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A reduced Galerkin finite element formulation based on proper orthogonal decomposition for the generalized KDV-RLW-Rosenau equation

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Abstract

This paper investigates reduced-order modeling of the Korteweg de Vries regularized long-wave Rosenau (KdV-RLW-Rosenau) equation using semi- and fully-discrete B-spline Galerkin approximations. The approach involves the application of a proper orthogonal decomposition (POD) method to a Galerkin finite element (GFE) formulation, resulting in a POD GFE formulation with lower dimensions and high accuracy. The error between the reduced POD GFE solution and the traditional GFE solution is analyzed using the Crank-Nicolson method. Numerical examples show that the theoretical conclusions are consistent with the results of the numerical computation, and that the POD method is effective and feasible.

Keywords: Generalized KdV-RLW-Rosenau equation; Proper orthogonal decomposition; Quadratic B-spline Galerkin finite element method; Crank-Nicolson method; Error estimates

1 Introduction

The study of nonlinear waves is helpful to clarify the law of motion changes of physical systems under nonlinear effects, and to reasonably explain related natural phenomena. Therefore, many models have been proposed to describe their behaviors. In fact, we can find various mathematical descriptions of wave dynamics, such as regular long waves (RLW), Korteweg-de Vries (KdV), and Rosenau equations; see, e.g., [1–3]. Among them Benjamin et al. [1] determined the exact expression of the RLW model under the constraints of initial conditions and boundary conditions and proved the existence and uniqueness of the solution of the RLW model. Bon et al. [4] adopted RLW modeling for small-amplitude long waves on the water surface; As pointed out by Abdulloev et al. [5], an important feature of the RLW problem is that the collision between two solitary waves either produces a sinusoidal solution or secondary solitary waves.

It is worth mentioning that the KdV equation has applications in many fields of physics and is widely used in the description of dynamic effects such as ion acoustics, magnetohydrodynamic waves, and longitudinal astigmatism; see [6–11] for details; moreover, the existence, regularity, and convergence of solutions for KdV-type equations were proved in [12–14]. Recently, Kaya and Aassila in [15] computed explicit solutions to the KdV equation with initial conditions using the Adomian decomposition method. Özer et al. [16] solved the KdV equation by an analytical-numerical method.

Rosenau proposed a new model to solve the KdV equation cannot describe the wave–full-wave and wave–wave interaction, it is called the Rosenau equation, used to describe the dynamic behavior of dense discrete systems; see, e.g., [3, 9]. Later, Zuo [8] solitary waves and periodic solutions of the Rosenau-KdV model were studied. Barreto et al. [17] discussed the existence of solutions of the Rosenau equation and using Galerkin, multipliers, and energy estimation techniques, obtained the Rosenau equation with a plus sign in the advection-like term in the motion domain.

It is known that Rosenau equation adds a dissipative effect term u_{xx}, which is usually called Rosenau-RLW (or Rosenau-Burgers) equation. A study of the Cauchy problem and the stability of traveling or diffusing waves for the Rosenau-RLW equations is given in [18–21]. Particularly, Piao et al. [22] reported the error of quadratic B-spline FEM solution for the Rosenau-RLW equation. Besides, many researchers have solved Rosenau-RLW equation by using different methods, such as B-spline collocation, GFE, finite difference, conservative difference scheme; see, e.g., [24–28].

On the other hand, many scholars have studied the (inviscid) Rosenau-KdV-RLW equation both analytically and numerically. Here are the following examples: Razborova et al. [29] and [30] determined their impulsive and kink (or topological soliton) solutions using sech and tanh ansatzs, and the singular solutions of the same equations using cosech ansatz. In addition to this, Razborova et al. [31] found a third invariant of the (inviscid) Rosenau-KdV-RLW equation under the condition that the mass and energy of the homogeneous boundary conditions are constant Lying symmetry. In addition, Wong saijai et al. used the cubic level method to solve the Rosenau-KdV-RLW equation in [32] and performed an implicit finite difference method for the equation and compared the numerical results obtained with the sine-cosine function approximate. In [33], the inviscid Rosennau-KdV-RLW equations were numerically solved using the pseudo-compact method and the trapezoidal scheme in time. In [34], A. Biswas et al. used the quintic B-spline finite element method, a local-structure preserving technique that includes multi-symplectic and energy- and momentum-preserving procedures [35], a linearly implicit conservative finite difference technique [36], and a local meshless technique and the finite difference method [37].

In this paper, we consider the higher order equation of wave propagation in unidirectional water:

$$u_t - \mu u_{xx} - \delta u_{xxt} + \eta u_{xxxx} + \nu u_{xxxxt} + \theta u_{xxx} + f_x(u) = 0, \quad x \in \Omega, t \in (0, T], \tag{1.1a}$$

with the initial condition:

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.1b}$$

and boundary conditions:

$$u(x, t) = 0, \qquad u_x(x, t) = 0, \quad x \in \partial\Omega, t \in (0, T], \tag{1.1c}$$

where $f(u) = \alpha u + \frac{\beta}{p+1}u^{p+1}$, $u(x,t)$, $t > 0$ and $\Omega \in (0,1)$ denote the (nondimensional) amplitude, time coordinate, and space coordinate of the wave, respectively. δ and ν are positive constants, and α, β, μ, η, θ are arbitrary parameters. $p \geq 1$ is a positive integer, which makes $f(u)$ a non-linear term. In equation (1.1a), different types of equations are obtained by assigning appropriate values to the parameters. For example, if $\mu = \eta = \nu = \theta = 0$, $\alpha = \beta = \delta = 1$ and $p = 1$, we can get the RLW equation; if $\alpha = \mu = \delta = \eta = \nu = 0$, $\beta = \theta = 1$ and $p = 1$, we can get the KdV equation; if $\mu = \delta = \eta = 0$, $\alpha = \beta = \nu = \theta = 1$ and $p = 1$, we can get the Rosenau-KdV equation, etc.

Note that solving numerical solutions of nonlinear complex systems requires high storage and CPU costs. Solving complex, turbulent, and chaotic systems remains a daunting task for us, even with optimal mesh generators, discretization schemes, and solution algorithms. Therefore, how to simplify the actual calculation load and save time and resource demand is an important problem. In the simplification process, it is also necessary to ensure that the numerical solution is accurate and efficient enough. In addition, it makes sense to use the reduced-order method if the underlying PDE must be solved multiple times, i.e. under optimal control or in the case of different parameter studies using the same PDE; see, e.g., [38–40].

Actually, among many reduced-order modeling techniques, the most frequently used method is based on a proper orthogonal decomposition (POD) technique. In other words, POD is a technique that provides sufficient approximation to represent fluid flows with fewer degrees of freedom, i.e., low-dimensional models that reduce computing load and save memory requirements(see [41]). Indeed, POD in combination with Galerkin projection has been used for many years to formulate reduced-order modelings for dynamic systems; see, e.g., [23, 42–56].

Compared with the work [57], in which the authors solved solution of the Eq. (1.1a)–(1.1c) with compact method in the domain $(-\infty, +\infty)$, in this article, we will mainly discuss the theoretical analysis and present a quadratic B-spline GFE approximation for Eq. (1.1a)–(1.1c), and further study the reduced-order modeling of Eq. (1.1a)–(1.1c) based on POD GFE approximation.

The remainder of this paper is organized as follows: The existence and uniqueness of weak solutions of Eq. (1.1a)–(1.1c) are proved in Sect. 2. In Sect. 3, the semidiscrete B-spline Galerkin approximation is discussed, and the second-order spatial error estimation in $L_\infty(\Omega)$ norm is derived. In Sect. 4, a fully discrete scheme based on the Crank-Nicolson method is proposed and the second-order accuracy of the scheme in the time direction is proved. In Sect. 5, the method for generating POD basis is introduced and the reduced-order modeling is constructed. The error estimates between the reduced-order modeling solution and the usual GFE solution are deduced below. In Sect. 6, some numerical experiments are given to verify the effectiveness and accuracy of the proposed method. Finally, the conclusion of this paper is given.

2 Existence and uniqueness

For open interval $\Omega = (0,1)$, $H^k(\Omega)$ be the standard Sobolev space of real-valued functions defined on Ω, so $H_0^k = \{v \in H^k(\Omega) : \frac{d^j v}{dx^j} = 0 \text{ on } \partial\Omega, j = 0, 1, \ldots, k-1\}$, where k is a non-negative integer. Let $L^2(\Omega)$ be the usual Hilbert space on Ω whose scaler product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The inner product of $L^2(\Omega)$ denoted by $(v,w) = \int_\Omega vw\,dx$. The norms of $L^\infty(\Omega)$ and $H^k(\Omega)$ are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_k$.

