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Analysis of JS-contractions with applications to fractional boundary value problems

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Abstract

In this article, we modify JS-contractions by weakening the conditions on the function θ, where $\theta : (0,\infty) \to (1,\infty)$ is a strictly increasing function. We prove fixed-point results for obtained contractions. Some examples are given to validate the results and modifications. We use our main theorem to establish the existence results for the solutions of the Atangana–Baleanu–Caputo fractional boundary value problem with integral boundary conditions. We also present a new definition of θ-Ulam stability and find the stability of our fractional boundary value problem.

Keywords: θ-contraction; Modified JS-contraction; Modified weak JS-contraction; Existence results; θ-Ulam stability

1 Introduction

The fixed point theory is important to numerous theoretical and applied fields, such as linear and variational inequalities, nonlinear analysis, differential equations and differential inclusions, the approximation principle, arithmetic of fractals, equilibrium issues, optimization issues and mathematical modeling. In fixed point theory, a fixed point theorem is a result showing that a mapping Γ (either linear or nonlinear) has at least one fixed point, under a few conditions on Γ. The space S involved in fixed point theorems can come from a variety of spaces, i.e., it could be a metric space, generalized metric space, normed linear space, uniform space, linear topological space, lattice, etc., while the conditions imposed on the operator Γ are generally metric or completeness, or compactness type conditions. The versatility of Banach contraction principle [11] allows its extensions and generalizations in different directions. As a mapping need not satisfy the Banach contractivity condition, new contractivity conditions are introduced to solve the problem.

Fixed point theory is also serving at its best for the existence theory of fractional and integer-order differential equations. The fractional derivative is a so-called generalization of the ordinary or integer-order derivative. Boundary conditions involved in the mathematical models are very important; among the most important boundary conditions are integral boundary conditions. Many types of fractional operators are available in the literature; for more details about fractional differential equations/operators and boundary conditions, we refer the readers to the articles [1–10, 16, 21, 22, 24–28] and the references therein.

In this era, fixed point theory provides a lot of methods for solving the existence problem in mathematical analysis. For finding the solutions of operator equations of the type

$$x = \Gamma x, \tag{A}$$

contractive operators are very important. For finding the solutions of operator equations (linear or nonlinear) $g(x) = 0$, the solutions of (A) are very important. Many celebrated contractive mappings were defined after the Banach contraction mapping. Some of famous contractive mappings are as follows:

Let (S, d) be a metric space, then a mapping $\Gamma : S \to S$ is said to be

(a_1) a Chatterjee contraction if there exit $\gamma_1 \in (0, \frac{1}{2})$ and $\forall a, b \in S$ such that

$$d(\Gamma a, \Gamma b) \leq \gamma_1 \big[d(a, \Gamma b) + d(b, \Gamma a)\big];$$

(a_2) a Kannan contraction if there exist $\gamma_2 \in (0, \frac{1}{2})$ and $\forall a, b \in S$ such that

$$d(\Gamma a, \Gamma b) \leq \gamma_2 \big[d(a, \Gamma a) + d(b, \Gamma b)\big];$$

(a_3) a Reich contraction if there exist $\gamma_3, \gamma_4, \gamma_5 \geq 0$ with $\gamma_3 + \gamma_4 + \gamma_5 < 1$ such that $\forall a, b \in S$ one has

$$d(\Gamma a, \Gamma b) \leq \gamma_3 d(a, b) + \gamma_4 d(a, \Gamma a) + \gamma_5 d(b, \Gamma b);$$

(a_4) a Ćirić contraction if there are $\gamma_6, \gamma_7, \gamma_8, \gamma_9 \geq 0$ with $\gamma_6 + \gamma_7 + \gamma_8 + 2\gamma_9 < 1$ such that $\forall a, b \in S$ one has

$$d(\Gamma a, \Gamma b) \leq \gamma_6 d(a, b) + \gamma_7 d(a, \Gamma a) + \gamma_8 d(b, \Gamma b) + \gamma_9 \big[d(a, \Gamma b) + d(b, \Gamma a)\big].$$

Remark 1 In the case of a complete metric space (S, d), any self-mapping satisfying any one of (a_1)–(a_4) has a unique fixed point.

Many generalizations and extensions of contracting mappings have appeared in the literature by weakening of the contractive conditions (both for single and multivalued maps) and also by weakening the structure (topology of the given space); for instance, see [2–16, 21]. In particular, in [20], Jleli et al. introduced the notion of θ-contractions and proved a variant of the Banach contraction principle in the setting of Branciari metric spaces [15]:

Theorem 1 ([20]) *Let $\Gamma : S \to S$ be a self-mapping on a complete metric space (S, d). Suppose there are $\theta \in \Theta$ and $\varsigma \in (0, 1)$ such that*

$$\forall a, b \in S, \quad d(\Gamma a, \Gamma b) \neq 0 \quad \Longrightarrow \quad \theta\big\{d(\Gamma a, \Gamma b)\big\} \leq \big[\theta\big\{d(a, b)\big\}\big]^{\varsigma}. \tag{1.1}$$

Then the mapping Γ has a unique fixed point.

Clearly, the Banach contraction principle can be deduced from the above theorem.

Many authors extended this work in various directions. We present the following definition to make our goal clearer.

Definition 1 Let Ξ denote the set of all mappings $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Ξ_1) θ is nondecreasing,

(Ξ_2) for each sequence $\{\alpha_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(\alpha_n) = 1 \Longleftrightarrow \lim_{n\to\infty} \alpha_n = 0^+$,

(Ξ_3) there exist $r \in (0,1)$ and $l \in (0,\infty)$ such that $\lim_{\alpha\to 0^+} \frac{\theta(\alpha)-1}{t^r} = l$,

(Ξ_4) θ is continuous on $(0,\infty)$,

(Ξ_5) θ is strictly increasing,

(Ξ_6) $\theta(\alpha+\beta) \le \theta(\alpha)\theta(\beta)$, $\forall \alpha,\beta \in (0,\infty)$.

In [23], Parvane et al. gave the idea of a modified θ-contraction (the class of mappings satisfying Ξ_2–Ξ_4 is denoted by Θ') and proved related fixed point results. Note that in [18], the class of mappings satisfying Ξ_1, Ξ_2, and Ξ_4 is denoted by Ω, whereas in [20] a similar class satisfying Ξ_1–Ξ_3 is denoted by Θ. However, in [17], the class of mappings, satisfying Ξ_1–Ξ_3 and Ξ_6, is denoted by Ψ, and in [19] the class fulfilling Ξ_1, Ξ_2, and Ξ_6 is denoted by Ψ_2.

The following contractive condition has been defined in [23]:

Definition 2 ([23]) Let $\Gamma : S \to S$ be a self-mapping on a metric space (S,d). Then Γ is said to be a modified JS-contraction ($\mathcal{P}$-contraction) whenever there are $\theta \in \Theta'$ and ς_1, ς_2, ς_3, $\varsigma_4 \ge 0$ with $\varsigma_1 + \varsigma_2 + \varsigma_3 + \varsigma_4 < 1$ such that the following condition holds:

$$\begin{aligned} \theta\{d(\Gamma a,\Gamma b)\} \le{}& [\theta\{d(a,b)\}]^{\varsigma_1}[\theta\{d(a,\Gamma a)\}]^{\varsigma_2}[\theta\{d(b,\Gamma b)\}]^{\varsigma_3} \\ &\times \left[\theta\left\{\left(\frac{d(a,\Gamma b)+d(b,\Gamma a)}{2}\right)\right\}\right]^{\varsigma_4}, \end{aligned} \tag{1.2}$$

for all $a,b \in S$.

The following theorem is a consequence of the above definition.

Theorem 2 ([23]) *Each $\mathcal{P}$-contraction mapping on a complete metric space has a unique fixed point.*

In this article, we consider the class of mappings satisfying only condition Ξ_5, denoted by χ. First, we present a comparison of this generalization with others by the following example.

