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Chlodowsky-type Szász operators via Boas–Buck-type polynomials and some approximation properties

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Abstract

In this paper, we construct the Chlodowsky-type Szász operators defined via Boas–Buck-type polynomials. We prove some approximation properties and obtain the rate of the convergence for these operators. We also study the Voronovskaya-type theorem and weighted approximation.

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1 Introduction and preliminaries

The basic sequence of Szász operators is given by

$$S_n(f,\xi)=e^{-n\xi}\sum_{\iota=0}^{\infty}\frac{(n\xi)^{\iota}}{\iota!}f\left(\frac{\iota}{n}\right)$$

for $x\in[0,\infty)$. Generalizations of these operators have been studied by many authors. In [21] the authors have obtained a generalization of Szász operators by means of the Appell polynomials defined as follows:

$$P_n(f,\xi)=\frac{e^{-n\xi}}{g(1)}\sum_{\iota=0}^{\infty}p_{\iota}(n\xi)f\left(\frac{\iota}{n}\right),$$

where $p_{\iota}(\xi)$, $\iota\geq 0$, are the Appell polynomials defined by

$$g(t)e^{\xi t}=\sum_{\iota=0}^{\infty}p_{\iota}(\xi)\frac{t^{\iota}}{\iota!}\quad\text{and}\quad g(t)=\sum_{\iota=0}^{\infty}a_{\iota}\frac{t^{\iota}}{\iota!},$$

and $g(t)$ is an analytic function in the disk $|t| < R$, $R > 1$, and $g(1) \neq 0$. A further generalization was given by Ismail [19] by using the Sheffer operators

$$T_n(f,\xi) = \frac{e^{-n\xi H(1)}}{g(1)} \sum_{\iota=0}^{\infty} s_\iota(n\xi) f\left(\frac{\iota}{n}\right)$$

for $n \in \mathbb{N}$, where $s_\iota(\xi)$, $\iota \geq 0$, are the Sheffer polynomials defined by

$$g(t)e^{\xi H(t)} = \sum_{\iota=0}^{\infty} s_\iota(\xi) \frac{t^\iota}{\iota!},$$

$H(t) = \sum_{\iota=0}^{\infty} h_\iota \frac{t^\iota}{\iota!}$ is an analytic function in the disk $|t| < R$, $R > 1$, $g(1) \neq 0$, and $H'(1) = 1$.

The multiple Sheffer polynomials $\{S_{k_1,k_2}(\xi)\}_{k_1,k_2=0}^{\infty}$ are defined as follows. The generating function is

$$A(t_1,t_2)e^{\xi H(t_1,t_2)} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} S_{k_1,k_2}(\xi) \frac{t_1^{k_1} t_2^{k_2}}{k_1!k_2!}, \tag{1.1}$$

where $A(t_1,t_2)$ and $H(t_1,t_2)$ are of the forms

$$A(t_1,t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1!k_2!} \tag{1.2}$$

and

$$H(t_1,t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} h_{k_1,k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1!k_2!}, \tag{1.3}$$

respectively, and satisfy the conditions $A(0,0) = a_{0,0} \neq 0$ and $H(0,0) = h_{0,0} \neq 0$. The positive linear operators involving multiple Sheffer polynomials for $\xi \in [0,\infty)$ were defined in [3] as follows:

$$G_n(f,\xi) = \frac{e^{-\frac{n\xi}{2}H(1,1)}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}(\frac{n\xi}{2})}{k_1!k_2!} f\left(\frac{k_1+k_2}{n}\right), \tag{1.4}$$

provided that the series in the above relations are convergent and the following conditions are satisfied:

(1) $S_{k_1,k_2}(\xi) \geq 0$, $k_1, k_2 \in \mathbb{N}$,

(2) $A(1,1) \neq 0$, $H_{t_1}(1,1) = 1$, $H_{t_2}(1,1) = 1$,

(3) Series (1.1), (1.2), and (1.3) are convergent for $|t_1| < R$, $|t_2| < R$, and $(R_1,R_2) > 1$.

In [12] the authors have studied the Kantorovich variant of Szász operators induced by multiple Sheffer polynomials for $\xi \in [0,\infty)$ as follows:

$$K_n^{(S)}(f,\xi) = \frac{ne^{-\frac{n\xi}{2}H(1,1)}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}(\frac{n\xi}{2})}{k_1!k_2!} \int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}} f(t)\,dt,$$

under the condition that the right side of the above relation exists. Szász-type operators involving Charlier polynomials were studied in [2].

We will treat the Chlodowsky variant of the Szász type operators induced by Boas-Buck-type polynomials. The generating functions for the Boas–Buck-type polynomials [20] are

$$A(t)B(\xi H(t)) = \sum_{k=0}^{\infty} p_k(\xi)t^k, \tag{1.5}$$

where A, B, and H are analytic functions given by the following expressions:

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \tag{1.6}$$

$$B(t) = \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0, r \geq 0, \tag{1.7}$$

$$H(t) = \sum_{r=0}^{\infty} h_r t^r, \quad h_1 \neq 0. \tag{1.8}$$

In what follows, we assume that the above polynomials satisfy the following conditions:

(1) $A(1) \neq 0$, $H'(1) = 1$, $p_k(\xi) \geq 0$, $k = 0, 1, 2, \ldots$,

(2) $B : \mathbb{R} \to (0, \infty)$,

(3) The power series (1.5), (1.6), (1.7), and (1.8) are convergent for $|t| < R$ ($R > 1$).

The Chlodowsky variant of the Szász-type operators induced by Boas–Buck-type polynomials given in [26] (see also [1]) is defined as follows:

$$B_n^*(f;\xi) = \frac{1}{A(1)B(\frac{n}{b_n}\xi H(1))} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}\xi\right) f\left(\frac{k}{n}b_n\right), \tag{1.9}$$

where (b_n) is a numerical positive increasing sequence such that

$$b_n \to \infty, \qquad \frac{b_n}{n} \to 0 \quad (n \to \infty).$$

The sequence $(b_n) = (\sqrt{n})$ satisfies the above conditions.

We assume the operators B_n^* to be positive. Also, we consider

$$\lim_{n\to\infty} \frac{B^{(k)}(y)}{B(y)} = 1 \quad \text{for } k \in \{1, 2, 3, \ldots, r\}, r \in \mathbb{N}.$$

In the recent years, different classes of operators were studied together with Korovkin- and Voronovskaja-type theorems (see [4–11, 13–18, 23, 27, 28, 30] and [22, 24, 25]).

2 Basic results

Here we calculate the moments and central moments for B_n^* (see [29]).

Lemma 2.1 [26] *For all* $\xi \in [0, \infty)$,

$$B_n^*(e_0;\xi) = 1,$$

$$B_n^*(e_1;\xi) = \frac{B'(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}x + \frac{b_n}{n}\frac{A'(1)}{A(1)},$$

$$B_n^*(e_2;\xi) = \frac{B''(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2 + \frac{b_n}{n}\frac{B'(\frac{n}{b_n}\xi H(1))[A(1)+2A'(1)+H''(1)A(1)]}{A(1)B(\frac{n}{b_n}\xi H(1))}x + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)},$$

$$B_n^*(e_3;\xi) = \frac{B'''(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^3 + \big(3A'(1)+3H''(1)A(1)+3A(1)\big)\frac{B''(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\frac{b_n}{n}\xi^2 + \big(3A''(1)+3H''(1)A'(1)+H'''(1)A(1)+6A'(1)+3H''(1)A(1)+A(1)\big) \cdot \frac{B'(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\frac{b_n^2}{n^2}\xi + \big(A'''(1)+3A''(1)+A'(1)\big)\frac{b_n^3}{A(1)n^3},$$

$$B_n^*(e_4;\xi) = \frac{B^{(4)}(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^4 + \big(4A'(1)+6H''(1)A(1)+6A(1)\big)\frac{B^{(3)}(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\frac{b_n}{n}\xi^3 + \big(6A''(t)+12H''(1)+A'(1)+4H'''(1)A(1)+3H''(1)^2A(1)+18A'(1) + 18H''(1)A(1)+7A(1)\big)\frac{B''(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\frac{b_n^2}{n^2}\xi^2 + \big(4A'''(1)+6A''(1)H''(1)+4A'(1)H'''(1)+A(1)H^{(4)}(1)+18A''(1) + 18H''(1)A'(1)+6H'''(1)A(1)+14A'(1)+7H''(1)A(1)+A(1)\big) \cdot \frac{B'(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\frac{b_n^3}{n^3}\xi + \big(A^{(4)}(1)+6A^{(3)}(1)+7A''(1)+A'(1)\big)\frac{b_n^4}{A(1)n^4}.$$