For simplicity, let $X = H_0^2(\Omega)$. At the same time, we assume that u is enough smooth in time. Find $u(\cdot,t) \in X$ for $t \in (0,T]$ such that the following weak formulation of equation (1.1a)–(1.1c) is given by:

$$\begin{cases} (u_t,\chi) + \mu(u_x,\chi_x) + \delta(u_{xt},\chi_x) + \eta(u_{xx},\chi_{xx}) \\ \quad + \nu(u_{xxt},\chi_{xx}) - \theta(u_{xx},\chi_x) = (f(u),\chi_x), \quad \forall \chi \in X, \\ u(0) = u_0. \end{cases} \tag{2.1}$$

And let C represent a positive constant independent of step size h and integer k, which may have different values in different cases.

Theorem 1 *For* $\forall T > 0$, *let* $u(x,0) = u_0 \in X$, (2.1) *exists a unique solution* u. *Let* $\|u\|_{L^\infty(H^2(\Omega))} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^2(\Omega)}$, *there exists a constant* C *depending on* T *so that*:

$$\|u\|_{L^\infty(H^2(\Omega))} \le C\|u_0\|_2. \tag{2.2}$$

Proof For the existence of (2.1), let $\{v_i\}_{i=1}^\infty$ be a orthogonal basis for X and $V^m = \operatorname{span}\{v_1, v_2, \ldots, v_m\}$. We define $u^m(t) = \sum_{i=1}^m g_i(t)v_i \in V^m$. Then for each $t > 0$, it satisfies:

$$\begin{cases} (u_t^m,\chi) + \mu(u_{x,}^m,\chi_x) + \delta(u_{xt}^m,\chi_x) + \eta(u_{xx}^m,\chi_{xx}) \\ \quad + \nu(u_{xxt}^m,\chi_{xx}) - \theta(u_{xx}^m,\chi_x) = (f(u^m),\chi_x), \\ u^m(0) = u_{0,m}, \end{cases} \tag{2.3}$$

where $u_{0,m}$ is the orthogonal projection of u_0 onto V^m and $u_{0,m}$ converges to u_0 in X. As a result, (2.3) becomes a system of nonlinear ODEs. Thus, using Picard's existence theorem, there is a positive time $t_m \in (0,T]$ such that the nonlinear system has a unique solution u^m in $(0,t_m]$.

The following a priori bounds are needed because we want to prove the global existence using the continuation argument. Setting $\chi = u^m$ in (2.3) and $u^m = 0$ on $\partial\Omega$, we have:

$$\begin{aligned} -\int_0^1 u_x^m f(u^m)\,dx &= \int_0^1 u^m f_x(u^m)\,dx \\ &= \int_0^1 u^m \left(\alpha u^m + \frac{\beta}{p+1}(u^m)^{p+1}\right)_x dx \\ &= \frac{\alpha}{2}\int_0^1 \left((u^m)^2\right)_x dx + \frac{\beta}{(p+1)(p+2)}\int_0^1 \left((u^m)^{p+2}\right)_x dx \\ &= 0. \end{aligned}$$

From (2.3) and $\delta, \nu > 0$, we get the following result:

$$\frac{d}{dt}\left[\|u^m\|^2 + \delta\|u_x^m\|^2 + \nu\|u_{xx}^m\|^2\right] \le C\left(\|u^m\|^2 + \|u_x^m\|^2 + \|u_{xx}^m\|^2\right). \tag{2.4}$$

Integrating (2.4) from 0 to t gives

$$\min\{1,\delta,\nu\}\left(\|u^m\|^2 + \|u_x^m\|^2 + \|u_{xx}^m\|^2\right)$$

$$\le \max\{1,\delta,\nu\}\left(\left\|u^m(0)\right\|^2+\left\|u_x^m(0)\right\|^2+\left\|u_{xx}^m(0)\right\|^2\right)$$
$$+C\int_0^t\left(\left\|u^m(s)\right\|^2+\left\|u_x^m(s)\right\|^2+\left\|u_{xx}^m(s)\right\|^2\right)ds.$$

Thus, we have:

$$\begin{aligned}&\left\|u^m\right\|^2+\left\|u_x^m\right\|^2+\left\|u_{xx}^m\right\|^2\\&\quad\le C\biggl(\left\|u^m(0)\right\|^2+\left\|u_x^m(0)\right\|^2+\left\|u_{xx}^m(0)\right\|^2\\&\qquad+\int_0^t\left(\left\|u^m(s)\right\|^2+\left\|u_x^m(s)\right\|^2+\left\|u_{xx}^m(s)\right\|^2\right)ds\biggr).\end{aligned}$$

By using Gronwall's inequality [16, 17], we get:

$$\left\|u^m(t)\right\|_2^2\le C\mathrm{e}^{Ct}\left\|u^m(0)\right\|_2^2. \tag{2.5}$$

For $t_m\in(0,T]$, using Sobolev's inequality theorem, we get:

$$\left\|u^m\right\|_{L^\infty(L^\infty(\Omega))}\le C\left\|u^m\right\|_{L^\infty(H^2(\Omega))}\le C(T)\|u_{0,m}\|_2. \tag{2.6}$$

Setting $\chi=u_t^m$ in (2.3), we get:

$$\begin{aligned}&\left(u_t^m,u_t^m\right)+\mu\left(u_x^m,u_{xt}^m\right)+\delta\left(u_{xt}^m,u_{xt}^m\right)+\eta\left(u_{xx}^m,u_{xxt}^m\right)\\&\quad+\nu\left(u_{xxt}^m,u_{xxt}^m\right)-\theta\left(u_{xx}^m,u_{xt}^m\right)=\left(f\left(u^m\right),u_{xt}^m\right).\end{aligned}$$

Using Young's inequality $ab\le\varepsilon a^2+\frac{1}{4\varepsilon}b^2$ with $a,b\in R$, $\varepsilon>0$, we have:

$$\begin{aligned}&\left\|u_t^m\right\|^2+\left\|u_{xt}^m\right\|^2+\left\|u_{xxt}^m\right\|^2\\&\quad\le C\biggl[\varepsilon(\theta+\mu+1)\left\|u_{xt}^m\right\|^2+\varepsilon\eta\left\|u_{xxt}^m\right\|^2+\frac{\mu}{4\varepsilon}\left\|u_x^m\right\|^2\\&\qquad+\frac{1}{4\varepsilon}(\theta+\eta)\left\|u_{xx}^m\right\|^2+\frac{1}{4\varepsilon}\left\|f\left(u^m\right)\right\|^2\biggr]\\&\quad\le C\left(\left\|u_x^m\right\|^2+\left\|u_{xx}^m\right\|^2+\left\|f\left(u^m\right)\right\|^2\right),\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}\left\|f\left(u^m\right)\right\|^2&=\int_\Omega\left(\alpha u^m+\frac{\varepsilon}{p+1}\left(u^m\right)^{p+1}\right)^2dx\\&\le 2\alpha^2\int_\Omega\left(u^m\right)^2dx+\frac{2\varepsilon^2}{(p+1)^2}\int_\Omega\left(u^m\right)^{2p+2}dx\\&\le C||u_{0,m}\|_2^2\left(1+\|u_{0,m}\|_2^{p+1}\right).\end{aligned} \tag{2.8}$$

With (2.5) and (2.8), the inequality (2.7) can be reduced to $\|u_t^m\|_{L^\infty(H^2)} \leq C\|u_{0,m}\|_2$. Then, let i be fixed and $m > i$, in $L^\infty(0,T)$ space, there exist the following convergence sequences:

$$\begin{aligned}&(u^m, v_i) \to (u, v_i), (u_t^m, v_i) \to (u_t, v_i), (u_x^m, v_{i,x}) \to (u_x, v_{i,x}), (u_{xt}^m, v_{i,x}) \to (u_{xt}, v_{i,x}),\\&(u_{xx}^m, v_{i,x}) \to (u_{xx}, v_{i,x}), (u_{xxt}^m, v_{i,xx}) \to (u_{xxt}, v_{i,xx}), (f(u^m), v_{i,x}) \to (f(u), v_{i,x}).\end{aligned}$$

Thus, we have:

$$(u_t, v_i) + \mu(u_x, v_{i,x}) + \delta(u_{xt}, v_{i,x}) + \eta(u_{xx}, v_{i,xx}) + \nu(u_{xxt}, v_{i,xx}) - \theta(u_{xx}, v_{i,x}) = (f(u), v_{i,x}).$$

The existence of (2.1) follows from the denseness of the basis v_i in X.

