Ibaraki and Saejung *Journal of Inequalities and Applications* (2023) 2023:92
https://doi.org/10.1186/s13660-023-03004-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On shrinking projection method for cutter type mappings with nonsummable errors

Takanori Ibaraki[1] and Satit Saejung[2*]

*Correspondence:
saejung@kku.ac.th
[2]Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, 40002, Thailand
Full list of author information is available at the end of the article

Abstract

We prove two key inequalities for metric and generalized projections in a certain Banach space. We then obtain some asymptotic behavior of a sequence generated by the shrinking projection method introduced by Takahashi et al. (J. Math. Anal. Appl. 341:276–286, 2008) where the computation allows some nonsummable errors. We follow the idea proposed by Kimura (Banach and Function Spaces IV (ISBFS 2012), pp. 303–311, 2014). The mappings studied in this paper are more general than the ones in (Ibaraki and Kimura in Linear Nonlinear Anal. 2:301–310, 2016; Ibaraki and Kajiba in Josai Math. Monogr. 11:105–120, 2018). In particular, the results in (Ibaraki and Kimura in Linear Nonlinear Anal. 2:301–310, 2016; Ibaraki and Kajiba in Josai Math. Monogr. 11:105–120, 2018) are both extended and supplemented. Finally, we discuss our results for finding a zero of maximal monotone operator and a minimizer of convex functions on a Banach space.

Mathematics Subject Classification: 47H09

Keywords: Fixed point; Shrinking projection method; Cutter type mapping; Metric projection; Generalized projection; Maximal monotone operator

1 Introduction

Iterative methods play an important role in approximation theory. Several problems can be transformed into a problem of finding a fixed point of certain mappings. Many iterative methods have been proposed and analyzed (for example, see [3, 4, 6, 18, 19, 25, 28]). In this paper, we are interested in one of the promising methods, namely, the shrinking projection method. It was proposed by Takahashi et al. [24] who proved that a sequence generated by this method converges strongly to a fixed point of a certain mapping in the Hilbert space setting. Kimura [15] modified this iterative scheme in the sense that the inexact value of the projection is allowed, while the asymptotic behavior of the iterative sequence performs well. Our paper concerns cutter type mappings in Banach spaces with certain geometric properties, and it can be regarded as an extension and a supplement to the recent results of Ibaraki and Kimura [12] and of Ibaraki and Kajiba [11].

Let $E := (E, \|\cdot\|)$ be a real Banach space with the dual space E^*. The strong and weak convergence in E are denoted by $\to$ and $\rightharpoonup$, respectively. The *normalized duality mapping*

$J : E \to 2^{E^*}$ is defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\| = \|x\|^2\} \quad \text{for all } x \in E.$$

Recall that E is

- *smooth* if $\lim_{t\to 0} \frac{1}{t}(\|x + ty\| - \|x\|)$ exists for all $x, y \in E$;
- *strictly convex* if $\frac{1}{2}\|x + y\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

If E is smooth, strictly convex, and reflexive, then J is single-valued and surjective. In this case, we can treat Jx as an element of E^*. For more details on the duality mappings, we refer to [5].

Let C be a nonempty subset of a smooth Banach space E. In this paper, we are interested in the following three different generalizations of cutter mappings in the Banach space setting [2, 16]. A mapping $T : C \to E$ with a nonempty *fixed point set* $\mathrm{Fix}(T) := \{z \in C : z = Tz\}$ is said to be

- a *cutter mapping of type (P)* if $\langle Tx - z, J(Tx - x)\rangle \le 0$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$;
- a *cutter mapping of type (Q)* if $\langle Tx - z, JTx - Jx\rangle \le 0$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$;
- a *cutter mapping of type (R)* if $\langle JTx - Jz, Tx - x\rangle \le 0$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$.

Suppose that $V : E \times E \to [0, \infty)$ is a function defined by

$$V(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in E.$$

It is known that

- $(\|x\| - \|y\|)^2 \le V(x, y) \le (\|x\| + \|y\|)^2$ for all $x, y \in E$;
- $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$.

Lemma 1 *Suppose that E is a smooth Banach space and C is a nonempty closed and convex subset of E. The following statements are true*:

(a) *If $T : C \to E$ is a cutter mapping of type* (P), *then* $\mathrm{Fix}(T)$ *is closed and convex*;

(b) *Suppose that E is strictly convex. If $T : C \to E$ is a cutter mapping of type* (Q), *then* $\mathrm{Fix}(T)$ *is closed and convex.*

Proof (a) Suppose that T is a cutter mapping of type (P) where C is closed and convex. We show that $\mathrm{Fix}(T)$ is closed. Assume that $\{z_n\}$ is a sequence in $\mathrm{Fix}(T)$ such that $z_n \to z \in C$; then $\langle Tz - z_n, J(Tz - z)\rangle \le 0$ for all $n \ge 1$. In particular, $\|Tz - z\|^2 = \langle Tz - z, J(Tz - z)\rangle \le 0$, that is, $z \in \mathrm{Fix}(T)$. Hence, $\mathrm{Fix}(T)$ is closed. Next, we prove that $\mathrm{Fix}(T)$ is convex. To see this, let $z, z' \in \mathrm{Fix}(T)$ and $\lambda \in [0, 1]$. We write $w := \lambda z + (1 - \lambda)z'$. Obviously, $w \in C$. Moreover, $\langle Tw - z, J(Tw - w)\rangle \le 0$ and $\langle Tw - z', J(Tw - w)\rangle \le 0$. This implies that $\|Tw - w\|^2 = \langle Tw - w, J(Tw - w)\rangle \le 0$, that is, $w \in \mathrm{Fix}(T)$.

(b) Suppose that E is strictly convex and $T : C \to E$ is a cutter mapping of type (Q), where C is closed and convex. We show that $\mathrm{Fix}(T)$ is closed. Assume that $\{z_n\}$ is a sequence in $\mathrm{Fix}(T)$ such that $z_n \to z \in C$; then $\langle Tz - z_n, JTz - Jz\rangle \le 0$ for all $n \ge 1$. In particular, $\langle Tz - z, JTz - Jz\rangle \le 0$. It follows from the strict convexity of E that $z \in \mathrm{Fix}(T)$. Hence, $\mathrm{Fix}(T)$ is closed. Next, we prove that $\mathrm{Fix}(T)$ is convex. To see this, let $z, z' \in \mathrm{Fix}(T)$ and $\lambda \in [0, 1]$. We write $w := \lambda z + (1 - \lambda)z'$. Obviously, $w \in C$. Moreover, $\langle Tw - z, JTw - Jw\rangle \le 0$ and $\langle Tw - z', JTw - Jw\rangle \le 0$. This implies that $\langle Tw - w, JTw - Jw\rangle \le 0$ and hence $w \in \mathrm{Fix}(T)$. □

In this paper, we also consider the following geometric properties. A Banach space E is

- *uniformly convex* if $\lim_n \|x_n - y_n\| = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying $\lim_n \|x_n\| = \lim_n \|y_n\| = \lim_n \frac{1}{2}\|x_n + y_n\| = 1$;
- *uniformly smooth* if $\lim_{t\to 0} \frac{1}{t}(\|x + ty\| - \|x\|)$ exists uniformly for all $x, y \in E$ with $\|x\| = \|y\| = 1$.

Note that every uniformly convex (uniformly smooth, respectively) space is reflexive and strictly convex (smooth, respectively). Moreover, uniform convexity and uniform smoothness are dual to each other, that is, E is uniformly convex (uniformly smooth, respectively) if and only if E^* is uniformly smooth (uniformly convex, respectively).

Lemma 2 ([5]) *If E is a uniformly smooth Banach space, then $J : E \to E^*$ is norm-to-norm uniformly continuous on bounded sets.*

Lemma 3 ([14]) *Suppose that E is a uniformly convex and smooth Banach space and suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E. If $\lim_n V(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_n \|x_n - y_n\| = 0$.*

In each subsection, we present some prototypes of cutter type mappings of types (P), (Q), and (R) together with their properties.

1.1 Metric projections

Suppose that E is a strictly convex and reflexive Banach space. Let C be a closed convex subset of E and $x \in E$. It is known that there exists a unique element $z \in C$ such

$$\|z - x\| = \min\big\{\|y - x\| : y \in C\big\}.$$

Such an element z is denoted by $P_C x$. Now, we call P_C the *metric projection* of E onto C. It is easy to see that $\mathrm{Fix}(P_C) = C$ and P_C is a cutter mapping of type (P). The following easy result is also needed in our study.

Lemma 4 *Suppose that C is a closed convex subset of a strictly convex and reflexive Banach space E and $u \in E$. If F is a nonempty closed convex subset of C such that $P_C u \in F$, then $P_F u = P_C u$.*

Proof It follows from the definition of P_C that $\|P_C u - u\| \le \|y - u\|$ for all $y \in C$. Since $F \subset C$, we have $\|P_C u - u\| \le \|y - u\|$ for all $y \in F$. It follows from $P_C u \in F$ that $P_C u = P_F u$. □

The following lemma (see [15, Lemma 2.1]) is easily deduced from the result of Tsukada [26]. Recall that E satisfies the *Kadec–Klee property* if $x_n \to x$ whenever $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$. It is known that every uniformly convex space satisfies the Kadec–Klee property.

