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Variational inequality arising from variable annuity with mean reversion environment

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Abstract

In this paper, we study a variational inequality arising from variable annuity (VA) to find the optimal surrender strategy for a VA investor when the underlying asset follows a mean reverting process. We formulate the problem as a free boundary partial differential equation (PDE) to obtain the optimal strategy. The PDE is solved analytically by the Mellin transform approach. Using the Mellin transform, we derive the integral equations for the value of the VA and the optimal surrender boundary. Since the solutions are obtained as the integral equations, we use the recursive integration method to determine the optimal surrender strategy. Finally, we provide the optimal surrender boundaries and values of VA with respect to some significant parameters to show the impacts of mean reversion.

Keywords: Variable annuity; Optimal surrender boundary; Variational inequality; Mean reversion

1 Introduction

A variable annuity (VA) is one of the most popular insurance contracts because it provides benefits that depend on the performance of an investment fund, unlike a traditional annuity. The policyholders in VA contracts can choose to pay in installments or a lump sum. However, they also have some risks because their payments are invested in various risky financial assets. To minimize the negative effects of financial market risk, most policyholders prefer VA contracts with various benefits such as guaranteed minimum death benefit (GMDB), guaranteed minimum accumulation benefit (GMAB), guaranteed minimum income benefit (GMIB), and guaranteed minimum withdrawal benefit (GMWB), etc.

Many researchers have recently focused on the pricing and hedging of VAs. In particular, we examine studies on VA contracts with surrender options. Bernard et al. [1] presented the fair price of a VA and proposed the optimal surrender strategy for it. They derived a formula by decomposing the price into an early exercise premium and the European part value. Shen et al. [2] addressed a guaranteed minimum maturity benefit (GMMB) embedded in a VA with a surrender option as a free boundary problem of an American put option. The authors derived a formula for the value of GMMB as a semi-closed form integral expression using a partial differential equation (PDE) approach and employed numerical integration techniques to examine the impacts of significant factors such as surrender fees and insurance charges. In fact, valuing a VA contract requires consideration of the optimal surrender boundary, which transforms the problem into a free boundary problem similar to pricing American options. Numerous researchers have recently studied different American-style options, including perpetual game options [3–5], American capped options [6], American strangle options [7–9] and cancellable American options [10–12], utilizing various approaches.

In addition, there have been studies that have expanded the investigation of VA contracts. Jeon and Kwak [13] considered path-dependent guarantees with the optimal surrender boundary in VA contracts. They also studied a VA contract with a surrender guarantee before maturity [14]. Additionally, Kang and Ziveyi [15] analyzed the surrender behavior of policyholders in a VA embedded with a guaranteed minimum maturity benefit rider under stochastic volatility and interest rate. The authors dealt with a 4-dimensional PDE to determine the optimal surrender strategy and numerically solved the PDE using the method of lines approach. Jeon and Kim [16] adopted the Laplace–Carson transform approach for the efficient valuation of a VA with a surrender option and found that the Gaver–Stehfest method is the most efficient for valuation. More recently, Jia et al. [17] proposed an innovative machine learning method for estimating a surrender charge for VAs, balancing human behavior and logical optimality.

This paper focuses on VA contracts with surrender options and provides the optimal surrender strategy for VA investors based on a PDE approach. Specifically, we first consider a mean-reverting model to develop the pricing model for a VA. Mean-reverting models have been widely used for the valuation of various derivatives. Schwartz [18] proposed a mean-reverting process for the pricing and hedging of commodity contingent claims. Sorensen [19] employed a mean-reverting process for currency exchange rates and considered American currency options under this process. Hui and Lo [20] studied the pricing of options with barriers under a mean-reverting model. Wong and Lau [21] addressed exotic path-dependent options based on the work of Hui and Lo [20] and provided an efficient and accurate approach for valuing options under the mean-reverting model. In this sense, we also study a VA contract with a surrender option under the mean-reverting model.

The Mellin transform approach has been widely utilized in recent years to solve PDEs with various derivatives since it has the advantage of transforming the given PDE into a simple ODE. There have been several research works on various derivatives such as vulnerable options [22–27], Asian options [28, 29], barrier options [30–32], commodity futures [33], and so on since Panini and Srivastav [34] first proposed the Mellin transforms for valuing European and American options. For this investigation, we also employ the Mellin transform technique. Specifically, we establish the integral equations for the value of a VA contract using the Mellin transform approach and demonstrate the optimal boundaries for the optimal surrender decisions in the VA contract when the underlying asset follows the model with mean reversion. Our main contribution is deriving an analytic solution, using the Mellin transform, for the partial differential equation arising from a variable annuity with the underlying state variable following an exponential Ornstein–Uhlenbeck process. To the best of my knowledge, utilizing the Mellin transform is the simplest approach among the methods available to derive such a solution.

The rest of this paper is organized as follows. In Sect. 2, we propose the problem for optimal surrender decision of a VA under the mean reversion model. In Sect. 3, we derive the integral equation for the optimal surrender boundaries based on the Mellin transform approach. In Sect. 4, using the results in Sect. 3, we provide some numerical examples of how mean reversion impacts the movement of optimal boundaries and VA values. We present the concluding remarks in Sect. 5. Finally, we provide the properties of the Mellin transform used in this paper in the Appendix.

2 Model

In this paper, we consider the variable annuity contract with a constant guaranteed minimum accumulation benefit at maturity T in the commodity market. It is well known that many underlying assets of option contracts, such as currencies, commodities, energy, temperature, and even some stocks, exhibit mean reversion (Schwartz [18], Wong and Lau [21], Chiu et al. [35]). In this paper, as in Knoller et al. [36], we assume that the underlying fund F_t satisfies the following stochastic differential equation (SDE) under the risk-neutral measure $\mathbb{Q}$:

$$dF_t = \left(\kappa\left(\theta + \frac{\sigma^2}{2\kappa} - \log F_t\right) - c\right)F_t\,dt + \sigma F_t\,dW_t, \tag{1}$$

where $r > 0$ is the instantaneous risk-free interest rate, κ is the mean reversion rate, θ is the long-term mean, $\sigma > 0$ is the volatility of the underlying asset, and c is the proportional insurance fee. We assume that all parameters $r, \kappa, \theta, \sigma$, and c are constants. $(W_t)_{t=0}^{T}$ is a one-dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where $\mathcal{G}$ is the natural filtration generated by $(W_t)_{t=0}^{T}$. Note that the insurance company withdraws a percentage c from the policyholder's account. Since the VA contract guarantees a minimum accumulation benefit, this assumption is reasonable.

For simplicity, let us denote μ by

$$\mu = \theta + \frac{\sigma^2}{2\kappa} - \frac{c}{\kappa}. \tag{2}$$

Since the VA contract guarantees the minimum accumulation benefit, the insurer will provide

$$\max(F_T, G),$$

where G is the guarantee level and constant, to the policyholder at the maturity T.

Similar as in Bernard et al. [1], we assume that the policyholder is able to surrender the VA contract at any time t before the maturity T. Then the surrender benefit of the VA contract is equal to

$$e^{-p(T-t)}F_t,$$

where $p \geq 0$ is the constant penalty rate.

Remark 1 There are several types of surrender benefit. In [1], the surrender benefit is defined as $e^{-p(T-t)}F_t$, whereas in [14] the surrender benefit is defined as $e^{-p(T-t)}\max\{F_t, G\}$. We adopt the surrender benefit in [1].

Let $V(t,F_t)$ be our VA contract's value at time t before surrender. Then the value $V(t,F_t)$ of the VA contract is written as the following optimal stopping problem.