Proposition 2.2 [26] *We have*

$$B_n^*\big((e_1-\xi);\xi\big) = \frac{B'(\frac{n}{b_n}\xi H(1)) - B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}x + \frac{b_n}{n}\frac{A'(1)}{A(1)},$$

$$B_n^*\big((e_1-\xi)^2;\xi\big) = \frac{B''(\frac{n}{b_n}\xi H(1)) - 2B'(\frac{n}{b_n}\xi H(1)) + B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2 + \frac{b_n}{n}\frac{(A(1)+2A'(1)+A(1)H''(1))B'(\frac{n}{b_n}\xi H(1)) - 2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\xi + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)},$$

$$B_n^*\big((e_1-\xi)^4;\xi\big) = \frac{\xi^4}{B(\frac{n}{b_n}\xi H(1))}\bigg[B^{(4)}\bigg(\frac{n}{b_n}\xi H(1)\bigg) - 4B^{(3)}\bigg(\frac{n}{b_n}\xi H(1)\bigg) + 6B''\bigg(\frac{n}{b_n}\xi H(1)\bigg) - 4B'\bigg(\frac{n}{b_n}\xi H(1)\bigg) + B\bigg(\frac{n}{b_n}\xi H(1)\bigg)\bigg] + \frac{2\xi^3 b_n}{nA(1)B(\frac{n}{b_n}\xi H(1))} \cdot \bigg[\big(2A'(1)+3A(1)H''(1)+3A(1)\big)B^{(3)}\bigg(\frac{n}{b_n}\xi H(1)\bigg) - 6\big(A'(1)+A(1)H''(1)+A(1)\big)$$

$$
\begin{aligned}
&\cdot B''\left(\frac{n}{b_n}\xi H(1)\right) + 3\big(2A'(1) + A(1)H''(1) + A(1)\big)B'\left(\frac{n}{b_n}\xi H(1)\right) \\
&- 2A'(1)B\left(\frac{n}{b_n}\xi H(1)\right)\bigg] + \frac{\xi^2 b_n^2}{n^2 A(1)B(\frac{n}{b_n}\xi H(1))}\big[\big(6A''(1) + 12A'(1)H''(1) \\
&+ 4A(1)H'''(1) + 21A(1)H''(1) + 18A'(1) + 7A(1)\big)B''\left(\frac{n}{b_n}\xi H(1)\right).
\end{aligned}
$$

3 Rates of convergence

By $BV[0,\infty)$ we denote the class of all functions of bounded variation on $[0,\infty)$, and by $\bigvee_a^b f$ we denote the total variation of a function f on $[a,b]$, i.e.,

$$
\bigvee_a^b f = V\big(f;[a,b]\big) = \sup_{P\in\mathbb{P}}\left(\sum_{i=1}^{n}\big|f(\xi_i) - f(\xi_{i-1})\big|\right),
$$

where $\mathbb{P}$ is the class of all partitions $P: a = \xi_0 < \xi_1 < \cdots < \xi_n = b$. We denote

$$
C_2[0,\infty) = \big\{f \in C[0,\infty) : \big|f(t)\big| \le M_2\big(1+t^2\big)\ \forall t \ge 0\big\},
$$

where M_2 is a constant, and

$$
D_{BV[0,\infty)} = \big\{f \in C_2[0,\infty) : f' \in BV[0,\infty)\big\}.
$$

Let

$$
f'_\xi(\theta) = \begin{cases} f'(\theta) - f'(\xi-) & \text{for } 0 \le \theta < \xi, \\ 0, & \text{for } \theta = \xi, \\ f'(\theta) - f'(\xi+) & \text{for } \xi < \theta < \infty. \end{cases} \tag{3.1}
$$

From the construction of operators $B_n^*(f;\xi)$ we obtain the following relation:

$$
B_n^*(f;\xi) = \int_0^\infty f(\theta)\frac{\partial\{K_n(\xi,\theta)\}}{\partial\theta}\,d\theta, \tag{3.2}
$$

where

$$
K_n(\xi,\theta) = \begin{cases} \sum_{k\le n\theta} P_{k,n}(\xi) & \text{for } 0 < \theta < \infty, \\ 0 & \text{for } \theta = 0, \end{cases}
$$

and

$$
P_{k,n}(\xi) = \frac{1}{A(1)B(\frac{n}{b_n}\xi H(1))}p_k\left(\frac{n}{b_n}\xi\right).
$$

Also, let

$$
\beta_n(\xi;\theta) = \int_0^\theta \frac{\partial\{K_n(\xi,u)\}}{\partial u}\,du. \tag{3.3}
$$

From the above relation it follows that

$$\beta_n(\xi;\theta) \le 1.$$

We provide the following result.

Theorem 3.1 *Let $f \in D_{BV[0,\infty)}$. Then for sufficiently large n,*

$$\begin{aligned}
&\left|B_n^*(f;\xi) - f(\xi)\right| \\
&\quad\le \left|\frac{1}{2}\bigl(f'(\xi+) + f'(\xi-)\bigr)\right| \cdot \left|B_n^*(t-\xi;\xi)\right| + \frac{B_n^*((\xi-t)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} f_x'\right) \\
&\qquad + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \left(\frac{M_2}{\xi^2} + 4M_2 + \frac{|f(\xi)|}{\xi^2}\right)B_n^*\bigl((t-\xi)^2;\xi\bigr) \\
&\qquad + \left|f'(\xi+)\right|\sqrt{B_n^*\bigl((t-\xi)^2;\xi\bigr)} + \frac{B_n^*((t-\xi)^2;\xi)}{\xi^2}\left|f(2\xi) - f(\xi) - \xi f'(\xi+)\right| \\
&\qquad + \left|\frac{1}{2}\bigl(f'(\xi+) - f'(\xi-)\bigr)\right| \cdot \sqrt{\left|B_n^*\bigl((t-\xi)^2;\xi\bigr)\right|}.
\end{aligned}$$

We need some auxiliary results. We start with the following:

Lemma 3.2 *For any $x \in (0,\infty)$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned}
(1)\quad \beta_n(\xi;t) &= \int_0^t \frac{\partial\{K_n(\xi,u)\}}{\partial u}\,du \\
&\le \Biggl(\frac{B''(\frac{n}{b_n}\xi H(1)) - 2B'(\frac{n}{b_n}\xi H(1)) + B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2 \\
&\quad + \frac{b_n}{n}\frac{(A(1) + 2A'(1) + A(1)H''(1))B'(\frac{n}{b_n}\xi H(1)) - 2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\xi\Biggr) \\
&\quad /(\xi-t)^2,
\end{aligned}$$

for $0 \le t < \xi$;

$$\begin{aligned}
(2)\quad & 1 - \beta_n(\xi;t) \\
&= \int_t^\infty \frac{\partial\{K_n(\xi,u)\}}{\partial u}\,du \\
&\le \Biggl(\frac{B''(\frac{n}{b_n}\xi H(1)) - 2B'(\frac{n}{b_n}\xi H(1)) + B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2 \\
&\quad + \frac{b_n}{n}\frac{(A(1) + 2A'(1) + A(1)H''(1))B'(\frac{n}{b_n}\xi H(1)) - 2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\xi\Biggr) \\
&\quad /(t-\xi)^2
\end{aligned}$$

for $\xi < t < \infty$.

Proof (1) Let $0 \le t < \xi$. Then Lemma 2.1 gives

$$\int_0^t \frac{\partial\{K_n(x,u)\}}{\partial u}\,du \le \int_0^t \left(\frac{\xi-u}{\xi-t}\right)^2 \frac{\partial\{K_n(\xi,u)\}}{\partial u}\,du \le \frac{B_n^*((\xi-u)^2;\xi)}{(\xi-t)^2}.$$

(2) In the case $\xi < t < \infty$, in a similar way, we obtain

$$\int_t^\infty \frac{\partial\{K_n(\xi,u)\}}{\partial u}\,du \le \int_0^\infty \left(\frac{\xi-u}{\xi-t}\right)^2 \frac{\partial\{K_n(\xi,u)\}}{\partial u}\,du \le \frac{B_n^*((\xi-u)^2;\xi)}{(t-\xi)^2}.$$

□

Lemma 3.3 *Let $f \in D_{BV[0,\infty)}$. Then for sufficiently large n,*

$$\begin{aligned}\left|B_n^*(f;\xi)-f(\xi)\right| \le{}& \left|\frac{1}{2}\bigl(f'(\xi+)+f'(\xi-)\bigr)\right| \cdot \left|B_n^*(t-\xi;\xi)\right| \\ &+ |I_1| + |I_2| + \left|\frac{1}{2}\bigl(f'(\xi+)-f'(\xi-)\bigr)\right| \cdot \sqrt{\left|B_n^*\bigl((t-\xi)^2;\xi\bigr)\right|},\end{aligned}$$

where $I_1 = \int_0^\xi [\int_t^\xi f_\xi'(u)\,du] \frac{\partial K_n(\xi,t)}{\partial t}\,dt$ and $I_2 = \int_\xi^\infty [\int_\xi^t f_\xi'(u)\,du] \frac{\partial K_n(\xi,t)}{\partial t}\,dt$.