Lemma 5 ([15]) *Suppose that E is a strictly convex reflexive Banach space and E satisfies the Kadec–Klee property. If $\{C_n\}$ is a sequence of nonempty closed convex subsets of E such that $C_{n+1} \subset C_n$ for all $n \ge 1$ and $C_0 := \bigcap_{n=1}^{\infty} C_n$ is nonempty, then $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$ for all $x \in E$.*

1.2 Generalized projections

Suppose that E is a smooth, strictly convex, and reflexive Banach space. Let C be a closed convex subset of E and $x \in E$. It is known that there exists a unique element $z \in C$ such

$$V(z,x) = \min\left\{V(y,x) : y \in C\right\} := V(C,x).$$

Such an element z is denoted by $\Pi_C x$. Now, we call Π_C the *generalized projection* of E onto C. It is easy to see that $\mathrm{Fix}(\Pi_C) = C$ and Π_C is a cutter mapping of type (Q). Moreover, we have the following result.

Lemma 6 ([1]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Suppose that $x \in E$ and $z \in C$. Then the following statements are equivalent*:

- $z = \Pi_C x$;
- $\langle y - z, Jz - Jx\rangle \geq 0$ *for all* $y \in C$;
- $V(y,z) + V(z,x) \leq V(y,x)$ *for all* $y \in C$.

Lemma 7 *Suppose that C is a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and $u \in E$. If F is a nonempty closed convex subset of C such that $\Pi_C u \in F$, then $\Pi_F u = \Pi_C u$.*

Proof It follows from the definition of Π_C that $V(\Pi_C u, u) \leq V(y,u)$ for all $y \in C$. Since $F \subset C$, we have $V(\Pi_C u, u) \leq V(y,u)$ for all $y \in F$. It follows from $\Pi_C u \in F$ that $\Pi_C u = \Pi_F u$. □

The following lemma is easily deduced from the result of Ibaraki et al. [13].

Lemma 8 ([13]) *Suppose that E is a strictly convex, smooth, and reflexive Banach space and E satisfies the Kadec–Klee property. If $\{C_n\}$ is a sequence of nonempty closed convex subsets of E such that $C_{n+1} \subset C_n$ for all $n \geq 1$ and $C_0 := \bigcap_{n=1}^{\infty} C_n$ is nonempty, then $\{\Pi_{C_n} x\}$ converges strongly to $\Pi_{C_0} x$ for all $x \in E$.*

1.3 Sunny generalized nonexpansive retractions

Suppose that E is a smooth Banach space and $C \subset E$. A mapping $R : C \to E$ is a *generalized nonexpansive* if $\mathrm{Fix}(R) \neq \emptyset$ and $V(Rx,p) \leq V(x,p)$ for all $x \in C$ and for all $p \in \mathrm{Fix}(R)$. A mapping $R : E \to C$ is

- a *retraction* if $R^2 = R$;
- *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and for all $t > 0$.

If $R : E \to C$ is a sunny generalized nonexpansive retraction from E onto C, then $\mathrm{Fix}(R) = C$ and R is a cutter mapping of type (R).

Kohsaka and Takahashi [17] proved the following result.

Lemma 9 ([17]) *Suppose that E is a smooth, strictly convex, and reflexive Banach space, and suppose that C^* is a nonempty closed convex subset of E^*. Then $R := J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction from E onto $J^{-1}C^*$, where Π_{C^*} is the generalized projection of E^* onto C^*.*

The purpose of this paper is to present some asymptotic behavior of an iterative sequence generated by the shrinking projection method for cutter type mappings.

2 Preliminaries

In this section, we collect some auxiliary results used in our main results. For a Banach space $E := (E, \|\cdot\|)$ and $r > 0$, we let $B_r := \{x \in E : \|x\| \leq r\}$.

Lemma 10 ([27]) *Suppose that E is a Banach space and $r > 0$. Then the following statements are true*:

(a) *If E is uniformly convex, then there is a continuous, strictly increasing, and convex function $\underline{g}_r : [0, 2r] \to [0, \infty)$ such that $\underline{g}_r(0) = 0$ and*

$$\|(1-\lambda)x + \lambda y\|^2 \leq (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - (1-\lambda)\lambda \underline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$ and for all $\lambda \in [0,1]$;

(b) *If E is uniformly smooth, then there is a continuous, strictly increasing, and convex function $\overline{g}_r : [0, 2r] \to [0, \infty)$ such that $\overline{g}_r(0) = 0$ and*

$$\|(1-\lambda)x + \lambda y\|^2 \geq (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - (1-\lambda)\lambda \overline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$ and for all $\lambda \in [0,1]$.

Lemma 11 ([15]) *Suppose that E is a Banach space and $r > 0$. Then the following statements are true*:

(a) *If E is uniformly convex and smooth, then the function $\underline{g}_r$ in Lemma 10(a) satisfies $\underline{g}_r(\|x-y\|) \leq V(x,y)$ for all $x, y \in B_r$*;

(b) *If E is uniformly smooth, then the function $\overline{g}_r$ in Lemma 10(b) satisfies $\overline{g}_r(\|x-y\|) \geq V(x,y)$ for all $x, y \in B_r$.*

3 Main results

The main results are presented according to the types of cutter mappings. The results of each subsection are given with respect to the functionals $\|\cdot\|^2$ and $V(\cdot,\cdot)$. The corresponding remark about the related results is presented. Our results are some extensions of the recent ones proved by Ibaraki and Kajiba [11] for mappings of types (P), (Q), and (R). Recall that for a nonempty subset C of a smooth, strictly convex, and reflexive Banach space E, a mapping $T : C \to E$ is of

- *type (P)* if $\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \geq 0$ for all $x, y \in C$;
- *type (Q)* if $\langle Tx - Ty, (Jx - JTx) - (Jy - JTy)\rangle \geq 0$ for all $x, y \in C$;
- *type (R)* if $\langle JTx - JTy, (x - Tx) - (y - Ty)\rangle \geq 0$ for all $x, y \in C$.

It is clear that every mapping of type (X) with a fixed point is a cutter mapping of type (X) where X = P, Q, R. Moreover, it is not hard to see that if $T : C \to E$ is a mapping of type (X) where X = P, Q, R, then $I - T$ is *strongly closed at zero*, that is, $p \in \mathrm{Fix}(T)$ whenever $\{x_n\}$ is a sequence in C such that $x_n \to p \in C$ and $Tx_n \to p$.

3.1 Two key inequalities for metric projections and generalized projections

We now prove two key inequalities of this paper. The first result is a Banach space version of the result in [23] concerning the generalized projection. The second one is for the metric projection.

Lemma 12 *Suppose that E is a smooth, strictly convex, and reflexive Banach space and C is a closed convex subset of E. Suppose that $u \in E$ and $\delta \geq 0$. If $x \in C$ satisfies*

$$V(x,u) \leq V(C,u) + \delta,$$

then $V(x, \Pi_C u) \leq \delta$.

Proof Note that $V(x,\Pi_C u) + V(C,u) = V(x,\Pi_C u) + V(\Pi_C u,u) \leq V(x,u) \leq V(C,u) + \delta$. Specifically, $V(x,\Pi_C u) \leq \delta$. □

Lemma 13 *Suppose that E is a uniformly convex Banach space, $u \in E$, and $\delta \geq 0$. Suppose that C is a closed convex subset of E such that $C - u \subset B_r$ for some $r > 0$. If $x \in C$ satisfies*

$$\|u - x\|^2 \leq d^2(u,C) + \delta,$$

then $\underline{g}_r(\|x - P_C u\|) \leq \delta$, where $\underline{g}_r$ is the function defined in Lemma 10(a).

Proof Let $p := P_C u$. Then $\|u-x\|^2 \leq \|u-p\|^2 + \delta$. Let $\lambda \in (0,1)$. It follows that $\lambda p + (1-\lambda)x \in C$. Note that $\{p-u, x-u\} \subset C - u \subset B_r$, and we make use of the function $\underline{g}_r$ in Lemma 10(a) to estimate the term $\|\lambda(p-u) + (1-\lambda)(x-u)\|^2$. Hence,

$$\begin{aligned}
\|p-u\|^2 &\leq \left\|\lambda p + (1-\lambda)x - u\right\|^2 \\
&= \left\|\lambda(p-u) + (1-\lambda)(x-u)\right\|^2 \\
&\leq \lambda\|p-u\|^2 + (1-\lambda)\|x-u\|^2 - \lambda(1-\lambda)\underline{g}_r\left(\|p-x\|\right) \\
&\leq \|p-u\|^2 + (1-\lambda)\delta - \lambda(1-\lambda)\underline{g}_r\left(\|p-x\|\right).
\end{aligned}$$

In particular, $\lambda\underline{g}_r(\|p-x\|) \leq \delta$. Letting $\lambda \uparrow 1$ gives the result. □

3.2 Cutter mappings of type (P)

Theorem 14 *Suppose that E is a smooth and uniformly convex Banach space and C is a closed convex subset of E. Suppose that $T : C \to E$ is a cutter mapping of type (P). Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_0 := \limsup_n \delta_n$. For given $u \in E$, a sequence $\{x_n\} \subset C$ is generated as follows: $x_1 \in C$, $C_1 := C$, and*

$$\begin{aligned}
&C_{n+1} := \left\{z \in C_n : \left\langle Tx_n - z, J(x_n - Tx_n)\right\rangle \geq 0\right\}; \\
&x_{n+1} \in C_{n+1} \text{ satisfies } \|x_{n+1} - u\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}
\end{aligned}$$

for all $n \geq 1$. Then the following statements are true:

(1) *If $C - u \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0)$, where $\underline{g}_r$ is defined by Lemma 10;*
(2) *If $\delta_0 = 0$ and $I - T$ is strongly closed at zero, then $x_n \to P_{\mathrm{Fix}(T)}u$.*

Proof First, we note that each C_n is nonempty because $\mathrm{Fix}(T) \subset C_n$. Moreover, it is clear that each C_n is closed and convex. For convenience, we write $p_n := P_{C_n}u$ and $p := P_{\bigcap_{n=1}^\infty C_n}u$.