For the uniqueness of (2.1), suppose u and v are two solutions of (2.1) and let $w = u - v$. We get:

$$\begin{aligned}&(w_t, \chi) + \mu(w_x, \chi_x) + \delta(w_{xt}, \chi_x) - \theta(w_{xx}, \chi_x)\\&\quad + \eta(w_{xx}, \chi_{xx}) + \nu(w_{xxt}, \chi_{xx}) = (f(u) - f(v), \chi_x), \quad \chi \in X.\end{aligned} \tag{2.9}$$

As $\|u\|_\infty$ and $\|v\|_\infty$ are bounded, we can get that $f(u)$ satisfies the Lipschitz condition. Setting $\chi = w$ in (2.9), we have:

$$\begin{aligned}&\|w(t)\|^2 + \|w_x(t)\|^2 + \|w_{xx}(t)\|^2\\&\quad \leq C\int_0^t \left(\|w(s)\|^2 + \|w_x(s)\|^2 + \|w_{xx}(s)\|^2\right) ds.\end{aligned} \tag{2.10}$$

According to Gronwall's inequality, we have $\|w(t)\|^2 + \|w_x(t)\|^2 + \|w_{xx}(t)\|^2 = 0$, that is to say, $w(t) = 0$, i.e., (2.1) has a unique solution. This concludes the proof. □

3 Semi discrete quadratic B-spline Galerkin approximation

The Galerkin finite element method is an effective method for solving differential equations. For fourth-order differential equations, the Galerkin finite element method with B-spline basis functions can achieve desired results. Divide the interval $\Omega = [0,1]$ uniformly and denote the partition as $0 = x_0 < x_1 < \cdots < x_I = 1$ with $J_i = (x_{i-1}, x_i)$ and $h = \max_{1\leq i\leq I}(x_i - x_{i-1})$. A subspace with respect to X is defined as follows:

$$X_h = \{v \in C^1(\Omega), v|_{J_i} \in P_2(J_i), i = 1, \ldots, I, v|_{\partial\Omega} = 0, v'|_{\partial\Omega} = 0\},$$

where $P_2(J_i)$ represents the set of polynomial degree less than or equal to 2 on J_i.

For the function $u_h : [0,T] \to X_h$, define the semidiscrete Galerkin approximation of (1.1.b) with initial value $u_h(0) = u_{h0}$:

$$\begin{aligned}&(u_{ht}, \chi) + \mu(u_{hx}, \chi_x) + \delta(u_{hxt}, \chi_x) - \theta(u_{hxx}, \chi_x)\\&\quad + \eta(u_{hxx}, \chi_{xx}) + \nu(u_{hxxt}, \chi_{xx}) = (f(u_h), \chi_x), \chi \in X_h,\end{aligned} \tag{3.1}$$

where $u_{h0} \in X_h$ is an appropriate approximation to u_0.

Then, the finite element space satisfies the following approximate properties, as shown in [58–60].

Lemma 1 *Assume $v \in H^4(\Omega) \cap X$ and $\chi \in X_h$, then there exists a constant C independent of h, such that*:

$$\|v - \chi\|_2 \leq Ch^2 \|v\|_4. \tag{3.2}$$

With $t > 0$, we use Picard's theorem and the continuation argument to prove the existence of a unique solution u^h to (3.1). Let $\chi = u_h$ in (3.1), with the same approach as the proof (2.6), the following priori bounds can be obtained:

$$\|u_h\|_{L^\infty(L^\infty(\Omega))} \leq C\|u_h\|_{L^\infty(H^2(\Omega))} \leq C\|u_{h0}\|_2. \tag{3.3}$$

To analyze the convergence of the semidiscrete scheme (3.1), we define the bilinear form:

$$\mathcal{A}(w,v) = (w,v) + \delta(w_x, v_x) + \nu(w_{xx}, v_{xx}), \quad \forall w, v \in X.$$

Let $\tilde{u}$ be an auxiliary projection of u on subspace X_h, as in [15], defined by the following formula:

$$\mathcal{A}(u - \tilde{u}, \chi) = 0, \quad \forall \chi \in X_h. \tag{3.4}$$

Now we use the standard error decomposition $e = u - u_h = \xi - \phi$, where $\xi = u - \tilde{u}$ and $\phi = u_h - \tilde{u}$.

In Lemma 2 below, we obtain error estimates for ξ and ξ_t.

Lemma 2 *Assume $u \in H^4(\Omega) \cap X$ and $t \in [0,T]$, then there exists a constant C independent of h, such that*

$$\|\xi\|_2 \leq Ch^2\|u\|_4, \qquad \|\xi_t\|_2 \leq Ch^2 \|u_t\|_4.$$

Proof By (3.4) and lemma 1, we have:

$$\begin{aligned} \min\{1,\delta,\nu\}\|u - \tilde{u}\|_2^2 &\leq \mathcal{A}(u - \tilde{u}, u - \tilde{u}) \\ &= \mathcal{A}(u - \tilde{u}, u - \chi) \\ &\leq \max\{1,\delta,\nu\}\|u - \tilde{u}\|_2 \|u - \chi\|_2. \end{aligned} \tag{3.5}$$

Thus, $\|u - \tilde{u}\|_2 \leq C\|u - \chi\|_2 \leq Ch^2\|u\|_4$, i.e.,

$$\|\xi\|_2 \leq Ch^2\|u\|_4. \tag{3.6}$$

Let $\hat{u}$ be the project of u_t on X_h defined as:

$$\mathcal{A}(u_t - \hat{u}, \chi) = 0, \quad \chi \in X_h. \tag{3.7}$$

Moreover, by (3.2), we know:

$$\|u_t - \hat{u}\|_2 \leq Ch^2\|u_t\|_4. \tag{3.8}$$

From (3.4) and (3.7), it follows

$$\begin{aligned}\min\{1,\delta,\nu\}\|\tilde{u}_t-\hat{u}\|_2^2 &\le \mathcal{A}(\tilde{u}_t-\hat{u},\tilde{u}_t-\hat{u})\\ &= \mathcal{A}(\tilde{u}_t-u_t,\tilde{u}_t-\hat{u})\\ &= -\frac{d}{dt}\mathcal{A}(u-\tilde{u},\tilde{u}_t-\hat{u})\\ &= 0.\end{aligned} \tag{3.9}$$

According to (3.8) and (3.9), we get $\|u_t-\tilde{u}_t\|_2 \le \|u_t-\hat{u}\|_2 + \|\hat{u}-\tilde{u}_t\|_2 \le Ch^2\|u_t\|_4$. Consequently, it holds

$$\|\xi_t\|_2 \le Ch^2\|u_t\|_4. \qquad \square$$

Theorem 2 *For $t\in[0,T]$, assume that u_{h0} is the X projection of u_0 onto X_h, there exists a positive constant C independent of h such that*:

$$\|u-u_h\|_\infty \le C(u,T)h^2.$$

Proof By subtracting (3.1) from (2.1), we can get:

$$\begin{aligned}&(\phi_t,\chi)+\mu(\phi_x,\chi_x)+\delta(\phi_{xt},\chi_x)-\theta(\phi_{xx},\chi_x)+\eta(\phi_{xx},\chi_{xx})+\nu(\phi_{xxt},\chi_{xx})\\ &\quad=(\xi_t,\chi)+\mu(\xi_x,\chi_x)+\delta(\xi_{xt},\chi_x)-\theta(\xi_{xx},\chi_x)+\eta(\xi_{xx},\chi_{xx})\\ &\qquad+\nu(\xi_{xxt},\chi_{xx})+\big(f(u_h)-f(u),\chi_x\big).\end{aligned} \tag{3.10}$$

Then, setting $\chi=\phi$ in (3.10), we obtain:

$$\begin{aligned}&(\phi_t,\phi)+\delta(\phi_{xt},\phi_x)+\nu(\phi_{xxt},\phi_{xx})\\ &\quad\le(\xi_t,\phi)+\mu(\xi_x,\phi_x)+\delta(\xi_{xt},\phi_x)-\theta(\xi_{xx},\phi_x)+\eta(\xi_{xx},\phi_{xx})\\ &\qquad+\nu(\xi_{xxt},\phi_{xx})+\theta(\phi_{xx},\phi_x)-\eta(\phi_{xx},\phi_{xx})+\big(f(u_h)-f(u),\phi_x\big).\end{aligned}$$

With the help of the Cauchy-Schwarz inequality and the Lipschitz continuity of f, we get:

$$\frac{d}{dt}\big(\|\phi\|^2+\delta\|\phi_x\|^2+\nu\|\phi_{xx}\|^2\big)\le C\big(\|\xi\|_2^2+\|\xi_t\|_2^2+\|\phi\|^2+\|\phi_x\|^2+\|\phi_{xx}\|^2\big). \tag{3.11}$$

