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New fixed point theorems via measure of noncompactness and its application on fractional integral equation involving an operator with iterative relations

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Abstract

In this paper, Darbo's fixed point theorem is generalized and it is applied to find the existence of solution of a fractional integral equation involving an operator with iterative relations in a Banach space. Moreover, an example is provided to illustrate the results.

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1 Introduction

In the nonlinear analysis, the concept of measure of noncompactness (MNC) technique plays an important role in addressing the problems in functional analysis. Kuratowski [1] was the first to describe the concept of MNC. After that, Darbo [2] developed a result on fixed point theory by using the concept of MNC. In recent times, the concept of MNC and its applications in the mathematical sciences have been generalized by many authors in various ways (see [3–16]).

In [17], the authors established some new generalizations of Darbo's fixed point theorem and studied the solvability of an infinite system of weighted fractional integral equations of a function with respect to another function. In [18], the authors studied the existence of solutions for an infinite system of Hilfer fractional differential equations in tempered sequence spaces via Meir–Keeler condensing operators. In [19], the authors first introduced the concept of a double sequence space and constructed a Hausdorff measure of noncompactness on this space. Furthermore, by employing this measure of noncompactness, they discussed the existence of solutions for infinite systems of third-order three-point boundary value problems in a double sequence space. In [20], the authors discussed an existence result for the solution of an infinite system of fractional differential equations with a three point boundary value condition. In [21], the authors studied the existence of solutions for an implicit functional equation involving a fractional integral with respect to a certain function, which generalizes the Riemann–Liouville fractional integral and the Hadamard fractional integral. In [22], the authors presented a generalization of Darbo's fixed point theorem, and they used it to investigate the solvability of an infinite system of fractional order integral equations. In [23], the authors work on the problem of the existence of positive solutions of a fractional integral equation via measures of noncompactness with the help of Darbo's fixed point theorem.

Motivated by the above mentioned works, in this article we establish a new fixed point theorem with the help of a newly defined condensing operator using a class of functions. The suggested fixed point theorem has the advantage of relaxing the constraint of the domain of compactness, which is necessary for several fixed point theorems. For particular cases of these classes of functions, the established fixed point theorems will reduce to Darbo's fixed point theorem.

Is there a contractive condition that ensures the existence of a fixed point but does not demand that the mapping be continuous at the fixed point? This is an open problem put up by Rhoades. The authors of [24] were inspired by [9] and used the MNC tool to investigate the existence of fixed points for mappings satisfying various contractive requirements. For JS-contractive type mappings in a Banach space, they also put forth expansions of Darbo's fixed point theorem.

Also, we investigate the existence of a solution for the following integral equation:

$$x(r) = T^m x(r), \tag{1}$$

where $r \in L = [0,1]$ and the operator T^m on the Banach space $\aleph = \mathbb{C}(L)$ is defined by the following iterative relation:

$$T^m x(r) = \begin{cases} \mathcal{D}(|\mathcal{K}(r,x(r))| + \int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1}|g(r,\hbar,x(\hbar))|\,d\hbar) & \text{for } m = 1, \\ \mathcal{D}(|\mathcal{K}(r,T^{m-1}x(r))| & \\ \quad + \int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1}|g(r,\hbar,T^{m-1}x(\hbar))|\,d\hbar) & \text{for } m = 2,3,\ldots, \end{cases}$$

where $0 < \mathcal{D} < 1$, $0 < \xi < 1$, $\mathcal{K}$ is a function from $L \times \mathbb{R}$ to $\mathbb{R}$, g is a function from $L \times L \times \mathbb{R}$ to $\mathbb{R}$, and Euler's gamma function is denoted by $\Gamma(,)$ and defined as follows:

$$\Gamma(\xi) = \int_0^\infty u^{\xi-1}e^{-u}\,du.$$

Fractional derivatives and integrals are quite flexible for describing the behaviors of different types of real life situations. They can be applied to study the heat transfer problem, non-Newtonian fluids, etc. In the above equation, we have used the Erdèlyi–Kober operator, which is a fractional integration operator introduced by Arthur Erdèlyi and Hermann Kober in 1940. This integral is given by [25]

$$I^\gamma_\beta f(t) = \frac{\beta}{\Gamma(\gamma)} \int_0^t \frac{s^{\beta-1}f(s)}{(t^\beta - s^\beta)^{1-\gamma}}\,ds, \quad \beta > 0, 0 < \gamma < 1,$$

where f is a continuous function. It is a generalization of the Riemann–Liouville fractional integral. This makes this integral a better choice for our problem.

In this note, we consider $\aleph$ as a Banach space and $B(\Delta, \varphi)$ represents the closed ball in the Banach space $\aleph$ with center Δ and with radius φ. Also, we use B_φ to represent $B(\theta, \varphi)$, (θ is the zero element), all nonempty bounded subsets of Banach space $\aleph$ are gathered in $\mathfrak{X}_\aleph$, Also, $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. To begin with, we have the following preliminaries.

2 Preliminaries

Definition 2.1 [10] Let $\mu : \mathfrak{X}_\aleph \to \mathbb{R}^+$ be a mapping. μ is called an MNC on the Banach space $\aleph$ provided that:

(1) for each $\mathcal{N} \in \mathfrak{X}_\aleph$, $\mu(\mathcal{N}) = 0$ if and only if $\mathcal{N}$ is a precompact set;

(2) for each pair $(\mathcal{N}, \mathcal{N}_1) \in \mathfrak{X}_\aleph \times \mathfrak{X}_\aleph$, we have

$$\mathcal{N} \subseteq \mathcal{N}_1 \quad \text{implies} \quad \mu(\mathcal{N}) \preceq \mu(\mathcal{N}_1);$$

(3) for each $\mathcal{N} \in \mathfrak{X}_\aleph$, one has

$$\mu(\mathcal{N}) = \mu(\overline{\mathcal{N}}) = \mu\bigl(\operatorname{conv}(\mathcal{N})\bigr),$$

where $\overline{\mathcal{N}}$ is the closure of $\mathcal{N}$ and $\operatorname{conv} \mathcal{N}$ is the convex hull of $\mathcal{N}$;

(4) $\mu(\lambda\mathcal{N} + (1-\lambda)\mathcal{N}_1) \leq \lambda\mu(\mathcal{N}) + (1-\lambda)\mu(\mathcal{N}_1)$ for $\lambda \in [0,1]$;

(5) if $\{\mathcal{N}_n = \overline{\mathcal{N}_n}\}_0^{+\infty} \subseteq \mathfrak{X}_\aleph$ is decreasing and $\lim_{n\to+\infty} \mu(\mathcal{N}_n) = 0$, then $\mathcal{N}_\infty = \bigcap_{n=0}^{\infty} \mathcal{N}_n \neq \emptyset$.

We have the following theorems from [26, 27] in this section.

Theorem 2.1 (*Schauder*) *Let $T : \mho \to \mho$ be a compact and continuous operator where $\mho$ is a nonempty, closed, and convex subset of the Banach space $\aleph$. As a result, T has at least one fixed point.*

Theorem 2.2 (*Darbo*) *Let $T : \mho \to \mho$ be a continuous operator where $\mho$ is a nonempty, bounded, closed, and convex subset of the Banach space $\aleph$. Assume that for each $X \subseteq \mho$, $\mu(TX) \leq \mathcal{H}\mu(X)$, where $\mathcal{H} \in [0,1)$. Consequently, T has a fixed point.*

Theorem 2.3 (*Brouwer*) *Let $\mho$ be a nonempty, compact, and convex subset of a finite dimensional normed space, and let $T : \mho \to \mho$ be a continuous operator. Then T has a fixed point.*

Now, some important definitions of functions are given below.