Note that $p_n \to p$ (see Lemma 5). It follows from $p \in C_{n+1}$ that

$$\langle Tx_n - p, J(x_n - Tx_n)\rangle \geq 0.$$

In particular,

$$\langle x_n - p, J(x_n - Tx_n)\rangle \geq \|x_n - Tx_n\|^2.$$

(1) Suppose that $C - u \subset B_r$ for some $r > 0$. It follows from Lemma 13 that

$$\|x_n - Tx_n\| \leq \|x_n - p\| \leq \|x_n - p_n\| + \|p_n - p\| \leq \underline{g}_r^{-1}(\delta_n) + \|p_n - p\|.$$

Hence, $\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0)$.

(2) We assume that $\delta_0 = 0$ and $I - T$ is strongly closed at zero. Note that $\|x_n - u\|^2 \leq \|p_n - u\|^2 + \delta_n$. In particular, $\{x_n\}$ is bounded. We prove that $x_n \rightharpoonup p$. Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ for some $q \in E$. It follows that $q \in \bigcap_{n=1}^\infty C_n$ and $\|q - u\|^2 \leq \liminf_k \|x_{n_k} - u\|^2 \leq \limsup_k \|x_{n_k} - u\|^2 \leq \lim_k \|p_{n_k} - u\|^2 = \|p - u\|^2$. Hence, $q = p$. Moreover, we have $\lim_n \|x_n - u\| = \|p - u\|$. It follows from the Kadec–Klee property and $x_n - u \rightharpoonup p - u$ that $x_n - u \to p - u$ and hence $x_n \to p$. It follows from a part of (1) that $\lim_n \|x_n - Tx_n\| = 0$. Since $I - T$ is strongly closed at zero, we have $p \in \operatorname{Fix}(T) \subset \bigcap_{n=1}^\infty C_n$. Note that $\operatorname{Fix}(T)$ is closed and convex (see Lemma 1). It follows from Lemma 4 that $x_n \to p = P_{\operatorname{Fix}(T)}u$. □

Theorem 15 *Suppose that E is a smooth and uniformly convex Banach space and C is a closed convex subset of E. Suppose that $T : C \to E$ is a cutter mapping of type (P). Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_0 := \limsup_n \delta_n$. For given $u \in E$, a sequence $\{x_n\} \subset C$ is generated as follows: $x_1 \in C$, $C_1 := C$, and*

$$C_{n+1} := \{z \in C_n : \langle Tx_n - z, J(x_n - Tx_n)\rangle \geq 0\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } V(x_{n+1}, u) \leq V(C_{n+1}, u) + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:

(1) *If $C \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0)$, where $\underline{g}_r$ is defined by Lemma 10.*
(2) *If $\delta_0 = 0$ and $I - T$ is strongly closed at zero, then $x_n \to \Pi_{\operatorname{Fix}(T)}u$.*

Proof First, we note that each C_n is nonempty because $\operatorname{Fix}(T) \subset C_n$. Moreover, it is clear that each C_n is closed and convex. For convenience, we write $\pi_n := \Pi_{C_n}u$ and $\pi := \Pi_{\bigcap_{n=1}^\infty C_n}u$. Note that $\pi_n \to \pi$ (see Lemma 8). It follows from $\pi \in C_{n+1}$ that

$$\langle Tx_n - \pi, J(x_n - Tx_n)\rangle \geq 0.$$

In particular,

$$\|x_n - Tx_n\|^2 \leq \langle x_n - \pi, J(x_n - Tx_n)\rangle \leq \|x_n - \pi\| \|x_n - Tx_n\|.$$

Note that $V(x_n, \pi_n) \leq \delta_n$ (see Lemma 12).

(1) Suppose that $C \subset B_r$ for some $r > 0$. It follows from Lemma 11 that $\|x_n - \pi_n\| \leq \underline{g}_r^{-1}(V(x_n, \pi_n)) \leq \underline{g}_r^{-1}(\delta_n)$, which implies that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - \pi\| \\ &\leq \|x_n - \pi_n\| + \|\pi_n - \pi\| \\ &\leq \underline{g}_r^{-1}(\delta_n) + \|\pi_n - \pi\|. \end{aligned}$$

Hence, $\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0)$.

(2) We assume that $\delta_0 = 0$ and $I - T$ is strongly closed at zero. Since $\delta_0 = 0$, we have $\lim_n V(x_n, \pi_n) = 0$. It follows from Lemma 3 that $\lim_n \|x_n - \pi_n\| = 0$. In particular, $\lim_n \|x_n - Tx_n\| \leq \lim_n \|x_n - \pi\| = \lim_n \|x_n - \pi_n\| = 0$. Since $I - T$ is strongly closed at zero, we have $\pi \in \operatorname{Fix}(T) \subset \bigcap_{n=1}^{\infty} C_n$. Note that $\operatorname{Fix}(T)$ is closed and convex (see Lemma 1). It follows from Lemma 7 that $x_n \to \pi = \Pi_{\operatorname{Fix}(T)} u$. □

Remark 16 Our Theorem 14 and Theorem 15 generalize Theorem 3.1 of [12] and Theorem 3.1 of [11], respectively. In fact, the mapping of type (P) in [11, 12] is replaced by the cutter mapping of type (P).

3.3 Cutter mappings of type (Q)

Theorem 17 *Suppose that E is a uniformly smooth and uniformly convex Banach space and C is a closed convex subset of E. Suppose that $T : C \to E$ is a cutter mapping of type (Q). Suppose that $\{\delta_n\}$ is a sequence in $[0, \infty)$ with $\delta_0 := \limsup_n \delta_n$. For given $u \in E$, a sequence $\{x_n\} \subset C$ is generated as follows: $x_1 \in C$, $C_1 := C$ and*

$$\begin{aligned} &C_{n+1} := \{z \in C_n : \langle Tx_n - z, Jx_n - JTx_n\rangle \geq 0\}; \\ &x_{n+1} \in C_{n+1} \text{ satisfies } \|x_{n+1} - u\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1} \end{aligned}$$

for all $n \geq 1$. Then the following statements are true:

1. *If $(C - u) \cup C \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$, where $\underline{g}_r$ and $\overline{g}_r$ are defined by Lemma 10;*
2. *If $\delta_0 = 0$ and $I - T$ is strongly closed at zero, then $x_n \to P_{\operatorname{Fix}(T)} u$.*

Proof First, we note that each C_n is nonempty because $\operatorname{Fix}(T) \subset C_n$. Moreover, it is clear that each C_n is closed and convex. For convenience, we write $p_n := P_{C_n} u$ and $p := P_{\bigcap_{n=1}^{\infty} C_n} u$. Note that $p_n \to p$ (see Lemma 8). It follows from $p \in C_{n+1}$ that $\langle Tx_n - p, Jx_n - JTx_n\rangle \geq 0$ and hence

$$V(Tx_n, x_n) \leq V(p, Tx_n) + V(Tx_n, x_n) \leq V(p, x_n).$$

(1) Assume that $(C - u) \cup C \subset B_r$ for some $r > 0$. It follows from Lemma 13 that $\underline{g}_r(\|p_n - x_n\|) \leq \delta_n$. By Lemma 11, we have $V(p, x_n) \leq \overline{g}_r(\|p - x_n\|)$. This implies that

$$\begin{aligned} V(Tx_n, x_n) &\leq V(p, x_n) \\ &\leq \overline{g}_r(\|p - x_n\|) \end{aligned}$$

$$\begin{aligned}&\leq \overline{g}_r\big(\|p-p_n\|+\|p_n-x_n\|\big)\\&\leq \overline{g}_r\big(\|p-p_n\|+\underline{g}_r^{-1}(\delta_n)\big).\end{aligned}$$

This implies that

$$\limsup_n V(Tx_n,x_n)\leq \overline{g}_r\big(\underline{g}_r^{-1}(\delta_0)\big).$$

In particular, $\limsup_n \|x_n-Tx_n\|\leq \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$.