By using Lemma 2 and $u_{h0}=\tilde{u}_0$, it further yields:

$$\|\phi\|_2^2 \le C\int_0^T\big(\|\xi\|_2^2+\|\xi_t\|_2^2+\|\phi\|_2^2\big)\,ds \le C(u,T)h^4 + C\int_0^T\|\phi\|_2^2\,ds.$$

Moreover, from the Gronwall's inequality, it holds

$$\|\phi\|_2^2 \le C(u,T)e^{CT}h^4 \le C(u,T)h^4.$$

From Sobolev inequality theorem, we can get:

$$\|\phi\|_\infty \le C(u,T)h^2.$$

By using Lemma 2 and the triangle inequality, the proof is completed. $\square$

4 Crank-Nicolson scheme

For a smooth function ζ on $[0,T]$, the time step is expressed as k such that $k = T/N$ with $t^n = nk$, $(n = 0,1,\dots,N)$. We denote $\zeta^n = \zeta(t^n)$, $\partial_t \zeta^n = \frac{\zeta^n - \zeta^{n-1}}{k}$, $\zeta^{n-\frac{1}{2}} = \frac{\zeta^n + \zeta^{n-1}}{2}$. The discrete-time finite element Galerkin approximation u_h^n of $u(t^n)$ is defined as a solution of

$$\begin{cases} (\partial_t u_h^n, \chi) + \delta(\partial_t u_{hx}^n, \chi_x) + \nu(\partial_t u_{hxx}^n, \chi_{xx}) + \mu(u_{hx}^{n-\frac{1}{2}}, \chi_x) \\ \quad + \eta(u_{hxx}^{n-\frac{1}{2}}, \chi_{xx}) - \theta(u_{hxx}^{n-\frac{1}{2}}, \chi_x) = (f(u_h^{n-\frac{1}{2}}), \chi_x), \quad \chi \in X_h, \\ u_h^0 = u_{h0}, \end{cases} \tag{4.1}$$

where $u_{h0} \in X_h$ is an appropriate approximation to u_0.

The following a priori bound is useful to prove error estimates of the fully-discrete Galerkin-Crank-Nicolson method.

Theorem 3 *Let u_h^n be a solution of* (4.1), *for $n \geq 1$, then there exists a positive constant C, such that*:

$$\|u_h^n\|_\infty \leq C\|u_h^0\|_2. \tag{4.2}$$

Proof Let $\chi = u_h^{n-\frac{1}{2}}$ in (4.1), it holds that:

$$\begin{aligned} &\bigl(\partial_t u_h^n, u_h^{n-\frac{1}{2}}\bigr) + \delta\bigl(\partial_t u_{hx}^n, u_{hx}^{n-\frac{1}{2}}\bigr) + \nu\bigl(\partial_t u_{hxx}^n, u_{hxx}^{n-\frac{1}{2}}\bigr) + \mu\bigl(u_{hx}^{n-\frac{1}{2}}, u_{hx}^{n-\frac{1}{2}}\bigr) \\ &\quad + \eta\bigl(u_{hxx}^{n-\frac{1}{2}}, u_{hxx}^{n-\frac{1}{2}}\bigr) - \theta\bigl(u_{hxx}^{n-\frac{1}{2}}, u_{hx}^{n-\frac{1}{2}}\bigr) = \bigl(f\bigl(u_h^{n-\frac{1}{2}}\bigr), u_{hx}^{n-\frac{1}{2}}\bigr). \end{aligned}$$

In fact, $(f(u_h^{n-\frac{1}{2}}), u_{hx}^{n-\frac{1}{2}}) = 0$, thus:

$$\|u_h^n\|^2 + \|u_{hx}^n\|^2 + \|u_{hxx}^n\|^2 \leq C\bigl(\|u_h^{n-1}\|^2 + \|u_{hx}^{n-1}\|^2 + \|u_{hxx}^{n-1}\|^2\bigr).$$

Then:

$$\|u_h^n\|_2^2 \leq C\|u_h^0\|_2^2.$$

By using the Sobolev imbedding theorem, we have completed the proof of (4.2). □

4.1 Existence and uniqueness

To prove the existence of the solution u_h^n of (4.1), we use the following variant of the Brouwer fixed point theorem.

Lemma 3 *Let H be a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Moreover, let $\mathcal{F}$ be a continuous mapping of H into itself, such that $(\mathcal{F}(W), W)_H > 0$, for all $W \in H$ with $\|W\|_H = \sigma > 0$. Then, there exists $W^* \in H$ with $\|W^*\|_H \leq \sigma$, satisfying that $\mathcal{F}(W^*) = 0$.*

Theorem 4 *Suppose that $u_h^0, u_h^1, \dots, u_h^{n-1}$ are given, and then there exists a unique solution u_h^n, $n > 1$, satisfying* (4.1).

Proof Let continuous mapping $\mathcal{F}: X_h \to X_h$, for $W, \chi \in X_h$, we denote:

$$\begin{aligned}\big(F(W), \chi\big) = {} & (W, \chi) + \delta(W_x, \chi_x) + \nu(W_{xx}, \chi_{xx}) - \big(u_h^{n-1}, \chi\big) - \delta\big(u_{hx}^{n-1}, \chi_x\big) \\ & - \nu\big(u_{hxx}^{n-1}, \chi_{xx}\big) + \frac{k\mu}{2}(W_x, \chi_x) + \frac{k\eta}{2}(W_{xx}, \chi_{xx}) \\ & - \frac{k\theta}{2}(W_{xx}, \chi_x) - \frac{k}{2}\big(f(W), \chi_x\big).\end{aligned}$$

Let $\chi = W$, using Schwarz's inequality and for appropriate k, it is easy to get that:

$$\begin{aligned}\big(F(W), W\big) \geq {} & \|W\|^2 + \left(\delta + \frac{k\mu}{2} - \frac{k\theta}{4}\right)\|W_x\|^2 + \left(\nu + \frac{k\eta}{2} - \frac{k\theta}{4}\right)\|W_{xx}\|^2 \\ & - \big(\|u_h^{n-1}\|\|W\| + \delta\|u_{hx}^{n-1}\|\|W_x\| + \nu\|u_{hxx}^{n-1}\|\|W_{xx}\|\big) \\ \geq {} & C\|W\|_2\big(\|W\|_2 - C\|u_h^{n-1}\|_2\big).\end{aligned}$$

For $\|W\|_2 = C\|u_h^{n-1}\|_2 + 1$, we have $(\mathcal{F}(W), W) > 0$. From Lemma 3, it can be seen that there exists W^* such that $\mathcal{F}(W^*) = 0$. Hence, $u_h^n = 2W^* - u_h^{n-1}$ satisfies (4.1). This completes the proof of existence.

Below we prove the uniqueness by induction. Suppose u_h^n and v_h^n are two solutions of (4.1). Let $W^n = u_h^n - v_h^n$, we get:

$$\begin{aligned}&\big(\partial_t W^n, \chi\big) + \delta\big(\partial_t W_x^n, \chi_x\big) + \nu\big(\partial_t W_{xx}^n, \chi_{xx}\big) + \mu\big(W_x^{n-\frac{1}{2}}, \chi_x\big) \\ &\quad + \eta\big(W_{xx}^{n-\frac{1}{2}}, \chi_{xx}\big) - \theta\big(W_{xx}^{n-\frac{1}{2}}, \chi_x\big) = \big(f\big(u_h^{n-\frac{1}{2}}\big) - f\big(v_h^{n-\frac{1}{2}}\big), \chi_x\big).\end{aligned}$$

Let $W^{n-1} = 0$ and choose $\chi = W^{n-\frac{1}{2}}$, it yields that:

$$\begin{aligned}\frac{\min\{1, \delta, \nu\}}{2}\partial_t\|W^n\|_2^2 \leq {} & -\mu\|W_x^{n-\frac{1}{2}}\|^2 + \theta\|W_{xx}^{n-\frac{1}{2}}\|\|W_x^{n-\frac{1}{2}}\| \\ & + \|f\big(u_h^{n-\frac{1}{2}}\big) - f\big(v_h^{n-\frac{1}{2}}\big)\|\|W_x^{n-\frac{1}{2}}\|.\end{aligned} \tag{4.3}$$

As $\|u_h^n\|_\infty$ and $\|v_h^n\|_\infty$ are bounded. we can get:

$$\|f\big(u_h^{n-\frac{1}{2}}\big) - f\big(v_h^{n-\frac{1}{2}}\big)\| \leq C\|W^{n-\frac{1}{2}}\|. \tag{4.4}$$

By (4.3) and (4.4), it can be inferred that:

$$\partial_t\|W^n\|_2^2 \leq C(\|W^{n-\frac{1}{2}}\|^2 + \|W_x^{n-\frac{1}{2}}\|^2 + \|W_{xx}^{n-\frac{1}{2}}\|^2 \leq C\big(\|W^{n-1}\|_2^2 + \|W^n\|_2^2\big).$$

Using discrete Gronwall's inequality, for sufficiently small k, we have $\|W^n\|_2^2 = 0$. That is to say, $W^n = 0$, i.e., $u_h^n = v_h^n$. The proof is completed. □

4.2 Error estimates

Letting $u^n = u(t^n)$ and using the elliptic projection $\tilde{u}^n$, we denote the error $e^n := u^n - u_h^n = \xi^n - \phi^n$, where $\xi^n = u^n - \tilde{u}^n$, $\phi^n = u_h^n - \tilde{u}^n$.