Example 1 If we define $\theta(a) = e^{-\frac{1}{a}}$ for $a \in (0,\infty)$, then $\theta'(a) = \frac{1}{a^2}e^{-\frac{1}{a}} > 0$ for all $a \in (0,\infty)$. Clearly, θ satisfies Ξ_4 and Ξ_5, but not Ξ_1. Also as $\theta(a+b) = e^{-\frac{1}{a+b}}$ and $\theta(a)\cdot\theta(b) = e^{-\frac{a+b}{ab}}$, we have $\theta(a+b) \nleq \theta(a)\cdot\theta(b)$. So θ does not satisfy Ξ_6. Now consider the sequences $\{a_n = \frac{1}{n}\}_{n\in\mathbb{N}}$ and $\{\theta(a_n) = e^{-\frac{1}{a_n}} = e^{-n}\}_{n\in\mathbb{N}}$, then clearly $\lim_{n\to\infty}\theta(a_n) = \lim_{n\to\infty} e^{-n} = 0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$. Hence, θ does not satisfy Ξ_2. We conclude that $\theta \in \chi$ but $\chi \not\subset \Psi$, $\chi \not\subset \Omega$, $\chi \not\subset \Theta$, $\chi \not\subset \Theta'$ and $\chi \not\subset \Psi_2$.

Example 2 In this example we take the usual metric on $X = \mathbb{R}$ or on its closed subset of type $[a,b]$ not including 0, say $[100, 10{,}000]$. Define $\Gamma(a) = \frac{1}{a}$ and $\theta(t) = e^t$, then we check

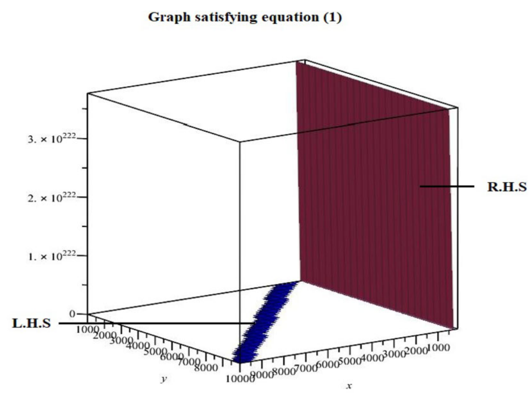

Figure 1 Clearly it satisfies the inequlity (1.2) of Definition 2

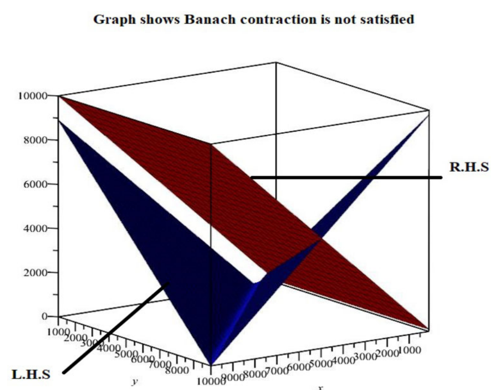

Figure 2 The contractive condition of Banach is not satisfied

that the Banach contractivity condition

$$d(\Gamma a, \Gamma b) \leq kd(a,b) \quad \text{for all } a, b \in X \text{ and for some } k \in (0,1)$$

is not satisfied but (1.1) holds; see Figs. 1 and 2.

The following lemma is an equivalent of the results in [12, Chap. 8, Sect. 1], and in [14, Proposition 1].

Lemma 1 *Suppose $\theta \in \chi$, then θ has only countably many discontinuities in $\mathbb{R}$; denote their set by Λ.*

Turinici in [29] used the idea of d-semi-Cauchy sequence, i.e., a sequence $(a_n; n \geq 0)$ is d-semi-Cauchy if $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$. The following proposition will be useful for the proof of our main results.

Proposition 1 ([29]) *Suppose that $(a_n; n \geq 0)$ is d-semi-Cauchy but not d-Cauchy. Further, let Λ be a countable subset of $\mathbb{R}_0^+$. Then there exist a number $\eta \in \mathbb{R}_0^+ \backslash \Lambda$, a rank $\nu(\eta) \geq 0$,* *and a couple of rank-sequences* $(m(n_1); n_1 \geq 0)$, $(n(n_2); n_2 \geq 0)$ *with*

$$n_1 \leq m(n_1) < n(n_1), \quad d(a_{m(n_1)}, a_{n(n_1)}) > \eta, \quad \forall n_1 \geq 0, \tag{1.3}$$

$$n(n_2) - m(n_2) \geq 2, \quad d(a_{m(n_2)}, a_{n(n_2)-1}) \leq \eta, \quad \forall n_2 \geq \nu(n_2), \tag{1.4}$$

$$d(a_{m(n_1)}, a_{n(n_1)}) \to \eta^+, \quad \text{as } n_1 \to \infty, \tag{1.5}$$

$$d(a_{m(n_2)+k_1}, a_{n(n_2)+k_2}) \to \eta, \quad \text{as } n_2 \to \infty, \forall k_1, k_2 \in \{0,1\}. \tag{1.6}$$

2 Main results

In this section we prove fixed point theorems for mappings involving contractions with a function θ related to the class χ defined above.

Theorem 3 *Let* $\Gamma : S \to S$ *be a self-mapping on a complete metric space* (S,d). *Suppose there exits* $\theta \in \chi$ *and* Γ *satisfies* (1.1). *Then* Γ *has a unique fixed point.*

Proof The proof of this theorem is included in the next theorem. □

Theorem 4 *Let* $\Gamma : S \to S$ *be a self-mapping on a complete metric space* (S,d). *Suppose there exits* $\theta \in \chi$ *and* Γ *satisfies* (1.2). *Then* Γ *has a unique fixed point.*

Proof Let $a_0 \in S$ be an arbitrary point. Consider a sequence $\{a_n\}_{n\in\mathbb{N}}$ in S, defined by $a_{n+1} = \Gamma a_n$. If $a_{n+1} = a_n$ for some n, then $a_n \in S$ is a fixed point of Γ. So we assume $a_{n+1} \neq a_n$ $\forall n \geq 0$ and consider, using (1.2),

$$\begin{aligned}
\theta\big(d(\Gamma a_{n-1}, \Gamma a_n)\big) &\leq \big[\theta\big\{d(a_{n-1}, a_n)\big\}\big]^{\varsigma_1}\big[\theta\big\{d(a_{n-1}, \Gamma a_{n-1})\big\}\big]^{\varsigma_2}\big[\theta\big\{d(a_n, \Gamma a_n)\big\}\big]^{\varsigma_3} \\
&\quad \times \left[\theta\left\{\left(\frac{d(a_{n-1}, \Gamma a_n) + d(a_n, \Gamma a_{n-1})}{2}\right)\right\}\right]^{\varsigma_4}, \\
\theta\big(d(\Gamma a_{n-1}, \Gamma a_n)\big) &\leq \big[\theta\big\{d(a_{n-1}, a_n)\big\}\big]^{\varsigma_1}\big[\theta\big\{d(a_{n-1}, a_n)\big\}\big]^{\varsigma_2}\big[\theta\big\{d(a_n, a_{n+1})\big\}\big]^{\varsigma_3} \\
&\quad \times \left[\theta\left\{\left(\frac{d(a_{n-1}, a_{n+1}) + d(a_n, a_n)}{2}\right)\right\}\right]^{\varsigma_4}, \\
\theta\big(d(\Gamma a_{n-1}, \Gamma a_n)\big) &\leq \big[\theta\big\{d(a_{n-1}, a_n)\big\}\big]^{\varsigma_1+\varsigma_2}\big[\theta\big\{d(a_n, a_{n+1})\big\}\big]^{\varsigma_3} \\
&\quad \times \left[\theta\left\{\left(\frac{d(a_{n-1}, a_n) + d(a_n, a_{n+1})}{2}\right)\right\}\right]^{\varsigma_4}, \\
\theta\big(d(\Gamma a_{n-1}, \Gamma a_n)\big) &\leq \big[\theta\big\{d(a_{n-1}, a_n)\big\}\big]^{\varsigma_1+\varsigma_2}\big[\theta\big\{d(a_n, a_{n+1})\big\}\big]^{\varsigma_3} \\
&\quad \times \big[\theta\big(\max\big\{d(a_{n-1}, a_n), d(a_n, a_{n+1})\big\}\big)\big]^{\varsigma_4}.
\end{aligned} \tag{2.1}$$