Proof For $f \in D_{BV[0,\infty)}$, we may write

$$\begin{aligned}f'(t) ={}& \frac{1}{2}\bigl[f'(\xi+)+f'(\xi-)\bigr] + f_\xi'(t) + \frac{1}{2}\bigl[f'(\xi+)-f'(\xi-)\bigr]\operatorname{sgn}(t-\xi) \\ &+ \delta_\xi(t)\left[f'(t) - \frac{1}{2}\bigl[f'(\xi+)+f'(\xi-)\bigr]\right],\end{aligned}$$

where

$$\delta_\xi(t) = \begin{cases} 1, & t = \xi, \\ 0, & t \ne \xi. \end{cases}$$

From the above facts we get

$$\begin{aligned}&\left|B_n^*(f;\xi)-f(\xi)\right| \\ &\quad= \int_0^\infty \bigl(f(t)-f(\xi)\bigr)\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt = \int_0^\infty \left[\int_\xi^t f'(u)\,du\right]\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt \\ &\quad= \int_0^\infty \left[\int_\xi^t \left\{\frac{1}{2}\bigl[f'(\xi+)+f'(\xi-)\bigr] + f_\xi'(u) + \frac{1}{2}\bigl[f'(\xi+)-f'(\xi-)\bigr]\operatorname{sgn}(u-\xi)\right.\right. \\ &\qquad \left.\left.+ \delta_\xi(u)\left[f'(u) - \frac{1}{2}\bigl[f'(\xi+)+f'(\xi-)\bigr]\right]du\right\}\frac{\partial\{K_n(\xi,t)\}}{\partial t}\right]dt.\end{aligned}$$

Since $\int_\xi^t \delta_\xi(u)\,du = 0$, we obtain

$$\begin{aligned}B_n^*(f;\xi)-f(\xi) ={}& \frac{1}{2}\bigl[f'(\xi+)+f'(\xi-)\bigr]\int_0^\infty (t-\xi)\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt \\ &+ \int_0^\infty \left[\int_\xi^t f_\xi'(u)\,du\right]\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt \\ &+ \frac{1}{2}\bigl[f'(\xi+)-f'(\xi-)\bigr]\int_0^\infty (t-\xi)\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt.\end{aligned}$$

Let us now break the second term in the above relation into two parts:

$$\int_0^\infty \left[\int_\xi^t f'(u)\,du\right]\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt = -\int_0^\xi \left[\int_t^\xi f'_\xi(u)\,du\right]\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt + \int_\xi^\infty \left[\int_\xi^t f'_\xi(u)\,du\right]\frac{\partial\{K_n(\xi,t)\}}{\partial t}\,dt = -I_1 + I_2.$$

Now we have the following estimate:

$$\begin{aligned}\left|B_n^*(f;\xi) - f(\xi)\right| \le{}& \left|\frac{1}{2}\left[f'(\xi+) + f'(\xi-)\right]\right| \cdot \left|B_n^*\big((t-\xi),\xi\big)\right| + |I_1| + |I_2| \\ &+ \left|\frac{1}{2}\left[f'(\xi+) - f'(\xi-)\right]\right| \cdot B_n^*\big(|t-\xi|,\xi\big).\end{aligned}$$

Applying the Cauchy–Schwarz inequality to the above relation, we get

$$\begin{aligned}\left|B_n^*(f;\xi) - f(\xi)\right| \le{}& \left|\frac{1}{2}\left[f'(\xi+) + f'(\xi-)\right]\right| \cdot \left|B_n^*\big((t-\xi),\xi\big)\right| + |I_1| + |I_2| \\ &+ \left|\frac{1}{2}\left[f'(\xi+) - f'(\xi-)\right]\right| \cdot \sqrt{B_n^*\big((t-\xi)^2,\xi\big)}.\end{aligned}$$

□

Lemma 3.4 *Let $f \in D_{BV[0,\infty)}$, and let n be sufficiently large. Then*

$$|I_1| \le \frac{B_n^*((\xi-u)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi\right) + \frac{\xi}{\sqrt{n}}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi\right),$$

where $I_1 = \int_0^\xi [\int_t^\xi f'_\xi(u)\,du]\frac{\partial K_n(\xi,t)}{\partial t}\,dt$.

Proof By integration by parts we have

$$\begin{aligned}|I_1| &= \left|\int_0^\xi \left[\int_\xi^t f'_\xi(u)\,du\right]\frac{\partial K_n(\xi,t)}{\partial t}\,dt\right| \\ &= \left|\int_0^\xi f_\xi(t)\frac{\partial K_n(\xi,t)}{\partial t}\,dt\right| + \left|\int_0^\xi \left[\int_0^t \frac{\partial K_n(\xi,u)}{\partial t}\,du\right] f'_\xi(t)\,dt\right| \\ &\le \int_0^{\xi-\frac{\xi}{\sqrt{n}}}\left|\int_0^t \frac{\partial K_n(\xi,u)}{\partial t}\,du\right|\left|f'_\xi(t)\right|dt + \int_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi}\left|\int_0^t \frac{\partial K_n(\xi,u)}{\partial t}\,du\right|\left|f'_\xi(t)\right|dt.\end{aligned}$$

Using Lemma 3.2, we obtain

$$\begin{aligned}|I_1| &\le B_n^*\big((\xi-u)^2;\xi\big)\int_0^{\xi-\frac{\xi}{\sqrt{n}}}\left(\bigvee_t^\xi f'_\xi\right)\frac{1}{(\xi-t)^2}\,dt + \int_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi\right)dt \\ &\le B_n^*\big((\xi-u)^2;\xi\big)\int_0^{\xi-\frac{\xi}{\sqrt{n}}}\left(\bigvee_t^\xi f'_\xi\right)\frac{1}{(\xi-t)^2}\,dt + \frac{\xi}{\sqrt{n}}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi\right).\end{aligned}$$

Substituting $u = \frac{\xi}{\xi - t}$, we get

$$\int_0^{\xi - \frac{\xi}{\sqrt{n}}} \left(\bigvee_t^{\xi} f'_\xi \right) \frac{1}{(\xi - t)^2}\, dt = \frac{1}{\xi} \int_1^{\sqrt{n}} \left(\bigvee_{\xi - \frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi \right) du \le \frac{1}{\xi} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\xi - \frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi \right),$$

which yields that

$$|I_1| \le \frac{B_n^*((\xi - u)^2; \xi)}{\xi} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\xi - \frac{\xi}{\sqrt{n}}}^{\xi} f'_x \right) + \frac{\xi}{\sqrt{n}} \left(\bigvee_{\xi - \frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi \right).$$

□

Lemma 3.5 *Let $f \in D_{BV[0,\infty)}$, and let n be sufficiently large. Then*

$$\begin{aligned} |I_2| \le{}& \left(\frac{M_2}{\xi^2} + 4M_2 + \frac{|f(\xi)|}{\xi^2} \right) B_n^*\big((t - \xi)^2; \xi\big) + \big|f'(\xi+)\big| \sqrt{B_n^*\big((t - \xi)^2; \xi\big)} \\ &+ \frac{B_n^*((t - \xi)^2; \xi)}{(t - \xi)^2} \big|f(2\xi) - f(\xi) - \xi f'(\xi+)\big| + \frac{\xi}{\sqrt{n}} \left(\bigvee_{\xi}^{\xi + \frac{\xi}{\sqrt{n}}} f'_\xi \right) \\ &+ B_n^*\big((t - \xi)^2; \xi\big) \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\xi}^{\xi + \frac{\xi}{k}} f'_\xi \right), \end{aligned}$$

where $I_2 = \int_\xi^\infty [\int_\xi^t f'_\xi(u)\, du] \frac{\partial K_n(\xi,t)}{\partial t}\, dt$.

Proof By the properties of integrals we have

$$\begin{aligned} &\left| \int_\xi^\infty \left[\int_\xi^t f'_\xi(u)\, du \right] \frac{\partial K_n(\xi, t)}{\partial t}\, dt \right| \\ &\quad \le \left| \int_{2\xi}^\infty \left[\int_\xi^t f'_\xi(u)\, du \right] \frac{\partial K_n(\xi, t)}{\partial t}\, dt \right| + \left| \int_\xi^{2\xi} \left[\int_\xi^t f'_\xi(u)\, du \right] \frac{\partial K_n(\xi, t)}{\partial t}\, dt \right| \\ &\quad = I'_2 + I''_2, \\ I'_2 &= \left| \int_{2\xi}^\infty \left[\int_\xi^t \big(f'_\xi(u) - f'_\xi(\xi+)\big)\, du \right] \frac{\partial K_n(\xi, t)}{\partial t}\, dt \right| \\ &\le \left| \int_{2\xi}^\infty \big(f(t) - f(\xi)\big) \frac{\partial K_n(\xi, t)}{\partial t}\, dt \right| + \big|f'(\xi+)\big| \left| \int_{2\xi}^\infty (t - \xi) \frac{\partial K_n(\xi, t)}{\partial t}\, dt \right| \\ &\le \int_{2\xi}^\infty \big|f(t)\big| \frac{\partial K_n(\xi, t)}{\partial t}\, dt + \big|f(\xi)\big| \int_{2\xi}^\infty \frac{\partial K_n(\xi, t)}{\partial t}\, dt + \big|f'(\xi+)\big| \int_{2\xi}^\infty |t - \xi| \frac{\partial K_n(\xi, t)}{\partial t}\, dt \\ &= A_1 + A_2 + A_3. \end{aligned}$$

Now we will estimate

$$A_1 \le M_2 \int_{2\xi}^\infty \big(1 + t^2\big) \frac{\partial K_n(\xi, t)}{\partial t}\, dt.$$