Definition 2.2 [28] Let $\mathfrak{F} : (\mathbb{R}^+)^4 \to \mathbb{R}$ be a function that satisfies:

(1) $\mathfrak{F}(1,1,s_2,s_3)$ is continuous;

(2) $0 \leq s_0 \leq 1$, $s_1 \geq 1 \Rightarrow \mathfrak{F}(s_0,s_1,s_2,s_3) \leq \mathfrak{F}(1,1,s_2,s_3) \leq s_2$;

(3) $\mathfrak{F}(1,1,s_2,s_3) = s_2 \Rightarrow s_2 = 0$ or $s_3 = 0$.

Also, we denote this class of functions by $\bar{Z}$.

For example, $\mathfrak{F}(s_0,s_1,s_2,s_3) = s_0s_2 - s_1s_3; s_0, s_1, s_2, s_3 \in \mathbb{R}^+$ is an element of $\bar{Z}$.

Definition 2.3 [29] Let $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be two functions. The pair (Ψ, Φ) is said to be a pair of shifting distance functions if the following conditions hold:

(1) For all $s, t \in \mathbb{R}^+$, if $\Psi(s) \leq \Phi(t)$, then $s \leq t$;

(2) For all $\{s_n\}, \{t_n\} \subseteq \mathbb{R}^+$ with $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = t_0$ and $\Psi(s_n) \leq \Phi(t_n) \forall n$, then $t_0 = 0$.

We denote the class of all these pairs (Ψ, Φ) of shifting distance function by Υ.

For example, assume that $\Psi(s_0) = \ln(\frac{1+2s_0}{2})$ and $\Phi(s_0) = \ln(\frac{1+s_0}{2})$, where $s_0 \in \mathbb{R}^+$.

Definition 2.4 [28] Let $\mathcal{A}$ be the collection of all functions $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathcal{E}(s_0) \geq s_0$$

for all $s_0 \in \mathbb{R}^+$.

For example, let $\mathcal{E}(s_0) = s_0$ for all $s_0 \in \mathbb{R}^+$.

Definition 2.5 Let $\mathcal{A}'$ be the collection of all functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$0 \leq \alpha(s_0) \leq 1$$

for all $s_0 \in \mathbb{R}^+$.

For example, let $\alpha(s_0) = |\sin(s_0)|$ for all $s_0 \in \mathbb{R}^+$.

Definition 2.6 Let $\bar{\mathcal{A}}$ be the collection of all functions $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\beta(s_0) \geq 1$$

for all $s_0 \in \mathbb{R}^+$.

For example, let $\beta(s_0) = 1 + s_0$ for all $s_0 \in \mathbb{R}^+$.

Definition 2.7 [30] Let Ł$'$ be the collection of all functions $\mathcal{S} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy:

(1) $\max\{s_0, s_1\} \leq \mathcal{S}(s_0, s_1)$ for all $s_0, s_1 \geq 0$;

(2) $\mathcal{S}$ is continuous and nondecreasing;

(3) $\mathcal{S}(s_0 + s_1, s_0' + s_1') \leq \mathcal{S}(s_0, s_0') + \mathcal{S}(s_1, s_1')$.

For example, let $\mathcal{S}(s_0, s_1) = s_0 + s_1$ for all $s_0, s_1 \in \mathbb{R}^+$.

Definition 2.8 [31] A continuous function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of class $\mathcal{C}$ if the following conditions are satisfied:

(1) $h(s_0, s_1) \leq s_0$,

(2) $h(s_0, s_1) = s_0$ implies that either $s_0 = 0$ or $s_1 = 0$. Also, $h(0, 0) = 0$.

Also, we denote this class of functions by $\bar{Ł}$.

For example, let

(1) $h(s_0, s_1) = s_0 - s_1$,

(2) $h(s_0, s_1) = \mathcal{V}s_0$; $0 < \mathcal{V} < 1$,

for all $s_0, s_1 \in \mathbb{R}^+$.

Definition 2.9 [32] A function $\upsilon : \mathbb{R} \to \mathbb{R}$ is an alternating function if:
(1) $\upsilon(s_0) = 0$ if and only if $s_0 = 0$,
(2) υ is continuous and increasing.

Also, we use $\mathcal{Y}$ to denote this class of functions.
For example, let $\upsilon(s_0) = (1 - \bar{a})s_0$, where $0 \le \bar{a} < 1$ and $s_0 \in \mathbb{R}$.

Definition 2.10 [31] Let $\bar{\mathcal{Y}}$ be the collection of all functions $\omega : \mathbb{R} \to \mathbb{R}$ such that $\omega(0) \ge 0$ and $\omega(s_0) > 0$ for all $s_0 > 0$.

Definition 2.11 Let $\mathcal{Y}'$ be the collection of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(s_0) > s_0$ for all $s_0 \in \mathbb{R}^+$ and $f(0) = 0$.

For example, let $f(s_0) = \mathcal{W} \cdot s_0$, where $\mathcal{W} > 1$ and $s_0 \in \mathbb{R}^+$.
With the help of these classes of functions, we obtain new generalizations of Darbo's fixed point theorem.

3 New fixed point theory

Theorem 3.1 *Let $T : \mho \to \mho$ be a continuous mapping where $\mho$ is a nonempty, bounded, closed, and convex subset of $\aleph$ such that*

$$\begin{aligned} &\mathcal{E}\big(\Psi\big(\mu\big(T^m X\big)\big)\big) \\ &\quad \le \mathfrak{F}\big[\alpha\big(\mathcal{M}_{m-1}(X)\big), \beta\big(\mathcal{M}_{m-1}(X)\big), \Phi\big(\mathcal{M}_{m-1}(X)\big), \gamma\big(\mathcal{M}_{m-1}(X)\big)\big], \end{aligned} \tag{2}$$

where

$$\mathcal{M}_{m-1}(X) = \max\big\{\mu(X), \mu(TX), \dots, \mu\big(T^{m-1}X\big)\big\}$$

for each $\emptyset \ne X \subseteq \mho$, where μ is an arbitrary MNC, $(\Psi, \Phi) \in \Upsilon$, $\mathfrak{F} \in \bar{\mathcal{Z}}$, $\mathcal{E} \in \mathcal{A}$, $\alpha \in \mathcal{A}'$, $\beta \in \bar{\mathcal{A}}$, and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$. Then there is at least one fixed point for T in $\mho$.

