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Some spectral domain in approximate point-spectrum-preserving maps on $\mathcal{B}(\mathcal{X})$

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Abstract

Let $\mathcal{X}$ be an infinite-dimensional complex Banach space, $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. Denote the spectral domain by $\sigma_\gamma(T) = \{\lambda \in \sigma_a(T) : T$ that is semi-Fredholm and $asc(T - \lambda I) < \infty\}$. In this paper, we characterize the structure of additive surjective maps $\varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ with $\sigma_\gamma(\varphi(T)) = \sigma_\gamma(T)$ for all $T \in \mathcal{B}(\mathcal{X})$.

Mathematics Subject Classification: Primary 47B48; secondary 47A10

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1 Introduction

The study of preserver problems has a long history and has established many remarkable results in past decades. Preserver problems aim to characterize those linear or nonlinear maps on operator algebras preserving certain properties, subsets or relations ([1–6, 9, 12, 15, 16, 22]). One of the most famous problems in this area is Kaplansky's problem [12] asking whether every surjective unital invertibility preserving linear map between two semisimple Banach algebras is a Jordan homomorphism. This problem was solved in some special cases of semisimple Banach algebras ([10, 13, 23]). Then, Aupetit in [2] solved it for von Neumann algebras. It is known that a spectrum is a very fundamental and key concept in operator theory. Some results about linear or additive maps preserving the spectrum as well as certain parts of the spectrum have been established by many authors ([2, 9, 20, 21]). Recently, many authors are interested in nonadditive preserver problems related to spectral domains of operators. For example, Hajighasemi and Hejazian in [11] characterized the nonlinear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the semi-Fredholm domain, the Fredholm domain, and the Weyl domain respectively. In [4], Bouramdane and Ech-Chérif El Kettani investigated the form of maps preserving some spectral domains of the skew product of operators. As is known, certain parts of a spectrum of operators are introduced to analyze the structure of operators, such as various spectra in the Weyl-type theorem. The Weyl-type theorem can reflect the connections between several spectra, which has been studied for more than one hundred years. There have been numerous significant results in terms of this. Note that some spectral domains play a key role in the research of the Weyl-type theorem and its perturbation [8, 17–19]. Moreover, these spectral domains are at most countable and very "small" subsets of a spectrum in general, such as normal eigenvalues, the semi-Fredholm domain in a spectrum, and so on. Thus, how may these spectral domains influence the structure of automorphisms on the algebra of all bounded linear operators on a Banach space? In [20], the authors characterized additive surjective maps φ on $\mathcal{B}(\mathcal{X})$ that preserve the semi-Fredholm domain in a spectrum, and showed that such a map is an automorphism or an antiautomorphism on $\mathcal{B}(\mathcal{X})$. In [7], Cao discussed the linear surjective maps preserving upper semi-Weyl operators, and showed that their induced maps on the Calkin algebra are Jordan automorphisms. In this paper, we combine the approximate point spectrum with a semi-Fredholm domain of operators, and consider an additive map that preserves the intersection of a semi-Fredholm domain with finite ascent and approximate point spectrum. How does the spectral domain influence the structure of automorphisms on the algebra of all bounded linear operators on a Banach (or Hilbert) space?

Throughout this paper let $\mathcal{X}$ be a complex infinite-dimensional Banach space and $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. For $T \in \mathcal{B}(\mathcal{X})$, we denote by $\mathcal{X}^*$, T^*, $\mathcal{N}(T)$, and $\mathcal{R}(T)$ the dual space of $\mathcal{X}$, the conjugate operator, the null space, and the range of T, respectively. Let $dim\mathcal{N}(T)$ be the dimension of $\mathcal{N}(T)$, and $codim\mathcal{R}(T)$ be the codimension of $\mathcal{R}(T)$. An operator T is called Fredholm if $\mathcal{R}(T)$ is closed, $dim\mathcal{N}(T) < \infty$ and $codim\mathcal{R}(T) < \infty$. Also, an operator T is called semi-Fredholm if $R(T)$ is closed and $dimN(T) < \infty$ or $codim\mathcal{R}(T) < \infty$. If T is a semi-Fredholm operator, then the index of T is denoted by $ind(T) = dimN(T) - codimR(T)$. An operator T is Weyl if it is Fredholm of index zero. Recall that a bounded operator T is said to be bounded below if it is injective and has closed range. For an operator T, the ascent of T is defined by

$$asc(T) = inf\left\{n \in \mathbb{N} : \mathcal{N}\left(T^n\right) = \mathcal{N}\left(T^{n+1}\right)\right\}.$$