Since $t \geq 2\xi$, that is, $t-\xi \geq \xi$, we get

$$A_1 \leq M_2 \int_{2\xi}^{\infty} \frac{(t-\xi)^2}{\xi^2} \frac{\partial K_n(\xi,t)}{\partial t}\,dt + 4M_2 \int_{2\xi}^{\infty} (t-\xi)^2 \frac{\partial K_n(\xi,t)}{\partial t}\,dt$$
$$\leq \left(\frac{M_2}{\xi^2} + 4M_2\right) B_n^*\big((t-\xi)^2;\xi\big).$$

Now we estimate

$$A_2 \leq \big|f(\xi)\big| \int_{2\xi}^{\infty} \frac{(t-\xi)^2}{\xi^2} \frac{\partial K_n(\xi,t)}{\partial t}\,dt \leq \frac{|f(\xi)|}{\xi^2} B_n^*\big((t-\xi)^2;\xi\big)$$

and

$$A_3 \leq \big|f'(\xi+)\big| \int_{2x}^{\infty} |t-\xi| \frac{\partial K_n(\xi,t)}{\partial t}\,dt \leq \big|f'(\xi+)\big| \sqrt{B_n^*\big((t-\xi)^2;\xi\big)}.$$

From last three relations we get the upper bound

$$\big|I_2'\big| \leq \left(\frac{M_2}{\xi^2} + 4M_2 + \frac{|f(\xi)|}{\xi^2}\right) B_n^*\big((t-\xi)^2;\xi\big) + \big|f'(\xi+)\big| \sqrt{B_n^*\big((t-\xi)^2;\xi\big)}.$$

Now we estimate

$$\big|I_2''\big| = \left| \int_{\xi}^{2\xi} \left[\int_{\xi}^{t} f_\xi'(u)\,du\right] \frac{\partial K_n(\xi,t)}{\partial t}\,dt \right| \leq \big|1-\beta_n(\xi,2\xi)\big| \left| \int_{\xi}^{2\xi} f_\xi'(u)\,du \right|$$
$$+ \left| \int_{\xi}^{2\xi} f_\xi'(t)\big(1-\beta_n(\xi,t)\big)\,dt \right|.$$

From Lemma 3.2 we have

$$\big|I_2''\big| \leq \frac{B_n^*((t-\xi)^2;\xi)}{\xi^2} \left| \int_{\xi}^{2\xi} \big(f'(u)-f'(\xi+)\big)\,du \right| + \left| \int_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} f_\xi{}'(t)\big(1-\beta_n(\xi,t)\big)\,dt \right|$$
$$+ \left| \int_{\xi+\frac{\xi}{\sqrt{n}}}^{2\xi} f_\xi{}'(t)\big(1-\beta_n(\xi,t)\big)\,dt \right|$$
$$= \frac{B_n^*((t-\xi)^2;\xi)}{\xi^2} \big|f(2\xi)-f(\xi)-\xi f'(\xi+)\big| \left| \int_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} f_\xi{}'(t)\big(1-\beta_n(\xi,t)\big)\,dt \right|$$
$$+ \left| \int_{\xi+\frac{\xi}{\sqrt{n}}}^{2\xi} f_\xi{}'(t)\big(1-\beta_n(\xi,t)\big)\,dt \right|,$$

$$\left| \int_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} f_\xi{}'(t)\big(1-\beta_n(\xi,t)\big)\,dt \right| \leq \int_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} \left(\bigvee_{\xi}^{t} f_\xi'\right) dt \leq \frac{\xi}{\sqrt{n}} \left(\bigvee_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} f_\xi'\right),$$

and

$$\left| \int_{\xi+\frac{\xi}{\sqrt{n}}}^{2\xi} f_\xi{}'(t)\big(1-\beta_n(\xi,t)\big)\,dt \right| \leq B_n^*\big((t-\xi)^2;\xi\big) \int_{\xi+\frac{\xi}{\sqrt{n}}}^{2\xi} \left(\bigvee_{\xi}^{t} f_\xi'\right) \frac{1}{(\xi-t)^2}\,dt.$$

For $u = \frac{\xi}{t-\xi}$, we get

$$\left|\int_{\xi+\frac{\xi}{\sqrt{n}}}^{2\xi} f_\xi{}'(t)\big(1-\beta_n(\xi,t)\big)\,dt\right| \le \frac{B_n^*((t-\xi)^2;\xi)}{\xi}\int_1^{\sqrt{n}}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{u}} f'_\xi\right)du \le \frac{B_n^*((t-\xi)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{k}} f'_\xi\right).$$

From the last relations we obtain that

$$\left|I_2''\right| \le \frac{B_n^*((t-\xi)^2;\xi)}{\xi^2}\left|f(2\xi)-f(\xi)-\xi f'(\xi+)\right| + \frac{\xi}{\sqrt{n}}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} f'_\xi\right) + \frac{B_n^*((t-\xi)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{k}} f'_\xi\right).$$

Hence

$$\begin{aligned}|I_2| \le \left|I_2'\right| + \left|I_2''\right| &\le \left(\frac{M_2}{\xi^2}+4M_2+\frac{|f(\xi)|}{\xi^2}\right)B_n^*\big((t-\xi)^2;\xi\big) + \left|f'(\xi+)\right|\sqrt{B_n^*\big((t-\xi)^2;\xi\big)}\\ &\quad + \frac{B_n^*((t-\xi)^2;\xi)}{\xi^2}\left|f(2\xi)-f(\xi)-\xi f'(\xi+)\right| + \frac{\xi}{\sqrt{n}}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} f'_\xi\right)\\ &\quad + \frac{B_n^*((t-\xi)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{k}} f'_\xi\right).\end{aligned}$$

□

Proof of Theorem 3.1 Based on Lemmas 3.2, 3.3, 3.4 and 3.5, we get the following estimate:

$$\begin{aligned}&\left|B_n^*(f;\xi)-f(\xi)\right|\\ &\quad\le \left|\frac{1}{2}\big(f'(\xi+)+f'(\xi-)\big)\right|\cdot\left|B_n^*(t-\xi;\xi)\right|\\ &\qquad + |I_1| + |I_2| + \left|\frac{1}{2}\big(f'(\xi+)-f'(\xi-)\big)\right|\cdot\sqrt{\left|B_n^*\big((t-\xi)^2;\xi\big)\right|}\\ &\quad\le \left|\frac{1}{2}\big(f'(\xi+)+f'(\xi-)\big)\right|\cdot\left|B_n^*(t-\xi;\xi)\right| + \frac{B_n^*((\xi-t)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi\right)\\ &\qquad + \frac{\xi}{\sqrt{n}}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi} f'_\xi\right) + \left(\frac{M_f}{\xi^2}+4M_f+\frac{|f(\xi)|}{\xi^2}\right)B_n^*\big((t-\xi)^2;\xi\big)\\ &\qquad + \left|f'(\xi+)\right|\sqrt{B_n^*\big((t-\xi)^2;\xi\big)} + \frac{B_n^*((t-\xi)^2;x)}{\xi^2}\left|f(2\xi)-f(\xi)-\xi f'(\xi+)\right|\\ &\qquad + \frac{\xi}{\sqrt{n}}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{\sqrt{n}}} f'_\xi\right) + \frac{B_n^*((t-\xi)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\xi}^{\xi+\frac{\xi}{k}} f'_\xi\right)\\ &\qquad + \left|\frac{1}{2}\big(f'(\xi+)-f'(\xi-)\big)\right|\cdot\sqrt{\left|B_n^*\big((t-\xi)^2;\xi\big)\right|}.\end{aligned}$$

Since

$$\left(\bigvee_a^b f\right) + \left(\bigvee_b^c f\right) = \left(\bigvee_a^c f\right),$$

we obtain

$$\begin{aligned}
&\left|B_n^*(f;\xi) - f(\xi)\right| \\
&\quad\le \left|\frac{1}{2}\bigl(f'(\xi+) + f'(\xi-)\bigr)\right| \cdot \left|B_n^*(t-\xi;\xi)\right| \\
&\qquad + |I_1| + |I_2| + \left|\frac{1}{2}\bigl(f'(\xi+) - f'(\xi-)\bigr)\right| \cdot \sqrt{\left|B_n^*\bigl((t-\xi)^2;\xi\bigr)\right|} \\
&\quad\le \left|\frac{1}{2}\bigl(f'(\xi+) + f'(\xi-)\bigr)\right| \cdot \left|B_n^*(t-\xi;\xi)\right| + \frac{B_n^*((\xi-t)^2;\xi)}{\xi}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi+\frac{\xi}{\sqrt{n}}} f'_\xi\right) \\
&\qquad + \frac{\xi}{\sqrt{n}}\left(\bigvee_{\xi-\frac{\xi}{\sqrt{n}}}^{\xi+\frac{\xi}{\sqrt{n}}} f'_\xi\right) + \left(\frac{M_f}{\xi^2} + 4M_f + \frac{|f(\xi)|}{\xi^2}\right)B_n^*\bigl((t-\xi)^2;\xi\bigr) \\
&\qquad + \left|f'(\xi+)\right|\sqrt{B_n^*\bigl((t-\xi)^2;\xi\bigr)} + \frac{B_n^*((t-\xi)^2;\xi)}{\xi^2}\left|f(2\xi) - f(\xi) - \xi f'(\xi+)\right| \\
&\qquad + \left|\frac{1}{2}\bigl(f'(\xi+) - f'(\xi-)\bigr)\right| \cdot \sqrt{\left|B_n^*\bigl((t-\xi)^2;\xi\bigr)\right|}. \qquad \square
\end{aligned}$$