Proof Take $X_0 = \mho$, $X_{n+m} = \overline{\operatorname{conv}(T^m X_n)}$ for $n = 0, 1, 2, \dots$.
Evidently, $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of nonempty, bounded, closed, and convex subsets such that

$$X_0 \supseteq X_m \supseteq X_{m+1} \supseteq \cdots \supseteq X_{m+n}.$$

If for an integer $N \in \mathbb{N}$ one has $\mu(X_N) = 0$, then X_N is relatively compact, and so Schauder's theorem guarantees the existence of a fixed point for T.
So, we can assume $\mu(X_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.
Now, from (2) we have

$$\begin{aligned} \mathcal{E}\big(\Psi\big(\mu(X_{n+m})\big)\big) &= \mathcal{E}\big(\Psi\big(\mu\big(\overline{\operatorname{conv}\big(T^m X_n\big)}\big)\big)\big) \\ &= \mathcal{E}\big(\Psi\big(\mu\big(T^m X_n\big)\big)\big) \\ &\le \mathfrak{F}\big[\alpha\big(\mathcal{M}_{m-1}(X_n)\big), \beta\big(\mathcal{M}_{m-1}(X_n)\big), \Phi\big(\mathcal{M}_{m-1}(X_n)\big), \gamma\big(\mathcal{M}_{m-1}(X_n)\big)\big] \end{aligned}$$

$$\begin{aligned}&\le \mathfrak{F}\big[\alpha\big(\mathcal{M}_{m-1}(X_n)\big),\beta\big(\mathcal{M}_{m-1}(X_n)\big),\Phi\big(\mathcal{M}_{m-1}(X_n)\big),\gamma\big(\mathcal{M}_{m-1}(X_n)\big)\big]\\&\le \Phi\big(\mu(X_n)\big)\big[\text{using (2) of Definition 2.5}\big]\end{aligned}\tag{3}$$

for $n = 0, 1, 2, \dots$, where

$$\mathcal{M}_{m-1}(X_n) = \max\big\{\mu(X_n),\mu(X_{n+1}),\dots,\mu(X_{n+m-1})\big\} = \mu(X_n).$$

On the other hand,

$$\mathcal{E}\big(\Psi\big(\mu(X_{n+m})\big)\big) \ge \Psi\big(\mu(X_{n+m})\big).\tag{4}$$

So, from equations (3) and (4), we get

$$\Psi\big(\mu(X_{n+m})\big) \le \Phi\big(\mu(X_n)\big).\tag{5}$$

Now, let

$$\lim_{n\to\infty}\mu(X_{n+m}) = \lim_{n\to\infty}\mu(X_n) = r.\tag{6}$$

Thus, by using condition (2) of Definition 2.3, equation (5), and equation (6), we get

$$r = 0,$$

i.e.,

$$\lim_{n\to\infty}\mu(X_{n+m}) = \lim_{n\to\infty}\mu(X_n) = 0.$$

Consequently, we conclude that $\mu(X_n) \to 0$ as $n \to \infty$. Therefore, by Definition 2.1 (6), $X_\infty = \bigcap_{n=0}^{\infty} X_n$ is nonempty, closed, and convex. The set X_∞ under the operation T is also invariant and $X_\infty \in \ker\mu$. Thus, by using Theorem 2.1 the proof is finished. □

Corollary 3.2 *Let $T : \mho \to \mho$ be a continuous mapping where $\mho$ is a nonempty, bounded, closed, and convex subset of $\aleph$ such that*

$$\begin{aligned}&\mathcal{E}\big(\Psi\big(\mu\big(T^m X\big)\big)\big)\\&\quad\le \big(\alpha\big(\mathcal{M}_{m-1}(X)\big).\Phi\big(\mathcal{M}_{m-1}(X)\big)\big) - \big(\beta\big(\mathcal{M}_{m-1}(X)\big).\gamma\big(\mathcal{M}_{m-1}(X)\big)\big),\end{aligned}\tag{7}$$

where

$$\mathcal{M}_{m-1}(X) = \max\big\{\mu(X),\mu(TX),\dots,\mu\big(T^{m-1}X\big)\big\}$$

for each $\emptyset \neq X \subseteq \mho$, where μ is an arbitrary MNC, $(\Psi,\Phi) \in \Upsilon$, $\mathcal{E} \in \mathcal{A}$, $\alpha \in \mathcal{A}'$, $\beta \in \bar{\mathcal{A}}$, and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$. Then there is at least one fixed point for T in $\mho$.

Proof Putting $\mathfrak{F}(s_0, s_1, s_2, s_3) = s_0s_2 - s_1s_3$ for all $s_0, s_1, s_2, s_3 \in \mathbb{R}^+$ in equation (2) of Theorem 3.1, we can get the above result. □

Corollary 3.3 *Let $T : \mho \to \mho$ be a continuous mapping where $\mho$ is a nonempty, bounded, closed, and convex subset of $\aleph$ such that*

$$\mu\bigl(T^m X\bigr) \le \mathcal{M}_{m-1}(X), \tag{8}$$

where

$$\mathcal{M}_{m-1}(X) = \max\bigl\{\mu(X), \mu(TX), \dots, \mu\bigl(T^{m-1}X\bigr)\bigr\}$$

for each $\emptyset \ne X \subseteq \mho$, where μ is an arbitrary MNC. Then there is at least one fixed point for T in $\mho$.

Proof Putting $\mathcal{E}(s_0) = s_0$, $\gamma(s_0) = 0$, $\alpha(s_0) = 1$, $\Psi(s_0) = s_0$, and $\Phi(s_0) = s_0$ for all $s_0 \in \mathbb{R}^+$ in equation (7) of Corollary 3.2, we can get the above result. □

Theorem 3.4 *Let $T : \mho \to \mho$ be a continuous mapping where $\mho$ is a nonempty, bounded, closed, and convex subset of $\aleph$ such that*

$$\begin{aligned} &f\bigl[\upsilon\bigl[\mathcal{S}\bigl(\mu(T^m X), \sigma(\mu(T^m X))\bigr)\bigr]\bigr] \\ &\quad \le h\bigl[\upsilon\bigl\{\mathcal{S}\bigl(\mathcal{M}_{m-1}(X), \sigma(\mathcal{M}_{m-1}(X))\bigr)\bigr\}, \omega\bigl\{\mathcal{S}\bigl(\mathcal{M}_{m-1}(X), \sigma(\mathcal{M}_{m-1}(X))\bigr)\bigr\}\bigr], \end{aligned} \tag{9}$$

where

$$\mathcal{M}_{m-1}(X) = \max\bigl\{\mu(X), \mu(TX), \dots, \mu\bigl(T^{m-1}X\bigr)\bigr\}$$

for each $\emptyset \ne X \subseteq \mho$, and μ is an arbitrary MNC, $\mathcal{S} \in Ł'$, $h \in \bar{Ł}$, $\upsilon \in \mathcal{Y}$, $\omega \in \bar{\mathcal{Y}}$, $f \in \mathcal{Y}'$, and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$. Then there is at least one fixed point for T in $\mho$.

Proof Taking $X_0 = \mho$, $X_{n+m} = \overline{\operatorname{conv}(T^m X_n)}$ for $n = 0, 1, 2, \dots$.

Evidently, $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of nonempty, bounded, closed, and convex subsets such that

$$X_0 \supseteq X_m \supseteq X_{m+1} \supseteq \cdots \supseteq X_{m+n} \supseteq \cdots.$$

If for an integer $N \in \mathbb{N}$ one has $\mu(X_N) = 0$, then X_N is relatively compact, and so Schauder's theorem guarantees the existence of a fixed point for T.