Problem 1 *Under the risk-neutral measure* $\mathbb{Q}$, *the value* $V(t,F_t)$ *of the VA contract with surrender benefit* $e^{-p(T-t)}\max(F_t,G)$ *is given by*

$$V(t,f)=\sup_{\theta\in\mathcal{S}(t,T)}\mathbb{E}^{\mathbb{Q}}\big[e^{-r(\theta-t)}\big(e^{-p(T-\theta)}F_\theta\mathbf{1}_{\{\theta<T\}}+\max(F_T,G)\mathbf{1}_{\{\theta=T\}}\big)|F_t=f\big],\tag{3}$$

where $\mathcal{S}(t,T)$ *is the set of all* $\mathcal{G}$*-stopping times taking values in* $[t,T]$ *and* $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_t]$ *is the conditional expectation under the risk-neutral measure* $\mathbb{Q}$.

3 Optimization problem with variational inequality

By a standard approach for the optimal stopping problem (see Peskir and Shiryaev [37]), the value function $V(t,f)$ solution to Problem 1 satisfies the following variational inequality:

$$\begin{cases}\partial_t V+\mathcal{L}V\le 0 & \text{if } V(t,f)=e^{-p(T-t)}f \text{ and } (t,f)\in\mathcal{D},\\ \partial_t V+\mathcal{L}V=0 & \text{if } V(t,f)>e^{-p(T-t)}f \text{ and } (t,f)\in\mathcal{D},\\ V(t,f)=\max\{f,G\} & \text{for } f>0,\end{cases}\tag{4}$$

where $\mathcal{L}$ is given by

$$\mathcal{L}=\frac{\sigma^2}{2}f^2\frac{\partial^2}{\partial f^2}+\kappa(\mu-\log f)f\frac{\partial}{\partial f}-r\tag{5}$$

and $\mathcal{D}\equiv\{(t,f)|0<t<T,0<f<\infty\}$.

In terms of the value function $V(t,f)$, we can define the *continuation region* **CR** and the *surrender region SR* as

$$\mathbf{CR}=\big\{(t,f)\in\mathcal{D}|V(t,f)>e^{-p(T-t)}f\big\}\quad\text{and}\quad\mathbf{SR}=\big\{(t,f)\in\mathcal{D}|V(t,f)=e^{-p(T-t)}f\big\}.\tag{6}$$

There exists the optimal surrender boundary $\mathcal{B}(t)$ that separates **SR** from **CR** and is defined as

$$\mathcal{B}(t)=\sup\big\{f>0|(t,f)\in\mathbf{CR}\big\}.\tag{7}$$

The continuation region **CR**, the surrender region **SR**, and the optimal surrender boundary $\mathcal{B}(t)$ are presented in Fig. 1.

Remark 2 Since

$$\log F_t=\log F_0e^{-\kappa t}+\mu\big(1-e^{-\kappa t}\big)+e^{-\kappa t}\int_0^t\sigma e^{\kappa s}\,dW_s,\tag{8}$$

the existence of the optimal surrender boundary $\mathcal{B}(t)$ can be shown in a similar way to Proposition 3.2 in [38].

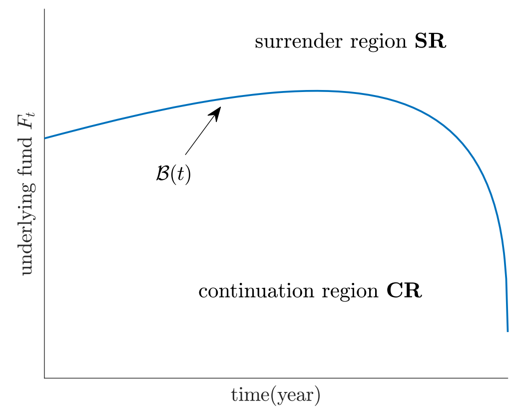

Figure 1 The continuation region **CR**, the surrender region **SR**, and the optimal surrender boundary $\mathcal{B}(t)$ of $V(t,f)$

Then we can rewrite the two regions **CR** and **SR** as

$$\mathbf{CR} = \{(t,f) \in \mathcal{D} | 0 < f < \mathcal{B}(t)\} \quad \text{and} \quad \mathbf{SR} = \{(t,f) \in \mathcal{D} | \mathcal{B}(t) \leq f < \infty\}. \tag{9}$$

At the boundary $f = \mathcal{B}(t)$, the *smooth-pasting* condition or the *super-contact* condition of $V(t,f)$ holds, i.e.,

$$V(t, \mathcal{B}(t)) = 0 \quad \text{and} \quad \partial_f V(t, \mathcal{B}(t)) = 0. \tag{10}$$

Since $\partial_t V + \mathcal{L}V = 0$ in **CR** and $V = e^{-p(T-t)} f$ in **SR**, $V(t,f)$ satisfies the following nonhomogeneous PDE:

$$\partial_t V + \mathcal{L}V = e^{-p(T-t)}(p + \kappa\mu - r - \log f) f \mathbf{1}_{\{\mathcal{B}(t)<f\}} \quad \text{and} \quad V(T,f) = \max\{f, G\}. \tag{11}$$

We now provide the analytic representation formula for the value function $V(t,f)$ and the optimal surrender boundary $\mathcal{B}(t)$ based on the Mellin transform approach.

Let us denote the time-reversed process of $V(t,f)$ and $\mathcal{B}(t)$ by

$$\widetilde{V}(\tau,f) = V(T-\tau,f) \quad \text{and} \quad \widetilde{\mathcal{B}}(\tau) = \mathcal{B}(T-\tau) \tag{12}$$

with $\tau = T - t$.

On the domain $\widetilde{D} = \{(\tau,f) | 0 < \tau \leq T, 0 < f < \infty\}$, the nonhomogeneous PDE (11) can be transformed into

$$-\partial_\tau \widetilde{V} + \mathcal{L}\widetilde{V} = \Psi(\tau,f) \quad \text{and} \quad \widetilde{V}(0,f) = \zeta(f), \tag{13}$$

where

$$\Psi(\tau,f) = e^{-p\tau}(p + \kappa\mu - r - \kappa \log f) f \mathbf{1}_{\{\widetilde{\mathcal{B}}(\tau)<f\}} \quad \text{and} \quad \zeta(f) = \max\{f, G\}. \tag{14}$$

Since the f-domain of the nonhomogeneous PDE (11) is given by $0 < f < \infty$, the Mellin transform to the PDE can be applied. The Mellin transform of $\widetilde{V}(\tau,f)$, $V_M(\tau,x)$, is defined as

$$V_M(\tau,x) \equiv \mathcal{M}\{\widetilde{V}(\tau,f)\} = \int_0^\infty \widetilde{V}(\tau,f)f^{x-1}\,dx,$$

where x is a complex variable. We provide the definition and the basic properties of the Mellin transform in the Appendix.

Then PDE (13) can be transformed as

$$\begin{cases} -\frac{\partial V_M}{\partial \tau} + \frac{\sigma^2}{2}x(x+1)V_M - \kappa\mu x V_M + \kappa\frac{\partial}{\partial x}(xV_M) - rV_M = \Psi_M(\tau,x), \\ V_M(0,x) = \zeta_M(x), \end{cases} \tag{15}$$

where we have used the properties of the Mellin transform in Proposition 4, and $\Psi_M(\tau,x)$ and $\zeta_M(x)$ are the Mellin transform of $\Psi(\tau,f)$ and $\zeta(f)$, respectively.