Theorem 5 *Take $u_{h0} = \tilde{u}(0)$, then there exists a positive constant C, for sufficiently small h and k, we have*:

$$\left\| u^n - u_h^n \right\|_\infty \le C(u,T)\left(h^2 + k^2\right), \quad 1 \le n \le N.$$

Proof If we subtract the equation (4.1) from (2.1), let $\rho^n = u_t^{n-\frac{1}{2}} - \partial_t u^n$, and use the auxiliary projection, we have:

$$\begin{aligned}
&\left(\partial_t\phi^n, \chi\right) + \delta\left(\partial_t\phi_x^n, \chi_x\right) + \nu\left(\partial_t\phi_{xx}^n, \chi_{xx}\right) \\
&\quad = \left(\partial_t\xi^n, \chi\right) + \delta\left(\partial_t\xi_x^n, \chi_x\right) + \nu\left(\partial_t\xi_{xx}^n, \chi_{xx}\right) - \mu\left(\phi_x^{n-\frac{1}{2}}, \chi_x\right) \\
&\qquad - \eta\left(\phi_{xx}^{n-\frac{1}{2}}, \chi_x\right) + \theta\left(\phi_{xx}^{n-\frac{1}{2}}, \chi_x\right) + \mu\left(\xi_x^{n-\frac{1}{2}}, \chi_x\right) + \eta\left(\xi_{xx}^{n-\frac{1}{2}}, \chi_{xx}\right) \\
&\qquad - \theta\left(\xi_{xx}^{n-\frac{1}{2}}, \chi_x\right) + \left(\rho^n, \chi\right) + \delta\left(\rho_x^n, \chi_x\right) + \nu\left(\rho_{xx}^n, \chi_{xx}\right) \\
&\qquad + \left(f\left(u_h^{n-\frac{1}{2}}\right) - f\left(u^{n-\frac{1}{2}}\right), \chi_x\right), \quad \forall \chi \in X_h.
\end{aligned} \tag{4.5}$$

Using $\chi = \phi^{n-\frac{1}{2}}$ in (4.5), we get that:

$$\begin{aligned}
\partial_t\left\|\phi^n\right\|_2^2 \le C\Big(&\left\|\phi^{n-\frac{1}{2}}\right\|_2^2 + \left\|\partial_t\xi^n\right\|^2 + \left\|\partial_t\xi_x^n\right\|^2 + \left\|\partial_t\xi_{xx}^n\right\|^2 + \left\|\xi_x^{n-\frac{1}{2}}\right\|^2 \\
&+ \left\|\xi_{xx}^{n-\frac{1}{2}}\right\|^2 + \left\|\rho^n\right\|_2^2 + \left\|f\left(u^{n-\frac{1}{2}}\right) - f\left(u_h^{n-\frac{1}{2}}\right)\right\|^2\Big).
\end{aligned} \tag{4.6}$$

Using the Lipschitz condition of f and the boundedness of $\|u_h^n\|_\infty$ and $\|u^n\|_\infty$ yields that:

$$\left\|f\left(u^{n-\frac{1}{2}}\right) - f\left(u_h^{n-\frac{1}{2}}\right)\right\| \le C\left(\left\|\xi^{n-\frac{1}{2}}\right\| + \left\|\phi^{n-\frac{1}{2}}\right\|\right). \tag{4.7}$$

Then, from (4.6) and (4.7) it follows:

$$\left\|\phi^n\right\|_2^2 \le Ck\left(\sum_{j=0}^n \left\|\phi^j\right\|_2^2 + \sum_{j=0}^n \left\|\xi^j\right\|_2^2 + \sum_{j=1}^n \left\|\rho^j\right\|^2\right). \tag{4.8}$$

Using discrete Gronwall's inequality, we have:

$$\left\|\phi^n\right\|_2^2 \le Ck\left(\sum_{j=0}^n \left\|\xi^j\right\|_2^2 + \sum_{j=1}^n \left\|\rho^j\right\|^2\right).$$

From Taylor's formula, we can get:

$$\left\|\rho^j\right\|^2 \le Ck^3 \int_{t_{j-1}}^{t_j} \left\|u_{ttt}(s)\right\|^2 ds. \tag{4.9}$$

Combining (4.8) with (4.9), we obtain:

$$\left\|\phi^n\right\|_2^2 \le c(u,T)\left(h^4 + k^4\right).$$

Consequently, using Lemma 2, the triangle inequality with the Sobolev inequality theorem, we complete the rest of the proof. □

Given the parameters α, β, μ, δ, η, ν, θ, the spatial step h, the time step increment k, and finite element space X_h, by solving Eq. (4.1), we can obtain a group of solution ensembles $\{u_h^n\}_{n=1}^N$ for Eq. (4.1). Thus we choose ℓ (in general, $\ell \ll N$, for example, $\ell = 32$, $N = 200$) instantaneous solutions $u_h^{n_i}(x)$ $(1 \le n_1 < n_2 < \cdots < n_\ell \le N)$ (they are usually evenly selected) from N solutions instantaneous $\{u_h^n(x)\}$ for Eq. (4.1), which are referred to as snapshots.

5 Error estimates for reduced order pod models of generalized KdV-RLW-Rosenau equation

5.1 Generation of POD basis and reduced GFE formulation

Define $\mathcal{V}$ as the space generated by the snapshots $\{U_i\}_{i=1}^\ell$, at least one of which is assumed to be nonzero. For $u_h^{n_i}(x)$ $(1 \le n_1 < n_2 < \cdots < n_\ell \le N)$ in Sect. 4, let $U_i(x) = u_h^{n_i}(x)$ $(1 \le i \le \ell)$, such that $\mathcal{V} = \operatorname{span}(U_1, U_2, \ldots, U_\ell)$. Let $\{\psi_j\}_{j=1}^l$ denote an orthogonal basis of $\mathcal{V}$ with $l = \dim \mathcal{V}$, for $(U_i, \psi_j)_X = (u_{hxx}^{n_i}, \psi_{jxx})$, each member of the ensemble can be expressed as:

$$U_i = \sum_{j=1}^{\ell} (U_i, \psi_j)_X \psi_j, \quad i = 1, 2, \ldots, \ell. \tag{5.1}$$

Definition 1 The POD method lies in finding the orthogonal basis ψ_j, $(j = 1, 2, \ldots, \ell)$ such that for every d $(1 \le d \le l)$, the mean square error between the elements U_i $(1 \le i \le \ell)$ and corresponding d-th partial sum of (5.1) is minimized on average:

$$\min_{\{\psi_j\}_{j=1}^d} \frac{1}{\ell} \sum_{i=1}^{\ell} \left\| U_i - \sum_{j=1}^{d} (U_i, \psi_j)_X \psi_j \right\|_X^2, \tag{5.2}$$

such that:

$$(\psi_i, \psi_j)_X = \delta_{ij}, \quad 1 \le i \le d, 1 \le j \le i, \tag{5.3}$$

where $\|U_i\|_X^2 = \|u_{hxx}^{n_i}\|_0^2$. A set of solutions $\{\psi_j\}_{j=1}^d$ of (5.2) and (5.3) is called the POD basis of rank d.