(2) We assume that $\delta_0=0$ and $I-T$ is strongly closed at zero. Note that $\|x_n-u\|^2\leq \|p_n-u\|^2+\delta_n$. In particular, $\{x_n\}$ is bounded. We prove that $x_n\rightharpoonup p$. Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k}\rightharpoonup q$ for some $q\in E$. It follows that $q\in\bigcap_{n=1}^\infty C_n$ and $\|q-u\|^2\leq \liminf_k\|x_{n_k}-u\|^2\leq \limsup_k\|x_{n_k}-u\|^2\leq \lim_k\|p_{n_k}-u\|^2=\|p-u\|^2$. Hence, $q=p$. Moreover, we have $\lim_n\|x_n-u\|=\|p-u\|$. It follows from the Kadec–Klee property and $x_n-u\rightharpoonup p-u$ that $x_n-u\to p-u$ and hence $x_n\to p$. It follows from the uniform convexity of E and $\lim_n V(p,x_n)=0$ that $\lim_n\|x_n-Tx_n\|=0$. Since $I-T$ is strongly closed at zero, we have $p\in \mathrm{Fix}(T)\subset\bigcap_{n=1}^\infty C_n$. Note that $\mathrm{Fix}(T)$ is closed and convex (see Lemma 1). It follows from Lemma 4 that $x_n\to p=P_{\mathrm{Fix}(T)}u$. □

Theorem 18 *Suppose that E is a uniformly smooth and uniformly convex Banach space and C is a closed convex subset of E. Suppose that $T:C\to E$ is a cutter mapping of type (Q). Suppose that $\{\delta_n\}$ is a sequence in $[0,\infty)$ with $\delta_0:=\limsup_n\delta_n$. For given $u\in E$, a sequence $\{x_n\}\subset C$ is generated as follows: $x_1\in C$, $C_1:=C$ and*

$$\begin{aligned}&C_{n+1}:=\big\{z\in C_n:\langle Tx_n-z,Jx_n-JTx_n\rangle\geq 0\big\};\\&x_{n+1}\in C_{n+1}\text{ satisfies } V(x_{n+1},u)\leq V(C_{n+1},u)+\delta_{n+1}\end{aligned}$$

for all $n\geq 1$. Then the following statements are true:

(1) *If $C\subset B_r$ for some $r>0$, then $\limsup_n\|x_n-Tx_n\|\leq \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$, where $\underline{g}_r$ and $\overline{g}_r$ are defined by Lemma 10;*

(2) *If $\delta_0=0$ and $I-T$ is strongly closed at zero, then $x_n\to \Pi_{\mathrm{Fix}(T)}u$.*

Proof First, we note that each C_n is nonempty because $\mathrm{Fix}(T)\subset C_n$. Moreover, it is clear that each C_n is closed and convex. For convenience, we write $\pi_n:=\Pi_{C_n}u$ and $\pi:=\Pi_{\bigcap_{n=1}^\infty C_n}u$. It follows from $\pi\in C_{n+1}$ that

$$\langle Tx_n-\pi,Jx_n-JTx_n\rangle\geq 0,$$

and hence

$$V(Tx_n,x_n)\leq V(\pi,Tx_n)+V(Tx_n,x_n)\leq V(\pi,x_n).$$

Note that $\pi_n\to\pi$ (see Lemma 8). By Lemma 12, we have $V(x_n,\pi_n)\leq\delta_n$.

(1) Suppose that $C \subset B_r$ for some $r > 0$. It follows from Lemma 11 that $V(\pi, x_n) \leq \overline{g}_r(\|\pi - x_n\|)$ and $\underline{g}_r(\|\pi - x_n\|) \leq V(x_n, \pi_n) \leq \delta_n$. In particular,

$$
\begin{aligned}
V(Tx_n, x_n) &\leq V(\pi, x_n) \\
&\leq \overline{g}_r\bigl(\|\pi - x_n\|\bigr) \\
&\leq \overline{g}_r\bigl(\|\pi - \pi_n\| + \|\pi_n - x_n\|\bigr) \\
&\leq \overline{g}_r\bigl(\|\pi - \pi_n\| + \underline{g}_r^{-1}(\delta_n)\bigr).
\end{aligned}
$$

This implies that

$$
\limsup_n V(Tx_n, x_n) \leq \overline{g}_r\bigl(\underline{g}_r^{-1}(\delta_0)\bigr).
$$

In particular, $\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$.

(2) We assume that $\delta_0 = 0$ and $I - T$ is strongly closed at zero. Since $\delta_0 = 0$, we have $\lim_n V(x_n, \pi_n) = 0$. It follows from Lemma 3 that $\lim_n \|x_n - \pi_n\| = 0$. In particular, $\lim_n \|x_n - \pi\| = 0$. Now, we have $\lim_n V(Tx_n, x_n) \leq \lim_n V(\pi, x_n) = 0$. It follows from the uniform convexity of E that $\lim_n \|x_n - Tx_n\| = 0$. Since $I - T$ is strongly closed at zero, we have $\pi \in \operatorname{Fix}(T) \subset \bigcap_{n=1}^{\infty} C_n$. Note that $\operatorname{Fix}(T)$ is closed and convex (see Lemma 1). It follows from Lemma 7 that $x_n \to \pi = \Pi_{\operatorname{Fix}(T)} u$. □

Remark 19 Our Theorem 17 and Theorem 18 generalize Theorem 4.1 of [12] and Theorem 4.1 of [11], respectively. In fact, the mapping of type (Q) in [11, 12] is replaced by the cutter mapping of type (Q).

3.4 Cutter mappings of type (R)

Suppose that $V^* : E^* \times E^* \to [0, \infty)$ is defined by

$$
V^*\bigl(x^*, y^*\bigr) := \bigl\|x^*\bigr\|^2 - 2\bigl\langle x^*, J^* y^*\bigr\rangle + \bigl\|y^*\bigr\|^2 \quad \text{for all } x^*, y^* \in E^*.
$$

Note that for $x, y \in E$ we have

$$
\begin{aligned}
V^*(Jy, Jx) &= \|Jy\|^2 - 2\bigl\langle Jy, J^* Jx\bigr\rangle + \|Jx\|^2 \\
&= \|y\|^2 - 2\langle x, Jy\rangle + \|x\|^2 \\
&= V(x, y).
\end{aligned}
$$

Inspired by the work of Honda et al. [7], we obtain the following results.

Lemma 20 *Suppose that C is a nonempty subset of a smooth, strictly convex, and reflexive Banach space E. Suppose that $T : C \to E$ is a mapping, and we define $T^* : JC \to E^*$ by*

$$
T^* x^* := JTJ^{-1} x^* \quad \text{for all } x^* \in JC.
$$

Then the following statements are true:

(1) $J \operatorname{Fix}(T) = \operatorname{Fix}(T^*)$;

(2) *T is a cutter mapping of type* (Q) *if and only if T^* is a cutter mapping of type* (R);
(3) *T is a cutter mapping of type* (R) *if and only if T^* is a cutter mapping of type* (Q).

Proof (1) Suppose that $p \in C$ and $p^* := Jp$. If $p = Tp$, then $T^*p^* = JTJ^{-1}(Jp) = Jp = p^*$. If $p^* = T^*p$, then $Tp = J^{-1}T^*Jp = J^{-1}T^*p^* = J^{-1}p^* = p$.

(2) Suppose that T is a cutter mapping of type (Q). To see that T^* is a cutter mapping of type (R), let $x^* \in JC$ and $p^* \in \operatorname{Fix}(T^*)$. Then $x^* = Jx$ and $p^* = Jp$ for some $(x,p) \in C \times \operatorname{Fix}(T)$. It follows that

$$\left\langle J^*p^* - J^*T^*x^*, T^*x^* - x^*\right\rangle = \langle p - Tx, JTx - Jx\rangle \geq 0.$$

On the other hand, we assume that T^* is a cutter mapping of type (R). To see that T is a cutter mapping of type (Q), let $(x,p) \in C \times \operatorname{Fix}(T)$. We write $x^* = Jx$ and $p^* = Jp$. It follows that $x^* \in JC$ and $p^* \in \operatorname{Fix}(T^*)$. Moreover, we have

$$\langle p - Tx, JTx - Jx\rangle = \left\langle J^*p^* - J^*T^*x^*, T^*x^* - x^*\right\rangle \geq 0.$$

(3) follows similarly. □

Based on Lemma 20 and Theorem 17, we obtain the following result.