To transform PDE (15) to an ODE, let us denote $V_M(\tau,x)$ by

$$V_M(\tau,x) = \Gamma(\xi,z) \quad \text{with } \xi = \tau, z = \log x + \kappa\tau. \tag{16}$$

It follows from PDE (15) and substitution (16) that $\Gamma(\xi,x)$ satisfies

$$\begin{cases} -\frac{\partial \Gamma}{\partial \xi} + \Phi(\xi,z)\Gamma = \Psi_M(\xi,e^{z-\kappa\xi}), \\ \Gamma(0,z) = \zeta_M(e^z), \end{cases} \tag{17}$$

where

$$\Phi(\xi,z) = \frac{\sigma^2}{2}e^{2(z-\kappa\xi)} + \left(\frac{\sigma^2}{2} - \kappa\mu\right)e^{z-\kappa\xi} - (r-\kappa). \tag{18}$$

By multiplying the integrating factor $\mathcal{J}(\xi,z) \equiv \exp\{\int_0^\xi \Phi(\eta,z)\,d\eta\}$ on both sides of the first-order ODE (17), we have

$$\Gamma(\xi,z) = \zeta_M(e^z)e^{\mathcal{J}(\xi,z)} - \int_0^\xi \Psi_M(y,e^{z-\kappa y})e^{\mathcal{J}(\xi-y,z)\,d\eta}\,dy, \tag{19}$$

where we have used the fact

$$\mathcal{J}(\xi,z) - \mathcal{J}(y,z) = \mathcal{J}(\xi-y,z).$$

From substitution (16), we obtain the solution to (15) in the following proposition.

Proposition 1 *The solution for $V_M(\tau,x)$ is given by*

$$V_M(\tau,x) = \zeta_M(e^{\kappa\tau}x)e^{\Xi(\tau,x)} - \int_0^\tau \Psi_M(y,xe^{\kappa(\tau-y)})e^{\Xi(\tau-y,x)}\,dy, \tag{20}$$

where $\Xi(\tau,x)$ is defined as

$$\begin{aligned} \Xi(\tau,x) &\equiv \mathcal{J}(\tau,e^{\kappa\tau}x) \\ &= \frac{\sigma^2}{2}\frac{1-e^{-2\kappa\tau}}{2\kappa}e^{2\kappa\tau}x^2 + \left(\frac{\sigma^2}{2} - \kappa\mu\right)\frac{1-e^{-\kappa\tau}}{\kappa}e^{\kappa\tau}x - (r-\kappa)\tau. \end{aligned} \tag{21}$$

3.1 Inverting the Mellin transform

Having now obtained $V_M(\tau,x)$, it is necessary to recover $V(\tau,f)$, the value of the VA contract in the (τ,f)-domain. Taking the inverse Mellin transformation of (20) in Proposition 1 yields

$$\begin{aligned} V(t,f) &= \mathcal{M}^{-1}\left\{\zeta_M\left(e^{\kappa\tau}x\right)e^{\Xi(\tau,x)}\right\} - \mathcal{M}^{-1}\left\{\int_0^\tau \Psi_M\left(y,xe^{\kappa(\tau-y)}\right)e^{\Xi(\tau-y,x)}\,dy\right\} \\ &\equiv V_{\mathrm{E}}(t,f) + V_{\mathrm{EP}}(t,f). \end{aligned} \tag{22}$$

To calculate (22), let us define $\mathcal{H}(t,f)$ by

$$\mathcal{H}(t,f) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{\Xi(t,x)} f^{-x}\,dx. \tag{23}$$

Note that $e^{\Xi(t,x)}$ is the Mellin transform of $\mathcal{H}(t,f)$.

Lemma 1

$$\mathcal{H}(t,f) = \exp\left\{-(r-\kappa)t - A(t)\left(\beta(t)\right)^2 - \frac{1}{4A(t)}(\log f)^2\right\} \times \frac{1}{2}\left(\pi A(t)\right)^{-\frac{1}{2}} f^{\beta(t)}, \tag{24}$$

where

$$\begin{aligned} A(t) &= \frac{1}{2}\left(v(t)\right)^2 e^{2\kappa t} \quad \text{with } v(t) = \sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}, \\ \beta(t) &= \left(1 - \frac{2\kappa\mu}{\sigma^2}\right)\frac{1-e^{-\kappa t}}{1-e^{-2\kappa t}}e^{-\kappa t}. \end{aligned} \tag{25}$$

Proof It follows from the definition of $\Xi(t,s)$ in (21) that

$$\begin{aligned} \Xi(t,x) &= \frac{\sigma^2}{2}\frac{1-e^{-2\kappa t}}{2\kappa}e^{2\kappa t}\left(x + \left(1-\frac{2\kappa\mu}{\sigma^2}\right)\frac{1-e^{-\kappa t}}{1-e^{-2\kappa t}}e^{-\kappa t}\right)^2 - (r-\kappa)t \\ &\quad - \frac{\sigma^2}{2}\frac{1-e^{-2\kappa t}}{2\kappa}\left(1-\frac{2\kappa t}{\sigma^2}\right)^2\left(\frac{1-e^{-\kappa t}}{1-e^{-2\kappa t}}\right)^2 \\ &= -(r-\kappa)t - A(t)\left(\beta(t)\right)^2 + A(t)\left(x+\beta(t)\right)^2. \end{aligned} \tag{26}$$

Since $A(t) \geq 0$ for all $t \geq 0$, it follows from Proposition 5 that

$$\begin{aligned} \mathcal{H}(t,f) &= \frac{1}{2\pi i}\int_{c-i\pi}^{c+i\pi} e^{\Xi(t,x)} f^{-x}\,dx \\ &= \exp\left\{-(r-\kappa)t - A(t)\left(\beta(t)\right)^2 + A(t)\right\}\frac{1}{2\pi i}\int_{c-i\pi}^{c+i\pi} e^{A(t)(x+\beta(t))^2} f^{-x}\,dx \\ &= \exp\left\{-(r-\kappa)t - A(t)\left(\beta(t)\right)^2 - \frac{1}{4A(t)}(\log f)^2\right\} \\ &\quad \times \frac{1}{2}\left(\pi A(t)\right)^{-\frac{1}{2}} f^{\beta(t)}. \end{aligned} \tag{27}$$

□

Lemma 2 *For any $K > 0$, the following relationships hold*:

$$\begin{aligned}
&\int_K^\infty e^{-\kappa t}\mathcal{H}\left(t,\frac{f}{u}\right)\frac{du}{u} = e^{-rt}\Phi\left(\frac{\log\frac{f}{K}-2A(t)\beta(t)}{\sqrt{2A(t)}}\right),\\
&\int_0^K e^{-\kappa t}\mathcal{H}\left(t,\frac{f}{u}\right)\frac{du}{u} = e^{-rt}\Phi\left(-\frac{\log\frac{f}{K}-2A(t)\beta(t)}{\sqrt{2A(t)}}\right),\\
&\int_K^\infty e^{-\kappa t}u^{e^{-\kappa t}}\mathcal{H}\left(t,\frac{f}{u}\right)\frac{du}{u}\\
&\quad = e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}}f^{e^{-\kappa t}}\Phi\left(\frac{\log\frac{f}{K}-2A(t)(\beta(t)-e^{-\kappa t})}{\sqrt{2A(t)}}\right)
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\int_K^\infty e^{-\kappa t}u^{e^{-\kappa t}}\log u^{e^{-\kappa t}}\mathcal{H}\left(t,\frac{f}{u}\right)\frac{du}{u}\\
&\quad = e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}}f^{e^{-\kappa t}}\left(e^{-\kappa t}\log f-2A(t)\beta(t)e^{-\kappa t}+2A(t)e^{-2\kappa t}\right)\\
&\qquad\times\Phi\left(\frac{\log\frac{f}{K}-2A(t)(\beta(t)-e^{-\kappa t})}{\sqrt{2A(t)}}\right)\\
&\qquad+e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}}f^{e^{-\kappa t}}e^{-\kappa t}\sqrt{2A(t)}\mathbf{n}\left(\frac{\log\frac{f}{K}-2A(t)(\beta(t)-e^{-\kappa t})}{\sqrt{2A(t)}}\right),
\end{aligned} \tag{29}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, $\mathbf{n}(\cdot)$ is the standard normal probability density function, and $A(t)$ and $\beta(t)$ are defined in Lemma 1.

