^{\theta_j}}{\Gamma(\theta_j+1)} V(s).\end{aligned}$$

This completes the proof. □

Theorem 4.8 *Assume that* (H_1) *and* (H_2) *are true, then the solution of* (1.1) *is UHR and GUHR stable if* $\frac{\Theta\Theta^*}{(1-\Theta)(1-\Theta^*)} \neq 1$.

Proof Using Lemma 4.7 and applying the same steps as the proof of Theorem 4.6, we arrive at the desired result. □

5 Illustrative examples

The following examples are addressed in order to support our main results:

Example 5.1 Consider the following CFODDEs:

$$\begin{cases} {}^C D_{0+}^{\rho(s)} w(s) = e^{-s}\big(\frac{|w(s-0.35)|}{15+|w(s-0.35)|} + \frac{|w(\frac{s}{4})|}{15+|w(\frac{s}{4})|} + \frac{|\nu(\frac{s}{4})|}{15+|\nu(\frac{s}{4})|}\big), & s \in [0,2], \\ {}^C D_{0+}^{\theta(s)} \nu(s) = e^{-s}\big(\frac{|\nu(s-0.35)|}{15+|\nu(s-0.35)|} + \frac{|\nu(\frac{s}{4})|}{15+|\nu(\frac{s}{4})|} + \frac{|w(\frac{s}{4})|}{15+|w(\frac{s}{4})|}\big), \\ w(0) = \frac{1}{\Gamma(0.7)} \int_0^1 (1-\eta)^{-0.3} \big(\frac{|w(\eta)-\nu(\eta)|}{12+|w(\eta)-\nu(\eta)|}\big)\, d\eta + 0.037, \\ \nu(0) = \frac{1}{\Gamma(0.8)} \int_0^1 (1-\eta)^{-0.4} \big(\frac{|\nu(\eta)-w(\eta)|}{12+|\nu(\eta)-w(\eta)|}\big)\, d\eta + 0.043. \end{cases}$$

It is clear that for $S = 2$, $\xi = 0.7$, $\lambda = 0.8$, and $\sigma = 0.25$, we have

$$\rho(s) = \begin{cases} 0.75, & s \in [0,1], \\ 0.5, & s \in (1,2], \end{cases} \quad \text{and} \quad \theta(s) = \begin{cases} 0.85, & s \in [0,1], \\ 0.6, & s \in (1,2]. \end{cases}$$

Obviously, for $j = 1, 2$, $w_0 = 0.037$, and $v_0 = 0.043$,

$$Q\big(s, w(s-\kappa), w(\sigma s), v(\sigma s)\big) = e^{-s}\left(\frac{w(s-0.35)}{15+|w(s-0.35)|} + \frac{w(\frac{s}{4})}{15+|w(\frac{s}{4})|} + \frac{v(\frac{s}{4})}{15+|v(\frac{s}{4})|}\right),$$
$$Q\big(s, v(s-\kappa), v(\sigma s), w(\sigma s)\big) = e^{-s}\left(\frac{v(s-0.35)}{15+|v(s-0.35)|} + \frac{v(\frac{s}{4})}{15+|v(\frac{s}{4})|} + \frac{w(\frac{s}{3})}{15+|w(\frac{s}{3})|}\right),$$
$$h\big(w(\eta), v(\eta)\big) = \left(\frac{|w(\eta)-v(\eta)|}{12+|w(\eta)-v(\eta)|}\right), \qquad h\big(v(\eta), w(\eta)\big) = \left(\frac{|v(\eta)-w(\eta)|}{12+|v(\eta)-w(\eta)|}\right).$$