4 Voronovskaya-type theorems

The Voronovskaya-type theorem for the Chlodowsky-type Szász operators based on Boas–Buck-type polynomials under certain conditions is known. First, we introduce following assumptions [26]:

$$\lim_{n\to\infty}\frac{n}{b_n}\frac{B'(\frac{n}{b_n}\xi H(1)) - B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))} = l_1(\xi); \tag{4.1}$$

$$\lim_{n\to\infty}\frac{n}{b_n}\frac{B''(\frac{n}{b_n}\xi H(1)) - 2B'(\frac{n}{b_n}\xi H(1)) + B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))} = l_2(\xi); \tag{4.2}$$

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{n}{b_n}\right)^2\frac{1}{B(\frac{n}{b_n}\xi H(1))}\left[B^{(4)}\left(\frac{n}{b_n}\xi H(1)\right) - 4B^{(3)}\left(\frac{n}{b_n}\xi H(1)\right) + 6B''\left(\frac{n}{b_n}\xi H(1)\right)\right. \\
&\quad\left. - 4B'\left(\frac{n}{b_n}\xi H(1)\right) + B\left(\frac{n}{b_n}\xi H(1)\right)\right] = l_3(\xi);
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\lim_{n\to\infty}\frac{n}{b_n}\frac{1}{B(\frac{n}{b_n}\xi H(1))A(1)}\left[\bigl(2A'(1) + 3A(1)H''(1) + 3A(1)\bigr)B^{(3)}\left(\frac{n}{b_n}\xi H(1)\right) - 6\bigl(A'(1)\right. \\
&\quad + A(1)H''(1) + A(1)\bigr)B''\left(\frac{n}{b_n}\xi H(1)\right) + 3\bigl(2A'(1) + A(1)H''(1) + A(1)\bigr)B'\left(\frac{n}{b_n}\xi H(1)\right) \\
&\quad\left. - 2A'(1)B\left(\frac{n}{b_n}\xi H(1)\right)\right] = l_4(\xi).
\end{aligned} \tag{4.4}$$

Remark 4.1 [26] As a consequence of the above assumption, we obtain

i) $\lim_{n\to\infty}\frac{n}{b_n}B_n^*(e_1 - \xi;\xi) = \eta_1(\xi)$,

ii) $\lim_{n\to\infty} \frac{n}{b_n} B_n^*((e_1-\xi)^2;\xi) = \eta_2(\xi)$,
iii) $\lim_{n\to\infty} (\frac{n}{b_n})^2 B_n^*((e_1-\xi)^4;\xi) = \eta_3(\xi)$,

where

$$\eta_1(\xi) = \xi l_1(\xi) + \frac{A'(1)}{A(1)}, \qquad \eta_2(\xi) = \xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big),$$
$$\eta_3(\xi) = \xi^4 l_3(\xi) + 2\xi^3 l_4(\xi) + 3\xi^2\big(H''(1)^2 + 2H''(1) + 1\big).$$

Theorem 4.2 [26] (*Voronovskaya-type theorem*) *For every $f \in C_E(\mathbb{R}_0^+)$ such that $f', f'' \in C_E(\mathbb{R}_0^+)$, we have*

$$\lim_{n\to\infty} \frac{n}{b_n}\big[B_n^*(f;\xi) - f(\xi)\big] = \left(\xi l_1(\xi) + \frac{A'(1)}{A(1)}\right) f'(\xi) + \frac{1}{2}\big(\xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big)\big) f''(\xi),$$

uniformly with respect to $\xi \in [0,a]$, $a > 0$, where $l_i(\xi)$, $i = 1,2$, are defined in (4.1) *and* (4.2).

Example 4.3 Write

$$NB_n^*(h,\xi) = (1 + u_n) B_n^*(h,\xi),$$

where

$$u_n = \begin{cases} \frac{b_m^2}{m^2}, & m^2 - m \le n \le m^2 - 1, \\ \frac{b_m^3}{m^3}, & n = m^2, m \in \mathbb{N}\setminus\{1\}, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 4.4 *For the fourth-order central moment, we have the following estimate*:

$$\left(\frac{n}{b_n}\right)^2 NB_n^*\big((y-\xi)^4;\xi\big) \to \eta_3(\xi) \quad \text{on } [0,M] \text{ as } n \to \infty.$$

Proof From Proposition 2.2 we have

$$\left(\frac{n}{b_n}\right)^2 NB_n^*\big((y-\xi)^4;\xi\big) = \left(\frac{n}{b_n}\right)^2 (1 + u_n) B_n^*\big((y-\xi)^4;\xi\big),$$

from which we obtain that

$$\lim_{n\to\infty} \left(\frac{n}{b_n}\right)^2 (1 + u_n) B_n^*\big((y-\xi)^4;\xi\big) = \eta_3(\xi) \quad \text{on } [0,M].$$

□

Theorem 4.5 *Let $f \in C^B[0,\infty)$, the space of bounded and continuous functions in* $[0,\infty)$, *and suppose that $f', f'' \in C^B[0,\infty)$. Then*

$$\begin{aligned} &\left(\frac{n}{b_n}\right)\big[NB_n^*(f;\xi) - f(\xi)\big] \\ &\quad \sim f'(\xi)\left(l_1(\xi)\xi + \frac{A'(1)}{A(1)}\right) \\ &\qquad + \frac{f''(\xi)}{2}\left(l_2(\xi)\xi^2 + \frac{(A(1) + 2A'(1) + A(1)H''(1)) - 2A'(1)}{A(1)B}\xi\right)(st_T) \end{aligned}$$

for each $x \in [0,M]$ and any finite M.

Proof Taylor's formula gives

$$f(y)=f(\xi)+(y-\xi)f'(\xi)+\frac{1}{2}(y-\xi)^2f''(\xi)+(y-\xi)^2\psi(y-\xi), \tag{4.5}$$

where $\psi(y-\xi)\to 0$ as $y-\xi\to 0$. Applying NB_n^* to both sides of relation (4.5), we get

$$\begin{aligned} NB_n^*(f) &= (1+u_n)f(\xi)+(1+u_n)f'(\xi)\left(\frac{B'(\frac{n}{b_n}\xi H(1))-B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}x+\frac{b_n}{n}\frac{A'(1)}{A(1)}\right) \\ &\quad +(1+u_n)\frac{f''(\xi)}{2}\left(\frac{B''(\frac{n}{b_n}\xi H(1))-2B'(\frac{n}{b_n}\xi H(1))+B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2\right. \\ &\quad +\frac{b_n}{n}\frac{(A(1)+2A'(1)+A(1)H''(1))B'(\frac{n}{b_n}\xi H(1))-2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}x \\ &\quad \left.+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}\right)+(1+u_n)NB_n^*\big(\Phi^2\psi(y-\xi);\xi\big). \end{aligned}$$

This yields

$$\begin{aligned} \left(\frac{n}{b_n}\right)NB_n^*(f) &= \left(\frac{n}{b_n}\right)(1+u_n)f(\xi) \\ &\quad +\left(\frac{n}{b_n}\right)(1+u_n)f'(\xi)\left(\frac{B'(\frac{n}{b_n}\xi H(1))-B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}x+\frac{b_n}{n}\frac{A'(1)}{A(1)}\right) \\ &\quad +\left(\frac{n}{b_n}\right)(1+u_n)\frac{f''(\xi)}{2}\left(\frac{B''(\frac{n}{b_n}\xi H(1))-2B'(\frac{n}{b_n}\xi H(1))+B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2\right. \\ &\quad +\frac{b_n}{n}\frac{(A(1)+2A'(1)+A(1)H''(1))B'(\frac{n}{b_n}\xi H(1))-2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\xi \\ &\quad \left.+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}\right)+\left(\frac{n}{b_n}\right)(1+u_n)NB_n^*\big(\Phi^2\psi(y-\xi);\xi\big). \end{aligned}$$

Therefore

$$\begin{aligned} &\left|\left(\frac{n}{b_n}\right)\left[NB_n^*(f;\xi)-f(\xi)-f'(\xi)\left(\frac{B'(\frac{n}{b_n}\xi H(1))-B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}x+\frac{A'(1)}{A(1)}\right)\right.\right. \\ &\quad -\frac{f''(\xi)}{2}\left(\frac{B''(\frac{n}{b_n}\xi H(1))-2B'(\frac{n}{b_n}\xi H(1))+B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2\right. \\ &\quad \left.\left.\left.+\frac{(A(1)+2A'(1)+A(1)H''(1))B'(\frac{n}{b_n}\xi H(1))-2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\xi\right)\right]\right| \\ &\le \left(\frac{n}{b_n}\right)Ku_n+\left(\frac{n}{b_n}\right)K_1u_n\left|\left(\frac{B'(\frac{n}{b_n}\xi H(1))-B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi+\frac{A'(1)}{A(1)}\right)\right| \\ &\quad +\left(\frac{b_n}{n}\right)\frac{K_2}{2}\left|\frac{A'(1)+A''(1)}{A(1)}\right| \\ &\quad +\left(\frac{n}{b_n}\right)u_n\frac{K_2}{2}\left|\frac{B''(\frac{n}{b_n}\xi H(1))-2B'(\frac{n}{b_n}\xi H(1))+B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2+\frac{b_n}{n}\frac{A'(1)+A''(1)}{A(1)}\right| \end{aligned}$$

$$+\left(\frac{n}{b_n}\right)\left|NB_n^*\big((y-\xi)^2\psi(y-\xi);\xi\big)\right| + u_n\left(\frac{n}{b_n}\right)\left|NB_n^*\big((y-\xi)^2\psi(y-\xi);\xi\big)\right|,$$

where $K = \sup_{\xi\in[0,M]}|f(\xi)|$, $K_1 = \sup_{\xi\in[0,M]}|f'(\xi)|$, and $K_2 = \sup_{\xi\in[0,M]}|f''(\xi)|$.