So, we can assume $\mu(X_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, from (9) we have

$$\begin{aligned} &f\bigl[\upsilon\bigl[\mathcal{S}\bigl(\mu(X_{n+m}), \sigma\bigl(\mu(X_{n+m})\bigr)\bigr)\bigr]\bigr] \\ &\quad = f\bigl[\upsilon\bigl[\mathcal{S}\bigl(\mu\bigl(\overline{\operatorname{conv}\bigl(T^m X_n\bigr)}\bigr), \sigma\bigl(\mu\bigl(\overline{\operatorname{conv}\bigl(T^m X_n\bigr)}\bigr)\bigr)\bigr)\bigr]\bigr] \\ &\quad = f\bigl[\upsilon\bigl[\mathcal{S}\bigl(\mu\bigl(T^m X_n\bigr), \sigma\bigl(\mu\bigl(T^m X_n\bigr)\bigr)\bigr)\bigr]\bigr] \\ &\quad \le h\bigl[\upsilon\bigl\{\mathcal{S}\bigl(\mathcal{M}_{m-1}(X_n), \sigma(\mathcal{M}_{m-1}(X_n))\bigr)\bigr\}, \omega\bigl\{\mathcal{S}\bigl(\mathcal{M}_{m-1}(X_n), \sigma(\mathcal{M}_{m-1}(X_n))\bigr)\bigr\}\bigr] \\ &\quad \le h\bigl[\upsilon\bigl\{\mathcal{S}\bigl(\mu(X_n), \sigma(\mu(X_n))\bigr)\bigr\}, \omega\bigl\{\mathcal{S}\bigl(\mu(X_n), \sigma(\mu(X_n))\bigr)\bigr\}\bigr] \end{aligned}$$

$$\leq \upsilon\bigl\{\mathcal{S}\bigl(\mu(X_n),\sigma(\mu(X_n))\bigr)\bigr\} \tag{10}$$

for $n = 0, 1, 2, \dots$, where

$$\mathcal{M}_{m-1}(X_n) = \max\bigl\{\mu(X_n),\mu(X_{n+1}),\dots,\mu(X_{n+m-1})\bigr\} = \mu(X_n).$$

Also,

$$f\bigl[\upsilon\bigl[\mathcal{S}\bigl(\mu(X_{n+m}),\sigma(\mu(X_{n+m}))\bigr)\bigr]\bigr] \geq \upsilon\bigl[\mathcal{S}\bigl(\mu(X_{n+m}),\sigma(\mu(X_{n+m}))\bigr)\bigr]. \tag{11}$$

Clearly, $\{\upsilon[\mathcal{S}(\mu(X_{n+m}),\sigma(\mu(X_{n+m})))]\}_{n=1}^{\infty}$ is a nonnegative and nonincreasing sequence. Hence, $\exists\, \delta \geq 0$ such that

$$\lim_{n\to\infty} \upsilon\bigl[\mathcal{S}\bigl(\mu(X_{n+m}),\sigma\bigl(\mu(X_{n+m})\bigr)\bigr)\bigr] = \delta. \tag{12}$$

If possible, let $\delta > 0$. As $n \to \infty$, we get

$$f(\delta) \leq \delta,$$

which is a contradiction. Hence, $\delta = 0$, i.e.,

$$\upsilon\Bigl\{\lim_{n\to\infty} \mathcal{S}\bigl(\mu(X_{n+m}),\sigma\bigl(\mu(X_{n+m})\bigr)\bigr)\Bigr\} = 0,$$

that is,

$$\lim_{n\to\infty} \mathcal{S}\bigl(\mu(X_{n+m}),\sigma\bigl(\mu(X_{n+m})\bigr)\bigr) = 0,$$

which gives

$$\lim_{n\to\infty} \mu(X_n) = 0.$$

Therefore, by Definition 2.1 (6), $X_\infty = \bigcap_{n=0}^{\infty} X_n$ is nonempty, closed, and convex.

Additionally, the set X_∞ under operator T is invariant and $X_\infty \in \ker\mu$. The proof is finished by using Theorem 2.1. □

Theorem 3.5 *Let $T : \mho \to \mho$ be a continuous mapping where $\mho$ is a nonempty, bounded, closed, and convex subset of $\aleph$ such that*

$$\begin{aligned} &f\bigl[\upsilon\bigl(\mu\bigl(T^m X\bigr)+\sigma\bigl(\mu\bigl(T^m X\bigr)\bigr)\bigr)\bigr] \\ &\quad\leq h\bigl[\upsilon\bigl(\mathcal{M}_{m-1}(X)+\sigma\bigl(\mathcal{M}_{m-1}(X)\bigr)\bigr),\omega\bigl(\mathcal{M}_{m-1}(X)+\sigma\bigl(\mathcal{M}_{m-1}(X)\bigr)\bigr)\bigr], \end{aligned} \tag{13}$$

where

$$\mathcal{M}_{m-1}(X) = \max\bigl\{\mu(X),\mu(TX),\dots,\mu\bigl(T^{m-1}X\bigr)\bigr\}$$

for each $\emptyset \neq X \subseteq \mho$, and μ is an arbitrary MNC, $h \in \bar{Ł}$, $\upsilon \in \mathcal{Y}$, $\omega \in \bar{\mathcal{Y}}$, $f \in \mathcal{Y}'$, and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$. Then there is at least one fixed point for T in $\mho$.

Proof Putting $\mathcal{S}(s_0, s_1) = s_0 + s_1$ ($s_0, s_1 \in \mathbb{R}^+$) in equation (9) of Theorem 3.4, we can get the above result. □

Theorem 3.6 *Let $T : \mho \to \mho$ be a continuous self-mapping on a nonempty, bounded, closed, and convex subset $\mho$ of $\aleph$ such that*

$$f\big[\upsilon\big(\mu\big(T^m X\big)\big)\big] \\ \le h\big[\upsilon\big(\mathcal{M}_{m-1}(X), \omega\big(\mathcal{M}_{m-1}(X)\big)\big)\big], \tag{14}$$

where

$$\mathcal{M}_{m-1}(X) = \max\big\{\mu(X), \mu(TX), \dots, \mu\big(T^{m-1}X\big)\big\}$$

for each $\emptyset \ne X \subseteq \mho$, and μ is an arbitrary MNC, $h \in \bar{Ł}$, $\upsilon \in \mathcal{Y}$, $\omega \in \bar{\mathcal{Y}}$, and $f \in \mathcal{Y}'$. Then there is at least one fixed point for T in $\mho$.

Proof Putting $\sigma(s_0) = 0$, ($s_0 \in \mathbb{R}^+$) in equation (13) of Theorem 3.5, we can get the above result. □

Theorem 3.7 *Let $T : \mho \to \mho$ be a continuous self-mapping on a nonempty, bounded, closed, and convex subset $\mho$ of $\aleph$ such that*

$$f\big[\upsilon\big[\mathcal{S}\big(\mu\big(T^m X\big), \sigma\big(\mu\big(T^m X\big)\big)\big)\big]\big] \\ \le \upsilon\big\{\mathcal{S}\big(\mathcal{M}_{m-1}(X), \sigma\big(\mathcal{M}_{m-1}(X)\big)\big)\big\}, \tag{15}$$

where

$$\mathcal{M}_{m-1}(X) = \max\big\{\mu(X), \mu(TX), \dots, \mu\big(T^{m-1}X\big)\big\}$$

for each $\emptyset \ne X \subseteq \mho$, and μ is an arbitrary MNC, $\upsilon \in \mathcal{Y}$, $\omega \in \bar{\mathcal{Y}}$, $f \in \mathcal{Y}'$, and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$. Then there is at least one fixed point for T in $\mho$.