If $d(a_{n-1}, a_n) \leq d(a_n, a_{n+1})$, then (2.1) gives

$$\theta\big(d(\Gamma a_{n-1}, \Gamma a_n)\big) \leq \big[\theta\big(d(a_{n-1}, a_n)\big)\big]^{\varsigma_1+\varsigma_2}\big[\theta\big(d(a_n, a_{n+1})\big)\big]^{\varsigma_3}\big[\theta\big(d(a_n, a_{n+1})\big)\big]^{\varsigma_4},$$

which implies

$$\theta\big(d(a_n, a_{n+1})\big)^{1-\varsigma_3-\varsigma_4} \leq \big[\theta\big(d(a_{n-1}, a_n)\big)\big]^{\varsigma_1+\varsigma_2},$$

$$\theta\big(d(a_n, a_{n+1})\big) \leq \big[\theta\big(d(a_{n-1}, a_n)\big)\big]^{\frac{\varsigma_1+\varsigma_2}{1-\varsigma_3-\varsigma_4}} < \theta\big(d(a_{n-1}, a_n)\big),$$

which is contrary to our supposition. Therefore we have $d(a_n, a_{n+1}) \leq d(a_{n-1}, a_n)$, and then we get

$$\theta\big(d(a_n, a_{n+1})\big) \leq \big[\theta\big(d(a_0, a_1)\big)\big]^{(\frac{\varsigma_1+\varsigma_2+\varsigma_4}{1-\varsigma_4})^n}. \tag{2.2}$$

Since $d(a_n, a_{n+1}) \leq d(a_{n-1}, a_n)$, $\{d(a_n, a_{n+1})\}$ is a decreasing sequence which is bounded from below. So there exists $\wp \in \mathbb{R}$ such that

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = \wp \geq 0. \tag{2.3}$$

If $\wp > 0$, then from (2.1) we have

$$\begin{aligned}\theta\big(d(\Gamma a_{n-1}, \Gamma a_n)\big) &\leq \big[\theta\big\{d(a_{n-1}, a_n)\big\}\big]^{\varsigma_1+\varsigma_2}\big[\theta\big\{d(a_n, a_{n+1})\big\}\big]^{\varsigma_3} \\ &\quad\times \big[\theta\big(\max\big\{d(a_{n-1}, a_n), d(a_n, a_{n+1})\big\}\big)\big]^{\varsigma_4},\end{aligned}$$

using (2.2) gives

$$\begin{aligned}\theta\big(d(a_n, a_{n+1})\big) &\leq \big[\theta\big(d(a_{n-1}, a_n)\big)\big]^{\varsigma_1}\big[\theta\big(d(a_{n-1}, a_n)\big)\big]^{\varsigma_2}\big[\theta\big(d(a_n, a_{n+1})\big)\big]^{\varsigma_3} \\ &\quad\times \big[\theta\big(d(a_{n-1}, a_n)\big)\big]^{\varsigma_4}, \\ \theta\big(d(a_n, a_{n+1})\big) &\leq \big[\theta\big\{d(a_{n-1}, a_n)\big\}\big]^{\frac{\varsigma_1+\varsigma_2+\varsigma_4}{1-\varsigma_4}} < \theta\big(d(a_{n-1}, a_n)\big),\end{aligned}$$

as θ fulfills (Ξ_5), then

$$d(a_n, a_{n+1}) < d(a_{n-1}, a_n),$$

so passing to the limit as $n \to \infty$ gives a contradiction, thus $\wp = 0$ and hence

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0. \tag{2.4}$$

As θ satisfies (Ξ_5), θ has a countable set Λ of discontinuity points by Lemma 1. Now to prove that (a_n) is a Cauchy sequence, to the contrary we assume that $\{a_n\}$ is not a Cauchy sequence. Therefore, using Proposition 1, there exist a number $\xi \in \mathbb{R}_0^+ \backslash \Lambda$, a rank $\rho(\xi) \geq 0$, and couple of rank sequences $\{m(\rho) : \rho \geq 0\}$, $\{n(\rho) : \rho \geq 0\}$ such that

$$d(a_{m(\rho)}, a_{n(\rho)}) > \xi, \quad \forall \rho \geq 0 \text{ and } n(\rho) > m(\rho) \geq \rho, \tag{2.5}$$

$$d(a_{m(\rho)}, a_{n(\rho)-1}) \leq \xi, \quad \text{for } n(\rho) - m(\rho) \geq 2 \text{ and } \rho \geq \rho(\xi). \tag{2.6}$$

Then, using Proposition 1 and (2.4)–(2.6), we have

$$\lim_{\rho\to\infty} d(a_{m(\rho)}, a_{n(\rho)}) = \xi \tag{2.7}$$

and

$$\lim_{\rho\to\infty} d(a_{m(\rho)+k_1}, a_{n(\rho)+k_2}) = \xi, \quad \forall k_1, k_2 \in \{0, 1\}. \tag{2.8}$$

From (2.2),

$$\theta\big(d(a_{m(\rho)+1},a_{n(\rho)+1})\big) \le \big[\theta\big\{d(a_{m(\rho)},a_{n(\rho)})\big\}\big]^{\frac{\varsigma_1+\varsigma_2+\varsigma_4}{1-\varsigma_4}}, \quad \forall\rho \ge 0. \tag{2.9}$$

Taking the limit as $\rho \to \infty$ and using the chosen $\xi \in \mathbb{R}_0^+ \backslash \Lambda$, (2.9) gives, using (2.7) and (2.8),

$$\lim_{\rho\to\infty} \theta\big(d(a_{m(\rho)+1},a_{n(\rho)+1})\big) \le \lim_{\rho\to\infty}\big[\theta\big\{d(a_{m(\rho)},a_{n(\rho)})\big\}\big]^{\frac{\varsigma_1+\varsigma_2+\varsigma_4}{1-\varsigma_4}},$$

thus, as θ is continuous at ξ,

$$\theta(\xi) \le \big[\theta(\xi)\big]^{\frac{\varsigma_1+\varsigma_2+\varsigma_4}{1-\varsigma_4}} < \theta(\xi),$$

which is not true. Hence $\{a_n\}$ is a Cauchy sequence in S. As S is complete, there exists $a \in S$ such that $a_n \to a$ as $n \to \infty$, i.e., $\lim_{n\to\infty} d(a_n,a) = 0$. Now we prove that $a = \Gamma a$. For this, consider

$$\begin{aligned}
\theta\big(d(\Gamma a_{n-1},\Gamma a)\big) &\le \big[\theta\big\{d(a_{n-1},a)\big\}\big]^{\varsigma_1}\big[\theta\big\{d(a_{n-1},\Gamma a_{n-1})\big\}\big]^{\varsigma_2}\big[\theta\big\{d(a,\Gamma a)\big\}\big]^{\varsigma_3} \\
&\quad \times \left[\theta\left\{\left(\frac{d(a_{n-1},\Gamma a)+d(a,\Gamma a_{n-1})}{2}\right)\right\}\right]^{\varsigma_4}, \\
\theta\big(d(a_n,\Gamma a)\big) &\le \big[\theta\big\{d(a_{n-1},a)\big\}\big]^{\varsigma_1}\big[\theta\big\{d(a_{n-1},a_n)\big\}\big]^{\varsigma_2}\big[\theta\big\{d(a,\Gamma a)\big\}\big]^{\varsigma_3} \\
&\quad \times \left[\theta\left\{\left(\frac{d(a_{n-1},\Gamma a)+d(a,a_n)}{2}\right)\right\}\right]^{\varsigma_4}.
\end{aligned} \tag{2.10}$$