Now we will prove that

$$\lim_{n\to\infty}\left(\frac{n}{b_n}\right)\left|NB_n^*\big((y-\xi)^2\psi(y-\xi);\xi\big)\right| = 0.$$

Applying the Cauchy–Schwartz inequality, we get

$$\left(\frac{n}{b_n}\right)\left|NB_n^*\big((y-\xi)^2\psi(y-\xi);\xi\big)\right| \le \left[\left(\frac{n}{b_n}\right)^2 NB_n^*\big((y-\xi)^4;\xi\big)\right]^{\frac{1}{2}}\cdot\left[NB_n^*\big(\psi^2;\xi\big)\right]^{\frac{1}{2}}. \quad (4.6)$$

Also, by setting $\eta_\xi(y) = (\psi(y-\xi))^2$ we have that $\eta_\xi(\xi) = 0$ and $\eta_\xi(\cdot)\in C[0,M]$. So

$$NB_n^*(\eta_\xi)\to 0(st_{\mathfrak{T}}) \quad \text{on } [0,M]. \quad (4.7)$$

Now from the last relation, (4.6), (4.7), and Lemma 4.4 we obtain that

$$\left(\frac{n}{b_n}\right)^2 NB_n^*\big((y-\xi)^2\psi(y-\xi);\xi\big)\to 0(st_{\mathfrak{T}}) \quad \text{on } [0,M]. \quad (4.8)$$

From the definition of the sequence (u_n) we obtain $(\frac{n}{b_n})u_n \to 0(st_{\mathfrak{T}})$ on $[0,M]$.

Let $\epsilon > 0$. Define the following sets:

$$\begin{aligned}
A = \Bigg|\{n: \Big|\left(\frac{n}{b_n}\right)\Bigg[&NB_n^*(f;\xi) - f(\xi) - f'(\xi)\left(\frac{B'(\frac{n}{b_n}\xi H(1)) - B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi + \frac{A'(1)}{A(1)}\right)\\
&-\frac{f''(\xi)}{2}\left(\frac{B''(\frac{n}{b_n}\xi H(1)) - 2B'(\frac{n}{b_n}\xi H(1)) + B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2\right.\\
&\left.+\frac{(A(1)+2A'(1)+A(1)H''(1))B'(\frac{n}{b_n}\xi H(1)) - 2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\xi\right)\Bigg]\Bigg|,
\end{aligned}$$

$$A_1 = \left|\left\{n: \left|\left(\frac{n}{b_n}\right)u_n\right| \ge \frac{\epsilon}{3K}\right\}\right|,$$

$$A_2 = \left|\left\{n: \left|\left(\frac{n}{b_n}\right)NB_n^*\big((y-\xi)^2\psi(y-\xi);\xi\big)\right| \ge \frac{\epsilon}{3}\right\}\right|,$$

$$A_3 = \left|\left\{n: \left|\left(\frac{n}{b_n}\right)u_n NB_n^*\big((y-\xi)^2\psi(y-\xi);\xi\big)\right| \ge \frac{\epsilon}{3}\right\}\right|.$$

From last relations we obtain that $A \le A_1 + A_2 + A_3$. Hence the result follows. □

Theorem 4.6 *Let $f, f', f'' \in C^B[0,\infty)$ and $\lim_{n\to\infty}(\frac{n}{b_n})^3 B_n^*((e_1-\xi)^6,\xi) = \eta_4(\xi)$. Then*

$$\begin{aligned}
\Bigg|\left(\frac{n}{b_n}\right)\big(B_n^*(f,\xi) - f(\xi)\big) - f'(\xi)\left(\frac{n}{b_n}\right)&\left(\frac{B'(\frac{n}{b_n}\xi H(1)) - B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}x + \frac{b_n}{n}\frac{A'(1)}{A(1)}\right)\\
-\frac{f''(\xi)}{2}\cdot\left(\frac{n}{b_n}\right)&\Bigg[\frac{B''(\frac{n}{b_n}\xi H(1)) - 2B'(\frac{n}{b_n}\xi H(1)) + B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2
\end{aligned}$$

$$+ \frac{b_n}{n}\frac{(A(1)+2A'(1)+A(1)H''(1))B'(\frac{n}{b_n}\xi H(1)) - 2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}x$$

$$+ \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}\Bigg]\Bigg| = O(1)\omega\left(f'', \left(\frac{b_n}{n}\right)^{-\frac{1}{2}}\right)$$

as $n \to \infty$ *for every* $\xi \in [0,\infty)$.

Proof By Taylor's theorem we get

$$f(u) = f(\xi) + f'(\xi)(u-\xi) + \frac{f''(\xi)}{2}(u-\xi)^2 + R(u,\xi),$$

where $R(u,\xi) = \frac{f''(\theta)-f''(\xi)}{2}(u-\xi)^2$ for $\theta \in (u,\xi)$. From this we have

$$\left|B_n^*(f,\xi) - f(\xi) - f'(\xi)B_n^*\big((u-\xi);\xi\big) - \frac{f''(\xi)}{2}B_n^*\big((u-\xi)^2;\xi\big)\right| \le B_n^*\big(\big|R(u,\xi)\big|,\xi\big),$$

from which we get that

$$\Bigg|\left(\frac{n}{b_n}\right)\big(B_n^*(f,\xi) - f(\xi)\big) - f'(\xi)\left(\frac{n}{b_n}\right)\left(\frac{B'(\frac{n}{b_n}\xi H(1)) - B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi + \frac{b_n}{n}\frac{A'(1)}{A(1)}\right)$$

$$- \frac{f''(\xi)}{2}\cdot\left(\frac{n}{b_n}\right)\Bigg[\frac{B''(\frac{n}{b_n}\xi H(1)) - 2B'(\frac{n}{b_n}\xi H(1)) + B(\frac{n}{b_n}\xi H(1))}{B(\frac{n}{b_n}\xi H(1))}\xi^2$$

$$+ \frac{b_n}{n}\frac{(A(1)+2A'(1)+A(1)H''(1))B'(\frac{n}{b_n}\xi H(1)) - 2A'(1)B(\frac{n}{b_n}\xi H(1))}{A(1)B(\frac{n}{b_n}\xi H(1))}\xi$$

$$+ \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}\Bigg]\Bigg|$$

$$\le \left(\frac{n}{b_n}\right)\cdot B_n^*\big(\big|R(u,\xi)\big|,\xi\big).$$

From the properties of modulus of continuity we obtain

$$\left|\frac{f''(\theta)-f''(\xi)}{2!}\right| \le \frac{1}{2!}\left(1 + \frac{|\theta-\xi|}{\delta}\right)\omega\big(f'',\delta\big).$$

We know that

$$\left|\frac{f''(\theta)-f''(\xi)}{2!}\right| \le \begin{cases} \omega(f'',\delta), & |u-\xi| \le \delta, \\ \frac{(t-\xi)^4}{\delta^4}\omega(f'',\delta), & |u-\xi| \ge \delta. \end{cases}$$

For $0 < \delta < 1$, we obtain that

$$\left|\frac{f''(\theta)-f''(\xi)}{2!}\right| \le \omega\big(f'',\delta\big)\left(1 + \frac{(u-\xi)^4}{\delta^4}\right),$$

which implies that

$$\big|R(u,\xi)\big| \le \omega\big(f'',\delta\big)\left(1 + \frac{(u-\xi)^4}{\delta^4}\right)(u-\xi)^2 = \omega\big(f'',\delta\big)\left((u-\xi)^2 + \frac{(u-\xi)^6}{\delta^4}\right).$$

By the linearity of B_n^* and the above relation we obtain

$$B_n^*\big(\big|R(u,\xi)\big|,\xi\big) \leq \omega\big(f'',\delta\big)\bigg(B_n^*\big((u-\xi)^2,\xi\big) + \frac{1}{\delta^4}B_n^*\big((u-\xi)^6,\xi\big)\bigg).$$

By Remark 4.1, for any $x \in [0,\infty)$, we obtain

$$B_n^*\big(\big|R(u,\xi)\big|,\xi\big) \leq \omega\big(f'',\delta\big)\bigg(O\bigg(\frac{b_n}{n}\bigg) + \frac{1}{\delta^4}O\bigg(\frac{b_n}{n}\bigg)^3\bigg) = O\bigg(\frac{b_n}{n}\bigg)\omega\big(f'',\delta_n\big).$$

We complete the proof by taking $\delta_n = (\frac{b_n}{n})^{-\frac{1}{2}}$. □

We prove the following results under the conditions given in the assumptions.