Proof Putting $h(s_0, s_1) \le s_0$, ($s_0, s_1 \in \mathbb{R}^+$) in equation (9) of Theorem 3.4, we can get the above result. □

Corollary 3.8 *Putting $\mathcal{S}(s_0, s_1) = s_0 + s_1$, $\sigma(s_0) = 0$, $\upsilon(t') = t'$, $h(s_0, s_1) = \mathcal{V} \cdot s_0$, and $f(s_0) = \mathcal{W} \cdot s_0$, where $0 < \mathcal{V} < 1$, $\mathcal{W} > 1$, $s_0, s_1 \in \mathbb{R}^+$, and $t' \in \mathbb{R}$ in equation (9) of Theorem 3.4, we obtain*

$$\mu\big(T^m X\big) \le \zeta \cdot \mathcal{M}_{m-1}(X), \tag{16}$$

where $\zeta = \frac{\mathcal{V}}{\mathcal{W}} \in (0, 1)$ and

$$\mathcal{M}_{m-1}(X) = \max\big\{\mu(X), \mu(TX), \dots, \mu\big(T^{m-1}X\big)\big\}.$$

4 Measure of noncompactness on $\mathbb{C}([0,1])$

Let $\aleph = \mathbb{C}(L)$ be the set of all real continuous functions on $L = [0,1]$. Then $\aleph$ is a Banach space with the norm

$$\|\pounds\| = \sup\left\{\left|\pounds(p)\right| : p \in L\right\}, \quad \pounds \in \aleph.$$

Assume that $J(\neq \emptyset) \subseteq \aleph$ is bounded. For given $\pounds \in J$ and arbitrary $d_0 > 0$, $\eta(\pounds, d_0)$ is the modulus of the continuity of $\pounds$ written as

$$\eta(\pounds, d_0) = \sup\left\{\left|\pounds(p_1) - \pounds(p_2)\right| : p_1, p_2 \in L, |p_2 - p_1| \leq d_0\right\}.$$

Also, we define

$$\eta(J, d_0) = \sup\left\{\eta(\pounds, d_0) : \pounds \in J\right\}$$

and

$$\eta_0(J) = \lim_{d_0 \to 0} \eta(J, d_0).$$

The MNC in $\aleph$ is denoted by the function η_0, and the Hausdorff MNC is denoted by $\mathfrak{Z}$, which is defined as $\mathfrak{Z}(J) = \frac{1}{2}\eta_0(J)$ (see [33]).

5 Solvability of a fractional integral equation

The aim of this section is to investigate the existence of a solution for the integral equation

$$x(r) = T^m x(r), \tag{17}$$

where $r \in L$ and the operator T^m on $\aleph = \mathbb{C}(L)$ is defined by the following iterative relation:

$$T^m x(r) = \begin{cases} \mathcal{D}(|\mathcal{K}(r, x(r))| + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r, \hbar, x(\hbar))|\, d\hbar) & \text{for } m = 1, \\ \mathcal{D}(|\mathcal{K}(r, T^{m-1}x(r))| & \\ \quad + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r, \hbar, T^{m-1}x(\hbar))|\, d\hbar) & \text{for } m = 2, 3, \ldots, \end{cases}$$

where $0 < \mathcal{D} < 1$.

For example, for case $m = 2$, the integral equation is as follows:

$$x(r) = \mathcal{D}\left(\left|\mathcal{K}\big(r, Tx(r)\big)\right| + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \left|g\big(r, \hbar, Tx(\hbar)\big)\right| d\hbar\right),$$

or equivalently

$$x(r) = \mathcal{D}\begin{pmatrix} |\mathcal{K}(r, \mathcal{D}(|\mathcal{K}(r, x(r))| + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r, \hbar, x(\hbar))|\, d\hbar))| \\ + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r, \hbar, \mathcal{D}(|\mathcal{K}(\hbar, x(\hbar))| \\ + \int_0^\hbar \frac{(\hbar^q - \varrho^q)^{\xi-1}}{\Gamma(\xi)} q\varrho^{q-1} |g(\hbar, \varrho, x(\varrho))|\, d\varrho))|\, d\hbar \end{pmatrix}.$$

We assume that

$$\mathfrak{D}_{b_0} = \left\{x \in \aleph = \mathbb{C}(L) : \|x\| \leq b_0\right\}.$$

We now take into account the following hypotheses:

(i) Let $\mathcal{K}: L \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\left|\mathcal{K}\bigl(r, x(r)\bigr) - \mathcal{K}\bigl(r, y(r)\bigr)\right| \leq \left|x(r) - y(r)\right|, \quad r \in L,\ x, y \in \mathbb{C}(L).$$

Furthermore, the function $r \to \mathcal{K}(r, 0)$ is a member of $\aleph = \mathbb{C}(L)$.

(ii) Let $g: L \times L \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function such that

$$\left|g(r, \hbar, x(\hbar))\right| \leq \mathfrak{P}.$$

(iii) There is a positive solution b_0 for

$$\mathcal{D}(b_0 + \mathcal{Q}m) \leq b_0,$$

where

$$\mathcal{Q} = \sup\left\{\left|\mathcal{K}(r, 0)\right| + \frac{\mathfrak{P}}{\Gamma(\xi + 1)} r^q : r \geq 0\right\}.$$

Theorem 5.1 *The integral equation* (17) *has at least one solution in* $\mathbb{C}(L)$ *according to hypotheses* (i)–(iii).

Proof First, we show that T^m is a self-mapping on $\mathbb{C}(L)$.

Because all the functions involved in the operator T^m are continuous, $T^m x(r): \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.

On the other hand, for an arbitrary fixed function $x \in \mathbb{C}(L)$, using the above hypotheses, we get