The procedure to handle (5.2) and (5.3) can be found in [39, 42, 43, 45, 49]. Then we introduce the positive semi-definite matrix $G = (G_{ij})_{\ell \times \ell} \in R^{\ell \times \ell}$ with rank l corresponding to the snapshots $\{U_i\}_{i=1}^\ell$ by

$$G_{ij} = \frac{1}{\ell} (U_i, U_j)_X. \tag{5.4}$$

Proposition 1 *Take $v_1, v_2, \ldots, v_l$ the associated orthogonal eigenvectors and let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0$ denote the positive eigenvalues of G. Let $(v_i)_j$ denote the jth component of the eigenvector v_i, thus, the POD basis of rank $d \le l$ is given by*:

$$\psi_i = \frac{1}{\sqrt{\ell \lambda_i}} \sum_{j=1}^{\ell} (v_i)_j U_j, \quad 1 \le i \le d \le l. \tag{5.5}$$

Moreover, the following error formula holds:

$$\sum_{i=1}^{\ell}\left\|U_i-\sum_{j=1}^{d}(U_i,\psi_j)_X\psi_j\right\|_X^2=\ell\sum_{j=d+1}^{l}\lambda_j. \tag{5.6}$$

Let $X^d=\operatorname{span}\{\psi_1,\psi_2,\dots,\psi_d\}$. *Define the Ritz-projection* $P^h:X\to X_h$, $P^h|_{X_h}=P^d:X_h\to X^d$ *and* $P^h:X\backslash X_h\to X_h\backslash X^d$, *for* $U\in X$, *we have*:

$$\left((P^hU)_{xx},v_{hxx}\right)=(U_{xx},v_{hxx}),\quad \forall v_h\in X_h. \tag{5.7}$$

Because of (5.7), *the linear operators* P^h *is well-defined and bounded*

$$\left\|(P^hU)_{xx}\right\|\le\|U_{xx}\|,\quad \forall U\in X. \tag{5.8}$$

Lemma 4 *For every* d $(d=1,2,\dots,l)$ *and* $u_h^{n_i}\in\mathcal{V}$ *is the solution of Eq.* (4.1), *the projection operators* P^d *satisfies*:

$$\sum_{i=1}^{\ell}\left\|\left(u_h^{n_i}-P^du_h^{n_i}\right)_{xx}\right\|^2\le\ell\sum_{j=d+1}^{l}\lambda_j, \tag{5.9}$$

$$\sum_{i=1}^{\ell}\left\|\left(u_h^{n_i}-P^du_h^{n_i}\right)_{x}\right\|^2\le Ch^2\ell\sum_{j=d+1}^{l}\lambda_j, \tag{5.10}$$

$$\sum_{i=1}^{\ell}\left\|u_h^{n_i}-P^du_h^{n_i}\right\|^2\le Ch^4\ell\sum_{j=d+1}^{l}\lambda_j. \tag{5.11}$$

Proof By (5.7), for every $U\in X$, it follows that:

$$\begin{aligned}\left\|(U-P^hU)_{xx}\right\|^2&=\left((U-P^hU)_{xx},(U-P^hU)_{xx}\right)\\&=\left((U-P^hU)_{xx},(U-v_h)_{xx}\right)\\&\le\left\|(U-P^hU)_{xx}\right\|\left\|(U-v_h)_{xx}\right\|,\quad\forall v_h\in X_h.\end{aligned} \tag{5.12}$$

That is to say:

$$\left\|(U-\mathrm{P}^hU)_{xx}\right\|\le\left\|(U-v_h)_{xx}\right\|,\quad \forall v_h\in X_h. \tag{5.13}$$

In fact, if $U=u_h^{n_i}$, we have $P^hu_h^{n_i}=P^du_h^{n_i}\in X^d$, from (5.6) and let $v_h=\sum_{j=1}^{d}(u_h^{n_i},\psi_j)_X\psi_j\in X^d\subset X_h$ in (5.13), then (5.9) can be proved.

To prove (5.11), for $\forall v\in X$, the following variational problem is considered:

$$(w_{xx},v_{xx})=\left(U-P^hU,v\right), \tag{5.14}$$

hence $U-P^hU\in X$, for $w\in X\cap H^4(\Omega)$, (5.14) has a unique solution, such that

$$\|w\|_4\le C\left\|U-P^hU\right\|.$$

Letting $v = U - P^h U$ in (5.14) and using (5.13) yields that:

$$\begin{aligned}\|U - P^h U\|^2 &= (w_{xx}, (U - P^h U)_{xx}) \\ &= ((w - w_h)_{xx}, (U - P^h U)_{xx}) \\ &\le \|(w - w_h)_{xx}\| \|(U - P^h U)_{xx}\|, \quad \forall w_h \in X_h.\end{aligned} \tag{5.15}$$

Using interpolation theory, taking $w_h = \pi_h w$ as interpolation function of w in X_h, we have:

$$\begin{aligned}\|U - P^h U\|^2 &\le Ch^2 \|w\|_4 \|(U - P^h U)_{xx}\| \\ &\le Ch^2 \|U - P^h U\| \|(U - P^h U)_{xx}\|,\end{aligned} \tag{5.16}$$

that is to say:

$$\|U - P^h U\| \le Ch^2 \|(U - P^h U)_{xx}\|. \tag{5.17}$$

If $U = u_h^{n_i}$, we have $P^h u_h^{n_i} = P^d u_h^{n_i} \in X^d$, (5.11) can be derived by using (5.17) and (5.9). Then we have to prove (5.10):

$$\begin{aligned}((U - P^h U)_x, (U - P^h U)_x) &= ((U - P^h U)_{xx}, P^h U - U) \\ &\le \|(U - P^h U)_{xx}\| \|U - P^h U\|.\end{aligned} \tag{5.18}$$

As we know, if $U = u_h^{n_i}$, $P^h u_h^{n_i} = P^d u_h^{n_i} \in X^d$, it follows that:

$$\|(u_h^{n_i} - P^d u_h^{n_i})_x\|^2 \le \|(u_h^{n_i} - P^d u_h^{n_i})_{xx}\| \|u_h^{n_i} - P^d u_h^{n_i}\|. \tag{5.19}$$

(5.10) can be derived by using (5.9), (5.11) and two fundamental inequality $\sum a_i b_i \le \sum a_i \sum b_i$, $\sum a_i \le C(\sum a_i^2)^{\frac{1}{2}}$. The proof is completed. □

Hence, the reduced-order modeling of Eq. (4.1) with initial value $u_d^0 = u_{d0}$ is denoted by:

$$\begin{aligned}&(\partial_t u_d^n, \chi) + \delta(\partial_t u_{dx}^n, \chi_x) + \nu(\partial_t u_{dxx}^n, \chi_{xx}) + \mu(u_{dx}^{n-\frac{1}{2}}, \chi_x) \\ &\quad + \eta(u_{dxx}^{n-\frac{1}{2}}, \chi_{xx}) - \theta(u_{dxx}^{n-\frac{1}{2}}, \chi_x) = (f(u_d^{n-\frac{1}{2}}), \chi_x), \quad \chi \in X^d.\end{aligned} \tag{5.20}$$

5.2 Error estimates

By using the same procedure as proofs in Sect. 4, we can easily prove that reduced formulation (5.20) has a unique group of solutions $u_d^n \in X^d$, such that stability holds. Below, we recur to usual finite element method to derive the error estimates for Eq. (5.20).

Theorem 6 *Let $u_h^n \in X_h$ be the solution of Eq.* (4.1) *and the $u_d^n \in X^d$ be the solution of Eq.* (5.20), *when $k = O(h^{2/3})$, $\ell = O(N^{2/3})$ and snapshots are uniformly selected, we have*:

$$\|u_h^n - u_d^n\|_2 \le Ck^2 + Ck\left(\sum_{j=d+1}^{\ell} \lambda_j\right)^{1/2}. \tag{5.21}$$

Proof Subtracting Eq. (5.20) from Eq. (4.1), letting $\chi = v_d \in X^d \subset X_h$ can get:

$$\begin{aligned} &\big(\partial_t\big(u_h^n - u_d^n\big), v_d\big) + \delta\big(\partial_t\big(u_{hx}^n - u_{dx}^n\big), v_{dx}\big) + \nu\big(\partial_t\big(u_{hxx}^n - u_{dxx}^n\big), v_{dxx}\big) \\ &\quad + \mu\big(u_{hx}^{n-\frac{1}{2}} - u_{dx}^{n-\frac{1}{2}}, v_{dx}\big) + \eta\big(u_{hxx}^{n-\frac{1}{2}} - u_{dxx}^{n-\frac{1}{2}}, v_{dxx}\big) + \theta\big(u_{hx}^{n-\frac{1}{2}} - u_{dx}^{n-\frac{1}{2}}, v_{dxx}\big) \\ &\quad = \big(f\big(u_h^{n-\frac{1}{2}}\big) - f\big(u_d^{n-\frac{1}{2}}\big), v_{dx}\big). \end{aligned} \tag{5.22}$$