Set $\Upsilon = \{d(a_{n-1},a), d(a_{n-1},a_n), d(a,\Gamma a), d(a_{n-1},\Gamma a), d(a,a_n)\}$. Then for $\max\Upsilon \le d(a,\Gamma a)$, (2.10) gives

$$\begin{aligned}
&\theta\big(d(a_n,\Gamma a)\big) \le \big(\theta\big(d(a,\Gamma a)\big)\big)^{\varsigma_1+\varsigma_2+\varsigma_3+\varsigma_4} < \theta\big(d(a,\Gamma a)\big), \\
&d(a_n,\Gamma a) < d(a,\Gamma a), \quad \text{since } \theta \in \chi, \\
&d(a,\Gamma a) < d(a,\Gamma a), \quad \text{as } n \to \infty,
\end{aligned}$$

which is not true. Now if $\max\Upsilon \le d(a_{n-1},\Gamma a)$, then (2.10) gives

$$\begin{aligned}
&\theta\big(d(a_n,\Gamma a)\big) \le \big(\theta\big(d(a_{n-1},\Gamma a)\big)\big)^{\varsigma_1+\varsigma_2+\varsigma_3+\varsigma_4} < \theta\big(d(a_{n-1},\Gamma a)\big), \\
&d(a_n,\Gamma a) < d(a_{n-1},\Gamma a), \quad \text{since } \theta \in \chi, \\
&d(a,\Gamma a) < d(a,\Gamma a), \quad \text{as } n \to \infty,
\end{aligned}$$

which again is not true. Now if $\max\Upsilon \le d(a,a_n)$, then (2.10) gives

$$\begin{aligned}
&\theta\big(d(a_n,\Gamma a)\big) \le \big(\theta\big(d(a,a_n)\big)\big)^{\varsigma_1+\varsigma_2+\varsigma_3+\varsigma_4} < \theta\big(d(a,a_n)\big), \\
&d(a_n,\Gamma a) < d(a,a_n), \quad \text{since } \theta \in \chi, \\
&d(a,\Gamma a) < 0, \quad \text{as } n \to \infty,
\end{aligned}$$

which implies

$$d(a,\Gamma a) = 0,$$
$$\Gamma a = a.$$

Similarly, for $\max \Upsilon \le d(a_{n-1},a_n)$ and $\max \Upsilon \le d(a_{n-1},a)$, (2.10) gives $\Gamma a = a$. Hence $a \in S$ is a fixed point of Γ. For the uniqueness of the fixed point of Γ, let $a,b \in S$ be two fixed points of Γ such that $a \neq b$. Using (1.2),

$$\begin{aligned}
\theta\bigl(d(\Gamma a,\Gamma b)\bigr) &\le \bigl[\theta\bigl\{d(a,b)\bigr\}\bigr]^{\varsigma_1}\bigl[\theta\bigl\{d(a,\Gamma a)\bigr\}\bigr]^{\varsigma_2}\bigl[\theta\bigl\{d(b,\Gamma b)\bigr\}\bigr]^{\varsigma_3} \\
&\quad \times \left[\theta\left\{\left(\frac{d(a,\Gamma b)+d(b,\Gamma a)}{2}\right)\right\}\right]^{\varsigma_4}, \\
\theta\bigl(d(a,b)\bigr) &\le \bigl[\theta\bigl\{d(a,b)\bigr\}\bigr]^{\varsigma_1}\bigl[\theta\bigl\{d(a,a)\bigr\}\bigr]^{\varsigma_2}\bigl[\theta\bigl\{d(b,b)\bigr\}\bigr]^{\varsigma_3} \\
&\quad \times \left[\theta\left\{\left(\frac{d(a,b)+d(b,a)}{2}\right)\right\}\right]^{\varsigma_4}, \\
\theta\bigl(d(a,b)\bigr) &\le \bigl[\theta\bigl(d(a,b)\bigr)\bigr]^{\varsigma_1+\varsigma_4} < \theta\bigl(d(a,b)\bigr),
\end{aligned}$$

which is not true. This proves the theorem. □

Corollary 1 *Let $\Gamma : S \to S$ be a self-mapping and (S,d) be a compact metric space. Suppose there is $\theta \in \chi$ and Γ satisfies (1.2). Then Γ has a unique fixed point.*

Proof The proof of this corollary is a direct consequence of Theorem 4, because it is well known that if S is compact then S is complete. Hence (S,d) is complete. Therefore, Γ has a unique fixed point in S. □

Example 3 Let (S,d) be a metric space with metric

$$d(a,b) = |a-b| \quad \text{for all } a,b \in S,$$

where $S = \{a_n = 1+3+5+\cdots+(2n-1) = n^2 : \forall n \in \mathbb{N}\}$. Define a mapping $\Gamma : S \to S$ by

$$\Gamma a = \frac{1}{a}, \quad \text{for all } a \in S.$$

Let $a_0 \in S$ be an arbitrary point. Consider a sequence $\{a_n\}_{n\in\mathbb{N}}$ in S, defined by $a_{n+1} = \Gamma a_n$. Clearly, (S,d) is a complete metric space. To check whether Γ is a Banach contraction or not, we write, for $1 \le m < n$,

$$\begin{aligned}
d(\Gamma a_n,\Gamma a_m) &= |\Gamma a_n - \Gamma a_m|, \\
d(\Gamma a_n,\Gamma a_m) &= |a_{n-1} - a_{m-1}|, \\
d(\Gamma a_n,\Gamma a_m) &= \bigl|(n-1)^2 - (m-1)^2\bigr|, \\
d(\Gamma a_n,\Gamma a_m) &< \bigl|n^2 - (m-1)^2\bigr|, \quad \text{since } n-1 < n, \\
d(\Gamma a_n,\Gamma a_m) &< \bigl|(m-1)^2 - n^2\bigr|,
\end{aligned}$$

$$
\begin{aligned}
&d(\Gamma a_n, \Gamma a_m) < \left|m^2 - n^2\right|, \quad \text{since } m-1 < m,\\
&d(\Gamma a_n, \Gamma a_m) < \left|n^2 - m^2\right|,\\
&d(\Gamma a_n, \Gamma a_m) < |a_n - a_m|.
\end{aligned}
$$

Hence Γ does not satisfy the Banach contractivity condition. Now to check whether Γ is a modified JS-contraction or not, consider the strictly increasing function $\theta : (0,\infty) \to (1,\infty)$ defined by $\theta(\tau) = e^{\tau}$. For some $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0,1)$ with $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 < 1$, one has

$$
\begin{aligned}
\theta\big(d(\Gamma a_n, \Gamma a_m)\big) &= \big[\theta\big(d(a_n, a_m)\big)\big]^{\zeta_1}\big[\theta\big(d(\Gamma a_n, a_n)\big)\big]^{\zeta_2}\big[\theta\big(d(\Gamma a_m, a_m)\big)\big]^{\zeta_3}\\
&\quad \times \left[\theta\left(\frac{d(a_n, \Gamma a_m) + d(\Gamma a_n, a_m)}{2}\right)\right]^{\zeta_4},\\
e^{d(\Gamma a_n, \Gamma a_m)} &\le e^{\zeta_1 d(a_n,a_m)} e^{\zeta_2 d(\Gamma a_n,a_n)} e^{\zeta_3 d(\Gamma a_m,a_m)} e^{\frac{\zeta_4}{2}(d(a_n,\Gamma a_m)+d(\Gamma a_n,a_m))},\\
e^{|\Gamma a_n - \Gamma a_m|} &\le e^{\zeta_1|a_n-a_m|+\zeta_2|\Gamma a_n-a_n|+\zeta_3|a_m-\Gamma a_m|+\frac{\zeta_4}{2}(|a_n-\Gamma a_m|+|\Gamma a_n-a_m|)},\\
e^{|\frac{1}{n^2}-\frac{1}{m^2}|} &\le e^{\zeta_1|n^2-m^2|+\zeta_2|n^2-\frac{1}{n^2}|+\zeta_3|m^2-\frac{1}{m^2}|+\frac{\zeta_4}{2}(|n^2-\frac{1}{m^2}|+|m^2-\frac{1}{n^2}|)},\\
e^{|\frac{m^2-n^2}{m^2n^2}|} &< e^{\zeta_1|n^2|+\zeta_2|n^2|+\zeta_3|m^2|+\frac{\zeta_4}{2}(|n^2|+|m^2|)},\\
e^{|\frac{n^2-m^2}{m^2n^2}|} &< e^{\zeta_1|n^2|+\zeta_2|n^2|+\zeta_3|n^2|+\frac{\zeta_4}{2}(|n^2|+|n^2|)}, \quad \text{since } m < n,\\
\left|\frac{n^2-m^2}{m^2n^2}\right| &< (\zeta_1+\zeta_2+\zeta_3+\zeta_4)\left|n^2\right|,\\
\frac{n^2-m^2}{m^2n^4} &< \zeta_1+\zeta_2+\zeta_3+\zeta_4.
\end{aligned}
$$