$$\begin{aligned}
&\left|T^m x(r)\right| \\
&\quad \leq \mathcal{D}\biggl(\left|\mathcal{K}\bigl(r, T^{m-1}x(r)\bigr) - \mathcal{K}(r, 0)\right| + \left|\mathcal{K}(r, 0)\right| \\
&\qquad + \int_0^r \frac{(r^q - \hbar^q)^{\xi - 1}}{\Gamma(\xi)} q\hbar^{q-1} \left|g\bigl(r, \hbar, T^{m-1}x(\hbar)\bigr)\right| \, d\hbar\biggr) \\
&\quad \leq \mathcal{D}\biggl(\left|T^{m-1}x(r)\right| + \left|\mathcal{K}(r, 0)\right| + \frac{\mathfrak{P}}{\Gamma(\xi).\xi}\bigl(-\bigl[\bigl(r^q - \hbar^q\bigr)^{\xi}\bigr]_0^r\bigr)\biggr) \\
&\quad \leq \mathcal{D}\biggl(\left|T^{m-1}x(r)\right| + \left|\mathcal{K}(r, 0)\right| + \frac{\mathfrak{P}}{\Gamma(\xi).\xi} r^{q\xi}\biggr) \\
&\quad \leq \mathcal{D}\biggl(\mathcal{D}(\left|\mathcal{K}\bigl(r, T^{m-2}x(r)\bigr)\right| + \int_0^r \frac{(r^q - \hbar^q)^{\xi - 1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r, \hbar, T^{m-2}x(\hbar))| \, d\hbar) \\
&\qquad + \left|\mathcal{K}(r, 0)\right| + \frac{\mathfrak{P}}{\Gamma(\xi + 1)} r^{q\xi}\biggr) \quad \bigl[\text{where, } \Gamma(\xi).\xi = \Gamma(\xi + 1)\bigr] \\
&\quad \leq \mathcal{D}\biggl(\left|\mathcal{K}\bigl(r, T^{m-2}x(r)\bigr)\right| + \int_0^r \frac{(r^q - \hbar^q)^{\xi - 1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r, \hbar, T^{m-2}x(\hbar)| \, d\hbar \\
&\qquad + \left|\mathcal{K}(r, 0)\right| + \frac{\mathfrak{P}}{\Gamma(\xi + 1)} r^{q\xi}\biggr)
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{D}\biggl(\bigl|\mathcal{K}\bigl(r, T^{m-2}x(r)\bigr) - \mathcal{K}(r,0)\bigr| + \bigl|\mathcal{K}(r,0)\bigr| \\
&\quad + \frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} + \bigl|\mathcal{K}(r,0)\bigr| + \frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} \biggr) \\
&\leq \mathcal{D}\biggl(\bigl|T^{m-2}x(r)\bigr| + 2\bigl|\mathcal{K}(r,0)\bigr| + 2\frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} \biggr) \\
&\vdots \\
&\leq \mathcal{D}\biggl(\bigl|Tx(r)\bigr| + (m-1)\bigl|\mathcal{K}(r,0)\bigr| + (m-1)\frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} \biggr) \\
&\leq \mathcal{D}\biggl(\bigl|\mathcal{K}\bigl(r, x(r)\bigr) - \mathcal{K}(r,0)\bigr| + m\bigl|\mathcal{K}(r,0)\bigr| + m\frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} \biggr) \\
&\leq \mathcal{D}\biggl(\bigl|x(r)\bigr| + m\bigl|\mathcal{K}(r,0)\bigr| + m\frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} \biggr) \\
&\leq \mathcal{D}\bigl(\|x\| + \mathcal{Q}m\bigr) \\
&\leq \|x\| + \mathcal{Q}m < \infty \quad [\text{since } 0 < \mathcal{D} < 1].
\end{aligned}
$$

Therefore, the above discussion shows that $T^m : \mathbb{C}(L) \to \mathbb{C}(L)$ is well defined. Moreover, applying assumption (iii) for each $x \in \mathfrak{D}_{b_0}$, we have

$$
\bigl|T^m x(r)\bigr| \leq \mathcal{D}\bigl(\|x\| + \mathcal{Q}m\bigr) \leq \mathcal{D}(b_0 + \mathcal{Q}m) \leq b_0, \quad [\text{since } 0 < \mathcal{D} < 1].
$$

So, the function T^m is a self-mapping on the ball $\mathfrak{D}_{b_0}$.

To show that T^m is continuous on $\mathfrak{D}_{b_0}$, we take $d_0 > 0$ and $x, y \in \mathfrak{D}_{b_0}$ such that $\|x - y\| < d_0$. So, we get

$$
\begin{aligned}
&\bigl|T^m x(r) - T^m y(r)\bigr| \\
&\quad\leq \biggl| \mathcal{D}\biggl(\bigl|\mathcal{K}(r, T^{m-1}x(r))\bigr| + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \bigl|g\bigl(r, \hbar, T^{m-1}x(\hbar)\bigr)\bigr|\, d\hbar \biggr) \\
&\qquad - \mathcal{D}\biggl(\bigl|\mathcal{K}(r, T^{m-1}y(r))\bigr| + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \bigl|g\bigl(r, \hbar, T^{m-1}y(\hbar)\bigr)\bigr|\, d\hbar \biggr) \biggr| \\
&\quad\leq \mathcal{D}\biggl(\biggl| \bigl|\mathcal{K}(r, T^{m-1}x(r))\bigr| + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \bigl|g(r, \hbar, T^{m-1}x(\hbar))\bigr|\, d\hbar \\
&\qquad - \bigl|\mathcal{K}(r, T^{m-1}y(r))\bigr| - \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \bigl|g\bigl(r, \hbar, T^{m-1}y(\hbar)\bigr)\bigr|\, d\hbar \biggr| \biggr) \\
&\quad\leq \mathcal{D}\biggl(\Bigl| \bigl|\mathcal{K}(r, T^{m-1}x(r))\bigr| - \bigl|\mathcal{K}(r, T^{m-1}y(r))\bigr| \Bigr| \\
&\qquad + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \bigl|g(r, \hbar, T^{m-1}x(\hbar))\bigr|\, d\hbar \\
&\qquad + \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \bigl|g\bigl(r, \hbar, T^{m-1}y(\hbar)\bigr)\bigr|\, d\hbar \biggr) \\
&\quad\leq \mathcal{D}\biggl(\bigl|\mathcal{K}(r, T^{m-1}x(r)) - \mathcal{K}(r, T^{m-1}y(r))\bigr| + \frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} + \frac{\mathfrak{P}}{\Gamma(\xi+1)} r^{q\xi} \biggr)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathcal{D}\left(\left|T^{m-1}x(r)-T^{m-1}y(r)\right|+2\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(\left|\mathcal{D}\left(\left|\mathcal{K}(r,T^{m-2}x(r))\right|+\int_0^r\frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)}q\hbar^{q-1}\left|g\big(r,\hbar,T^{m-2}x(\hbar)\big)\right|d\hbar\right)\right.\right.\\
&\quad\left.-\mathcal{D}\left(\left|\mathcal{K}\big(r,T^{m-2}y(r)\big)\right|+\int_0^r\frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)}q\hbar^{q-1}\left|g\big(r,\hbar,T^{m-2}y(\hbar)\right|d\hbar\right)\right|\\
&\quad\left.+2\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(\mathcal{D}\left(\left|\mathcal{K}(r,T^{m-2}x(r))-\mathcal{K}(r,T^{m-2}y(r))\right|\right.\right.\\
&\quad+\int_0^r\frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)}q\hbar^{q-1}\left|g(r,\hbar,T^{m-1}x(\hbar))\right|d\hbar\\
&\quad\left.\left.+\int_0^r\frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)}q\hbar^{q-1}\left|g(r,\hbar,T^{m-1}y(\hbar))\right|d\hbar\right)+2\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(\mathcal{D}\left(|\mathcal{K}(r,T^{m-2}x(r))-\mathcal{K}(r,T^{m-2}y(r))|+\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}+\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\right.\\
&\quad\left.+2\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(\left|T^{m-2}x(r)-T^{m-2}y(r)\right|+4\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\ \vdots\\
&\leq \mathcal{D}\left(\left|Tx(r)-Ty(r)\right|+2(m-1)\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(\left|\mathcal{K}\big(r,x(r)\big)-\mathcal{K}\big(r,y(r)\big)\right|+2(m-1)\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}+2\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(\left|x(r)-y(r)\right|+2m\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(\|x-y\|+2m\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right)\\
&\leq \mathcal{D}\left(d_0+2m\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\right),
\end{aligned}
\tag{18}
$$

i.e.,

$$
\left|T^m x(r)-T^m y(r)\right|\leq d_0+2m\frac{\mathfrak{P}}{\Gamma(\xi+1)}r^{q\xi}\quad[\text{since } 0<\mathcal{D}<1].
$$

The operator T^m on the ball $\mathfrak{D}_{b_0}$ is therefore continuous.