The error decomposition is given as follows:

$$e := u_h^n - u_d^n = \xi^n - \phi^n,$$

with $\xi^n = u_h^n - P^d u_h^n$, $\phi^n = u_d^n - P^d u_h^n$. Letting $v_d = \phi^n + \phi^{n-1}$ and substituting ξ^n and ϕ^n into (5.22) gives

$$\begin{aligned} &\|\phi^n\|^2 + \delta\|\phi_x^n\|^2 + \nu\|\phi_{xx}^n\|^2 \\ &\quad = \|\phi^{n-1}\|^2 + \delta\|\phi_x^{n-1}\|^2 + \nu\|\phi_{xx}^{n-1}\|^2 \\ &\qquad + \big(\xi^n - \xi^{n-1}, \phi^n + \phi^{n-1}\big) + \delta\big(\xi_x^n - \xi_x^{n-1}, \phi_x^n + \phi_x^{n-1}\big) \\ &\qquad + \nu\big(\xi_{xx}^n - \xi_{xx}^{n-1}, \phi_{xx}^n + \phi_{xx}^{n-1}\big) \\ &\qquad + \frac{k\mu}{2}\big(\xi_x^n + \xi_x^{n-1}, \phi_x^n + \phi_x^{n-1}\big) - \frac{k\mu}{2}\big(\phi_x^n + \phi_x^{n-1}, \phi_x^n + \phi_x^{n-1}\big) \\ &\qquad + \frac{k\eta}{2}\big(\xi_{xx}^n + \xi_{xx}^{n-1}, \phi_{xx}^n + \phi_{xx}^{n-1}\big) \\ &\qquad - \frac{k\eta}{2}\big(\phi_{xx}^n + \phi_{xx}^{n-1}, \phi_{xx}^n + \phi_{xx}^{n-1}\big) + \frac{k\theta}{2}\big(\xi_x^n + \xi_x^{n-1}, \phi_{xx}^n + \phi_{xx}^{n-1}\big) \\ &\qquad - \frac{k\theta}{2}\big(\phi_x^n + \phi_x^{n-1}, \phi_{xx}^n + \phi_{xx}^{n-1}\big) \\ &\qquad + k\big(f\big(u_d^{n-\frac{1}{2}}\big) - f\big(u_h^{n-\frac{1}{2}}\big), \phi_x^n + \phi_x^{n-1}\big). \end{aligned} \tag{5.23}$$

By the Lipschitz condition of f, the boundedness of $\|u_h^n\|_\infty$ and $\|u_d^n\|_\infty$, and Young's inequality, we can get:

$$\begin{aligned} &\big|\big(f\big(u_d^{n-\frac{1}{2}}\big) - f\big(u_h^{n-\frac{1}{2}}\big), \phi_x^n + \phi_x^{n-1}\big)\big| \\ &\quad \le \big\|f\big(u_d^{n-\frac{1}{2}}\big) - f\big(u_h^{n-\frac{1}{2}}\big)\big\| \big\|\phi_x^n + \phi_x^{n-1}\big\| \\ &\quad \le C\big\|u_d^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}\big\| \big\|\phi_x^n + \phi_x^{n-1}\big\| \\ &\quad \le C\big(\|\xi^n\| + \|\phi^n\| + \|\xi^{n-1}\| + \|\phi^{n-1}\|\big)\big\|\phi_x^n + \phi_x^{n-1}\big\| \\ &\quad \le C\big(\|\xi^n\|^2 + \|\phi^n\|^2 + \|\xi^{n-1}\|^2 + \|\phi^{n-1}\|^2 + \|\phi_x^n\|^2 + \|\phi_x^{n-1}\|^2\big). \end{aligned} \tag{5.24}$$

Combining (5.7), (5.24), and Young's inequality, (5.23) can be rewritten as:

$$\begin{aligned} &\min\{1, \delta, \nu\}\|\phi^n\|^2 \\ &\quad \le \max\{1, \delta, \nu\}\|\phi^{n-1}\|^2 \end{aligned}$$

$$+ (1 + Ck)\left(\|\xi^n\|^2 + \|\xi^{n-1}\|^2\right) + \left(\delta + \frac{k(\mu+\theta)}{2}\right)\|\xi_x^n\|^2 + \left(\frac{k(\delta+\mu+\theta)}{2}\right)\|\xi_x^{n-1}\|^2$$
$$+ (1 + Ck)\left(\|\phi^n\|^2 + \|\phi^{n-1}\|^2\right) + \left(\delta + Ck + \frac{k(\theta+3\mu)}{2}\right)\left(\|\phi_x^n\|^2 + \|\phi_x^{n-1}\|^2\right)$$
$$+ k(\eta+\theta)\left(\|\phi_{xx}^n\|^2 + \|\phi_{xx}^{n-1}\|^2\right).$$

Choosing the sufficiently small k yields that:

$$\|\phi^n\|_2^2 \le C\|\phi^{n-1}\|_2^2 + C\left(\|\xi^n\|_1^2 + \|\xi^{n-1}\|_1^2\right). \tag{5.25}$$

Summing the above inequality (5.25) from 1 to n gives

$$\|\phi^n\|_2^2 \le C\sum_{j=0}^{n-1}\|\phi^j\|_2^2 + C\left(\sum_{j=1}^{n}\|\xi^j\|_1^2 + \sum_{j=0}^{n-1}\|\xi^j\|_1^2\right).$$

With the discrete Growall's inequality, (5.17) and (5.18), we have:

$$\|\phi^n\|_2^2 \le Ch^2\left(\sum_{j=1}^{n}\|\xi_{xx}^j\|^2 + \sum_{j=0}^{n-1}\|\xi^j\|_1^2\right). \tag{5.26}$$

Without loss of generality, let $n_{i-1} \le n \le n_i \le N$ $(i = 1,2,\dots,\ell)$ $(n = 1,2,\dots,N)$ and $n_0 = 0$. Expanding u_h^n into Tayor series with respect to t_{n_i} yields that $u_h^n = u_h^{n_i} \pm \epsilon_i k u_{th}(\xi_i)$, $t_{n_{i-1}} \le \xi_i \le t_{n_i}$, $i = 1,2,\dots,\ell$, where ϵ_i is the step number from t_n to t_{n_i} or to $t_{n_{i-1}}$ $(i = 1,2,\dots,\ell)$. If the snapshot interval is the same, then $\epsilon_i \le N/\ell$. By (5.26), we obtain that:

$$\|\phi^n\|_2^2 \le Ck^2h^2\left(\frac{N}{\ell}\right)^3 + Ch^2\frac{N}{\ell}\sum_{j=n_1}^{n_i}\|\xi_{xx}^j\|^2, \quad 1 \le n \le N.$$

Hence, if $\ell = O(N^{2/3})$ and $k = O(h^{2/3})$, by using Lemma 4, we can get:

$$\|\phi^n\|_2 \le Ck^2 + Ck\left(\sum_{j=d+1}^{\ell}\lambda_j\right)^{1/2}.$$

With Lemma 4, (5.17) and the triangle inequality, the proof is completed. □

Using Theorems 5, 6. and Sobolev inequality yields the following result.

Theorem 7 *With the hypotheses of Theorem 6, the following error estimates between the solutions of Eq.* (2.1) *and Eq.* (5.20) *hold*:

$$\|u^n - u_d^n\|_\infty \le Ch^2 + Ck^2 + Ck\left(\sum_{j=d+1}^{\ell}\lambda_j\right)^{1/2}.$$

6 Numerical experiments

We divide the interval $[0,1]$ into m equidistant subintervals such that $0 = x_0 < x_1 < \cdots < x_m = 1$, and denote the subinterval length as $h = x_{i+1} - x_i$. Let the spline set $B_{-1}, B_0, \dots, B_m$

constitute the basis function on the interval $[0,1]$. Referring to [61, 62], a quadratic B-spline $B_i(x)$ with the desired properties is defined as:

$$B_i(x) = \begin{cases} (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2 + 3(x_i - x)^2, & [x_{i-1}, x_i], \\ (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2, & [x_i, x_{i+1}], \\ (x_{i+2}-x)^2, & [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases}$$

The values of the B-spline functions and their first derivatives at the knots are given by:

$$\begin{cases} B_i(x_{i-1}) = B_i(x_{i+2}) = 0, & B_i(x_i) = B_i(x_{i+1}) = 1, \\ B_i'(x_{i-1}) = B_i'(x_{i+2}) = 0, & B_i'(x_i) = -B_i'(x_i+1) = \frac{h}{2}. \end{cases} \tag{6.1}$$

Therefore, the approximate solution can be written in terms of the quadratic spline functions as:

$$u_h(x,t) = \sum_{i=-1}^{m} a_i(t) B_i(x), \tag{6.2}$$

where $a_i(t)$ are yet undetermined coefficients.