If the infimum does not exist, then the $asc(T)$ is defined as ∞. It is known that

$$asc(T) \leq p \Leftrightarrow \mathcal{R}\left(T^p\right) \cap \mathcal{N}(T) = \{0\}$$

for some $p \geq 0$.

Denote the spectrum, the point spectrum, and the approximate point spectrum of T, respectively, by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\},$$

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

In [24, 25], Cao considered property (R), which is a variant of Weyl's theorem. Also, there is a spectral domain that plays an important role in the study of property (R). Now, we define the spectral domain by

$$\sigma_\gamma(T) = \left\{\lambda \in \sigma_a(T) : T \text{ is semi-Fredholm and } asc(T - \lambda I) < \infty\right\}.$$

Note that if $\lambda \in \sigma_\gamma(T)$, then $T - \lambda I$ is a semi-Fredholm operator. Thus, $T - \lambda I$ has closed range. It follows from $\lambda \in \sigma_a(T)$ that $\mathcal{N}(T - \lambda I) \neq \{0\}$. This implies that $\sigma_\gamma(T) \subseteq \sigma_p(T)$.

Moreover, since $asc(T-\lambda I)<\infty$, we have that there exists $\epsilon>0$ such that $T-\lambda I$ is bounded below for all λ with $0<|\lambda-\lambda_0|<\epsilon$. Then, $\lambda\in iso\sigma_a(T)$, which induces that

$$\sigma_\gamma(T)\subseteq iso\sigma_a(T)\subseteq\sigma(T).$$

Thus, the spectral domain $\sigma_\gamma(T)$ is at most countable and is a very "small" subset of a spectrum in general.

In this paper, we will characterize additive surjective maps φ on $\mathcal{B}(\mathcal{X})$ preserving the spectral domain $\sigma_\gamma(T)$ in both directions. The main result is the following Theorem:

Theorem 1.1 *Let $\varphi:\mathcal{B}(\mathcal{X})\to\mathcal{B}(\mathcal{X})$ be a surjective additive map. If $\sigma_\gamma(\varphi(T))=\sigma_\gamma(T)$ for all $T\in\mathcal{B}(\mathcal{X})$, then one of the following assertions holds*:
(1) *there is an invertible operator $A\in\mathcal{B}(\mathcal{X})$ such that*

$$\varphi(T)=ATA^{-1}\quad\textit{for all } T\in\mathcal{B}(\mathcal{X});$$

(2) *there is a bounded invertible linear operator $C:\mathcal{X}^*\to\mathcal{X}$ such that*

$$\varphi(T)=CT^*C^{-1}\quad\textit{for all } T\in\mathcal{B}(\mathcal{X}).$$

In this case, $\mathcal{X}$ must be a reflexive space.

2 Preliminaries

Let $z\in\mathcal{X}$ and $f\in\mathcal{X}^*$, we denote by $z\otimes f$ the bounded linear rank-one operator if both z and f are nonzero. The rank-one operator $z\otimes f$ is defined by $(z\otimes f)x=f(x)z$ for all $x\in\mathcal{X}$. For a subset $\mathcal{M}$ of $\mathcal{X}$, $\bigvee\{\mathcal{M}\}$ denotes the closed subspace spanned by $\mathcal{M}$. We first establish some useful results that are needed for the proof of our main Theorem.

Proposition 2.1 *Let $T\in\mathcal{B}(\mathcal{X})$ be such that $0\in\sigma_\gamma(T)$. If $K\in\mathcal{B}(\mathcal{X})$ is a rank-one operator such that $0\in\sigma_\gamma(T+2K)$, then $0\in\sigma_\gamma(T+K)$ or $0\in\sigma_\gamma(T-K)$.*

Proof From the fact that the class of Fredholm operators is invariant under compact perturbations and $0\in\sigma_\gamma(T)$, we derive that the operators $T+K$, $T-K$ are Fredholm.