Next, we elect an arbitrary nonempty subset $\mathcal{G}$ of the ball $\mathfrak{D}_{b_0}$. We consider a constant number $d_0>0$. We take arbitrary numbers $r,r''\in[0,1]$ such that $|r-r''|\leq d_0$. It can be assumed that $r''<r$ without losing generality. So, for $x\in\mathcal{G}$ we obtain

$$
\begin{aligned}
&\left|T^m\big(T^m x(r)\big)-T^m\big(T^m x(r'')\big)\right|\\
&\quad=\left|T^{2m}x(r)-T^{2m}x\big(r''\big)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left| \begin{array}{c} \mathcal{D}\Big(\big|\mathcal{K}\big(r, T^{2m-1}x(r)\big)\big| + \int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \big|g\big(r,\hbar,T^{2m-1}x(\hbar)\big)\big|\, d\hbar \Big) \\ -\mathcal{D}\Big(\big|\mathcal{K}\big(r'', T^{2m-1}x(r'')\big)\big| + \int_0^{r''} \frac{((r'')^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \big|g\big(r'',\hbar,T^{2m-1}x(\hbar)\big)\big|\, d\hbar \Big) \end{array} \right| \\
&\leq \mathcal{D}\left(\begin{array}{c} |\mathcal{K}(r, T^{2m-1}x(r)) - \mathcal{K}(r'', T^{2m-1}x(r''))| \\ + \,|\int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r,\hbar,T^{2m-1}x(\hbar))|\, d\hbar \\ - \int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r'',\hbar,T^{2m-1}x(\hbar))|\, d\hbar| \\ + \,|\int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r'',\hbar,T^{2m-1}x(\hbar))|\, d\hbar \\ - \int_0^{r''} \frac{((r'')^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r'',\hbar,T^{2m-1}x(\hbar))|\, d\hbar| \end{array} \right) \\
&\leq \mathcal{D}\Big(\big|T^{2m-1}x(r) - T^{2m-1}x\big(r''\big)\big| + \frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + \Omega\big(r,r''\big) \Big) \\
&\leq \mathcal{D}\left(\left| \begin{array}{c} \mathcal{D}(|\mathcal{K}(r, T^{2m-2}x(r))| + \int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r,\hbar,T^{2m-2}x(\hbar))|\, d\hbar) \\ - \mathcal{D}(|\mathcal{K}(r'', T^{2m-2}x(r''))| + \int_0^{r''} \frac{((r'')^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} |g(r'',\hbar,T^{2m-2}x(\hbar))|\, d\hbar) \end{array} \right| \begin{array}{c} \\ \\ + \frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + \Omega(r,r'') \end{array} \right) \\
&\leq \mathcal{D}\left(\mathcal{D}\left(\left| \begin{array}{c} \big|\mathcal{K}\big(r, T^{2m-2}x(r)\big) - \mathcal{K}\big(r'', T^{2m-2}x\big(r''\big)\big)\big| \\ + \frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + \Omega\big(r,r''\big) \end{array} \right| \right) + \frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + \Omega\big(r,r''\big) \right) \\
&\leq \mathcal{D}\Big(\mathcal{D}\Big(\Big| \big|T^{2m-2}x(r) - T^{2m-2}x\big(r''\big)\big| + \frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + \Omega\big(r,r''\big) \Big| \Big) \\
&\quad + \frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + \Omega\big(r,r''\big) \Big) \\
&\leq \mathcal{D}\Big(\big|T^{2m-2}x(r) - T^{2m-2}x\big(r''\big)\big| + 2\frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + 2\Omega\big(r,r''\big) \Big) \\
&\vdots \\
&\leq \mathcal{D}\Big(\big|\mathcal{K}\big(r, Tx(r)\big) - \mathcal{K}\big(r'', Tx\big(r''\big)\big)\big| + (2m-1)\frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + (2m-1)\Omega\big(r,r''\big) \Big) \\
&\leq \mathcal{D}\Big(\big|\mathcal{K}\big(r, x(r)\big) - \mathcal{K}\big(r'', x\big(r''\big)\big)\big| + 2m\frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + 2m\Omega\big(r,r''\big) \Big) \\
&\leq \mathcal{D}\Big(\big|x(r) - x\big(r''\big)\big| + 2m\frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + 2m\Omega\big(r,r''\big) \Big) \\
&\leq \mathcal{D}\Big(\max\big\{\eta(x,d_0),\dots,\eta\big(T^{m-1}x,d_0\big)\big\} + 2m\frac{\eta_{b_0}(g,d_0)}{\Gamma(\xi+1)} r^{q\xi} + 2m\Omega\big(r,r''\big) \Big),
\end{aligned}
$$

where

$$
\eta_{b_0}(g,d_0) = \sup\big\{ \big|g(r,\hbar,x(\hbar)) - g\big(r'',\hbar,x(\hbar)\big)\big| : r, r'', \hbar \in [0,1], x \in [-b_0,b_0], \big|r - r''\big| \leq d_0 \big\}
$$

and

$$
\begin{aligned}
\Omega\big(r,r''\big) = \bigg| &\int_0^r \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \Big| g\big(r'',\hbar,T^{2m-1}x(\hbar)\big) \Big|\, d\hbar \\
&- \int_0^{r''} \frac{(r^q-\hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \Big| g\big(r'',\hbar,T^{2m-1}x(\hbar)\big) \Big|\, d\hbar \bigg|
\end{aligned}
$$

$$\begin{aligned}
&\leq \left| \int_0^r \frac{(r^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \left| g\left(r'', \hbar, T^{2m-1}x(\hbar)\right) \right| d\hbar \right. \\
&\quad - \int_0^r \frac{((r'')^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \left| g\left(r'', \hbar, T^{2m-1}x(\hbar)\right) \right| d\hbar \Bigg| \\
&\quad + \left| \int_{r''}^r \frac{((r'')^q - \hbar^q)^{\xi-1}}{\Gamma(\xi)} q\hbar^{q-1} \left| g\left(r'', \hbar, T^{2m-1}x(\hbar)\right) \right| d\hbar \right| \\
&\leq \left| \frac{\mathfrak{P}}{\Gamma(\xi)\xi} \left(\left[\left(r^q - \hbar^q\right)^{\xi} \right]_0^r - \left[\left(r''\right)^{q\xi} - \hbar^{q\xi} \right) \right]_0^r \right) \right| \\
&\quad + \left| -\frac{\mathfrak{P}}{\Gamma(\xi)\xi} \left(\left[\left((r'')^q - \hbar^q \right)^{\xi} \right]_{r''}^r \right) \right| \\
&\leq \frac{\mathfrak{P}}{\Gamma(\xi+1)} \left(\left(r^{q\xi} - (r'')^{q\xi} \right) + \left((r'')^q - r^q \right)^{\xi} \right) \\
&\quad + \frac{\mathfrak{P}}{\Gamma(\xi+1)} \left(\left((r'')^q - r^q \right)^{\xi} \right) \\
&\leq \frac{\mathfrak{P}}{\Gamma(\xi+1)} \left(\left(r^{q\xi} - (r'')^{q\xi} \right) + 2\left((r'')^q - r^q \right)^{\xi} \right).
\end{aligned}$$