Proof It follows from Lemma 1 that

$$\begin{aligned}
&\int_K^\infty e^{-\kappa t}u^{e^{-\kappa t}}\mathcal{H}\left(t,\frac{f}{u}\right)\frac{du}{u}\\
&\quad = e^{-rt-A(t)(\beta(t))^2}f^{e^{-\kappa t}}\int_K^\infty\frac{1}{2}\left(\pi A(t)\right)^{-\frac{1}{2}}\left(\frac{f}{u}\right)^{\beta(t)-e^{-\kappa t}}e^{-\frac{1}{4A(t)}(\log(f/u))^2}\frac{du}{u}\\
&\quad = -e^{-rt-A(t)(\beta(t))^2}f^{e^{-\kappa t}}\int_{\log\frac{f}{K}}^{-\infty}\frac{1}{2}\left(\pi A(t)\right)^{-\frac{1}{2}}e^{\nu(\beta(t)-e^{-\kappa t})}e^{-\frac{1}{4A(t)}\nu^2}\,d\nu\left(\nu=\log\frac{f}{u}\right)\\
&\quad = e^{-rt-2A(t)\beta(t)+A(t)e^{-2\kappa t}}f^{e^{-\kappa t}}\int_{-\infty}^{(\log\frac{f}{K}-2A(t)(\beta(t)-e^{-\kappa t}))/\sqrt{2A(t)}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\nu^2}\,d\nu\\
&\quad = e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}}f^{e^{-\kappa t}}\Phi\left(\frac{\log\frac{f}{K}-2A(t)(\beta(t)-e^{-\kappa t})}{\sqrt{2A(t)}}\right).
\end{aligned} \tag{30}$$

Similarly, we can easily derive

$$\begin{aligned}
&\int_K^\infty e^{-\kappa t}\mathcal{H}\left(t,\frac{f}{u}\right)\frac{du}{u} = e^{-rt}\Phi\left(\frac{\log\frac{f}{K}-2A(t)\beta(t)}{\sqrt{2A(t)}}\right),\\
&\int_0^K e^{-\kappa t}\mathcal{H}\left(t,\frac{f}{u}\right)\frac{du}{u} = e^{-rt}\Phi\left(-\frac{\log\frac{f}{K}-2A(t)\beta(t)}{\sqrt{2A(t)}}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_K^{\infty} e^{-\kappa t} u^{e^{-\kappa t}} \log u^{e^{-\kappa t}} \mathcal{H}\left(t, \frac{f}{u}\right) \frac{du}{u} \\
&= \int_K^{\infty} e^{-\kappa t} u^{e^{-\kappa t}} e^{-\kappa t} \left(\log \frac{u}{f} + \log f\right) \mathcal{H}\left(t, \frac{f}{u}\right) \frac{du}{u} \\
&= e^{-\kappa t} \log f \int_K^{\infty} e^{-\kappa t} u^{e^{-\kappa t}} \mathcal{H}\left(t, \frac{f}{u}\right) \frac{du}{u} + e^{-\kappa t} \int_K^{\infty} e^{-\kappa t} u^{e^{-\kappa t}} \log \frac{u}{f} \mathcal{H}\left(t, \frac{f}{u}\right) \frac{du}{u} \\
&= e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}} f^{e^{-\kappa t}} \Phi\left(\frac{\log \frac{f}{K} - 2A(t)(\beta(t) - e^{-\kappa t})}{\sqrt{2A(t)}}\right) \times e^{-\kappa t} \log f \\
&\quad + e^{-\kappa t} \int_K^{\infty} e^{-\kappa t} u^{e^{-\kappa t}} \log \frac{u}{f} \mathcal{H}\left(t, \frac{f}{u}\right) \frac{du}{u}.
\end{aligned} \tag{31}$$

The second term in the last equality in (31) can be written as

$$\begin{aligned}
&e^{-\kappa t} \int_K^{\infty} e^{-\kappa t} u^{e^{-\kappa t}} \log \frac{u}{f} \mathcal{H}\left(t, \frac{f}{u}\right) \frac{du}{u} \\
&= -e^{-rt-A(t)(\beta(t))^2} f^{e^{-\kappa t}} e^{-\kappa t} \int_K^{\infty} \frac{1}{2} \left(\pi A(t)\right)^{-\frac{1}{2}} \left(\frac{f}{u}\right)^{\beta(t)-e^{-\kappa t}} \log \frac{f}{u} e^{-\frac{1}{4A(t)}(\log(f/u))^2} \frac{du}{u} \\
&= e^{-rt-A(t)(\beta(t))^2} f^{e^{-\kappa t}} e^{-\kappa t} \int_{\log \frac{f}{K}}^{-\infty} \frac{1}{2} \left(\pi A(t)\right)^{-\frac{1}{2}} \nu e^{\nu(\beta(t)-e^{-\kappa t})} e^{-\frac{1}{4A(t)}\nu^2} \, d\nu \left(\nu = \log \frac{f}{u}\right) \\
&= e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}} f^{e^{-\kappa t}} e^{-\kappa t} \int_{\log \frac{f}{K}}^{-\infty} \frac{1}{2} \left(\pi A(t)\right)^{-\frac{1}{2}} \nu e^{-\frac{1}{4A(t)}(\nu - 2A(t)(\beta(t)-e^{-\kappa t}))^2} \, d\nu \\
&= e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}} f^{e^{-\kappa t}} \left(-2A(t)\beta(t)e^{-\kappa t} + 2A(t)e^{-2\kappa t}\right) \\
&\quad \times \Phi\left(\frac{\log \frac{f}{K} - 2A(t)(\beta(t) - e^{-\kappa t})}{\sqrt{2A(t)}}\right) \\
&\quad + e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}} f^{e^{-\kappa t}} e^{-\kappa t} \sqrt{2A(t)} \mathbf{n}\left(\frac{\log \frac{f}{K} - 2A(t)(\beta(t) - e^{-\kappa t})}{\sqrt{2A(t)}}\right).
\end{aligned} \tag{32}$$

It follows from (31) and (32) that

$$\begin{aligned}
&\int_K^{\infty} e^{-\kappa t} u^{e^{-\kappa t}} \log u^{e^{-\kappa t}} \mathcal{H}\left(t, \frac{f}{u}\right) \frac{du}{u} \\
&= e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}} f^{e^{-\kappa t}} \left(e^{-\kappa t} \log f - 2A(t)\beta(t)e^{-\kappa t} + 2A(t)e^{-2\kappa t}\right) \\
&\times \Phi\left(\frac{\log \frac{f}{K} - 2A(t)(\beta(t) - e^{-\kappa t})}{\sqrt{2A(t)}}\right) \\
&+ e^{-rt-2A(t)\beta(t)e^{-\kappa t}+A(t)e^{-2\kappa t}} f^{e^{-\kappa t}} e^{-\kappa t} \sqrt{2A(t)} \mathbf{n}\left(\frac{\log \frac{f}{K} - 2A(t)(\beta(t) - e^{-\kappa t})}{\sqrt{2A(t)}}\right).
\end{aligned}$$