Since each spline covers three intervals, three splines $B_{i-1}(x)$, $B_i(x)$, $B_{i+1}(x)$ cover each finite element $[x_i, x_{i+1}]$. From formula (6.2) and spline function properties (6.1), we can get the nodal value of function $u^h(x,t)$ and its derivative at node x_i and fixed time $(\tilde{t})$, which can be expressed by coefficient $a_i(\tilde{t})$ as follows:

$$u_h(x_i, \tilde{t}) = a_{i-1}(\tilde{t}) + a_i(\tilde{t}), \qquad \left.\frac{\partial u_h(x,\tilde{t})}{\partial x}\right|_{x=x_i} = \frac{2(a_i(\tilde{t} - a_{i-1}(\tilde{t}))}{h}. \tag{6.3}$$

Combining (6.3) and homogeneous boundary conditions (1.1c), we get $a_{-1}(\tilde{t}) = a_0(\tilde{t}) = 0$, $a_{m-1}(\tilde{t}) = a_m(\tilde{t}) = 0$. And then (6.2) is modified as:

$$u_h(x,t) = \sum_{i=1}^{m-2} a_i(t) B_i(x).$$

That is, we must determine the $m-2$ unknowns $a_i(t)$ $(i = 1, 2, \ldots, m-2)$ at each moment of t.

6.1 The test problem

In this part, we consider the numerical solutions of the generalized KdV-RLW-Rosenau equation for the following test problems.

Example Consider the following KdV-RLW-Rosenau equation:

$$\begin{cases} u_t - u_{xx} - u_{xxt} + u_{xxxx} + u_{xxxxt} + u_{xxx} \\ \quad + u_x + uu_x = g(x,t), \quad (x,t) \in (0,1) \times (0,10], \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0, \quad t \in (0,10], \\ u(x,0) = x^2(1-x)^2, \quad x \in [0,1], \end{cases} \tag{6.4}$$

where

$$\begin{cases} g(x,t) = -e^{-2t}(12e^t + 7x^2e^t - 6x^3e^t + x^4e^t - 26xe^t \\ \qquad\qquad - 2x^3 + 10x^4 - 18x^5 + 14x^6 - 4x^7). \end{cases}$$

The exact solution is:

$$u(x,t) = e^{-t}x^2(x-1)^2.$$

Table 1 gives the error and second-order convergence of the scheme (4.1) in the time and space directions, which proves the theoretical results in Theorem 5.

Assuming the space step is $h = 0.01$ and the time step is $k = 0.05$, we have $k = O(h^{2/3})$. When $t = 200k$, we first find the usual GFE solution u_h^n of Eq. (6.4) by scheme (4.1), which is depicted graphically on the left-hand side in Fig. 1. We output one group of 200 values at time $t = 1k, 2k, \dots, 200k$, and choose 32 values from 200 so that every 6 values consist of one group of snapshots. Finally, using Matlab software, we find one group of 32 eigenvalues, which are arranged in a non-decreasing order. We construct one group of POD bases by using (5.5), take the first 8 POD bases from a set of 32 POD bases to expand into subspace X^d and find a set of numerical solutions at $t = 200k$ with Eq. (6.4) by scheme (5.20), which are depicted graphically on the right-hand side in Fig. 1.

In Table 2, d, λ_j, and ratio are number of POD basis, eigenvector, and $\|u^n - u_d^n\|_\infty/(h^2 + k^2 + k(\sum_{j=d+1}^{l}\lambda_j)^{1/2})$, respectively. Figure 2 shows the errors between reduced order solutions u_d^n with different number of POD bases and exact solution u^n (blue piecewise line), and errors between usual GFE solutions u_h^n and exact solution (red line), respectively. Table 2 and Fig. 2 demonstrate that the result for numerical example is consistent with those

Table 1 Error estimates and convergence order for the Galerkin-Crank-Nicolson scheme (4.1) to the Eq. (6.4) at $t = 10$

k	$\|u^n - u_h^n\|_\infty$	Order	$\|u^n - u_h^n\|_\infty/(h^2 + k^2)$
1/10	2.8080×10^{-8}	–	1.4040×10^{-6}
1/20	5.9795×10^{-9}	2.1670	1.1959×10^{-6}
1/40	1.4312×10^{-9}	2.0440	1.1450×10^{-6}
1/80	3.5376×10^{-10}	2.0114	1.1320×10^{-6}
1/160	8.6987×10^{-11}	2.0166	1.1134×10^{-6}

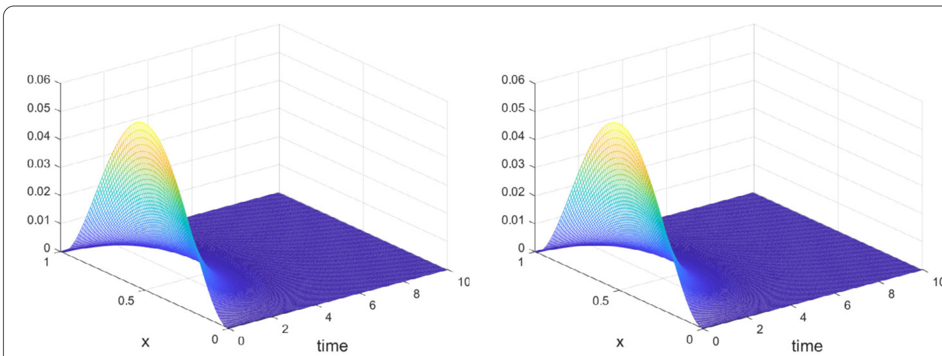

Figure 1 Evolutions of the usual GFE solution (on left-hand side) and POD (with 8 bases) GFE solution (on right-hand side) of the KdV-RLW-Rosenau equation (6.4) with $h = 0.01$, $k = 0.05$

Table 2 Error estimates for the POD-Galerkin-Crank-Nicolson scheme (5.20) to the Eq. (6.4) at $t = 10$ with $h = 0.01$, $k = 0.05$

d	$k(\sum_{j=d+1}^{\ell} \lambda_j)^{1/2}$	$\Vert u^n - u_d^n \Vert_\infty$	Ratio
2	2.8783×10^{-4}	5.8492×10^{-9}	2.0255×10^{-6}
4	1.8863×10^{-4}	5.8043×10^{-9}	2.0814×10^{-6}
6	1.3610×10^{-4}	5.7050×10^{-9}	2.0851×10^{-6}
8	1.0902×10^{-4}	5.6895×10^{-9}	2.1001×10^{-6}
10	8.1327×10^{-5}	5.6828×10^{-9}	2.1194×10^{-6}
12	5.3441×10^{-5}	5.6779×10^{-9}	2.1398×10^{-6}
14	3.9221×10^{-5}	5.6678×10^{-9}	2.1475×10^{-6}
16	2.5628×10^{-5}	5.6641×10^{-9}	2.1532×10^{-6}

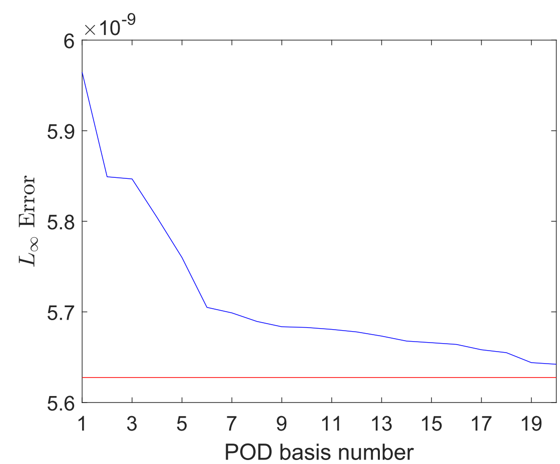

Figure 2 The errors between reduced order modeling with different number of POD bases solution and exact solution (blue piecewise line) compare with the error between usual GFE solution and exact solution (red line) at $t = 10$ with $h = 0.01$, $k = 0.05$

obtained for the theoretical case. Obviously, it takes much less time to compute the POD GFE solution compared to the usual GFE solution, for details see [45].

7 Conclusions

We summarize the reduced-order modeling of the KdV-RLW-Rosenau equation discussed in the paper from two aspects. First, we analyze the semidiscrete Galerkin method and obtain L_∞-error estimates using the Galerkin-Crank-Nicolson finite element method. Numerical examples confirm the theoretical results and demonstrate the efficiency of the fully discrete method. Second, we introduce a proper orthogonal decomposition (POD) method to derive a reduced POD GFE formulation of the equation. We analyze the error between the traditional GFE solution and the reduced GFE solution using POD and investigate the relation between the number of snapshots and the number of solutions in all time instances. Our approach can also be used for more general cases.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Second author wrote the main manuscript text and first author reviewed the manuscript.

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