Let $K=x\otimes f$, where $x\in\mathcal{X}$ and $f\in\mathcal{X}^*$. Then, it follows from $0\in\sigma_\gamma(T+2K)\subseteq\sigma_p(T+2K)$ that $\mathcal{N}(T+2K)\neq\{0\}$. Moreover, note that $0\in\sigma_\gamma(T)\subseteq\sigma_p(T)$, we also have $\mathcal{N}(T)\neq\{0\}$. We claim that

$$\mathcal{N}(T+K)\neq\{0\}\quad\text{and}\quad\mathcal{N}(T-K)\neq\{0\}.$$

Note that $\mathcal{N}(T)\cap\mathcal{N}(f)\subseteq(\mathcal{N}(T+K)\cap\mathcal{N}(T-K))$. Thus, we assume that $\mathcal{N}(T)\cap\mathcal{N}(f)=\{0\}$. Then, there exist nonzero vectors $u\in\mathcal{N}(T)$ and $v\in\mathcal{N}(T+K)$ such that $f(u)=f(v)=1$. Take $w=u+v$ and $z=3u-v$, then $f(w)=f(z)=2$. Thus, we obtain that w, z are nonzero vectors such that

$$w\in\mathcal{N}(T+K)\quad\text{and}\quad z\in\mathcal{N}(T-K).$$

Thus, we have that the operators $T-K$ and $T+K$ are Fredholm with $\mathcal{N}(T-K)\neq\{0\}$ and $\mathcal{N}(T+K)\neq\{0\}$.

Moreover, note that $0 \in \sigma_\gamma(T)$, then $asc(T) < \infty$. By the Proposition 2.7 in [14], we obtain that $asc(T-K) < \infty$ or $asc(T+K) < \infty$. This implies that $0 \in \sigma_\gamma(T+K)$ or $0 \in \sigma_\gamma(T-K)$. □

The following result will give a necessary condition for operators with the rank not less than one.

Proposition 2.2 *Let $T \in \mathcal{B}(\mathcal{X})$ and $dimR(T) \geq 2$. Then, there exists an operator S such that $0 \in \sigma_\gamma(S)$ and $\sigma_\gamma(S+2T)$, but $0 \notin \sigma_\gamma(S-T)$ and $0 \notin \sigma_\gamma(S+T)$.*

Proof Assume that $dimR(T) \geq 2$. Then, there exist two linearly independent vectors x_1, x_2 such that Tx_1, Tx_2 are linearly independent. Let $\mathcal{L} = \bigvee\{x_1, x_2\}$, $\mathcal{M} = \bigvee\{Tx_1, Tx_2\}$. Then, there are two infinite-dimensional closed subspaces $\mathcal{K}$, $\mathcal{R}$ such that $\mathcal{X} = \mathcal{L} \oplus \mathcal{K} = \mathcal{M} \oplus \mathcal{R}$, with respect to this decomposition, the operator T can be expressed as follows:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}.$$

Note that $dim\mathcal{L} = dim\mathcal{M} = 2$. This implies that

$$\mathcal{R} \cong \mathcal{X}/\mathcal{L} \cong \mathcal{X}/\mathcal{M} \cong \mathcal{K},$$

where "$\cong$" denotes isomorphism. Then, the two closed subspaces $\mathcal{K}$, $\mathcal{R}$ are isomorphic. Thus, we can find an invertible operator $A : \mathcal{R} \to \mathcal{K}$. For the operator $T_{22}A : \mathcal{R} \to \mathcal{R}$, there exists a complex number $\mu \in \mathbb{C}$ such that the operators $\mu I + T_{22}A$, $\mu I + 2T_{22}A$, and $\mu I - 2T_{22}A$ are invertible. It follows that the operators $\mu A^{-1} + T_{22}$, $\mu A^{-1} + 2T_{22}$ and $\mu A^{-1} - 2T_{22}$ are invertible. With respect to the decomposition $\mathcal{X} = \mathcal{L} \oplus \mathcal{K} = \mathcal{M} \oplus \mathcal{R}$, we now define an operator $S \in \mathcal{B}(\mathcal{X})$ by

$$S = \begin{pmatrix} S_{11} & 0 \\ 0 & \mu A^{-1} \end{pmatrix},$$

where

$$\begin{cases} S_{11}x_1 = 0, \\ S_{11}x_2 = -2Tx_2. \end{cases}$$

Then,

$$S + 2T = \begin{pmatrix} S_{11} + 2T_{11} & 2T_{12} \\ 0 & \mu A^{-1} + 2T_{22} \end{pmatrix}.$$