□

Theorem 1 *The value $V(t,s)$ of the VA contract is given by*

$$V(t,f) = V_{\mathrm{E}}(t,f) + V_{\mathrm{EP}}(t,f), \tag{33}$$

where

$$
\begin{aligned}
V_{\mathrm{E}}(t,f) &= e^{-r\tau+\theta(1-e^{-\kappa\tau})+(\nu(\tau))^2} f^{e^{-\kappa\tau}} \Phi\left(d_1\left(\tau, \frac{f}{G}\right)\right) + Ge^{-r\tau}\Phi\left(-d_2\left(\tau, \frac{f}{G}\right)\right), \\
V_{\mathrm{EP}}(t,f) &= e^{-p\tau}\int_0^\tau e^{-(r-p)y+\theta(1-e^{-\kappa y})+\frac{1}{2}(\nu(y))^2} f^{e^{-\kappa y}}\left[(r-p-\kappa\mu)\Phi\left(d_1\left(y, \frac{f}{\mathcal{B}(t+y)}\right)\right)\right. \\
&\quad + \kappa\left(e^{-\kappa y}\log f + \theta\left(1-e^{-\kappa y}\right) + \left(\nu(y)\right)^2\right)\Phi\left(d_1\left(y, \frac{f}{\mathcal{B}(t+y)}\right)\right) \\
&\quad \left. + \kappa\nu(y)\mathbf{n}\left(d_1\left(y, \frac{f}{\mathcal{B}(t+y)}\right)\right)\right] dy
\end{aligned}
\tag{34}
$$

and $\tau = T - t$, μ *defined in* (2),

$$
\begin{aligned}
d_1(t,f) &= \frac{e^{-\kappa t}\log f + \theta(1-e^{-\kappa t}) + (\nu(t))^2}{\nu(t)}, \qquad d_2(t,f) = d_1(t,f) - \nu(t), \\
\nu(t) &= \sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}, \qquad \theta = \mu - \frac{\sigma^2}{2\kappa}.
\end{aligned}
\tag{35}
$$

Proof By Proposition 4, we can easily deduce that $\zeta_M(e^{\kappa\tau}x)$ and $\Psi_M(y, e^{\kappa(\tau-y)}x)$ are the Mellin transforms of $\zeta(f^{e^{-\kappa\tau}})e^{-\kappa\tau}$ and $\Psi(y, f^{e^{-\kappa(\tau-y)}})e^{-\kappa(\tau-y)}$, respectively. It follows from the Mellin convolution theorem in Proposition 6 that

$$
\begin{aligned}
V(t,f) &= \mathcal{M}^{-1}\left\{\zeta_M\left(e^{\kappa\tau}x\right)e^{\Xi(\tau,x)}\right\} - \mathcal{M}^{-1}\left\{\int_0^\tau \Psi_M\left(y, xe^{\kappa(\tau-y)}\right)e^{\Xi(\tau-y,x)}\,dy\right\} \\
&= \int_0^\infty \zeta\left(u^{e^{-\kappa\tau}}\right)e^{-\kappa\tau}\mathcal{H}\left(\tau, \frac{f}{u}\right)\frac{du}{u} \\
&\quad - \int_0^\tau\int_0^\infty \Psi\left(y, u^{e^{-\kappa(\tau-y)}}\right)e^{-\kappa(\tau-y)}\mathcal{H}\left(\tau-y, \frac{f}{u}\right)\frac{du}{u}\,dy \\
&= \int_0^\infty \zeta\left(u^{e^{-\kappa\tau}}\right)e^{-\kappa\tau}\mathcal{H}\left(\tau, \frac{f}{u}\right)\frac{du}{u} \\
&\quad - \int_0^\tau\int_0^\infty \Psi\left(\tau-y, u^{e^{-\kappa y}}\right)e^{-\kappa y}\mathcal{H}\left(y, \frac{f}{u}\right)\frac{du}{u}\,dy \\
&= \int_0^\infty \max\{u, G\}e^{-\kappa\tau}\mathcal{H}\left(\tau, \frac{f}{u}\right)\frac{du}{u} \\
&\quad + e^{-p\tau}\int_0^\tau\int_0^\infty \left(r-p-\kappa\mu+\kappa\log u^{e^{-\kappa y}}\right)u^{e^{-\kappa y}} \\
&\quad \times \mathbf{1}_{\{\widetilde{\mathcal{B}}(\tau-y)<u^{e^{-\kappa y}}\}}e^{py}e^{-\kappa y}\mathcal{H}\left(y, \frac{f}{u}\right)\frac{du}{u}\,dy \\
&= \int_{G^{e^{\kappa t}}}^\infty u^{e^{-\kappa\tau}}e^{-\kappa\tau}\mathcal{H}\left(\tau, \frac{f}{u}\right)\frac{du}{u} \\
&\quad + G\int_0^{G^{e^{\kappa t}}} e^{-\kappa\tau}\mathcal{H}\left(\tau, \frac{f}{u}\right)\frac{du}{u} \\
&\quad + e^{-p\tau}\int_0^\tau\int_{(\widetilde{\mathcal{B}}(\tau-y))^{e^{\kappa y}}}^\infty \left(r-p-\kappa\mu+\kappa\log u^{e^{-\kappa y}}\right)u^{e^{-\kappa y}}e^{py}e^{-\kappa y}\mathcal{H}\left(y, \frac{f}{u}\right)\frac{du}{u}\,dy.
\end{aligned}
$$

Since $\widetilde{\mathcal{B}}(\tau - y) = \mathcal{B}(T - (\tau - y)) = \mathcal{B}(t + y)$, it follows from Lemma 2 and the definitions of $A(t)$ and $\beta(t)$ that we obtain the desired results. □

From Theorem 1 and the smooth pasting condition (10), we can directly obtain the integral equation of the optimal stopping boundary $\mathcal{B}(t)$.

Corollary 1 *The optimal stopping boundary $z(t)$ satisfies the following integral equation*:

$$e^{-p(T-t)}\mathcal{B}(t) = V_{\mathrm{E}}\big(t, \mathcal{B}(t)\big) + V_{\mathrm{EP}}\big(t, \mathcal{B}(t)\big). \tag{36}$$

Proposition 2 *When the time to maturity $T - t$ goes to zero, the optimal stopping boundary $\mathcal{B}(t)$ goes to G, i.e.*,

$$\lim_{t\to T-} \mathcal{B}(t) = \max\big\{G, e^{\frac{p+\kappa\mu-r}{\kappa}}\big\}. \tag{37}$$

Proof We first show $\mathcal{B}(T-) \geq G$. If $\mathcal{B}(T-) < G$, then

$$\begin{aligned}\lim_{t\to T-} V_{\mathrm{E}}\big(t, \mathcal{B}(t)\big) &= \lim_{t\to T-}\bigg[e^{-r\tau+\theta(1-e^{-\kappa\tau})+(\nu(\tau))^2}\big(\mathcal{B}(t)\big)^{e^{-\kappa\tau}}\Phi\bigg(d_1\bigg(\tau, \frac{\mathcal{B}(t)}{G}\bigg)\bigg) \\ &\quad + Ge^{-r\tau}\Phi\bigg(-d_2\bigg(\tau, \frac{\mathcal{B}(t)}{G}\bigg)\bigg)\bigg] = G\end{aligned} \tag{38}$$

and

$$\lim_{t\to T-} V_{\mathrm{EP}}\big(t, \mathcal{B}(t)\big) = 0. \tag{39}$$

It follows from

$$\begin{aligned}\mathcal{B}(T-) &= \lim_{t\to T-} e^{-p(T-t)}\mathcal{B}(t) = \lim_{t\to T-} V\big(t, \mathcal{B}(t)\big) \\ &= \lim_{t\to T-} V_{\mathrm{E}}\big(t, \mathcal{B}(t)\big) + \lim_{t\to T-} V_{\mathrm{EP}}\big(t, \mathcal{B}(t)\big)\end{aligned} \tag{40}$$

that

$$B(T-) = G.$$

Since this contradicts $\mathcal{B}(T-) < G$, we conclude that $\mathcal{B}(T-) \geq G$.

From the variational inequality in (4), for any $(t,f) \in \mathbf{SR}$,

$$\partial_t V + \mathcal{L}V \leq 0.$$

Since $V(t,f) = e^{-p(T-t)}f$ in $\mathbf{SR}$, we have $f \geq e^{\frac{p+\kappa\mu-r}{\kappa}}$ for all $(t,f) \in \mathbf{SR}$.