So, we have

$$\begin{aligned}
&\left| T^m\left(T^m x(r)\right) - T^m\left(T^m x\left(r''\right)\right) \right| \\
&\quad \leq \mathcal{D}\left(\max\left\{ \eta(x, d_0), \ldots, \eta\left(T^{m-1}x, d_0\right) \right\} + 2m \frac{\eta_{b_0}(g, d_0)}{\Gamma(\xi+1)} r^{q\xi} + 2m\Omega\left(r, r''\right) \right).
\end{aligned} \tag{19}$$

As $d_0 \to 0$, one has $r \to r''$, $\eta_{b_0}(g, d_0) \to 0$ and $\Omega(r, r'') \to 0$, and so we get

$$\eta_0\left(T^m\left(T^m X\right)\right) \leq \lim_{d_0 \to 0} \mathcal{D}\left(\max\left\{\eta(X, d_0), \ldots, \eta\left(\Re^{m-1}X, d_0\right)\right\}\right),$$

i.e.,

$$\eta_0\left(T^m\left(T^m X\right)\right) \leq \mathcal{D}\left(\max\left\{\eta_0(X), \ldots, \eta_0\left(T^{m-1}X\right)\right\}\right). \tag{20}$$

So, from (16) and using Corollary 3.8, the result is obtained. □

Example 5.1 Let us define the functional integral equation, a special type of equation (17), as follows:

$$x(r) = T^2 x(r), \tag{21}$$

where

$$Tx(r) = \frac{1}{2}\left(\frac{x(r)+1}{7+r^2} + \int_0^r \frac{(r^3 - \hbar^3)^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})} \cos\left(\frac{r^2 x^2(\hbar)}{1+\hbar^2} \right) d\hbar \right).$$

In other words,

$$x(r) = T\left(Tx(r)\right) = \frac{1}{2}\left(\frac{T(x(r))+1}{7+r^2} + \int_0^r \frac{(r^3 - \hbar^3)^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})} \cos\left(\frac{r^2 T(x^2(\hbar))}{1+\hbar^2} \right) d\hbar \right),$$

where

$$Tx(r) = \frac{1}{2}\left(\frac{x(r)+1}{7+r^2} + \int_0^r \frac{(r^3-\hbar^3)^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})} \cos\left(\frac{r^2 x^2(\hbar)}{1+\hbar^2}\right) d\hbar\right).$$

In fact, we check the existence of the solution to the following integral equation:

$$x(r) = \frac{1}{2}\left(\begin{array}{c} \frac{\frac{1}{2}(\frac{x(r)+1}{7+r^2}+\int_0^r \frac{(r^3-\hbar^3)^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})}\cos(\frac{r^2 x(\hbar)^2}{1+\hbar^2})\,d\hbar)+1}{7+r^2} \\ + \int_0^r \frac{(r^3-\hbar^3)^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})}\cos(\frac{r^2 \frac{1}{4}(\frac{x(\hbar)+1}{7+\hbar^2}+\int_0^{\hbar} \frac{(\hbar^3-\varrho^3)^{-\frac{3}{4}}}{\Gamma(\frac{1}{4})}\cos(\frac{\hbar^2 x(\varrho)^2}{1+\varrho^2})\,d\varrho)^2}{1+\hbar^2})\,d\hbar \end{array}\right). \tag{22}$$

We have

$$\begin{aligned}
&\mathcal{K}\big(r,x(r)\big) = \frac{x(r)+1}{7+r^2}, \\
&g\big(r,\hbar,x(\hbar)\big) = \cos\left(\frac{r^2 x(\hbar)^2}{1+\hbar^2}\right), \\
&\mathcal{D} = \frac{1}{2}.
\end{aligned}$$

Now we examine the conditions of Theorem 5.1.

(i) $\mathcal{K}(r,x(r)) = \frac{x(r)+1}{7+r^2}$ is a continuous function such that

$$\big|\mathcal{K}\big(r,x(r)\big) - \mathcal{K}\big(r,y(r)\big)\big| = \left|\frac{x(r)+1}{7+r^2} - \frac{y(r)+1}{7+r^2}\right| \le \frac{|x-y|}{7} \le |x-y|$$

for each $r \in L$ and $x, y \in \mathbb{R}$.

Furthermore, $\mathcal{K}(r,0) = \frac{1}{7+r^2}$ is continuous and $|\mathcal{K}(r,0)| \le \frac{1}{7}$.

So, the function $r \to \mathcal{K}(r,0)$ is a member of $\mathbb{C}(L)$.

(ii) $g(r,\hbar,x(\hbar)) = \cos(\frac{r^2 x(\hbar)^2}{1+\hbar^2})$ is a continuous and bounded function, as

$$\left|\cos\left(\frac{r^2 x(\hbar)^2}{1+\hbar^2}\right)\right| \le 1 = \mathfrak{P}[\text{say}].$$

(iii) Now, we calculate the constant $\mathcal{Q}$:

$$\begin{aligned}
\mathcal{Q} &= \sup\left\{\big|\mathcal{K}(r,0)\big| + \frac{\mathfrak{P}}{\Gamma(\frac{1}{4}+1)} r^q : r \ge 0\right\} \\
&= \sup\left\{\frac{1}{7+r^2} + \frac{\mathfrak{P}}{\Gamma(\frac{5}{4})} r^q : r \ge 0\right\} \\
&= \frac{1}{7} + \frac{1}{\Gamma(\frac{5}{4})} \\
&= \frac{7+\Gamma(\frac{5}{4})}{7\Gamma(\frac{5}{4})},
\end{aligned}$$

where the inequality

$$\mathcal{D}(b_0 + \mathcal{Q}m) = \frac{1}{2}(b_0 + 2\left(\frac{7 + \Gamma(\frac{5}{4})}{7\Gamma(\frac{5}{4})}\right) \leq b_0$$

holds for every $b_0 \geq 2(\frac{7+\Gamma(\frac{5}{4})}{7\Gamma(\frac{5}{4})})$. Therefore, as a number b_0, we can catch $b_0 = 2(\frac{7+\Gamma(\frac{5}{4})}{7\Gamma(\frac{5}{4})})$.

Therefore, we draw the conclusion that the integral equation (5.1) has at least one solution, which is in the ball $\mathfrak{D}_{b_0}$ in $\mathbb{C}(L)$ in accordance with the Theorem 5.1.

6 Conclusion

In this article, we have established a new fixed point theorem with the help of newly defined condensing operators using a class of functions. This newly established theorem is a generalization of the Darbo's fixed point theorem. We have applied this theorem to find the existence of a solution of a fractional integral equation involving an operator with iterative relations in a Banach space, which is an extension of simple fractional integral equations. Finally, we have justified our findings with the help of a suitable example. So, our work is an extension of some previous generalizations in fixed point theory.

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