Clearly, $\frac{n^2-m^2}{m^2n^4} > 0$, for $m < n$. Hence $0 < \frac{n^2-m^2}{m^2n^4} < \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 < 1$. Hence Γ satisfies all the conditions of Theorem 4. Therefore, Γ has a unique fixed point in S, that is, $1 \in S$ is such that $\Gamma(1) = 1$.

Remark 2 (1) Assuming $\varsigma_2 = \varsigma_3 = \varsigma_4 = 0$ in Theorem 4, we obtain Theorem 2.1 of [20].
(2) Taking $\theta(a) = e^a$ and $\varsigma_4 = 0$ in Theorem 4, Reich contraction (a_3) is obtained.
(3) Taking $\theta(a) = e^{\sqrt{a}}$ and $\varsigma_1 = \varsigma_4 = 0$ in Theorem 4, Theorem 2.6 of [17] is obtained.
(4) Taking $\theta(a) = e^{\sqrt[n]{a}}$ in Theorem 4 gives Corollary 2 of [23].

3 Modified weak JS-contractions

In this section we prove fixed point theorems for mappings involving contractions with a function $\theta \in \chi$ as defined above and a function $\delta \in \varrho$ as defined below.

Let κ be the class of mappings $\vartheta : [1,\infty) \to [0,\infty)$ satisfying the following conditions:

(κ_1) ϑ is continuous,

(κ_2) $\vartheta(1) = 0$,

(κ_3) for each $\{z_n\} \subseteq (1,\infty)$, $\lim_{n\to\infty} \vartheta(z_n) = 1 \Longleftrightarrow \lim_{n\to\infty} z_n = 0$.

The class κ was introduced in [23]. By using class κ, Parvane et al. in [23] proved the following theorem.

Theorem 5 *Let (S,d) be a complete metric space. Let Γ be a self-mapping on S such that*

(i) Γ *is a weak JS-contraction*,
(ii) Γ *is continuous*.
Then Γ *has a unique fixed point*.

We introduce a new large class ϱ of mappings $\delta : [1,\infty) \to [0,\infty)$ satisfying only (κ_1). First, with the help of an example, we show that this class is larger than κ.

Example 4 Consider the set of polynomials of order at most n, $\{P_n(a) | P_n(a) = c_0 + c_1 a + c_2 a^2 + \cdots + c_n a^n, c_i \in \mathbb{R}^+\} \subset \varrho$ because $P_n(a)$ satisfies (ϱ_1). For example, a function $P_2(a) = a^2 + 3a - 1 \in \varrho$, but it does not belong to class κ because δ only satisfies (κ_1) but not either (κ_2) or (κ_3). Hence $\kappa \subset \varrho$, but $\varrho \nsubseteq \kappa$.

Definition 3 Let (S,d) be a metric space and Γ be a self-mapping on S. Then Γ is called a modified weak JS-contraction if for all $a,b \in S$, Γ satisfies

$$d(\Gamma a, \Gamma b) > 0 \quad \Longrightarrow \quad \theta\bigl(d(\Gamma a, \Gamma b)\bigr) \leq \theta\bigl(d(a,b)\bigr) - \delta\bigl(\theta\bigl(d(a,b)\bigr)\bigr), \tag{3.1}$$

where $\theta \in \chi$ and $\delta \in \varrho$.

Theorem 6 *Let* (S,d) *be a complete metric space*. *If* Γ *satisfies* (3.1), *then* Γ *has a unique fixed point*.

Proof Let $a_0 \in S$ be an arbitrary point. Consider a sequence $\{a_n\}$ defined by $a_{n+1} = \Gamma a_n = \Gamma^n a_0$. Without loss of generality, suppose $a_{n+1} \neq a_n$ $\forall n \geq 0$. As Γ satisfies the modified weak JS-contractivity condition,

$$\begin{aligned}
&\theta\bigl(d(a_n,a_{n+1})\bigr) \leq \theta\bigl(d(a_{n-1},a_n)\bigr) - \delta\bigl(\theta\bigl(d(a_{n-1},a_n)\bigr)\bigr), \\
&\theta\bigl(d(a_n,a_{n+1})\bigr) \leq \theta\bigl(d(a_{n-1},a_n)\bigr) - \delta\bigl(\theta\bigl(d(a_{n-1},a_n)\bigr)\bigr) < \theta\bigl(d(a_{n-1},a_n)\bigr), \\
&\theta\bigl(d(a_n,a_{n+1})\bigr) < \theta\bigl(d(a_{n-1},a_n)\bigr).
\end{aligned} \tag{3.2}$$

As θ is an increasing mapping,

$$d(a_n,a_{n+1}) < d(a_{n-1},a_n).$$

Clearly, $\{d(a_n,a_{n+1})\}$ is a decreasing and bounded below sequence. Then $\exists\, \alpha \geq 0$ such that

$$\lim_{n\to\infty} d(a_n,a_{n+1}) = \alpha.$$

Letting $0 < a$ and using (3.2),

$$\begin{aligned}
&\theta\bigl(d(a_n,a_{n+1})\bigr) \leq \theta\bigl(d(a_{n-1},a_n)\bigr) - \delta\bigl(\theta\bigl(d(a_{n-1},a_n)\bigr)\bigr), \\
&\theta\bigl(d(a_n,a_{n+1})\bigr) < \theta\bigl(d(a_{n-1},a_n)\bigr), \\
&d(a_n,a_{n+1}) < d(a_{n-1},a_n), \quad \text{because } \theta \text{ is strictly increasing} \\
&\lim_{n\to\infty} d(a_n,a_{n+1}) < \lim_{n\to\infty} d(a_{n-1},a_n), \\
&\alpha < \alpha,
\end{aligned}$$

which is not true. So $\alpha = 0$ and hence

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0. \tag{3.3}$$

As θ fulfills (Ξ_5), by Lemma 1, θ has a countable set of discontinuity points, say Λ. Also assume that the sequence $\{a_n\}$ is not a Cauchy sequence, then let a number $\xi \in (0,\infty)\backslash\Lambda$, a rank $\rho(\xi) \geq 0$, and couple of rank sequences $\{m(\rho) : \rho \geq 0\}$, $\{n(\rho) : \rho \geq 0\}$ be such that Proposition 1 holds. Then

$$\lim_{\rho\to\infty} d(a_{m(\rho)}, a_{n(\rho)}) = \xi, \tag{3.4}$$

$$\lim_{\rho\to\infty} d(a_{m(\rho)+k_1}, a_{n(\rho)+k_2}) = \xi, \quad \forall k_1, k_2 \in \{0,1\}. \tag{3.5}$$