We can verify that $0 \in \sigma_\gamma(S)$ and $0 \in \sigma_\gamma(S+2T)$. Moreover,

$$S \pm T = \begin{pmatrix} S_{11} \pm T_{11} & \pm T_{12} \\ 0 & \alpha A^{-1} \pm T_{22} \end{pmatrix}.$$

Note that $(S_{11} \pm T_{11})x_1 = \pm Tx_1$, $(S_{11} + T_{11})x_2 = -Tx_2$ and $(S_{11} - T_{11})x_2 = -3Tx_2$. This implies that the operators $S + T$ and $S - T$ are invertible operators. Therefore,

$$0 \notin \sigma_\gamma(S + T) \quad \text{and} \quad 0 \notin \sigma_\gamma(S - T). \qquad \square$$

The next Proposition gives a criterion of two operators being equal by rank-one operators and the spectral domain $\sigma_\gamma(\bullet)$.

Proposition 2.3 *Let $A, B \in \mathcal{B}(\mathcal{X})$. If $\sigma_\gamma(A + F) = \sigma_\gamma(B + F)$ for all rank-one operator $F \in \mathcal{B}(\mathcal{X})$. Then, $A = B$.*

Proof For any nonzero vector $x \in \mathcal{X}$, let $\mathcal{G} = \{f \in \mathcal{X}^* | f(x) = 1\}$. We choose a scalar $\alpha \in \mathbb{C}$ such that $\alpha > \|A\| + \|B\|$. For any $f \in \mathcal{G}$, we define an operator

$$F_f = (A - \alpha I)x \otimes f.$$

Then, $F_f x = Ax - \alpha x$. Thus, $\alpha \in \sigma_p(A - F_f)$. Note that

$$\|A - F_x\| \geq \alpha > \|A\| \geq \|A\|_e = \|A - F_x\|_e,$$

where $\|A\|_e$ is the essential norm of A. We derive that $\alpha \in \sigma_\gamma(A - F_f)$, and so $\alpha \in \sigma_\gamma(B - F_f)$. It follows that $\alpha \in \sigma_p(B - F_f)$. Then, there exists a nonzero vector $y_f \in \mathcal{X}$ such that $(B - F_f)y_f = \alpha y_f$. We can obtain

$$y_f = f(y_f)(B - \alpha I)^{-1}(A - \alpha I)x.$$

Putting $y = (B - \alpha I)^{-1}(A - \alpha I)x$, it follows from $(B - F_f)y_f = \alpha y_f$ that $(B - F_f)y = \alpha y$ for any $f \in \mathcal{G}$. We claim that x and y are linearly dependent. Indeed, if x and y are linearly independent, then there exists some $f' \in \mathcal{G}$ such that $f'(y) = 0$. This implies that $By = \alpha y$. This is a contradiction with the fact $\alpha > \|A\| + \|B\|$. It follows that $(B - F_f)x = \alpha x$. Therefore, we obtain $Ax = Bx$. From the arbitrariness of x, we have $A = B$. $\square$

3 Proof of main result

In the following, we will give the proof of the main theorem and show the result in four steps.

Step 1. φ is injective.

Let $\varphi(T) = 0$. If $T \neq 0$, then we can choose $z_0 \in \mathcal{X}$ such that $Tz_0 \neq 0$. It follows that there is $f \in \mathcal{X}^*$ such that $f(z_0) = 1$ and $f(Tz_0) \neq 0$. Choose a scalar $\mu_0 \in \mathbb{C}$ such that $|\mu_0| > \|T\|$. Now, we define a rank-one operator $U \in \mathcal{B}(\mathcal{X})$ as follows

$$U = (Tz_0 + \mu_0 z_0) \otimes f.$$

Then, we have $(U - T)(z_0) = \mu_0 z_0$. This implies that $\mu_0 \in \sigma_\gamma(U - T)$. Thus,

$$\mu_0 \in \sigma_\gamma\big(\varphi(U - T)\big) = \sigma_\gamma\big(\varphi(U)\big) = \sigma_\gamma(U).$$

However, $\sigma_\gamma(U) = \{\mu_0 + f(Tz_0)\}$, which is a contradiction. Thus, φ is injective.