Theorem 4.7 *Let $f \in C^B[0,\infty)$ and $f', f'' \in C[0,\infty)$. Then*

$$\lim_{n\to\infty}\frac{n}{b_n}\big[B_n^*(fg,\xi) - B_n^*(f,\xi)B_n^*(g,\xi)\big] = \frac{1}{2}\big(\xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big)f'(\xi)g'(\xi)\big)$$

for any $x \in [0,M]$, where $M > 0$.

Proof After some calculations, we obtain

$$\begin{aligned}
&\frac{n}{b_n}\big[B_n^*(fg,\xi) - B_n^*(f,\xi)B_n^*(g,\xi)\big] \\
&\quad= \bigg[\frac{n}{b_n}\big(B_n^*(fg,\xi) - fg\big)\big) - \bigg(\xi l_1(\xi) + \frac{A'(1)}{A(1)}\bigg)(fg)'(\xi) \\
&\qquad - \frac{1}{2}\big(\xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big)\frac{(fg)''(\xi)}{2}\bigg] \\
&\qquad - g(\xi)\bigg[\frac{n}{b_n}\big(B_n^*(f,\xi) - f(\xi)\big) - \bigg(\xi l_1(\xi) + \frac{A'(1)}{A(1)}\bigg)f'(\xi) \\
&\qquad - \frac{1}{2}\big(\xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big)\frac{f''(\xi)}{2}\bigg] \\
&\qquad - B_n^*(f,\xi)\bigg[\frac{n}{b_n}\big(B_n^*(g,\xi) - g(\xi)\big) - \bigg(\xi l_1(\xi) + \frac{A'(1)}{A(1)}\bigg)g'(\xi) \\
&\qquad - \frac{1}{2}\big(\xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big)\frac{g''(\xi)}{2}\bigg] + \frac{1}{2}\big(\xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big)f'(\xi)g'(\xi) \\
&\qquad + \frac{1}{2}\big(\xi^2 l_2(\xi) + \xi\big(1 + H''(1)\big)\frac{g''(\xi)}{2}\big[f(\xi) - B_n^*(f,\xi)\big] \\
&\qquad + \bigg(\xi l_1(\xi) + \frac{A'(1)}{A(1)}\bigg)g'(\xi)\big[f(\xi) - B_n^*(f,\xi)\big].
\end{aligned}$$

Now the proof follows from Theorem 4.2 and Proposition 2.2. □

5 Weighted approximation

Now we will study some properties of B_n^* in weighted spaces. Also, we will suppose that

$$\lim_{n\to\infty}\frac{B^{(k)}(y)}{B(y)} = 1 \quad \text{for every } k = 1,2,\ldots,r; r \in \mathbb{N}.$$

Let $\rho(x) = x^2 + 1$ be the weight function, and let M_f be a positive constant. We write

(i) $B_\rho[0,\infty)$ for the space of bounded functions $|f(x)| \leq M_f \rho(x)$ with $\|f\|_\rho = \sup_{x\geq 0} \frac{|f(x)|}{\rho(x)}$.

(ii) $C_\rho[0,\infty)$ for the subspace of continuous functions in $B_\rho[0,\infty)$.

(iii) $C_\rho^*[0,\infty)$ for the space of functions $f \in C_\rho[0,\infty)$ with fn ite $\lim_{x\to\infty} \frac{f(x)}{\rho(x)}$.

The weighted modulus of continuity $\Omega(f;\delta)$ is defined by

$$\Omega(f;\delta) = \sup_{x\geq 0, 0<|h|\leq\delta} \frac{|f(x+h) - f(x)|}{(1+h^2)\rho(x)} \quad \text{for all } f \in C_\rho^*[0,\infty).$$

For any $\mu \in [0,\infty)$,

$$\Omega(f;\mu\delta) \leq 2(1+\mu)\bigl(1+\delta^2\bigr)\Omega(f;\delta),$$

and

$$\bigl|f(t) - f(x)\bigr| \leq 2\left(\frac{|t-x|}{\delta} + 1\right)\bigl(1+\delta^2\bigr)\Omega(f;\delta)\bigl(1+x^2\bigr)\bigl(1+(t-x)^2\bigr), \quad f \in C_\rho^*[0,\infty).$$

Theorem 5.1 *For $f \in C_\rho^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \bigl\|B_n^*(f;x) - f(x)\bigr\|_\rho = 0.$$

Proof It suffices to check that $B_n^*(e_i;x)$ uniformly converges to e_i as $n \to \infty$, where $e_i(x) = x^i$, $i = 0,1,2$, and apply the weighted Korovkin-type theorem. Using Lemma 2.1, the case $i = 0$ is trivial. Now

$$\bigl\|B_n^* e_1 - e_1\bigr\|_\rho = \sup_{x\geq 0}\left\{\frac{|B_n^* e_1 - e_1|}{\rho(x)}\right\} \leq \sup_{x\geq 0} \frac{|\alpha_1(n,x)|}{\rho(x)},$$

and by a similar consideration, we have

$$\bigl\|B_n^* e_2 - e_2\bigr\|_\rho = \sup_{x\geq 0}\left\{\frac{|B_n^* e_2 - e_2|}{\rho(x)}\right\} \leq \sup_{x\geq 0}\left\{\frac{|\alpha_2(n,x)|}{\rho(x)}\right\},$$

where

$$\begin{aligned}
\alpha_1(n,x) &= \left(\frac{B'(\frac{n}{b_n}xH(1))}{B(\frac{n}{b_n}xH(1))} - 1\right)x + \frac{b_n}{n}\cdot\frac{A'(1)}{A(1)}, \\
\alpha_2(n,x) &= \left(\frac{B''(\frac{n}{b_n}xH(1))}{B(\frac{n}{b_n}xH(1))} - 1\right)x^2 + \frac{b_n}{n}\frac{B'(\frac{n}{b_n}xH(1))[A(1) + 2A'(1) + H''(1)A(1)]}{A(1)B(\frac{n}{b_n}xH(1))}x \\
&\quad + \frac{b_n^2}{n^2}\frac{A'(1) + A''(1)}{A(1)}.
\end{aligned}$$

We conclude that

$$\lim_n \bigl\|B_n^* e_i - e_i\bigr\|_\rho = \lim_{n\to\infty} \bigl\|B_n^* e_i - e_i\bigr\|_\rho = 0 \quad (i = 0,1,2),$$

which finishes the proof. □

Theorem 5.2 *Let* $f \in C^*_\rho[0,\infty)$. *Then*

$$\sup_{x\in[0,\infty)} \frac{|B^*_n(f;x) - f(x)|}{(1+x^2)(A(n,x)+B(n,x)x+C(n,x)x^2+D(n,x)x^3+E(n,x)x^4)} \le K\Omega\bigl(f;n^{-\frac{1}{4}}\bigr)$$

for sufficiently large n, $A(n,x)$, $B(n,x)$, $C(n,x)$, $D(n,x)$, *and* $E(n,x)$ *depend on* n *and* x, *and* K *is a positive constant.*

Proof For $x \in [0,\infty)$, we have

$$B^*_n(f;x) - f(x) = \frac{1}{A(1)B(\frac{n}{b_n}xH(1))} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left[f\left(\frac{k}{n}b_n\right) - f(x)\right].$$

Using the properties of the weighted modulus, we obtain

$$\begin{aligned} &\bigl|B^*_n(f;x) - f(x)\bigr| \\ &\quad\le \frac{1}{A(1)B(\frac{n}{b_n}xH(1))} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) 2\bigl(1+\delta_n^2\bigr)\Omega(f;\delta_n)\bigl(1+x^2\bigr) \\ &\quad\quad \cdot \left(\frac{|(\frac{k}{n}b_n) - x|}{\delta_n} + 1\right)\bigl(1+(t-x)^2\bigr). \end{aligned}$$

Let us denote by $S(t,x) = (\frac{|(\frac{k}{n}b_n)-x|}{\delta_n} + 1)(1+(t-x)^2)$. Then

$$S(t,x) \le \begin{cases} 2(1+\delta_n^2) & \text{if } |\frac{k}{n}b_n - x| \le \delta_n, \\ 2(1+\delta_n^2)\frac{(\frac{k}{n}b_n - x)^4}{\delta_n^4} & \text{if } |\frac{k}{n}b_n - x| \ge \delta_n. \end{cases}$$

From last relation we get that

$$S(x,t) \le 2\bigl(1+\delta_n^2\bigr)\left(1 + \frac{(\frac{k}{n}b_n - x)^4}{\delta_n^4}\right).$$

So

$$\begin{aligned} &\bigl|B^*_n(f;x) - f(x)\bigr| \\ &\quad\le 4\frac{1}{A(1)B(\frac{n}{b_n}xH(1))} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\bigl(1+\delta_n^2\bigr)\Omega(f;\delta_n)\bigl(1+x^2\bigr) \\ &\quad\quad \cdot \bigl(1+\delta_n^2\bigr)\left(1 + \frac{(\frac{k}{n}b_n - x)^4}{\delta_n^4}\right). \end{aligned}$$