Using a modified weak JS-contraction and letting $\rho \to \infty$,

$$\lim_{\rho\to\infty} \theta\bigl(d(a_{m(\rho)+1}, a_{n(\rho)+1})\bigr) \leq \lim_{\rho\to\infty} \theta\bigl(d(a_{m(\rho)}, a_{n(\rho)})\bigr) - \lim_{\rho\to\infty} \delta\bigl(\theta\bigl(d(a_{m(\rho)}, a_{n(\rho)})\bigr)\bigr).$$

Using the fact that we chose $\xi \in (0,\infty)\backslash\Lambda$, the function θ is continuous, and using the continuity of δ gives

$$\theta\Bigl(\lim_{\rho\to\infty} d(a_{m(\rho)+1}, a_{n(\rho)+1})\Bigr) \leq \theta\Bigl(\lim_{\rho\to\infty} d(a_{m(\rho)}, a_{n(\rho)})\Bigr) - \delta\Bigl(\theta\Bigl(\lim_{\rho\to\infty} d(a_{m(\rho)}, a_{n(\rho)})\Bigr)\Bigr),$$

from (3.4) and (3.5), the above relation gives

$$\theta(\xi) \leq \theta(\xi) - \delta\bigl(\theta(\xi)\bigr), \tag{3.6}$$

$$\theta(\xi) \leq \theta(\xi) - \delta\bigl(\theta(\xi)\bigr) < \theta(\xi), \tag{3.7}$$

which is not true. So our supposition is wrong and $\{a_n\}$ is a Cauchy sequence. Using the fact that (S,d) is complete, $\exists\, a \in S$ such that $a_n \to a$. That is,

$$\lim_{n\to\infty} d(a_n, a) = 0. \tag{3.8}$$

For $a \in S$ to be a fixed point, we observe that

$$\begin{aligned}
&\theta\bigl(d(a_{n+1}, \Gamma a)\bigr) \leq \theta\bigl(d(a_n, a)\bigr) - \delta\bigl(\theta\bigl(d(a_n, a)\bigr)\bigr),\\
&\theta\bigl(d(a_{n+1}, \Gamma a)\bigr) < \theta\bigl(d(a_n, a)\bigr),\\
&d(a_{n+1}, \Gamma a) < d(a_n, a),\\
&\lim_{n\to\infty} d(a_{n+1}, \Gamma a) < \lim_{n\to\infty} d(a_n, a),\\
&d(a, \Gamma a) < 0,\\
&d(a, \Gamma a) = 0,\\
&\Gamma a = a.
\end{aligned}$$

Hence, $a \in S$ is a fixed point of Γ. For the uniqueness of the fixed point, suppose on the contrary that a fixed point of Γ is not unique. Let $a, b \in S$ be fixed points of Γ such that $a \neq b$. Then $\Gamma a = a$ and $\Gamma b = b$. Using a modified weak JS-contraction gives

$$\begin{aligned}
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq \theta\big(d(a,b)\big) - \delta\big(\theta\big(d(a,b)\big)\big),\\
&\theta\big(d(a,b)\big) \leq \theta\big(d(a,b)\big) - \delta\big(\theta\big(d(a,b)\big)\big) < \theta\big(d(a,b)\big),\\
&\theta\big(d(a,b)\big) < \theta\big(d(a,b)\big),
\end{aligned}$$

which is a contradiction. So our supposition is wrong and hence the fixed point of Γ is unique. □

Example 5 Let $S = (0,\infty)$ be a nonempty set and $d : S \times S \to [0,\infty)$ a metric defined by

$$d(a,b) = |a-b| \quad \text{for all } a, b \in S.$$

Consider two mappings $\delta \in \varrho$ and $\Gamma : S \to S$, respectively defined by

$$\begin{aligned}
&\delta a = \ln(a),\\
&\Gamma a = \cos\frac{a}{90}.
\end{aligned}$$

Take $\theta \in \chi$ such that $\theta(\tau) = e^{\tau}$. Now $\forall a, b \in S$,

$$\begin{aligned}
&\theta\big(d(\Gamma a,\Gamma b)\big) = e^{d(\Gamma a,\Gamma b)},\\
&\theta\big(d(\Gamma a,\Gamma b)\big) = e^{|\cos\frac{a}{90} - \cos\frac{b}{90}|}.
\end{aligned} \tag{3.9}$$

By the mean value theorem, there exits at least one $c \in (\frac{b}{90}, \frac{a}{90})$ such that $|\cos\frac{a}{90} - \cos\frac{b}{90}| \leq |\frac{a}{90} - \frac{b}{90}|$. From (3.9),

$$\begin{aligned}
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq e^{|\frac{a}{90} - \frac{b}{90}|},\\
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq e^{\frac{|a-b|}{90}}.
\end{aligned} \tag{3.10}$$

As $e^{\frac{\alpha}{90}} \leq e^{\alpha} - \alpha$, $\forall \alpha > 0$, from (3.10) we get

$$\begin{aligned}
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq e^{\frac{|a-b|}{90}} \leq e^{|a-b|} - |a-b|,\\
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq e^{d(a,b)} - d(a,b),\\
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq \theta\big(d(a,b)\big) - \ln\big(e^{d(a,b)}\big),\\
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq \theta\big(d(a,b)\big) - \delta\big(e^{d(a,b)}\big),\\
&\theta\big(d(\Gamma a,\Gamma b)\big) \leq \theta\big(d(a,b)\big) - \delta\big(\theta\big(d(a,b)\big)\big).
\end{aligned}$$

Therefore, Γ satisfies (3.1). Hence Γ has a unique fixed point, i.e., using an iterative sequence $a_{n+1} = \Gamma a_n = \cos\frac{a_n}{90}$ with an initial guess $a_0 \in (0,\infty)$ gives the fixed point $a_\circ \approx 0.999938 \in S$ for Γ.

In the next section, solution of a nonlinear BVP is discussed by using our main results.

4 Applications in nonlinear boundary value problems

In this section we consider the following fractional order boundary value problem:

$$\left({}_{0}^{ABC}D^{\alpha}u\right)(t)=\digamma\left(u(t),t\right),\quad 1<\alpha\leq 2, t\in[0,1],\tag{4.1}$$

with boundary conditions

$$u(0)=0,\qquad u'(1)=\eta\int_0^1 u(\chi)\,d\chi,\tag{4.2}$$

where $\digamma:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $\eta>0$.

Lemma 2 *Assume that $\kappa\in(C[0,1],\mathbb{R})$. Then the solution of the following linear AB-Caputo BVP*:

$$\left({}_{0}^{ABC}D^{\alpha}u\right)(t)=K(t),\quad 1<\alpha\leq 2, t\in[0,1],\tag{4.3}$$

with boundary conditions (4.2), *is given by*

$$u(t)=\delta(t)+\int_0^1 G(t,\chi)K(\chi)\,d\chi,$$

where

$$\delta(t)=-\frac{2(2-\alpha)K(1)}{B(\alpha-1)(2-\eta)}t,\qquad (2-\eta)\neq 0,\qquad \eta\in\mathbb{R}^{+}\backslash\{2\},$$

and

$$\begin{aligned}&\int_0^1 G(t,\chi)K(\chi)\,d\chi\\&=\begin{cases}\frac{1}{(2-\eta)B(\alpha-1)\alpha\Gamma(\alpha)}[\alpha(2-\eta)\{(2-\alpha)\Gamma(\alpha)+(\alpha-1)(t-\chi)^{\alpha-1}\}\\\quad+2t\{-\alpha(\alpha-1)^2(1-\chi)^{\alpha-2}\\\quad+\eta(1-\chi)(\alpha\Gamma(\alpha)(2-\alpha)+(\alpha-1)(1-\chi)^{\alpha-1})\}], & 0\leq\chi\leq t,\\\frac{1}{(2-\eta)B(\alpha-1)\alpha\Gamma(\alpha)}[2t\{-\alpha(\alpha-1)^2(1-\chi)^{\alpha-2}\\\quad+\eta(1-\chi)(\alpha\Gamma(\alpha)(2-\alpha)+(\alpha-1)(1-\chi)^{\alpha-1})\}], & t\leq\chi\leq 1.\end{cases}\end{aligned}$$