Step 2. φ preserves rank-one operators in both directions.

Let $T \in \mathcal{B}(\mathcal{X})$ be such that $dimR(T) \geq 2$. From Proposition 2.2, we derive that there exists an operator S such that

$$0 \in \sigma_\gamma(S) \quad \text{and} \quad 0 \in \sigma_\gamma(S + 2T).$$

However,

$$0 \notin \sigma_\gamma(S - T) \quad \text{and} \quad 0 \notin \sigma_\gamma(S + T).$$

Then,

$$0 \in \sigma_\gamma\big(\varphi(S)\big) \quad \text{and} \quad 0 \in \sigma_\gamma\big(\varphi(S) + 2\varphi(T)\big).$$

Moreover,

$$0 \notin \sigma_\gamma\big(\varphi(S) + \varphi(T)\big) \quad \text{and} \quad 0 \notin \sigma_\gamma\big(\varphi(S) - \varphi(T)\big).$$

Then, by Proposition 2.1, we have that $dimR(\varphi(T)) \geq 2$. Since φ is bijective and φ^{-1} has the same property as φ, it follows that φ preserves the set of operators of rank one in both directions.

Step 3. φ preserves idempotents of rank one and their linear spans in both directions.

Let $x \otimes f$ be an idempotent of rank one and $\varphi(x \otimes f) = y \otimes g$. Then, $\sigma_\gamma(x \otimes f) = \{1\}$, and so $\sigma_\gamma(y \otimes g) = \{1\}$. This implies that $\sigma(y \otimes g) = \{0, 1\}$. Then, $y \otimes g$ is an idempotent of rank one. That is, $g(y) = 1$. Thus, φ preserves idempotents of rank one in both directions. In the following, we will prove $\varphi(\mathbb{C}x \otimes f) \subseteq \mathbb{C}y \otimes g$. We choose a nonzero vector $z \in \mathcal{X}$ and a linear function $h \in \mathcal{X}^*$ such that

$$g(z) = 0, \qquad h(y) = 0 \quad \text{and} \quad h(z) = 1.$$

For the operator $y \otimes h$ and $z \otimes g$, since φ is surjective, there are two rank-one operators $y_0 \otimes h_0$ and $z_0 \otimes g_0$ such that

$$\varphi(y_0 \otimes h_0) = y \otimes h \quad \text{and} \quad \varphi(z_0 \otimes g_0) = z \otimes g.$$

Note that

$$\varphi(x \otimes f + y_0 \otimes h_0) = y \otimes g + y \otimes h, \qquad \varphi(x \otimes f + z_0 \otimes g_0) = y \otimes g + z \otimes g.$$

Then, $x \otimes f + y_0 \otimes h_0$ and $x \otimes f + z_0 \otimes g_0$ are two rank-one operators. Thus, for any nonzero $\lambda \in \mathbb{C}$, $\lambda x \otimes f + y_0 \otimes h_0$ and $\lambda x \otimes f + z_0 \otimes g_0$ are also rank-one operators. Fix a nonzero complex number λ, we let $\varphi(\lambda x \otimes f) = \lambda y_\lambda \otimes g_\lambda$, where $y_\lambda \otimes g_\lambda$ is a rank-one idempotent. Then, $\varphi(\lambda x \otimes f + y_0 \otimes h_0) = \lambda y_\lambda \otimes g_\lambda + y \otimes h$ is also rank one. We obtain that y_λ and y are linearly dependent or the same is true for g_λ and h. We claim that y_λ and y are linearly dependent. Indeed, suppose on the contrary, we can obtain g_λ and h are linearly dependent. Then, there exists a nonzero $\alpha_\lambda \in \mathbb{C}$ such that $g_\lambda = \overline{\alpha_\lambda} h$. Thus,

$$\varphi(\lambda x \otimes f + z_0 \otimes g_0) = \alpha_\lambda \lambda y_\lambda \otimes h + z \otimes g.$$