Considering $w, w^* \in \mho_j$, $j = 1, 2$, we have

$$\begin{aligned}
&\big|Q\big(s, w(s-\kappa), w(\sigma s), v(\sigma s)\big) - Q\big(s, w^*(s-\kappa), w^*(\sigma s), v^*(\sigma s)\big)\big| \\
&\quad\le \big|e^{-s}\big|\left|\frac{w(s-0.35)}{15+|w(s-0.35)|} - \frac{w^*(s-0.35)}{15+|w^*(s-0.35)|}\right| \\
&\qquad + \left|\frac{w(\frac{s}{4})}{15+|w(\frac{s}{4})|} - \frac{w^*(\frac{s}{4})}{1+|w^*(\frac{s}{4})|}\right| + \left|\frac{v(\frac{s}{4})}{15+|v(\frac{s}{4})|} - \frac{v^*(\frac{s}{4})}{1+|v^*(\frac{s}{4})|}\right| \\
&\quad\le \frac{1}{15}\big|w(s-0.35) - w^*(s-0.35)\big| + \left|w\left(\frac{s}{4}\right) - w^*\left(\frac{s}{4}\right)\right| + \left|v\left(\frac{s}{4}\right) - v^*\left(\frac{s}{4}\right)\right|
\end{aligned}$$

and

$$\begin{aligned}
\big|h\big(w(\eta), v(\eta)\big) - h\big(w^*(\eta), v^*(\eta)\big)\big| &= \left|\frac{|w(\eta)-v(\eta)|}{12+|w(\eta)-v(\eta)|} - \frac{|w^*(\eta)-v^*(\eta)|}{12+|w^*(\eta)-v^*(\eta)|}\right| \\
&\le \frac{1}{12}\big|\big(w(\eta)-v(\eta)\big) - \big(w^*(\eta)-v^*(\eta)\big)\big| \\
&\le \frac{1}{12}\big(\big|w(\eta)-w^*(\eta)\big| + \big|v(\eta)-v^*(\eta)\big|\big).
\end{aligned}$$

In the first case ($j = 1$, $S = 1$), one has

$$\begin{cases}
{}^{C}D_0^{\rho_1} w(s) = e^{-s}\big(\frac{|w(s-0.35)|}{15+|w(s-0.35)|} + \frac{|w(\frac{s}{4})|}{15+|w(\frac{s}{4})|} + \frac{|v(\frac{s}{4})|}{15+|v(\frac{s}{4})|}\big), \quad s \in [0,1], \\
{}^{C}D_0^{\theta_1} v(s) = e^{-s}\big(\frac{|v(s-0.35)|}{15+|v(s-0.35)|} + \frac{|v(\frac{s}{4})|}{15+|v(\frac{s}{4})|} + \frac{|w(\frac{s}{4})|}{15+|w(\frac{s}{4})|}\big), \\
w(0) = \frac{1}{\Gamma(0.7)}\int_0^1 (1-\eta)^{-0.3}\big(\frac{|w(\eta)-v(\eta)|}{12+|w(\eta)-v(\eta)|}\big)\,d\eta + 0.037, \\
v(0) = \frac{1}{\Gamma(0.8)}\int_0^1 (1-\eta)^{-0.4}\big(\frac{|v(\eta)-w(\eta)|}{12+|v(\eta)-w(\eta)|}\big)\,d\eta + 0.043.
\end{cases} \tag{5.1}$$

Hence, $L_Q = \frac{1}{15}$ and $L_h = \frac{1}{12}$. Therefore, the hypotheses (H_1) and (H_2) are satisfied. Also, we have

$$\frac{2L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{3L_Q S^{\rho_1}}{\Gamma(\rho_1+1)} \approx 0.40104 < 1, \qquad \frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{3L_Q S^{\theta_1}}{\Gamma(\theta_1+1)} \approx 0.39045 < 1.$$

Consequently, the system (5.1) possesses a unique solution according to Theorem 3.3. Further, we have

$$\frac{\Theta\Theta^*}{(1-\Theta)(1-\Theta^*)} \approx \frac{(0.23679)(0.24945)}{(1-0.23679)(1-0.24945)} \approx 0.10312 \neq 1.$$

Hence, the solution of the considered problem (5.1) is UH and GUH stable. If we consider the function $\varphi(s) = \frac{s}{3}$, $s \in [0,1]$, we conclude that the stated problem (5.1) is UHR and GHR stable.