Proof Using [1, Prop. 3.1], we get

$$u(t)=c_1+c_2t+\frac{(2-\alpha)}{B(\alpha-1)}\int_0^t K(\chi)\,d\chi+\frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)}\int_0^t(t-\chi)^{\alpha-1}K(\chi)\,d\chi\tag{4.4}$$

and

$$u'(t)=c_2+\frac{(2-\alpha)}{B(\alpha-1)}K(t)+\frac{(\alpha-1)^2}{B(\alpha-1)\Gamma(\alpha)}\int_0^t(t-\chi)^{\alpha-2}K(\chi)\,d\chi.\tag{4.5}$$

Using $u(0)=0$ implies $c_1=0$. Putting the value of c_1 into (4.4), we get

$$u(t)=c_2t+\frac{(2-\alpha)}{B(\alpha-1)}\int_0^t K(\chi)\,d\chi+\frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)}\int_0^t(t-\chi)^{\alpha-1}K(\chi)\,d\chi.\tag{4.6}$$

Now using $u'(1) = \eta \int_0^1 u(\chi)\,d\chi$ in (4.5), we get

$$c_2 = -\frac{(2-\alpha)}{B(\alpha-1)}K(1) - \frac{(\alpha-1)^2}{B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha-2}K(\chi)\,d\chi + \eta \int_0^1 u(\chi)\,d\chi.$$

Putting the value of c_2 into (4.6), we have

$$\begin{aligned} u(t) = {}& -\frac{(2-\alpha)}{B(\alpha-1)}tK(1) - \frac{(\alpha-1)^2 t}{B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha-2}K(\chi)\,d\chi + \eta t \int_0^1 u(\chi)\,d\chi \\ & + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^t K(\chi)\,d\chi + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-\chi)^{\alpha-1}K(\chi)\,d\chi. \end{aligned} \tag{4.7}$$

Letting $A = \int_0^1 u(\chi)\,d\chi$, we deduce

$$\begin{aligned} A ={}& \int_0^1 u(t)\,dt \\ ={}& -\frac{(2-\alpha)}{B(\alpha-1)}K(1) \int_0^1 t\,dt - \frac{(\alpha-1)^2}{B(\alpha-1)\Gamma(\alpha)} \int_0^1 \int_0^1 t(1-\chi)^{\alpha-2}K(\chi)\,d\chi\,dt \\ & + \eta \int_0^1 \int_0^1 tu(\chi)\,d\chi\,dt + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^1 \int_0^t K(\chi)\,d\chi\,dt \\ & + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^1 \int_0^t (t-\chi)^{\alpha-1}K(\chi)\,d\chi\,dt \\ ={}& -\frac{(2-\alpha)}{B(\alpha-1)}\frac{1}{2}K(1) - \frac{(\alpha-1)^2}{B(\alpha-1)\Gamma(\alpha)}\frac{1}{2} \int_0^1 (1-\chi)^{\alpha-2}K(\chi)\,d\chi + \frac{\eta A}{2} \\ & + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^1 (1-\chi)K(\chi)\,d\chi + \frac{(\alpha-1)}{\alpha B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha}K(\chi)\,d\chi\,dt \\ ={}& -\frac{(2-\alpha)}{(2-\eta)B(\alpha-1)}K(1) - \frac{(\alpha-1)^2}{(2-\eta)B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha-2}K(\chi)\,d\chi \\ & + \frac{2(2-\alpha)}{(2-\eta)B(\alpha-1)} \int_0^1 (1-\chi)K(\chi)\,d\chi \\ & + \frac{2(\alpha-1)}{(2-\eta)B(\alpha-1)\alpha\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha}K(\chi)\,d\chi. \end{aligned}$$

Putting this value into (4.7), we get

$$\begin{aligned} u(t) = {}& -\frac{(2-\alpha)}{B(\alpha-1)}tK(1) - \frac{(\alpha-1)^2 t}{B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha-2}K(\chi)\,d\chi \\ & - \frac{\eta t(2-\alpha)}{(2-\eta)B(\alpha-1)}K(1) - \frac{\eta t(\alpha-1)^2}{(2-\eta)B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha-2}K(\chi)\,d\chi \\ & + \frac{\eta t 2(2-\alpha)}{(2-\eta)B(\alpha-1)} \int_0^1 (1-\chi)K(\chi)\,d\chi \\ & + \frac{\eta t 2(\alpha-1)}{(2-\eta)B(\alpha-1)\alpha\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha}K(\chi)\,d\chi\} \\ & + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^t K(\chi)\,d\chi + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-\chi)^{\alpha-1}K(\chi)\,d\chi. \end{aligned}$$

After simplification, we get

$$\begin{aligned}
u(t) &= -\frac{(2-\alpha)}{B(\alpha-1)} tK(1) - \frac{(\alpha-1)^2 t}{B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha-2} K(\chi)\,d\chi \\
&\quad - \frac{\eta t(2-\alpha)}{(2-\eta)B(\alpha-1)} K(1) - \frac{\eta t(\alpha-1)^2}{(2-\eta)B(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha-2} K(\chi)\,d\chi \\
&\quad + \frac{\eta t 2(2-\alpha)}{(2-\eta)B(\alpha-1)} \int_0^1 (1-\chi) K(\chi)\,d\chi \\
&\quad + \frac{\eta t 2(\alpha-1)}{(2-\eta)B(\alpha-1)\alpha\Gamma(\alpha)} \int_0^1 (1-\chi)^{\alpha} K(\chi)\,d\chi \} \\
&\quad + \frac{(2-\alpha)}{B(\alpha-1)} \int_0^t K(\chi)\,d\chi + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-\chi)^{\alpha-1} K(\chi)\,d\chi \\
&= -\frac{2(2-\alpha)}{B(\alpha-1)(2-\eta)} tK(1) \\
&\quad + \frac{1}{B(\alpha-1)\Gamma(\alpha)} \int_0^t \bigl\{(2-\alpha)\Gamma(\alpha) + (\alpha-1)(t-\chi)^{\alpha-1}\bigr\} K(\chi)\,d\chi \\
&\quad + \frac{2t}{\alpha(2-\eta)B(\alpha-1)\Gamma(\alpha)} \int_0^1 \bigl[-\alpha(\alpha-1)^2(1-\chi)^{\alpha-2} \\
&\quad + \eta(1-\chi)\bigl\{\alpha\Gamma(\alpha)(2-\alpha) + (\alpha-1)(1-\chi)^{\alpha-1}\bigr\}\bigr] K(\chi)\,d\chi \\
&= \delta(t) + \int_0^1 G(t,\chi) K(\chi)\,d\chi,
\end{aligned}$$

where

$$\delta(t) = -\frac{2(2-\alpha)}{B(\alpha-1)(2-\eta)} tK(1)$$

and

$$\begin{aligned}
&\int_0^1 G(t,\chi)K(\chi)\,d\chi \\
&= \begin{cases}
\frac{1}{(2-\eta)B(\alpha-1)\alpha\Gamma(\alpha)} [\alpha(2-\eta)\{(2-\alpha)\Gamma(\alpha) + (\alpha-1)(t-\chi)^{\alpha-1}\} \\
\quad + 2t\{-\alpha(\alpha-1)^2(1-\chi)^{\alpha-2} \\
\quad + \eta(1-\chi)(\alpha\Gamma(\alpha)(2-\alpha) + (\alpha-1)(1-\chi)^{\alpha-1})\}], & 0 \le \chi \le t, \\
\frac{1}{(2-\eta)B(\alpha-1)\alpha\Gamma(\alpha)} [2t\{-\alpha(\alpha-1)^2(1-\chi)^{\alpha-2} \\
\quad + \eta(1-\chi)(\alpha\Gamma(\alpha)(2-\alpha) + (\alpha-1)(1-\chi)^{\alpha-1})\}], & t \le \chi \le 1,
\end{cases}
\end{aligned}$$

proving the lemma. □

Finally, we have

$$u(t) = \delta(t) + \int_0^1 G(t,\chi) \digamma\bigl(u(\chi),\chi\bigr)\,d\chi.$$

Suppose $\digamma$ and G satisfy the following conditions:

$(C1)$ $\theta[|\digamma(u(t),t)-\digamma(v(t),t)|] \le \theta[|u(t)-v(t)|]$; we will take $\theta(t)=e^t$ in this paper as a special case.