Since h and g are linearly independent, we choose $\beta_\lambda \in \mathbb{C}$ such that $y_\lambda = \beta_\lambda z$, and so $\varphi(\lambda x \otimes f) = \alpha_\lambda \beta_\lambda \lambda z \otimes h$, where $z \otimes h$ is a rank-one idempotent. Note that $\varphi(\lambda x \otimes f) = \lambda y_\lambda \otimes g_\lambda$, where $y_\lambda \otimes g_\lambda$ is a rank-one idempotent. Thus, $\alpha_\lambda \beta_\lambda \lambda = \lambda$. It follows that

$$\varphi(\lambda x \otimes f) = \lambda z \otimes h.$$

Since φ is surjective, there is a rank-one operator $e_0 \otimes k_0$ such that $\varphi(e_0 \otimes k_0) = z \otimes h$ and $k_0(e_0) = 1$. Since $y \otimes g + z \otimes h$ is a projection of rank two, we have $x \otimes f + e_0 \otimes k_0$ is a rank-two operator. Thus, $\lambda x \otimes f + e_0 \otimes k_0$ is also a rank-two operator. However,

$$\varphi(\lambda x \otimes f + e_0 \otimes k_0) = (\lambda + 1) z \otimes h.$$

This contradicts the fact that φ preserves the set of operators of rank one in both directions. Therefore, we have that y_λ and y are linearly dependent. Then, there exists a nonzero $\gamma_\lambda \in \mathbb{C}$ such that $y_\lambda = \gamma_\lambda y$. Since $\varphi(\lambda x \otimes f + z_0 \otimes g_0) = \gamma_\lambda \lambda y \otimes g_\lambda + z \otimes g$ and the two vectors y and z are linearly independent, we can find a nonzero $\mu_\lambda \in \mathbb{C}$ such that $g_\lambda = \overline{\mu_\lambda} g$. Then, $\varphi(\lambda x \otimes f) = \gamma_\lambda \mu_\lambda \lambda y \otimes g$, where $y \otimes g$ is a rank-one idempotent. We know that $\varphi(\lambda x \otimes f) = \lambda y_\lambda \otimes g_\lambda$, where $y_\lambda \otimes g_\lambda$ is also a rank-one idempotent. Thus, $\gamma_\lambda \mu_\lambda \lambda = \lambda$. It follows that

$$\varphi(\lambda x \otimes f) = \lambda y \otimes g.$$

Therefore, φ preserves idempotents of rank one and their linear spans in both directions.

From the main result of [16] it gives that

(1) There is an invertible operator $A \in \mathcal{B}(\mathcal{X})$ such that $\varphi(T) = AFA^{-1}$ for all finite-rank operators $F \in \mathcal{B}(\mathcal{X})$, or

(2) There is a bounded invertible linear operator $C : \mathcal{X}^* \to \mathcal{X}$ such that $\varphi(T) = CF^*C^{-1}$ for all finite-rank operators $F \in \mathcal{B}(\mathcal{X})$. In this case $\mathcal{X}$ must be a reflexive space.

Step 4. φ takes the desired from.

Assume that (1) holds. Let $T \in \mathcal{B}(\mathcal{X})$ and for any rank-one operator F, we have

$$\begin{aligned} \sigma_\gamma(T + F) &= \sigma_\gamma\big(\varphi(T) + \varphi(F)\big) \\ &= \sigma_\gamma\big(\varphi(T) + AFA^{-1}\big) \\ &= \sigma_\gamma\big(A\big(A^{-1}\varphi(T)A + F\big)A^{-1}\big) \\ &= \sigma_\gamma\big(A^{-1}\varphi(T)A + F\big). \end{aligned}$$

Then, we obtain that $T = A^{-1}\varphi(T)A$ by the Proposition 2.3. Consequently, $\varphi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(\mathcal{X})$.

If (2) holds, then we similarly have that $\varphi(T) = CT^*C^{-1}$ for all $T \in \mathcal{B}(\mathcal{X})$. □

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The author characterize the structure of additive surjective maps on B(X) preserving the spectral domain in both directions.

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