In the second case ($j = 2$, $S = 2$), we get

$$\begin{cases} {}^{C}D_{0}^{\rho_2} w(s) = e^{-s}(\frac{|w(s-0.35)|}{15+|w(s-0.35)|} + \frac{|w(\frac{s}{4})|}{15+|w(\frac{s}{4})|} + \frac{|\nu(\frac{s}{4})|}{15+|\nu(\frac{s}{4})|}), \quad s \in]1,2], \\ {}^{C}D_{0}^{\theta_2} \nu(s) = e^{-s}(\frac{|\nu(s-0.35)|}{15+|\nu(s-0.35)|} + \frac{|\nu(\frac{s}{4})|}{15+|\nu(\frac{s}{4})|} + \frac{|w(\frac{s}{4})|}{15+|w(\frac{s}{4})|}), \\ w(0) = \frac{1}{\Gamma(0.7)} \int_0^1 (1-\eta)^{-0.3} (\frac{|w(\eta)-\nu(\eta)|}{12+|w(\eta)-\nu(\eta)|})\, d\eta + 0.037, \\ \nu(0) = \frac{1}{\Gamma(0.8)} \int_0^1 (1-\eta)^{-0.4} (\frac{|\nu(\eta)-w(\eta)|}{12+|\nu(\eta)-w(\eta)|})\, d\eta + 0.043. \end{cases} \tag{5.2}$$

With the same steps, for (5.2), we may demonstrate when $S = 2$ that

$$\frac{2L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{3L_Q S^{\rho_2}}{\Gamma(\rho_2+1)} \approx 0.61713 < 1, \qquad \frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{3L_Q S^{\theta_2}}{\Gamma(\theta_2+1)} \approx 0.65083 < 1.$$

Hence, the system (5.2) has a unique solution based on Theorem 3.3. Additionally, we get

$$\frac{\Theta\Theta^*}{(1-\Theta)(1-\Theta^*)} \approx \frac{(0.36176)(0.42465)}{(1-0.36176)(1-0.42465)} \approx 0.41835 \neq 1.$$

Thus, the solution of the stated problem (5.2) is UH and GUH stable. Also, if we consider the function $\varphi(s) = \frac{s}{3}$ with $s \in (1,2]$, then we conclude that the problem (5.2) is UHR and GHR stable.

We also provide the following example for further analysis:

Example 5.2 Consider the following CFODDEs:

$$\begin{cases} {}^{C}D_{0+}^{\rho(s)} w(s) = \sin(e^{-s})(\frac{|w(s-0.25)|}{150+|w(s-0.25)|} + \frac{|w(\frac{s}{3})|}{150+|w(\frac{s}{3})|} + \frac{|\nu(\frac{s}{3})|}{150+|\nu(\frac{s}{3})|}), \quad s \in [0,3], \\ {}^{C}D_{0+}^{\theta(s)} \nu(s) = \sin(e^{-s})(\frac{|\nu(s-0.25)|}{150+|\nu(s-0.25)|} + \frac{|\nu(\frac{s}{3})|}{150+|\nu(\frac{s}{3})|} + \frac{|w(\frac{s}{3})|}{150+|w(\frac{s}{3})|}), \\ w(0) = \frac{1}{\Gamma(0.6)} \int_0^1 (1-\eta)^{-0.4} (\frac{|w(\eta)-\nu(\eta)|}{30+|w(\eta)-\nu(\eta)|})\, d\eta + 0.02, \\ \nu(0) = \frac{1}{\Gamma(0.5)} \int_0^1 (1-\eta)^{-0.5} (\frac{|\nu(\eta)-w(\eta)|}{30+|\nu(\eta)-w(\eta)|})\, d\eta + 0.03. \end{cases}$$

Obviously, for $S = 3$, $\xi = 0.6$, $\lambda = 0.5$, $\sigma = \frac{1}{3}$, one has

$$\rho(s) = \begin{cases} 0.8, & s \in [0,1], \\ 0.875, & s \in (1,3], \end{cases} \quad \text{and} \quad \theta(s) = \begin{cases} 0.7, & s \in [0,1], \\ 0.785, & s \in (1,3]. \end{cases}$$