This implies that

$$B(T-) \geq \widehat{\mathcal{B}} \equiv \max\big\{G, e^{\frac{p+\kappa\mu-r}{\kappa}}\big\}.$$

Let us temporarily denote $\widetilde{V}$ by

$$\widetilde{V}(t,f) = e^{p(T-t)}V(t,f).$$

Then $\widetilde{V}(t,f)$ satisfies

$$\begin{cases} \partial_t \widetilde{V} + \mathcal{L}_p \widetilde{V} \leq 0 & \text{if } \widetilde{V}(t,f) = f \text{ and } (t,f) \in \mathcal{D}, \\ \partial_t \widetilde{V} + \mathcal{L}_p \widetilde{V} = 0 & \text{if } \widetilde{V}(t,f) > f \text{ and } (t,f) \in \mathcal{D}, \\ \widetilde{V}(t,f) = \max\{f, G\} & \text{for } f > 0, \end{cases} \tag{41}$$

where $\mathcal{L}_p$ is given by

$$\mathcal{L}_p = \frac{\sigma^2}{2} f^2 \frac{\partial^2}{\partial f^2} + \kappa(\mu - \log f) f \frac{\partial}{\partial f} - (r - p). \tag{42}$$

If $\mathcal{B}(T-) > \widehat{\mathcal{B}}$, then there exists a domain $\mathcal{D}_\epsilon = \{T - \epsilon \leq t \leq T, \widehat{\mathcal{B}} < f < \mathcal{B}(T-)\} \subset \mathbf{CR}$ for any sufficiently small $\epsilon > 0$ such that

$$\partial_t \widetilde{V}(t,f) + \mathcal{L}_p \widetilde{V}(t,f) = 0 \quad \text{for } (t,f) \in \mathcal{D}_\epsilon.$$

At $t = T$ in the domain $\mathcal{D}_\epsilon$, we deduce that

$$\left.\frac{\partial \widetilde{V}}{\partial t}\right|_{t=T} = -\big[\mathcal{L}\widetilde{V}(t,f)\big]_{t=T} > 0, \tag{43}$$

where we have used that $\widetilde{V}(T,f) = \max\{f, G\} = f$ for $(T,f) \in \mathcal{D}_\epsilon$. Hence, we have $\widetilde{V}(t,f) < \widetilde{V}(T,f) = f$ in the domain $\mathcal{D}_\epsilon$, which contradicts $\widetilde{V}(t,f) > f$ in the domain $\mathcal{D}_\epsilon \subset \mathbf{CR}$. This implies that

$$\lim_{t \to T-} \mathcal{B}(t) = \mathcal{B}(T-) = \max\big\{G, e^{\frac{p+\kappa\mu-r}{\kappa}}\big\}. \qquad \square$$

In the following proposition, we provide the delta for VA contract, which is one of the Greeks and essential for delta hedging strategy.

Proposition 3 *The delta Δ of VA contract can be presented as*

$$\Delta(t,f) = \Delta_{\mathrm{E}}(t,f) + \Delta_{\mathrm{P}}(t,f), \tag{44}$$

where

$$\begin{aligned} \Delta_{\mathrm{E}}(t,f) &= e^{-r\tau+\gamma(1-e^{-\kappa\tau})+(\nu(\tau))^2} e^{-\kappa\tau} f^{e^{-\kappa\tau}-1} \Phi\left(d_1\left(\tau, \frac{f}{G}\right)\right) \\ &\quad + e^{-r\tau+\gamma(1-e^{-\kappa\tau})+(\nu(\tau))^2} f^{e^{-\kappa\tau}} \mathbf{n}\left(d_1\left(\tau, \frac{f}{G}\right)\right) \frac{e^{-\kappa\tau}}{\nu(\tau) f} \\ &\quad - G e^{-r\tau} \mathbf{n}\left(-d_2\left(\tau, \frac{f}{G}\right)\right) \frac{e^{-\kappa\tau}}{\nu(\tau) f} \end{aligned} \tag{45}$$

and

$$
\begin{aligned}
\Delta_{\mathrm{P}}(t,f) = {} & e^{-p\tau}\int_0^{\tau} e^{-(r-p)y+\gamma(1-e^{-\kappa y})+\frac{1}{2}(\nu(y))^2} \\
& \times e^{-\kappa y} f^{e^{-\kappa y}-1}\bigg[(r-p-\kappa\mu)\Phi\bigg(d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\bigg) \\
& + \kappa\big(\log f e^{-\kappa y} + \gamma\big(1-e^{-\kappa y}\big) + \big(\nu(y)\big)^2\big)\big)\Phi\bigg(d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\bigg) \\
& + \kappa\nu(y)\mathbf{n}\bigg(d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\bigg)\bigg]\,dy \\
& + e^{-p\tau}\int_0^{\tau} e^{-(r-p)y+\gamma(1-e^{-\kappa y})+\frac{1}{2}(\nu(y))^2} f^{e^{-\kappa y}}(r-p-\kappa\mu) \\
& \times \mathbf{n}\bigg(d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\bigg)\frac{e^{-\kappa\tau}}{\nu(y)f}\,dy \\
& + e^{-p\tau}\int_0^{\tau} e^{-(r-p)y+\gamma(1-e^{-\kappa y})+\frac{1}{2}(\nu(y))^2} f^{e^{-\kappa y}} \\
& \times \kappa\bigg(\frac{1}{f}e^{-\kappa y} + \gamma\big(1-e^{-\kappa y}\big) + \big(\nu(y)\big)^2\bigg)\big)\Phi\bigg(d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\bigg)\,dy \\
& + e^{-p\tau}\int_0^{\tau} e^{-(r-p)y+\gamma(1-e^{-\kappa y})+\frac{1}{2}(\nu(y))^2} f^{e^{-\kappa y}}\kappa\big(\log f e^{-\kappa y} + \gamma\big(1-e^{-\kappa y}\big) \\
& + \big(\nu(y)\big)^2\big)\big)\mathbf{n}\bigg(d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\bigg)\frac{e^{-\kappa\tau}}{\nu(y)f}\,dy \\
& - e^{-p\tau}\int_0^{\tau} e^{-(r-p)y+\gamma(1-e^{-\kappa y})+\frac{1}{2}(\nu(y))^2} f^{e^{-\kappa y}}\kappa\nu(y) \\
& \times \mathbf{n}\bigg(d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\bigg)d_1\bigg(y,\frac{f}{\mathcal{B}(t+y)}\bigg)\frac{e^{-\kappa\tau}}{\nu(y)f}\,dy.
\end{aligned}
\tag{46}
$$

Proof Since

$$
\Delta(t,f) = \frac{\partial V}{\partial f} = \frac{\partial V_{\mathrm{E}}}{\partial f} + \frac{\partial V_{\mathrm{P}}}{\partial f}, \tag{47}
$$

the proof is clear. □

4 Numerical examples

In this section, we present the example for fair fee rate and several numerical examples to demonstrate the impact of mean reversion and to show the sensitivity of the optimal surrender boundary of VA contract with respect to significant parameters. Since the optimal surrender boundary is derived as the integral equation in Corollary 1, we use the recursive integration method, which is proposed by Huang and Subrahmanyam [39], to solve the integral equation efficiently. For the numerical examples, we employ the baseline parameters in Table 1.