Theorem 21 *Suppose that E is a uniformly smooth and uniformly convex Banach space. Suppose that C is a subset of E such that JC is closed and convex. Suppose that $T : C \to E$ is a cutter mapping of type* (R). *Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_0 := \limsup_n \delta_n$. For given $u \in E$, a sequence $\{x_n\} \subset C$ is generated as follows: $x_1 \in C$, $C_1 := C$, and*

$$C_{n+1} := \bigl\{z \in C_n : \langle JTx_n - Jz, x_n - Tx_n\rangle \geq 0\bigr\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } \|Ju - Jx_{n+1}\|^2 \leq d^2(Ju, JC_{n+1}) + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:

(1) *If $(JC - Ju) \cup JC \subset B_r^*$ for some $r > 0$, then $\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))$, where $\underline{g}_r$ is defined by Lemma 10 (as E is uniformly convex) and $\underline{g}_r^*$ and $\overline{g}_r^*$ are defined by Lemma 10 (as E^* is uniformly convex and uniformly smooth, respectively). Here, $B_r^* := \{x^* \in E^* : \|x^*\| \leq r\}$.*
(2) *If $\delta_0 = 0$ and $I - T$ is strongly closed at zero, then $x_n \to J^{-1}P^*_{J\operatorname{Fix}(T)}Ju$, where $P^*_{J\operatorname{Fix}(T)}$ is a metric projection of E^* onto $J\operatorname{Fix}(T)$.*

Proof Set $T^* := JTJ^{-1}$. Then $T^* : JC \to E^*$ is a cutter mapping of type (Q). Let $u^* = Ju$. Define two sequences $\{x_n^*\} \subset JC$ and $\{C_n^*\}$ by

$$x_n^* := Jx_n \quad \text{and} \quad C_n^* := JC_n \quad \text{for all } n \geq 1.$$

It follows that

$$C_{n+1}^* = \bigl\{z^* \in C_n^* : \left\langle T^*x_n^* - z^*, Jx_n^* - JT^*x_n^*\right\rangle \geq 0\bigr\}$$

and

$$x_{n+1}^* \in C_{n+1}^* \text{ satisfies } \|x_{n+1}^* - u^*\|^2 \le d^2(u^*, C_{n+1}^*) + \delta_{n+1}.$$

(1) Suppose that $(JC - Ju) \cup JC \subset B_r^*$ for some $r > 0$. Using Theorem 17 for cutter mappings of type (Q) gives

$$\limsup_n V(x_n, Tx_n) = \limsup_n V^*(T^* x_n, x_n^*) \le \overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)).$$

This implies that

$$\limsup_n \|x_n - Tx_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))).$$

(2) We assume that $\delta_0 = 0$ and $I - T$ is strongly closed at zero. It follows from the uniform convexity and uniform smoothness of E that $I^* - T^*$ is strongly closed at zero, where I^* is an identity mapping of E^*. Using Theorem 17 for cutter mappings of type (Q) gives $x_n^* \to P_{\mathrm{Fix}(T^*)}^* u^*$. Hence, $x_n = J^{-1}x_n^* \to J^{-1}P_{J\mathrm{Fix}(T)}^* Ju$. □

Based on Lemma 20 and Theorem 18, we obtain the following result.

Theorem 22 *Suppose that E is a uniformly smooth and uniformly convex Banach space. Suppose that C is a subset of E such that JC is closed and convex. Suppose that $T : C \to E$ is a cutter mapping of type (R). Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_0 := \limsup_n \delta_n$. For given $u \in E$, a sequence $\{x_n\} \subset C$ is generated as follows: $x_1 \in C$, $C_1 := C$, and*

$$C_{n+1} := \{z \in C_n : \langle JTx_n - Jz, x_n - Tx_n\rangle \ge 0\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } V(u, x_{n+1}) \le V(u, C_{n+1}) + \delta_{n+1}$$

for all $n \ge 1$. Then the following statements are true:

(1) *If $C \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - Tx_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))$, where $\underline{g}_r$ is defined by Lemma 10 (as E is uniformly convex) and $\underline{g}_r^*$ and $\overline{g}_r^*$ are defined by Lemma 10 (as E^* is uniformly convex and uniformly smooth, respectively);*
(2) *If $\delta_0 = 0$ and $I - T$ is strongly closed at zero, then $x_n \to R_{\mathrm{Fix}(T)}u$.*

Proof Set $T^* := JTJ^{-1}$. Then $T^* : JC \to E^*$ is a cutter mapping of type (Q). Let $u^* = Ju$. Define two sequences $\{x_n^*\} \subset JC$ and $\{C_n^*\}$ by

$$x_n^* := Jx_n \quad \text{and} \quad C_n^* := JC_n \quad \text{for all } n \ge 1.$$

It follows that

$$C_{n+1}^* = \{z^* \in C_n^* : \langle T^* x_n^* - z^*, Jx_n^* - JT^* x_n^*\rangle \ge 0\}$$

and

$$x_{n+1}^* \in C_{n+1}^* \text{ satisfies } V^*(x_{n+1}^*, u^*) \le V^*(C_{n+1}^*, u^*) + \delta_{n+1}.$$

(1) Suppose that $C \subset B_r$ for some $r > 0$. Using Theorem 18 for cutter mappings of type (Q) gives

$$\limsup_n V(x_n, Tx_n) = \limsup_n V^*\left(T^* x_n, x_n^*\right) \leq \overline{g}_r^*\left(\underline{g}_r^{*-1}(\delta_0)\right).$$

This implies that

$$\limsup_n \|x_n - Tx_n\| \leq \underline{g}_r^{-1}\left(\overline{g}_r^*\left(\underline{g}_r^{*-1}(\delta_0)\right)\right).$$

(2) We assume that $\delta_0 = 0$ and $I - T$ is strongly closed at zero. Note that $I^* - T^*$ is strongly closed at zero, where I^* is the identity mapping of E^*. Using Theorem 18 for cutter mappings of type (Q) gives $x_n^* \to \Pi^*_{\mathrm{Fix}(T^*)} u^*$, where $\Pi^*_{\mathrm{Fix}(T^*)}$ is the generalized projection from E^* onto $\mathrm{Fix}(T^*)$. It follows from Lemma 9 that

$$x_n = J^{-1} J x_n \to J^{-1} \Pi^*_{\mathrm{Fix}(T^*)} u^* = J^{-1} \Pi^*_{\mathrm{Fix}(T^*)} J u = R_{\mathrm{Fix}(T)} u. \qquad \square$$

Remark 23 Our Theorem 21 and Theorem 22 generalize Theorem 5.1 of [12] and Theorem 5.1 of [11], respectively. In fact, the mapping of type (R) in [11, 12] is replaced by the cutter mapping of type (R). It is worth mentioning that the bound of the limit superior $\limsup_n \|x_n - Tx_n\|$ in [11, 12] is $\underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))))$.

4 Deduced results for maximal monotone operators

We now discuss the problem of finding a zero of maximal monotone operators in a Banach space ([8–10]). Suppose that E is a smooth, strictly convex, and reflexive Banach space. An operator $A \subset E \times E^*$ with domain $\mathrm{dom}(A) := \{x \in E : Ax \neq \varnothing\}$ and range $\mathrm{ran}(A) := \bigcup\{Ax : x \in E\}$ is *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is *maximal* if $A' = A$ whenever $A' \subset E \times E^*$ is monotone and $A \subset A'$. We are interested in finding a *zero* of a maximal monotone operator, that is, an element $u \in E$ such that $(u, 0) \in A$. The set of zeros of A is denoted by $\mathrm{Zer}(A)$.

Suppose that $A \subset E \times E^*$ and $B \subset E^* \times E (= E^* \times (E^*)^*)$ are maximal monotone. It is known that $\overline{\mathrm{dom}(A)}$ and $\overline{\mathrm{dom}(B)}$ are convex; and

$$\mathrm{ran}\left(I + \lambda J^{-1} A\right) = J^{-1}\left(\mathrm{ran}(J + \lambda A)\right) = \mathrm{ran}(I + \lambda BJ) = E$$

for all $\lambda > 0$. Here, I is the identity operator. In particular, the following three single-valued operators are well defined:

$$P_\lambda := \left(I + \lambda J^{-1} A\right)^{-1} : \overline{\mathrm{dom}(A)} \to E;$$
$$Q_\lambda := (J + \lambda A)^{-1} J : \overline{\mathrm{dom}(A)} \to E;$$
$$R_\lambda := (I + \lambda BJ)^{-1} : J^{-1}\overline{\mathrm{dom}(B)} \to E.$$

It is known that

(a) $\mathrm{Fix}(P_\lambda) = \mathrm{Fix}(Q_\lambda) = \mathrm{Zer}(A)$ and $\mathrm{Fix}(R_\lambda) = \mathrm{Zer}(BJ)$.

(b) P_λ, Q_λ, and Q_λ are mappings of types (P), (Q), and (R), respectively.

(c) If $\mathrm{Zer}(A) \neq \emptyset$, then P_λ and Q_λ are cutter mappings of types (P) and (Q), respectively, and both $I - P_\lambda$ and $I - Q_\lambda$ are strongly closed at zero.
(d) If $\mathrm{Zer}(BJ) \neq \emptyset$, then R_λ is a cutter mapping of type (R), and $I - R_\lambda$ is strongly closed at zero.

We immediately obtain the following corollaries.