It is clear that for $j = 1, 2$, $w_0 = 0.02$, and $\nu_0 = 0.03$,

$$\begin{aligned} Q\big(s, w(s-\kappa), w(\sigma s), \nu(\sigma s)\big) &= \sin\big(e^{-s}\big)\bigg(\frac{|w(s-0.25)|}{150+|w(s-0.25)|} \\ &\quad + \frac{|w(\frac{s}{3})|}{150+|w(\frac{s}{3})|} + \frac{|\nu(\frac{s}{3})|}{150+|\nu(\frac{s}{3})|}\bigg), \\ Q\big(s, \nu(s-\kappa), \nu(\sigma s), w(\sigma s)\big) &= \sin\big(e^{-s}\big)\bigg(\frac{|\nu(s-0.25)|}{150+|\nu(s-0.25)|} \end{aligned}$$

$$+ \frac{|v(\frac{s}{3})|}{150 + |v(\frac{s}{3})|} + \frac{|w(\frac{s}{3})|}{150 + |w(\frac{s}{3})|}\Bigg),$$

$$h\big(w(\eta), v(\eta)\big) = \left(\frac{|w(\eta) - v(\eta)|}{30 + |w(\eta) - v(\eta)|}\right),$$

$$h\big(v(\eta), w(\eta)\big) = \left(\frac{|v(\eta) - w(\eta)|}{30 + |v(\eta) - w(\eta)|}\right).$$

Considering $w, w^* \in \mho_j$, $j = 1, 2$, we have

$$\begin{aligned}
&\big|Q\big(s, w(s-\kappa), w(\sigma s), v(\sigma s)\big) - Q\big(s, w^*(s-\kappa), w^*(\sigma s), v^*(\sigma s)\big)\big| \\
&\quad\le \big|\sin\big(e^{-s}\big)\big|\left|\frac{w(s-0.25)}{150 + |w(s-0.25)|} - \frac{w^*(s-0.25)}{150 + |w^*(s-0.25)|}\right| \\
&\qquad + \left|\frac{w(\frac{s}{3})}{150 + |w(\frac{s}{3})|} - \frac{w^*(\frac{s}{3})}{1 + |w^*(\frac{s}{3})|}\right| + \left|\frac{v(\frac{s}{3})}{150 + |v(\frac{s}{3})|} - \frac{v^*(\frac{s}{3})}{150 + |v^*(\frac{s}{3})|}\right| \\
&\quad\le \frac{1}{150}\big|w(s-0.25) - w^*(s-0.25)\big| + \left|w\left(\frac{s}{3}\right) - w^*\left(\frac{s}{3}\right)\right| + \left|v\left(\frac{s}{3}\right) - v^*\left(\frac{s}{3}\right)\right|
\end{aligned}$$

and

$$\begin{aligned}
\big|h\big(w(\eta), v(\eta)\big) - h\big(w^*(\eta), v^*(\eta)\big)\big| &= \left|\frac{|w(\eta) - v(\eta)|}{40 + |w(\eta) - v(\eta)|} - \frac{|w^*(\eta) - v^*(\eta)|}{40 + |w^*(\eta) - v^*(\eta)|}\right| \\
&\le \frac{1}{30}\big|\big(w(\eta) - v(\eta)\big) - \big(w^*(\eta) - v^*(\eta)\big)\big| \\
&\le \frac{1}{30}\big(\big|w(\eta) - w^*(\eta)\big| + \big|v(\eta) - v^*(\eta)\big|\big).
\end{aligned}$$

In the first case ($j = 1$, $S = 1$), we get

$$\begin{cases}
{}^C D_0^{\rho_1(s)} w(s) = \sin(e^{-s})\big(\frac{|w(s-0.25)|}{150+|w(s-0.25)|} + \frac{|w(\frac{s}{3})|}{150+|w(\frac{s}{3})|} + \frac{|v(\frac{s}{3})|}{150+|v(\frac{s}{3})|}\big), \quad s \in [0,1], \\
{}^C D_0^{\theta_1(s)} v(s) = \sin(e^{-s})\big(\frac{|v(s-0.25)|}{150+|v(s-0.25)|} + \frac{|v(\frac{s}{3})|}{150+|v(\frac{s}{3})|} + \frac{|w(\frac{s}{3})|}{150+|w(\frac{s}{3})|}\big), \\
w(0) = \frac{1}{\Gamma(0.6)} \int_0^1 (1-\eta)^{-0.4}\big(\frac{|w(\eta)-v(\eta)|}{30+|w(\eta)-v(\eta)|}\big)\, d\eta + 0.02, \\
v(0) = \frac{1}{\Gamma(0.5)} \int_0^1 (1-\eta)^{-0.5}\big(\frac{|v(\eta)-w(\eta)|}{30+|v(\eta)-w(\eta)|}\big)\, d\eta + 0.03,
\end{cases} \tag{5.3}$$