After some calculations, we get

$$\begin{aligned} &\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left(1 + \frac{(\frac{k}{n}b_n - x)^4}{\delta_n^4}\right) \\ &\quad= \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) + \frac{1}{\delta_n^4}\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left[\left(\frac{k}{n}\right)^4 b_n^4 - 4\left(\frac{k}{n}\right)^3 b_n^3 x + 6\left(\frac{k}{n}\right)^2 b_n^2 x^2\right.\end{aligned}$$

$$
\begin{aligned}
&-4\left(\frac{k}{n}\right)b_n x^3+x^4\Big] \\
&=\left(1+\frac{x^4}{\delta_n^4}\right)A(1)B\left(\frac{n}{b_n}xH(1)\right)+\frac{b_n^4}{n^4\delta_n^4}\sum_{k=0}^{\infty}k^4p_k\left(\frac{n}{b_n}x\right)-4\frac{xb_n^3}{n^3\delta_n^4}\sum_{k=0}^{\infty}k^3p_k\left(\frac{n}{b_n}x\right) \\
&\quad+6\frac{x^2b_n^2}{n^2\delta_n^4}\sum_{k=0}^{\infty}k^2p_k\left(\frac{n}{b_n}x\right)-4\frac{x^3b_n}{n\delta_n^4}\sum_{k=0}^{\infty}kp_k\left(\frac{n}{b_n}x\right).
\end{aligned}
$$

From these relations and Lemma 2.1 of [26]) we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\left(1+\frac{(\frac{k}{n}b_n-x)^4}{\delta_n^4}\right) \\
&=\left(1+\frac{x^4}{\delta_n^4}\right)A(1)B\left(\frac{n}{b_n}xH(1)\right) \\
&\quad-4\frac{x^3b_n}{n\delta_n^4}\left[A'(1)B\left(\frac{n}{b_n}xH(1)\right)+\frac{n}{b_n}xA(1)B'\left(\frac{n}{b_n}xH(1)\right)\right] \\
&\quad+6\frac{x^2b_n^2}{n^2\delta_n^4}\left[\frac{n^2}{b_n^2}x^2A(1)B''\left(\frac{n}{b_n}xH(1)\right)\right. \\
&\quad+\frac{n}{b_n}x\big(A(1)+2A'(1)+H''(1)A(1)\big)B'\left(\frac{n}{b_n}xH(1)\right) \\
&\quad\left.+\big(A'(1)+A''(1)\big)B\left(\frac{n}{b_n}xH(1)\right)\right]-4\frac{xb_n^3}{n^3\delta_n^4}\left[\frac{n^3}{b_n^3}x^3A(1)B'''\left(\frac{n}{b_n}xH(1)\right)\right. \\
&\quad+\frac{n^2}{b_n^2}x^2\big(3A'(1)+3H''(1)A(1)+3A(1)\big)B''\left(\frac{n}{b_n}xH(1)\right) \\
&\quad+\frac{n}{b_n}x\big(3A''(1)+3H''(1)A'(1)+H'''(1)A(1)+6A'(1) \\
&\quad\left.+3H''(1)A(1)+A(1)\big)B'\left(\frac{n}{b_n}xH(1)\right)+\big(A'''(1)+3A''(1)+A'(1)\big)B\left(\frac{n}{b_n}xH(1)\right)\right] \\
&\quad+\frac{b_n^4}{n^4\delta_n^4}\left[\frac{n^4}{b_n^4}x^4A(1)B^{(4)}\left(\frac{n}{b_n}xH(1)\right)\right. \\
&\quad+\frac{n^3}{b_n^3}x^3\big(4A'(1)+6H''(1)A(1)+6A(1)\big)B'''\left(\frac{n}{b_n}xH(1)\right) \\
&\quad+\frac{n^2}{b_n^2}x^2\big(6A''(t)+12H''(1)+A'(1)+4H'''(1)A(1)+3H''(1)^2A(1)+18A'(1) \\
&\quad+18H''(1)A(1)+7A(1)\big)B''\left(\frac{n}{b_n}xH(1)\right)+\big(4A'''(1)+6A''(1)H''(1)\big] \\
&\quad+4A'(1)H'''(1)+A(1)H^{(4)}(1)+18A''(1) \\
&\quad+18H''(1)A'(1)+6H'''(1)A(1)+14A'(1)+7H''(1)A(1)+A(1)\big)\frac{n}{b_n}xB'\left(\frac{n}{b_n}xH(1)\right) \\
&\quad\left.+\big(A^{(4)}(1)+6A^{(3)}(1)+7A''(1)+A'(1)\big)B\left(\frac{n}{b_n}xH(1)\right)\right].
\end{aligned}
$$

From last two relations we get

$$
\begin{aligned}
&\left|B_n^*(f;x)-f(x)\right| \\
&\quad\le 4\frac{(1+\delta_n^2)^2\Omega(f;\delta_n)(1+x^2)}{A(1)B(\frac{n}{b_n}xH(1))}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\left(1+\frac{(\frac{k}{n}b_n-x)^4}{\delta_n^4}\right) \\
&\quad\le 4\frac{(1+\delta_n^2)^2\Omega(f;\delta_n)(1+x^2)}{A(1)B(\frac{n}{b_n}xH(1))}\left\{\left(1+\frac{x^4}{\delta_n^4}\right)A(1)B\left(\frac{n}{b_n}xH(1)\right)\right. \\
&\qquad -4\frac{x^3b_n}{n\delta_n^4}\left[A'(1)B\left(\frac{n}{b_n}xH(1)\right)+\frac{n}{b_n}xA(1)B'\left(\frac{n}{b_n}xH(1)\right)\right] \\
&\qquad +6\frac{x^2b_n^2}{n^2\delta_n^4}\left[\frac{n^2}{b_n^2}x^2A(1)B''\left(\frac{n}{b_n}xH(1)\right)\right. \\
&\qquad +\frac{n}{b_n}x\bigl(A(1)+2A'(1)+H''(1)A(1)\bigr)B'\left(\frac{n}{b_n}xH(1)\right) \\
&\qquad \left.+\bigl(A'(1)+A''(1)\bigr)B\left(\frac{n}{b_n}xH(1)\right)\right] \\
&\qquad -4\frac{xb_n^3}{n^3\delta_n^4}\left[\frac{n^3}{b_n^3}x^3A(1)B'''\left(\frac{n}{b_n}xH(1)\right)\right. \\
&\qquad +\frac{n^2}{b_n^2}x^2\bigl(3A'(1)+3H''(1)A(1)+3A(1)\bigr)B''\left(\frac{n}{b_n}xH(1)\right) \\
&\qquad +\frac{n}{b_n}x\bigl(3A''(1)+3H''(1)A'(1)+H'''(1)A(1)+6A'(1)+3H''(1)A(1) \\
&\qquad \left.+A(1)\bigr)B'\left(\frac{n}{b_n}xH(1)\right)+\bigl(A'''(1)+3A''(1)+A'(1)\bigr)B\left(\frac{n}{b_n}xH(1)\right)\right] \\
&\qquad +\frac{b_n^4}{n^4\delta_n^4}\left[\frac{n^4}{b_n^4}x^4A(1)B^{(4)}\left(\frac{n}{b_n}xH(1)\right)+\frac{n^3}{b_n^3}x^3\bigl(4A'(1)+6H''(1)A(1)\right. \\
&\qquad +6A(1)\bigr)B'''\left(\frac{n}{b_n}xH(1)\right)+\frac{n^2}{b_n^2}x^2\bigl(6A''(t)+12H''(1)+A'(1)+4H'''(1)A(1) \\
&\qquad +3H''(1)^2A(1)+18A'(1)+18H''(1)A(1)+7A(1)\bigr)B''\left(\frac{n}{b_n}xH(1)\right) \\
&\qquad +\bigl(4A'''(1)+6A''(1)H''(1)\bigr]+4A'(1)H'''(1)+A(1)H^{(4)}(1)+18A''(1) \\
&\qquad +18H''(1)A'(1)+6H'''(1)A(1)+14A'(1)+7H''(1)A(1)+A(1)\bigr)\frac{n}{b_n}xB'\left(\frac{n}{b_n}xH(1)\right) \\
&\qquad \left.\left.+\bigl(A^{(4)}(1)+6A^{(3)}(1)+7A''(1)+A'(1)\bigr)B\left(\frac{n}{b_n}xH(1)\right)\right]\right\}.
\end{aligned}
$$

For $\delta_n = n^{-\frac{1}{4}}$, we have

$$
\begin{aligned}
&\left|B_n^*(f;x)-f(x)\right| \\
&\quad\le 16\Omega(f;\delta_n)\bigl(1+x^2\bigr)\bigl(A(n,x)+B(n,x)x+C(n,x)x^2+D(n,x)x^3+E(n,x)x^4\bigr),
\end{aligned}
$$

where $A(n,x)$, $B(n,x)$, $C(n,x)$, $D(n,x)$, and $E(n,x)$ depend on n and x.

Now from last relation we obtain

$$\sup_{x\in[0,\infty)} \frac{|B_n^*(f;x)-f(x)|}{(1+x^2)(A(n,x)+B(n,x)x+C(n,x)x^2+D(n,x)x^3+E(n,x)x^4)} \le K\Omega\bigl(f;n^{-\frac{1}{4}}\bigr). \qquad \square$$

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Author contributions
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