$(C2)$ $\max|G(t,\chi)| \le \beta$ for some $\beta \in (0,1)$.

Theorem 7 *If $\digamma$ and G satisfy $(C1)$ and $(C2)$, then the unique solution of problem (4.1) with boundary conditions (4.2) exists.*

Proof Define $\Pi : W \to W$ by

$$\Pi\bigl(u(t)\bigr)=\delta(t)+\int_0^1 G(t,\chi)\digamma\bigl(u(\chi),\chi\bigr)\,d\chi.$$

Clearly, the fixed point of Π is the solution of given problem (4.1)–(4.2).

To find the fixed point of Π, we will show that it satisfies all the conditions of Theorem 3. For this purpose, consider

$$\begin{aligned} e^{|\Pi(u(t))-\Pi(v(t))|} &= e^{|\{\delta(t)+\int_0^1 G(t,\chi)\digamma(u(\chi),\chi)\,d\chi\}-\{\delta(t)+\int_0^1 G(t,\chi)\digamma(u(\chi),\chi)\,d\chi\}|} \\ &= e^{|\int_0^1 G(t,\chi)\digamma(u(\chi),\chi)\,d\chi-\int_0^1 G(t,\chi)\digamma(v(\chi),\chi)\,d\chi|} \\ &= e^{|\int_0^1 G(t,\chi)\{\digamma(u(\chi),\chi)\,d\chi-\digamma(v(\chi),\chi)\}\,d\chi|} \\ &\le e^{\beta\int_0^1 |\{\digamma(u(\chi),\chi)-\digamma(v(\chi),\chi)\}|\,d\chi}, \end{aligned}$$

which finally gives us

$$e^{\|\Pi(u)-\Pi(v)\|} \le \bigl[e^{\|u-v\|}\bigr]^{\beta},$$

or equivalently

$$\theta\bigl[d\bigl(\Pi(u),\Pi(v)\bigr)\bigr] \le \bigl(\theta\bigl[d(u,v)\bigr]\bigr)^{\beta}.$$

Hence all the conditions of Theorem 3 are satisfied, so we obtain the existence of a unique solution of the fractional boundary value problem (4.1)–(4.2). □

4.1 θ-Ulam stability

In this section we define θ-Ulam stability of the given FBVP to ensure that our solution is stable.

We have

$$\bigl|G(t,\chi)\bigr| \le \beta \quad \text{for some } \beta > 0. \tag{4.8}$$

Definition 4 The fractional BVP (4.1)–(4.2) is θ-Ulam stable if there exists a positive function λ such that for each $\epsilon > 0$ and for each solution $\vartheta(t) \in C([0,1],\mathbb{R})$ of

$$\theta\bigl[\bigl|\bigl({}_0^{ABC}D^{\sigma}\upsilon\bigr)(t)-\zeta\bigl(t,\vartheta(t)\bigr)\bigr|\bigr] \le \epsilon, \quad \text{for all } t \in [0,T], \tag{4.9}$$

there exists a solution $\upsilon(t) \in X$ of (4.1)–(4.2) such that

$$\theta\bigl[\bigl|\vartheta(t)-\upsilon(t)\bigr|\bigr] \le \lambda(\epsilon), \quad \text{for all } t \in [0,T]. \tag{4.10}$$

Remark 3 A function $\nu(t) \in X$ is a solution of (4.9) if and only if there exists a function $\tau \in X$ such that

(i) $|\tau(t)| \leq \epsilon$, for all $t \in [0, T]$,

(ii) $({}_0^{ABC}D^\sigma \upsilon)(t) = \digamma(t, \vartheta(t)) + \tau(t)$, for all $t \in [0, T]$.

In the next theorem, we find conditions under which the solution of the given fractional BVP (4.1)–(4.2) is θ-Ulam stable.

Theorem 8 *Assume that a function* $\zeta : [0, T] \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfying* $(C1)$–$(C2)$. *Then the solution of the fractional BVP* (4.1)–(4.2) *is* θ*-Ulam stable.*

Proof Suppose $\upsilon(t) \in X$ is any solution of (4.9). Then from Remark 3,

$$\left({}_0^{ABC}D^\sigma \upsilon\right)(t) = \zeta\left(t, \upsilon(t)\right) + \tau(t), \quad \text{for all } t \in [0, T].$$

Now using Theorem 7, we can write

$$\upsilon(t) = \varrho(t) + \int_0^T \psi(t, \varkappa)\zeta\left(t, \upsilon(\varkappa)\right) d\varkappa + \int_0^T \psi(t, \varkappa)\tau(\varkappa)\, d\varkappa,$$

which implies

$$\left|\upsilon(t) - \varrho(t) - \int_0^T \psi(t, \varkappa)\zeta\left(t, \upsilon(\varkappa)\right) d\varkappa\right| \leq \beta\epsilon, \tag{4.11}$$

where ω is defined in Remark 3.

Now assume $\vartheta(t) \in X$ is a unique solution of the fractional BVP (4.1)–(4.2). We consider

$$\begin{aligned} e^{|\upsilon(t)-\vartheta(t)|} &= e^{|\upsilon(t)-\varrho(t)-\int_0^T \psi(t,\varkappa)\zeta(t,\vartheta(\varkappa))\,d\varkappa|} \\ &\leq e^{|\upsilon(t)-\varrho(t)-\int_0^T \psi(t,\varkappa)\zeta(t,\upsilon(\varkappa))\,d\varkappa|} e^{|\int_0^T \psi(t,\varkappa)\zeta(t,\upsilon(\varkappa))\,d\varkappa-\int_0^T \psi(t,\varkappa)\zeta(t,\vartheta(\varkappa))|}, \end{aligned}$$

and from (4.11) we get

$$e^{|\upsilon(t)-\vartheta(t)|} \leq e^{\beta\epsilon} e^{\beta|\upsilon(t)-\vartheta(t)|}.$$

Now taking the supremum over $t \in [0, T]$, we get

$$e^{\|\upsilon-\vartheta\|} \leq e^{\beta\epsilon} e^{\beta\|\upsilon-\vartheta\|},$$

which implies

$$e^{\|\upsilon-\vartheta\|} \leq e^{\beta(1+\epsilon)-1} = \lambda(\epsilon),$$

where $\lambda(t) = e^{\beta(1+t)-1}$. Hence the fractional BVP (4.1)–(4.2) is θ-Ulam stable. $\square$

5 Conclusion

The first section of this article was of introductory nature, whereas in the second section, the existence and uniqueness of a fixed point of a mapping satisfying the JS-contractivity condition was proved. This has been proved using a strictly increasing function $\theta \in \chi$, in the setting of a complete metric space. An example which supports the main result of the second section has been established, in which an example of a function θ has been taken which satisfies only the condition of our class and not of other classes. In the third section, the existence and uniqueness of a fixed point of the mappings satisfying a modified weak JS-contractivity condition has been proved, in which a function $\delta \in \varrho$ is used along with a function $\theta \in \chi$. Another example has been given to validate all the conditions of Theorem 6 in this section. In the last section of this paper, the existence results of a nonlinear ABC-fractional order BVP with integral boundary conditions have been proved. At the end, θ-Ulam stability has been introduced and used to establish the stability of an ABC-factional order boundary value problem. The results of this article use weaker assumptions than the existing results in the literature.

Modified JS-contractions have been studied primarily in the context of metric spaces. However, there is room to extend these contractions to more general spaces, such as partial metric spaces, quasimetric spaces, and fuzzy metric spaces. This could lead to new results and applications in various areas such as nonlinear analysis.

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