We first consider the example for fair fee rate. As suggested by Bernard et al. [1], the fair fee rate c^* can be determined by the following equation:

$$
F_0 = V_{\mathrm{E}}\big(0, F_0^{c^*}\big), \tag{48}
$$

Table 1 The baseline parameters

Parameter	Value	Parameter	Value
$T-t$	5	r	0.04
κ	0.02	σ	0.2
θ	log 20	G	20

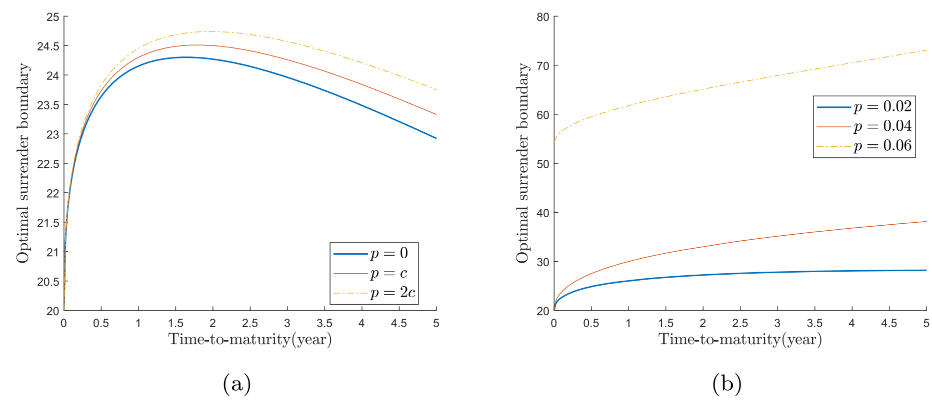

Figure 2 Optimal surrender boundary $\mathcal{B}(t)$ varying with parameter p

where V_E is the European benefit of the VA contract in Theorem 1, F_0 is the initial single premium, and F_t^c denotes the policyholder's account value at time t with fee rate c. To compute this fair fee, it is possible to use the root finding techniques such as the bisection method.

Suppose that the initial single premium F_0 is equal to the constant guarantee G, i.e.,

$$F_0 = G = 20.$$

Then, under the baseline parameters given in Table 1, the fair fee c^* is 0.0019 or 0.19%.

Figure 2 presents the optimal surrender boundaries for different values of the penalty rate p as time-to-maturity increases. We note that the optimal surrender is determined if the account value process touches the optimal surrender boundary. As expected, a high value of p leads to a high surrender boundary at all times. We also find the effect and sensitivity of the penalty rate p on the optimal surrender boundaries. Moreover, in contrast to Bernard et al. [1], the optimal surrender boundaries exist even when the penalty rate p is greater than or equal to r in our model (see Fig. 2(b)).

Figure 3 presents the sensitivity of optimal surrender boundaries for different values of the guarantee level G. As one can see, the optimal surrender boundaries are increasing or decreasing depending on the remaining time-to-maturity and the guarantee level G. More concretely, if the time-to-maturity ($T-t$) is short (roughly $T-t < 0.5$), the optimal boundaries increases as $T-t$ increases. In contrast, if the time-to-maturity ($T-t$) is long (roughly $T-t > 0.5$), the boundaries decrease as $T-t$ increases. We also find that a high value of G leads to a high surrender boundary in Fig. 3.

We provide Fig. 4 and Fig. 5 to show the impacts of mean reversion. In particular, Fig. 4(b) and Fig. 5(b) present the illustrations of sensitivities of optimal surrender boundaries with respect to the parameters for mean reverting model. Figure 4 illustrates how the

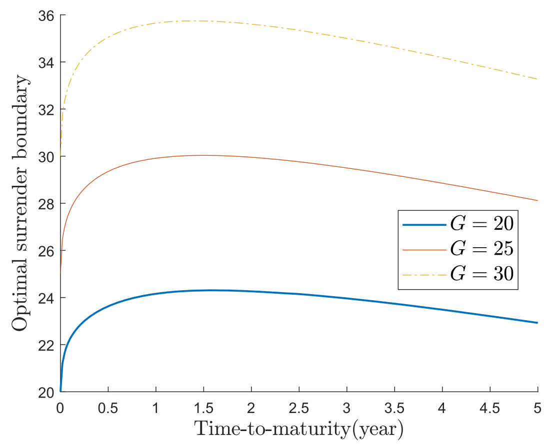

Figure 3 Optimal surrender boundary $\mathcal{B}(t)$ varying with parameters G

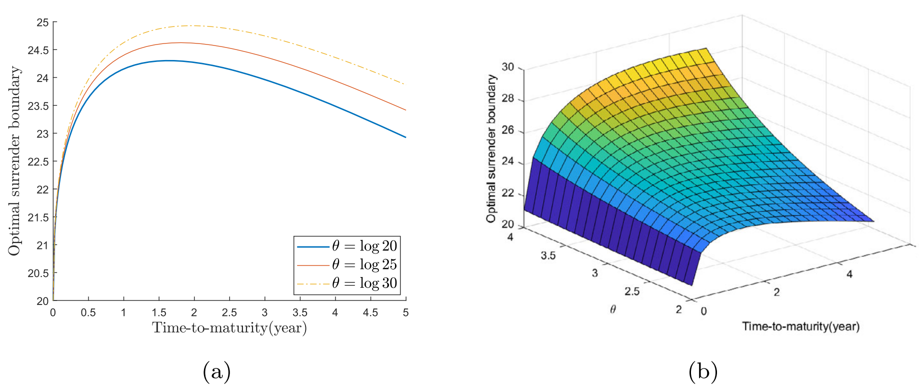

Figure 4 Optimal surrender boundary $\mathcal{B}(t)$ varying with parameter θ

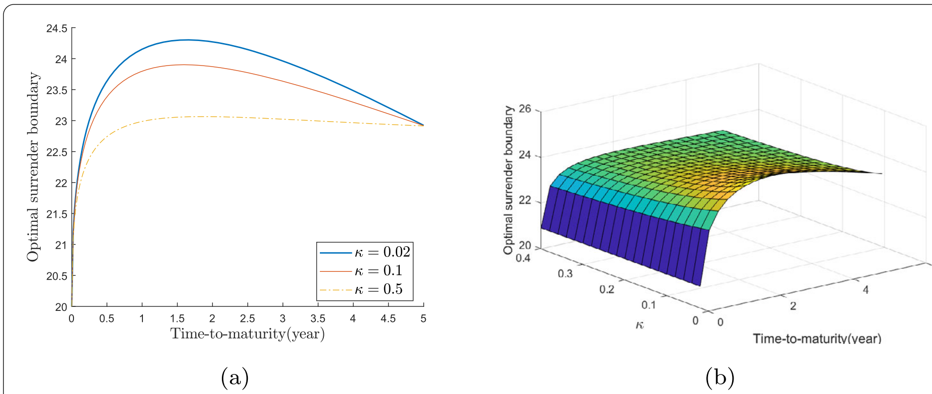

Figure 5 Optimal surrender boundary $\mathcal{B}(t)$ varying with parameters κ

optimal surrender boundaries vary with respect to the long-term mean θ. Figure 4 shows that a lower value of θ induces a lower boundary. As time-to-maturity increases, in Fig. 4, we also observe that the surrender boundaries increase logarithmically, and that the dif-