4.1 Results for the resolvent P_λ

Corollary 24 *Suppose that E is a smooth and uniformly convex Banach space and $A \subset E \times E^*$ is a maximal monotone operator such that $\mathrm{Zer}(A) \neq \emptyset$. Suppose that $u \in E$ and $\lambda \in (0, \infty)$. Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_0 := \limsup_n \delta_n$. Construct a sequence $\{x_n\} \subset \overline{\mathrm{dom}(A)}$ as follows: $x_1 \in \overline{\mathrm{dom}(A)}$, $C_1 := \overline{\mathrm{dom}(A)}$, and*

$$C_{n+1} := \{z \in C_n : \langle P_\lambda x_n - z, J(x_n - P_\lambda x_n)\rangle \geq 0\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } \|x_{n+1} - u\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:
(1) *If $\overline{\mathrm{dom}(A)} - u \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - P_\lambda x_n\| \leq \underline{g}_r^{-1}(\delta_0)$, where $\underline{g}_r$ is defined by Lemma 10;*
(2) *If $\delta_0 = 0$, then $x_n \to P_{\mathrm{Zer}(A)} u$.*

Corollary 25 *Suppose that E is a smooth and uniformly convex Banach space and $A \subset E \times E^*$ is a maximal monotone operator such that $\mathrm{Zer}(A) \neq \emptyset$. Suppose that $u \in E$ and $\lambda \in (0, \infty)$. Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_0 := \limsup_n \delta_n$. Construct a sequence $\{x_n\} \subset \overline{\mathrm{dom}(A)}$ as follows: $x_1 \in \overline{\mathrm{dom}(A)}$, $C_1 := \overline{\mathrm{dom}(A)}$, and*

$$C_{n+1} := \{z \in C_n : \langle P_\lambda x_n - z, J(x_n - P_\lambda x_n)\rangle \geq 0\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } V(x_{n+1}, u) \leq V(C_{n+1}, u) + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:
(1) *If $\overline{\mathrm{dom}(A)} \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - P_\lambda x_n\| \leq \underline{g}_r^{-1}(\delta_0)$, where $\underline{g}_r$ is defined by Lemma 10;*
(2) *If $\delta_0 = 0$, then $x_n \to \Pi_{\mathrm{Zer}(A)} u$.*

4.2 Results for the resolvent Q_λ

Corollary 26 *Suppose that E is a uniformly smooth and uniformly convex Banach space and that $A \subset E \times E^*$ is a maximal monotone operator such that $\mathrm{Zer}(A) \neq \emptyset$. Suppose that $u \in E$ and $\lambda \in (0, \infty)$. Suppose that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\delta_0 := \limsup_n \delta_n$. Construct a sequence $\{x_n\} \subset \overline{\mathrm{dom}(A)}$ as follows: $x_1 \in \overline{\mathrm{dom}(A)}$, $C_1 := \overline{\mathrm{dom}(A)}$, and*

$$C_{n+1} := \{z \in C_n : \langle Q_\lambda x_n - z, Jx_n - JQ_\lambda x_n\rangle \geq 0\}$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } \|x_{n+1} - u\| \leq d(u, C_{n+1}) + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:

(1) *If* $(\overline{\mathrm{dom}(A)} - u) \cup \overline{\mathrm{dom}(A)} \subset B_r$ *for some* $r > 0$, *then* $\limsup_n \|x_n - Q_\lambda x_n\| \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$, *where* $\underline{g}_r$ *and* $\overline{g}_r$ *are defined by Lemma* 10;
(2) *If* $\delta_0 = 0$, *then* $x_n \to P_{\mathrm{Zer}(A)} u$.

Corollary 27 *Suppose that* E *is a uniformly smooth and uniformly convex Banach space and that* $A \subset E \times E^*$ *is a maximal monotone operator such that* $\mathrm{Zer}(A) \ne \emptyset$. *Suppose that* $u \in E$ *and* $\lambda \in (0,\infty)$. *Suppose that* $\{\delta_n\}$ *is a sequence of nonnegative real numbers such that* $\delta_0 := \limsup_n \delta_n$. *Construct a sequence* $\{x_n\} \subset \overline{\mathrm{dom}(A)}$ *as follows*: $x_1 \in \overline{\mathrm{dom}(A)}$, $C_1 := \overline{\mathrm{dom}(A)}$, *and*

$$C_{n+1} := \bigl\{z \in C_n : \langle Q_\lambda x_n - z, Jx_n - JQ_\lambda x_n\rangle \ge 0\bigr\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } V(x_{n+1}, u) \le V(C_{n+1}, u) + \delta_{n+1}$$

for all $n \ge 1$. *Then the following statements are true*:
(1) *If* $\overline{\mathrm{dom}(A)} \subset B_r$ *for some* $r > 0$, *then* $\limsup_n \|x_n - Q_\lambda x_n\| \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$, *where* $\underline{g}_r$ *and* $\overline{g}_r$ *are defined by Lemma* 10;
(2) *If* $\delta_0 = 0$, *then* $x_n \to \Pi_{\mathrm{Zer}(A)} u$.

4.3 Results for the resolvent R_λ

Corollary 28 *Suppose that* E *is a uniformly smooth and uniformly convex Banach space and* $B \subset E^* \times E$ *such that* $\mathrm{Zer}(BJ) \ne \emptyset$. *Suppose that* $u \in E$ *and* $\lambda \in (0,\infty)$. *Suppose that* $\{\delta_n\}$ *is a sequence of nonnegative real numbers such that* $\delta_0 := \limsup_n \delta_n$. *Construct a sequence* $\{x_n\} \subset \overline{\mathrm{dom}(BJ)}$ *as follows*: $x_1 \in \overline{\mathrm{dom}(BJ)}$, $C_1 := \overline{\mathrm{dom}(BJ)}$, *and*

$$C_{n+1} := \bigl\{z \in C_n : \langle JR_\lambda x_n - Jz, x_n - R_\lambda x_n\rangle \ge 0\bigr\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } \|Ju - Jx_{n+1}\|^2 \le d^2(Ju, JC_{n+1}) + \delta_{n+1}$$

for all $n \ge 1$. *Then the following statements are true*:
(1) *If* $(J\overline{\mathrm{dom}(BJ)} - Ju) \cup J\overline{\mathrm{dom}(BJ)} \subset B_r^*$ *for some* $r > 0$, *then* $\limsup_n \|x_n - R_\lambda x_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))$, *where* $\underline{g}_r$ *is defined by Lemma* 10 (*as* E *is uniformly convex*) *and* $\underline{g}_r^*$ *and* $\overline{g}_r^*$ *are defined by Lemma* 10 (*as* E^* *is uniformly convex and uniformly smooth, respectively*);
(2) *If* $\delta_0 = 0$, *then* $x_n \to J^{-1}P^*_{J\mathrm{Zer}(BJ)} Ju$, *where* $P^*_{J\mathrm{Zer}(BJ)}$ *is a metric projection of* E^* *onto* $J\,\mathrm{Zer}(BJ)$.

Corollary 29 *Suppose that* E *is a uniformly smooth and uniformly convex Banach space and* $B \subset E^* \times E$ *such that* $\mathrm{Zer}(BJ) \ne \emptyset$. *Suppose that* $u \in E$ *and* $\lambda \in (0,\infty)$. *Suppose that* $\{\delta_n\}$ *is a sequence of nonnegative real numbers such that* $\delta_0 := \limsup_n \delta_n$. *Construct a sequence* $\{x_n\} \subset \overline{\mathrm{dom}(BJ)}$ *as follows*: $x_1 \in \overline{\mathrm{dom}(BJ)}$, $C_1 := \overline{\mathrm{dom}(BJ)}$, *and*

$$C_{n+1} := \bigl\{z \in C_n : \langle JR_\lambda x_n - Jz, x_n - R_\lambda x_n\rangle \ge 0\bigr\};$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } V(u, x_{n+1}) \le V(u, C_{n+1}) + \delta_{n+1}$$

for all $n \ge 1$. *Then the following statements are true*:

(1) *If* $\overline{\mathrm{dom}(BJ)} \subset B_r$ *for some* $r > 0$, *then* $\limsup_n \|x_n - R_\lambda x_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))$, *where* $\underline{g}_r$ *is defined by Lemma* 10 (*as E is uniformly convex*) *and* $\underline{g}_r^*$ *and* $\overline{g}_r^*$ *are defined by Lemma* 10 (*as* E^* *is uniformly convex and uniformly smooth, respectively*);
(2) *If* $\delta_0 = 0$, *then* $x_n \to R_{\mathrm{Zer}(BJ)}u$.

4.4 Applications to convex minimization problems

We discuss the convex minimization problem in a Banach space. This problem is to find a minimizer of a proper lower semicontinuous convex function in a Banach space. Suppose that E is a reflexive, smooth, and strictly convex Banach space with its dual E^* and $f : E \to (-\infty, \infty]$ and $f^* : E^* \to (-\infty, \infty]$ are proper lower semicontinuous convex functions. Then the subdifferentials of f and f^* are defined as follows:

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \le f(y),\ \forall y \in E\} \quad (\forall x \in E),$$
$$\partial f^*(x^*) = \{x \in E : f^*(x^*) + \langle x, y^* - x^* \rangle \le f^*(y^*),\ \forall y^* \in E^*\} \quad (\forall x^* \in E^*).$$

By Rockafellar's theorem [20, 21], the subdifferentials $\partial f \subset E \times E^*$ and $\partial f^* \subset E^* \times E$ are maximal monotone. It is easy to see that $\mathrm{Zer}(\partial f) = \mathrm{argmin}\{f(x) : x \in E\}$ and $\mathrm{Zer}(\partial f^*) = \mathrm{argmin}\{f^*(x^*) : x^* \in E^*\}$.