Clearly, $L_Q = \frac{1}{150}$ and $L_h = \frac{1}{30}$. Hence, the hypotheses (H_1) and (H_2) hold. Also, we get

$$\frac{2L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{3L_Q S^{\rho_1}}{\Gamma(\rho_1+1)} \approx 0.0782 < 1, \qquad \frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{3L_Q S^{\theta_1}}{\Gamma(\theta_1+1)} \approx 0.0972 < 1.$$

As a result, problem (5.3) possesses a unique solution according to Theorem 3.3. Additionally, one has

$$\frac{\Theta\Theta^*}{(1-\Theta)(1-\Theta^*)} \approx \frac{(0.05878)(0.08256)}{(1-0.05878)(1-0.08256)} \approx 0.00562 \neq 1.$$

Thus, the solution of the given problem (5.3) is UH and GUH stable. If we consider the function $\varphi(s) = \frac{s}{3}$, $s \in [0,1]$, then we conclude that the considered problem (5.3) is UHR and GHR stable.

In the second case ($j = 2$, $S = 3$), we have

$$\begin{cases} {}^{C}D_0^{\rho_2(s)}w(s) = \sin(e^{-s})(\frac{|w(s-0.25)|}{150+|w(s-0.25)|} + \frac{|w(\frac{s}{3})|}{150+|w(\frac{s}{3})|} + \frac{|v(\frac{s}{3})|}{150+|v(\frac{s}{3})|}), \quad s \in (1,3], \\ {}^{C}D_0^{\theta_2(s)}v(s) = \sin(e^{-s})(\frac{|v(s-0.25)|}{150+|v(s-0.25)|} + \frac{|v(\frac{s}{3})|}{150+|v(\frac{s}{3})|} + \frac{|w(\frac{s}{3})|}{150+|w(\frac{s}{3})|}), \\ w(0) = \frac{1}{\Gamma(0.6)} \int_0^1 (1-\eta)^{-0.4}(\frac{|w(\eta)-v(\eta)|}{30+|w(\eta)-v(\eta)|})\, d\eta + 0.02, \\ v(0) = \frac{1}{\Gamma(0.5)} \int_0^1 (1-\eta)^{-0.5}(\frac{|v(\eta)-w(\eta)|}{30+|v(\eta)-w(\eta)|})\, d\eta + 0.03. \end{cases} \tag{5.4}$$

With the same steps, for (5.3), we can calculate

$$\frac{2L_h S^{\xi}}{\Gamma(\xi+1)} + \frac{3L_Q S^{\rho_2}}{\Gamma(\rho_2+1)} < 1, \qquad \frac{2L_h S^{\lambda}}{\Gamma(\lambda+1)} + \frac{3L_Q S^{\theta_2}}{\Gamma(\theta_2+1)} < 1.$$

Thus, system (5.4) possesses a unique solution based on Theorem 3.3. Additionally, we have

$$\frac{\Theta\Theta^*}{(1-\Theta)(1-\Theta^*)} \neq 1.$$

Thus, the solution of the stated problem (5.4) is UH and GUH stable. Also, if we consider the function $\varphi(s) = \frac{s}{3}$ with $s \in (1,3]$, then we conclude that the problem (5.4) is UHR and GHR stable.

6 Conclusion and future works

Variable-order differential operators have been extensively used in the literature in order to solve a wide range of different real-world problems. Several studies have established that variable-order fractional derivatives may successfully represent many complicated physical phenomena, which are necessary for looking at memory characteristics arising through space and time. These related operators are considered useful tools with various applications in electromagnetism, fluids, diffusion, etc. For instance, great attention has been paid to studying the existence, uniqueness, and stability of solutions of fractional differential equations with variable order. In this work, we considered coupled boundary value problems with mixed-type delays under variable-order differentiation. Under suitable conditions and hypotheses, we showed the existence and uniqueness of solutions via fixed point techniques and nonlinear functional approaches. We also gave some Ulam–Hyers and Ulam–Hyers–Rassias stability results. The obtained results were supported by concrete examples. In future works, we will aim to study further types of stabilities for such problems under different boundary conditions and for different types of fractional derivatives.

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