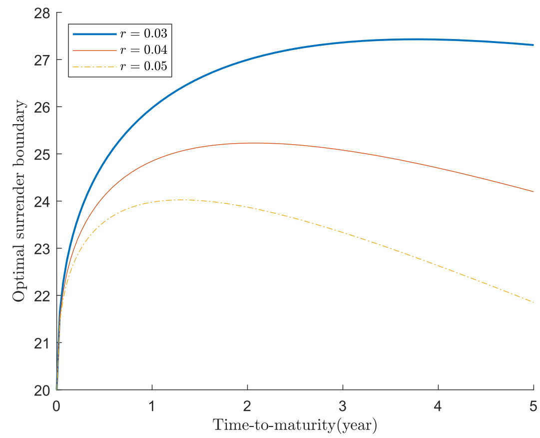

Figure 6 Optimal surrender boundary $\mathcal{B}(t)$ varying with parameters r

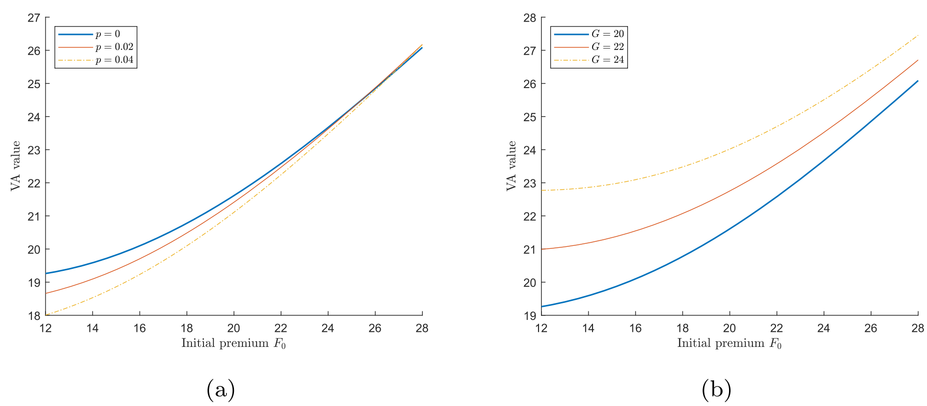

Figure 7 Value V of the VA contract varying with parameters p and G

ference in values of boundaries increases. Figure 5 illustrates how the optimal surrender boundaries vary with respect to mean reversion rate κ. We observe that the boundaries converge similar values for sufficiently small or large time-to-maturity. In other words, we may conclude that mean reversion rate κ becomes a significant parameter if time-to-maturity of a VA contract is in the appropriate range. Figure 6 shows the effects of risk-free interest rate r. We find that high interest rate has a large impact on the optimal surrender boundary for VA contract when time-to-maturity is long. Specifically, we can see that there is a large difference in values for different interest rate and long time-to-maturity.

Figure 7 presents how the values of VA contract move with respect to the parameters p and G. As shown in Fig. 7, the values increase exponentially as the initial premium F_0 increases. For large F_0 in Fig. 7(a), we can see that values are roughly the same for different values of the penalty rate p. However, for different values of the guarantee level G in Fig. 7(b), we find that the values remain different. Finally, Fig. 8 plots the delta of VA for deep-in-money guarantees based on the result in Proposition 3. From Fig. 8, we observe a positive linear relation between the delta and the initial premium.

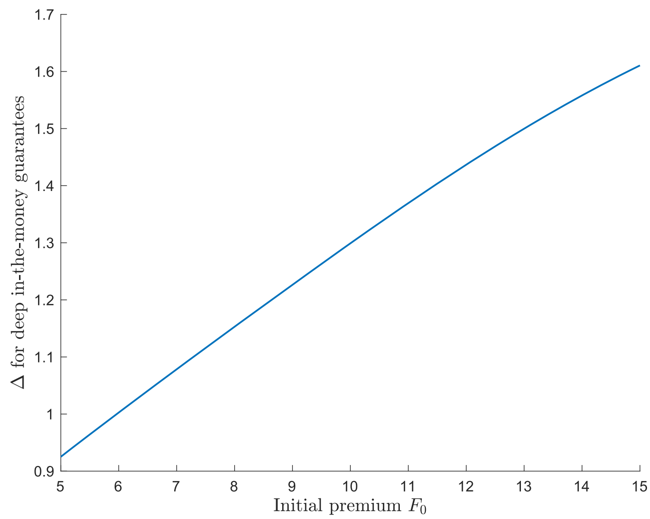

Figure 8 Delta of the VA contract for deep-in-money guarantees

5 Concluding remarks

We study how to determine the optimal surrender decision of a VA contract in the mean reversion environment. We first deal with the optimal surrender boundary and the value of a VA in the mean reverting model, which is considered as a more realistic market model. Under the mean reversion environment, the policyholder's account value is kept in a small range around the mean level during the lifetime of the VA contract. Since the account value may include the derivatives such as currencies and commodities that seem to exhibit some mean reversion property, the proposed model is suitable for modeling the account value.

To obtain the boundary for the optimal surrender decision, we adopt the PDE approach and derive an integral equation for the optimal stopping boundary via the Mellin transform approach. The integral equation is solved numerically using the recursive integration method. We perform numerical examples with respect to some significant parameters and present the optimal surrender boundaries as nonmonotonic functions. From the examples for sensitivity analyses of the optimal surrender boundaries, we find the meaningful movements of the boundaries with respect to the significant parameters. In particular, we show how the optimal surrender strategy for a VA investor depends on the account value process with mean reversion. That is, we conclude that these results help to make optimal decisions when investing in VA contracts. We also believe that our method can be applied to other exotic VA contracts when the underlying asset follows the mean reverting model.

Appendix: Summary: properties of the Mellin transform

In this appendix we briefly review the definition and properties of the Mellin transformation. The readers can refer to Sneddon [40] for more details.

Definition 1 For a locally integrable function $g(z)$ in $(0,+\infty)$, the Mellin transform of $\mathcal{M}[g](x)$ of $g(z)$ is defined by

$$\mathcal{M}[g](x) = \int_0^\infty g(z)z^{x-1}\,dx, \quad x \in \mathbb{C},$$

and if this integral converges for $a_1 < \mathbf{Re}(x) < a_2$, then the inverse Mellin transform is given by

$$g(z) = \mathcal{M}^{-1}\big[\mathcal{M}[g]\big](z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}[g](x) z^{-x}\, dz.$$

Here, $\mathbf{Re}(x)$ is the real part of a complex number x.

Proposition 4 *Let $g(z)$ be a locally integrable function in $(0, +\infty)$. Suppose that the Mellin transform $M[f](x)$ of $g(z)$ exists for $a_1 < \mathbf{Re}(x) < a_2$.*

(a) *For any positive integer n,*

$$\begin{aligned} &\mathcal{M}\left[\left(z\frac{\partial}{\partial z}\right)^n g\right](x) = (-x)^n M[g](x), \\ &\mathcal{M}\big[(\log z)^n g\big](x) = \frac{\partial^n M[g]}{\partial x^n}(x). \end{aligned} \tag{49}$$

(b) *For any constant $\xi \neq 0$,*

$$\mathcal{M}\big[g\big(x^{\delta}\big)\big](x) = \begin{cases} \frac{1}{\xi}\mathcal{M}[g(z)](\frac{x}{\xi}) & \text{for } \xi > 0, \\ -\frac{1}{\xi}\mathcal{M}[g(z)](\frac{x}{\xi}) & \text{for } \xi < 0. \end{cases} \tag{50}$$

Proposition 5 *For $\alpha \in \mathbb{C}$ with $\mathbf{Re}(\alpha) > 0$ and $b \in \mathbb{R}$, the inverse Mellin transform of $g(x) = e^{\alpha(x+b)^2}$ is given by*

$$\mathcal{M}^{-1}[g](z) = \frac{1}{2\sqrt{\pi\alpha}} z^b e^{-\frac{1}{4\alpha}(\log z)^2}.$$

Proposition 6 (The Mellin convolution theorem) *Suppose that $g(z)$ and $h(z)$ are locally integrable functions in $(0, \infty)$ and the Mellin transforms $\mathcal{M}[g](x)$ and $\mathcal{M}[h](x)$ exist for $a_1 < \mathbf{Re}(x) < a_2$, where $\mathbf{Re}(x)$ denotes the real part of a complex number x. Then*

$$\begin{aligned} g(z) * h(z) &\triangleq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}[g](x)\mathcal{M}[h](x) z^{-x}\, dx \\ &= \int_0^{\infty} g\left(\frac{z}{u}\right) h(u)\frac{du}{u}, \quad \text{where } a_1 < c < a_2. \end{aligned}$$

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Junkee Jeon and Geonwoo Kim designed the model. Junkee Jeon contributed analysis of the mathematical model. Junkee Jeon proved the Theorems in the paper. Geonwoo Kim carried out the numerical experiments. Junkee Jeon and Geonwoo Kim wrote the paper.

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