Fix $\lambda > 0$ and $z \in E$. Let P_λ and Q_λ be the resolvent of ∂f, and let R_λ be the resolvent of ∂f^*, then we know that

$$P_\lambda z = (I + \lambda J^{-1}\partial f)^{-1} = \operatorname*{argmin}_{y \in E}\left\{f(y) + \frac{1}{2\lambda}\|y - z\|^2\right\},$$
$$Q_\lambda z = (J + \lambda \partial f)^{-1} J = \operatorname*{argmin}_{y \in E}\left\{f(y) + \frac{1}{2\lambda}\|y\|^2 - \frac{1}{\lambda}\langle y, Jz\rangle\right\},$$
$$R_\lambda z = (I + \lambda \partial f^*)^{-1} = J^{-1}\operatorname*{argmin}_{y^* \in E^*}\left\{f^*(y^*) + \frac{1}{2\lambda}\|y^*\|^2 - \frac{1}{\lambda}\langle z, y^*\rangle\right\}.$$

See, for instance, [8, 10, 22]. As a direct consequence of our theorems, we can show the following applications.

Corollary 30 *Suppose that E is a smooth and uniformly convex Banach space and* $f : E \to (-\infty, \infty]$ *is a proper lower semicontinuous convex function such that* $\mathrm{Zer}(\partial f) \neq \emptyset$. *Suppose that* $u \in E$ *and* $\lambda \in (0, \infty)$. *Suppose that* $\{\delta_n\}$ *is a nonnegative real sequence such that* $\delta_0 := \limsup_n \delta_n$. *Construct a sequence* $\{x_n\} \subset \overline{\mathrm{dom}(\partial f)}$ *as follows*: $x_1 \in \overline{\mathrm{dom}(\partial f)}$, $C_1 = \overline{\mathrm{dom}(\partial f)}$, *and*

$$y_n := \operatorname*{argmin}_{y \in E}\left\{f(y) + \frac{1}{2\lambda}\|y - x_n\|^2\right\},$$
$$C_{n+1} := \{z \in C_n : \langle y_n - z, J(x_n - y_n)\rangle \ge 0\},$$
$$x_{n+1} \in C_{n+1} \text{ satisfies } \|x_{n+1} - u\|^2 \le d(u, C_{n+1})^2 + \delta_{n+1}$$

for all $n \ge 1$. *Then the following statements are true*:
(1) *If* $\overline{\mathrm{dom}(\partial f)} - u \subset B_r$ *for some* $r > 0$, *then* $\limsup_n \|x_n - y_n\| \le \underline{g}_r^{-1}(\delta_0)$, *where* $\underline{g}_r$ *is defined by Lemma* 10;
(2) *If* $\delta_0 = 0$, *then* $x_n \to P_{\mathrm{Zer}(\partial f)}u$.

Corollary 31 *Suppose that E is a smooth and uniformly convex Banach space and $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function such that $\mathrm{Zer}(\partial f) \neq \emptyset$. Suppose that $u \in E$ and $\lambda \in (0, \infty)$. Suppose that $\{\delta_n\}$ is a nonnegative real sequence such that $\delta_0 := \limsup_n \delta_n$. Construct a sequence $\{x_n\} \subset \overline{\mathrm{dom}(\partial f)}$ as follows: $x_1 \in \overline{\mathrm{dom}(\partial f)}$, $C_1 = \overline{\mathrm{dom}(\partial f)}$, and*

$$y_n := \operatorname*{argmin}_{y \in E} \left\{ f(y) + \frac{1}{2\lambda} \|y - x_n\|^2 \right\},$$

$$C_{n+1} := \left\{ z \in C_n : \left\langle y_n - z, J(x_n - y_n) \right\rangle \geq 0 \right\},$$

$$x_{n+1} \in C_{n+1} \text{ satisfies } V(x_{n+1}, u) \leq V(C_{n+1}, u) + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:

(1) *If $\overline{\mathrm{dom}(\partial f)} \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - y_n\| \leq \underline{g}_r^{-1}(\delta_0)$, where $\underline{g}_r$ is defined by Lemma 10;*

(2) *If $\delta_0 = 0$, then $x_n \to \Pi_{\mathrm{Zer}(\partial f)} u$.*

Corollary 32 *Suppose that E is a uniformly smooth and uniformly convex Banach space and $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function such that $\mathrm{Zer}(\partial f) \neq \emptyset$. Suppose that $u \in E$ and $\lambda \in (0, \infty)$. Suppose that $\{\delta_n\}$ is a nonnegative real sequence such that $\delta_0 := \limsup_n \delta_n$. Construct a sequence $\{x_n\} \subset \overline{\mathrm{dom}(\partial f)}$ as follows: $x_1 \in \overline{\mathrm{dom}(\partial f)}$, $C_1 = \overline{\mathrm{dom}(\partial f)}$, and*

$$y_n := \operatorname*{argmin}_{y \in E} \left\{ f(y) + \frac{1}{2\lambda} \|y\|^2 - \frac{1}{\lambda} \langle y, Jx_n \rangle \right\},$$

$$C_{n+1} := \left\{ z \in C_n : \langle y_n - z, Jx_n - Jy_n \rangle \geq 0 \right\},$$

$$x_{n+1} \in C_{n+1} \text{ satisfies } \|x_{n+1} - u\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:

(1) *If $(\overline{\mathrm{dom}(\partial f)} - u) \cup \overline{\mathrm{dom}(\partial f)} \subset B_r$ for some $r > 0$, then $\limsup_n \|x_n - y_n\| \leq \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$, where $\underline{g}_r$ and $\overline{g}_r$ are defined by Lemma 10;*

(2) *If $\delta_0 = 0$, then $x_n \to P_{\mathrm{Zer}(\partial f)} u$.*

Corollary 33 *Suppose that E is a uniformly smooth and uniformly convex Banach space and $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function such that $\mathrm{Zer}(\partial f) \neq \emptyset$. Suppose that $u \in E$ and $\lambda \in (0, \infty)$. Suppose that $\{\delta_n\}$ is a nonnegative real sequence such that $\delta_0 := \limsup_n \delta_n$. Construct a sequence $\{x_n\} \subset \overline{\mathrm{dom}(\partial f)}$ as follows: $x_1 \in \overline{\mathrm{dom}(\partial f)}$, $C_1 = \overline{\mathrm{dom}(\partial f)}$, and*

$$y_n := \operatorname*{argmin}_{y \in E} \left\{ f(y) + \frac{1}{2\lambda} \|y\|^2 - \frac{1}{\lambda} \langle y, Jx_n \rangle \right\},$$

$$C_{n+1} := \left\{ z \in C_n : \langle y_n - z, Jx_n - Jy_n \rangle \geq 0 \right\},$$

$$x_{n+1} \in C_{n+1} \text{ satisfies } V(x_{n+1}, u) \leq V(C_{n+1}, u) + \delta_{n+1}$$

for all $n \geq 1$. Then the following statements are true:

(1) *If* $\overline{\mathrm{dom}(\partial f)} \subset B_r$ *for some* $r > 0$, *then* $\limsup_n \|x_n - y_n\| \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$, *where* $\underline{g}_r$ *and* $\overline{g}_r$ *are defined by Lemma* 10;

(2) *If* $\delta_0 = 0$, *then* $x_n \to \Pi_{\mathrm{Zer}(\partial f)} u$.

Corollary 34 *Suppose that* E *is a uniformly smooth and uniformly convex Banach space and* $f^* : E^* \to (-\infty, \infty]$ *is a proper lower semicontinuous convex function such that* $\mathrm{Zer}(\partial f^*) \neq \emptyset$. *Suppose that* $u \in E$ *and* $\lambda \in (0, \infty)$. *Suppose that* $\{\delta_n\}$ *is a nonnegative real sequence such that* $\delta_0 := \limsup_n \delta_n$. *Construct a sequence* $\{x_n\} \subset \overline{\mathrm{dom}(\partial f^* J)}$ *as follows*: $x_1 \in \overline{\mathrm{dom}(\partial f^* J)}$, $C_1 = \overline{\mathrm{dom}(\partial f^* J)}$, *and*

$$y_n := J^{-1} \operatorname*{argmin}_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2\lambda} \|y^*\|^2 - \frac{1}{\lambda} \langle x_n, y^* \rangle \right\},$$

$$C_{n+1} := \{ z \in C_n : \langle Jy_n - Jz, x_n - y_n \rangle \ge 0 \},$$

$$x_{n+1} \in C_{n+1} \text{ satisfies } \|Ju - Jx_{n+1}\|^2 \le d(u, C_{n+1})^2 + \delta_{n+1}$$

for all $n \ge 1$. *Then the following statements are true*:

(1) *If* $(J\overline{\mathrm{dom}(\partial f^* J)} - Ju) \cup J\overline{\mathrm{dom}(\partial f^* J)} \subset B_r^*$ *for some* $r > 0$, *then* $\limsup_n \|x_n - y_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))$, *where* $\underline{g}_r$ *is defined by Lemma* 10 (*as* E *is uniformly convex*) *and*; $\underline{g}_r^*$ *and* $\overline{g}_r^*$ *are defined by Lemma* 10 (*as* E^* *is uniformly convex and uniformly smooth, respectively*);

(2) *If* $\delta_0 = 0$, *then* $x_n \to J^{-1} P^*_{J\mathrm{Zer}(\partial f^* J)} Ju$, *where* $P^*_{J\mathrm{Zer}(\partial f^* J)}$ *is a metric projection of* E^* *onto* $J\,\mathrm{Zer}(\partial f^* J)$.

Corollary 35 *Suppose that* E *is a uniformly smooth and uniformly convex Banach space and* $f^* : E^* \to (-\infty, \infty]$ *is a proper lower semicontinuous convex function such that* $\mathrm{Zer}(\partial f^*) \neq \emptyset$. *Suppose that* $u \in E$ *and* $\lambda \in (0, \infty)$. *Suppose that* $\{\delta_n\}$ *is a nonnegative real sequence such that* $\delta_0 := \limsup_n \delta_n$. *Construct a sequence* $\{x_n\} \subset \overline{\mathrm{dom}(\partial f^* J)}$ *as follows*: $x_1 \in \overline{\mathrm{dom}(\partial f^* J)}$, $C_1 = \overline{\mathrm{dom}(\partial f^* J)}$, *and*

$$y_n := J^{-1} \operatorname*{argmin}_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2\lambda} \|y^*\|^2 - \frac{1}{\lambda} \langle x_n, y^* \rangle \right\},$$

$$C_{n+1} := \{ z \in C_n : \langle Jy_n - Jz, x_n - y_n \rangle \ge 0 \},$$

$$x_{n+1} \in C_{n+1} \text{ satisfies } V(u, x_{n+1}) \le V(u, C_{n+1}) + \delta_{n+1}$$

for all $n \ge 1$. *Then the following statements are true*:

(1) *If* $\overline{\mathrm{dom}(\partial f^* J)} \subset B_r$ *for some* $r > 0$, *then* $\limsup_n \|x_n - y_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))$, *where* $\underline{g}_r$ *is defined by Lemma* 10 (*as* E *is uniformly convex*) *and*; $\underline{g}_r^*$ *and* $\overline{g}_r^*$ *are defined by Lemma* 10 (*as* E^* *is uniformly convex and uniformly smooth, respectively*);

(2) *If* $\delta_0 = 0$, *then* $x_n \to R_{\mathrm{Zer}(\partial f^* J)} u$.

5 Conclusions

In this paper, we prove two key inequalities for metric projections and generalized projections in a certain Banach space. Using them, we obtain some asymptotic behavior of a sequence generated by the shrinking projection method introduced by Takahashi et al. [24] for cutter mappings of types (P), (Q), and (R). The mappings studied in this paper are more general than the ones in [8, 10]. In particular, the results in [8, 10] are both extended and supplemented. Finally, we discuss our results for finding a zero of maximal monotone operator and a minimizer of convex function defined on a Banach space. It would be interesting to extend our work to the class of quasinonexpansive mappings of Bregman type.

Acknowledgements
The authors would like to thank the reviewers for their comments and letting them know of some interesting articles.

Funding
TI was supported by JSPS KAKENHI Grant Numbers JP19K03632, JP19H01479. SS was supported by the Fundamental Fund of Khon Kaen University and the National Science, Research and Innovation Fund (NSRF), Thailand.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
TI and SS have equally made contributions. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics Education, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama, 240-8501, Japan. [2]Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, 40002, Thailand.

Received: 29 April 2022 Accepted: 27 June 2023 Published online: 07 July 2023

References

1. Alber, Y.I.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lecture Notes in Pure and Appl. Math., vol. 178, pp. 15–50. Dekker, New York (1996)
2. Aoyama, K., Kohsaka, F., Takahashi, W.: Three generalizations of firmly nonexpansive mappings: their relations and continuity properties. J. Nonlinear Convex Anal. **10**, 131–147 (2009)
3. Atalan, Y.: On a new fixed point iterative algorithm for general variational inequalities. J. Nonlinear Convex Anal. **20**, 2371–2386 (2019)
4. Chairatsiripong, C., Yambangwai, D., Thianwan, T.: New iterative methods for nonlinear operators as concerns convex programming applicable in differential problems, image deblurring, and signal recovering problems. Math. Methods Appl. Sci. **46**, 3332–3355 (2023). https://doi.org/10.1002/mma.8693
5. Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Mathematics and Its Applications, vol. 62. Kluwer Academic, Dordrecht (1990)
6. Hacıoğlu, E., Gürsoy, F., Maldar, S., Atalan, Y., Milovanović, G.V.: Iterative approximation of fixed points and applications to two-point second-order boundary value problems and to machine learning. Appl. Numer. Math. **167**, 143–172 (2021). https://doi.org/10.1016/j.apnum.2021.04.020
7. Honda, T., Ibaraki, T., Takahashi, W.: Duality theorems and convergence theorems for nonlinear mappings in Banach spaces and applications. Int. J. Math. Stat. **6**, 46–64 (2010)
8. Ibaraki, T.: Approximation of a zero point of monotone operators with nonsummable errors. Fixed Point Theory Appl. **2016** 48 (2016). https://doi.org/10.1186/s13663-016-0535-2
9. Ibaraki, T.: Iterative approximation with errors of zero points of maximal monotone operators in a Hilbert space. Linear Nonlinear Anal. **3**, 171–178 (2017)
10. Ibaraki, T.: Iterative approximation with errors of zero points of monotone operators in a Banach space. Yokohama Math. J. **63**, 91–109 (2017). http://hdl.handle.net/10131/00011742
11. Ibaraki, T., Kajiba, S.: A shrinking projection method for generalized firmly nonexpansive mappings with nonsummable errors. Josai Math. Monogr. **11**, 105–120 (2018)
12. Ibaraki, T., Kimura, Y.: Approximation of a fixed point of generalized firmly nonexpansive mappings with nonsummable errors. Linear Nonlinear Anal. **2**, 301–310 (2016)
13. Ibaraki, T., Kimura, Y., Takahashi, W.: Convergence theorems for generalized projections and maximal monotone operators in Banach spaces. Abstr. Appl. Anal. **2003**, 621–629 (2003). https://doi.org/10.1155/S1085337503207065
14. Kamimura, S., Takahashi, W.: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**, 938–945 (2003). https://doi.org/10.1137/S105262340139611X
15. Kimura, Y.: Approximation of a fixed point of nonlinear mappings with nonsummable errors in a Banach space. In: Banach and Function Spaces IV (ISBFS 2012), pp. 303–311. Yokohama Publishers, Yokohama (2014)
16. Kimura, Y., Saejung, S.: Strong convergence for a common fixed point of two different generalizations of cutter operators. Linear Nonlinear Anal. **1**, 53–65 (2015)

17. Kohsaka, F., Takahashi, W.: Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces. J. Nonlinear Convex Anal. **8**, 197–209 (2007)
18. Maldar, S.: Iterative algorithms of generalized nonexpansive mappings and monotone operators with application to convex minimization problem. J. Appl. Math. Comput. **68**, 1841–1868 (2022). https://doi.org/10.1007/s12190-021-01593-y
19. Maldar, S., Gürsoy, F., Atalan, Y., Abbas, M.: On a three-step iteration process for multivalued Reich-Suzuki type α-nonexpansive and contractive mappings. J. Appl. Math. Comput. **68**, 863–883 (2022). https://doi.org/10.1007/s12190-021-01552-7
20. Rockafellar, R.T.: Characterization of the subdifferentials of convex functions. Pac. J. Math. **17**, 497–510 (1966)
21. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209–216 (1970)
22. Rockafellar, R.T.: On the virtual convexity of the domain and range of a nonlinear maximal monotone operator. Math. Ann. **185**, 81–90 (1970). https://doi.org/10.1007/BF01359698
23. Saejung, S.: Fixed Point Algorithms and Related Topics. Yokohama Publishers, Yokohama (2017)
24. Takahashi, W., Takeuchi, Y., Kubota, R.: Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **341**, 276–286 (2008). https://doi.org/10.1016/j.jmaa.2007.09.062
25. Thianwan, T.: Mixed type algorithms for asymptotically nonexpansive mappings in hyperbolic spaces. Eur. J. Pure Appl. Math. **14**(3), 650–665 (2021). https://doi.org/10.29020/nybg.ejpam.v14i3.4005
26. Tsukada, M.: Convergence of best approximations in a smooth Banach space. J. Approx. Theory **40**, 301–309 (1984). https://doi.org/10.1016/0021-9045(84)90003-0
27. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127–1138 (1991). https://doi.org/10.1016/0362-546X(91)90200-K
28. Yambangwai, D., Chairatsiripong, C., Thianwan, T.: Iterative manner involving sunny nonexpansive retractions for nonlinear operators from the perspective of convex programming as applicable to differential problems, image restoration and signal recovery. AIMS Math. **8**, 7163–7195 (2023). https://doi.org/10.3934/math.2023361

Publisher's Note