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Fixed point theorems in graphical cone metric spaces and application to a system of initial value problems

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Abstract

In this paper, we introduce the notion of graphical cone metric spaces over Banach algebra and prove some fixed point results for a particular type of contractive mappings defined on such spaces. These results extend and generalize several results from metric, graphical metric, and cone metric spaces. Some examples that demonstrate the results proved herein are provided. An application of our results to the existence of solution of a pair of initial value problems is provided.

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1 Introduction

The concept of metric spaces, which was of abstract nature and provided a tool to identify the distance between two points, was first initiated by Fréchet [7] in 1906. Banach [3] proved his famous result known as "Banach contraction principle" in metric spaces. Several generalizations of metric spaces and the results of Banach [3] have been considered so far (see, e.g., [1, 2, 6, 8–11, 13, 15, 18, 21]). Huang and Zhang [11] introduced the concept of cone metric spaces as a generalization of metric spaces. They showed that the class of contractive mappings in such spaces is wider than that in metric spaces. Liu and Xu [16] improved the notion of cone metric spaces and the fixed point results of Huang and Zhang [11] by introducing the cone metric spaces over Banach algebra. The approach of Liu and Xu [16] allows us to introduce the "contractive vectors" instead of "contractive scalars" in the conditions imposed on mappings. The benefit of using a contractive vector instead of a contractive scalar in contractive conditions is illustrated in [16]. Shukla et al. [24] worked on the notion of graphical metric spaces and generalized the concept of metric over sets involving a graphical structure. They proved a version of the Banach contraction principle in graphical metric spaces and showed that the fixed point results in graphical metric spaces can be applied to a wider class of contraction mappings in comparison to the usual metric case. They also showed the applicability of their results to the solution of integral equations. In a graphical metric space, the triangular inequality is weakened with the help of the underlying graph structure. Some generalizations of fixed point results of Shukla et al. [24] are considered in [22, 23]. In [22], some topological properties of graphical metric spaces were considered, and it was established that the topologies induced by a graphical metric are T_1 but not necessarily T_2 (i.e., Hausdorff), hence a graphical metric has a scope in the situations where non-Hausdorff topologies are useful.

As the cone metric spaces and graphical metric spaces have their own advantages over the usual metric spaces, in this paper, we initiate an amalgamation of these two concepts possessing the advantages of both the spaces, called the graphical cone metric spaces over Banach algebra. This notion generalizes and unifies the notions of metric, cone metric, and graphical metric spaces. Hence this generalized approach distinguishes our notion from both the notions of cone metric spaces and graphical metric spaces. Some fixed point results, which generalize several known fixed point results in this new setting, are also proved. The obtained fixed point results are applied to find the solution of a pair of initial value problems.

2 Preliminaries

Let $\mathfrak{B}$ be a real Banach algebra, i.e., $\mathfrak{B}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (see [19]): for all $x, y, z \in \mathfrak{B}$, $a \in \mathbb{R}$,

(1) $x(yz) = (xy)z$;
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(3) $a(xy) = (ax)y = x(ay)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

Throughout the paper, we assume that $\mathfrak{B}$ is a Banach algebra with a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in \mathfrak{B}$. An element $x \in \mathfrak{B}$ is said to be invertible if there is an inverse element $y \in \mathfrak{B}$ such that $xy = yx = e$. The inverse of $x \in \mathfrak{B}$ is denoted by x^{-1}. The spectral radius of an element $x \in \mathfrak{B}$ is denoted by $\rho(x)$ and

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}}.$$

Definition 1 (Liu and Xu [16]) Let $\mathfrak{B}$ be a Banach algebra with a unit e and a zero element θ. A nonempty closed subset C of $\mathfrak{B}$ is called a cone if the following conditions hold:

(I) $\{\theta, e\} \subset C$;
(II) if $\alpha_1, \alpha_2 \in [0, \infty)$, then $\alpha_1 C + \alpha_2 C \subseteq C$;
(III) $C^2 = CC \subseteq C$;
(IV) $C \cap (-C) = \{\theta\}$.

A cone C is called a solid cone if $C^\circ \neq \emptyset$, where C° stands for the interior of C. We always assume that the cone under consideration is solid. Every cone in a Banach algebra $\mathfrak{B}$ induces a partial order $\preceq$ on $\mathfrak{B}$ defined by $x \preceq y$ if and only if $y - x \in C$ for all $x, y \in \mathfrak{B}$. Also, we write $x \ll y$ if and only if $y - x \in C^\circ$ for all $x, y \in \mathfrak{B}$.

Remark 1 (Jungck et al. [13]) Let C be a cone in a Banach space $\mathfrak{B}$, and $a, b, c \in C$.

(i) If $a \preceq b$ and $b \ll c$, then $a \ll c$.
(ii) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(iii) If $\theta \preceq u \ll c$ for every $c \in C^\circ$, then $u = \theta$.

Definition 2 (Dordević et al. [4]) A sequence $\{u_n\}$ in a cone C is said to be a c-sequence if, for each $c \in \mathfrak{B}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n > n_0$.

Lemma 1 (Huang and Radenović [10]) *Let C be a solid cone in a Banach algebra $\mathfrak{B}$. Then*:

(i) *If the sequences $\{u_n\}$ and $\{v_n\}$ are two c-sequences in $\mathfrak{B}$ and $\alpha, \beta \in C$ are vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence in $\mathfrak{B}$.*

(ii) *If $h \in \mathfrak{B}$, $u_n = h^n$ and $\rho(h) < 1$, then $\{u_n\}$ is a c-sequence.*

Definition 3 (Liu and Xu [16]) Let X be a nonempty set and $\mathfrak{B}$ be a Banach algebra. A mapping $d : X \times X \to \mathfrak{B}$ is called a cone metric if it satisfies: for all $x, y, z \in X$,

(i) $\theta \preceq d(x,y)$;

(ii) $d(x,y) = \theta$ if and only if $x = y$;

(iii) $d(x,y) = d(y,x)$;

(iv) $d(x,y) \preceq d(x,z) + d(z,y)$.

In this case, the pair (X,d) is called a cone metric space over Banach algebra $\mathfrak{B}$.

We adapt some notions about graphs from Jachymski [12] and Shukla et al. [24].

If X is a nonempty set, then the diagonal of $X \times X$ is denoted by Δ and $\Delta = \{(x,x) : x \in X\}$. We denote by $\mathcal{G}$ a directed graph such that its set of vertices is $V(\mathcal{G}) = X$ and the set of edges $E(\mathcal{G})$ contains no parallel edges and $\Delta \subseteq E(\mathcal{G})$. In this case we say that X is endowed with the graph $\mathcal{G}$. By $\widetilde{\mathcal{G}}$ we denote a symmetric graph defined by $\widetilde{\mathcal{G}} = E(\mathcal{G}) \cup E(\mathcal{G}^{-1})$, where $\mathcal{G}^{-1}$ is defined by $V(\mathcal{G}^{-1}) = X$ and $E(\mathcal{G}^{-1}) = \{(x,y) : (y,x) \in E(\mathcal{G})\}$.

Let $x, y \in V(\mathcal{G})$. A sequence of vertices $\{x_i\}_{i=0}^{n}$ in $V(\mathcal{G})$ is called a path of length $n+1$ from x to y if $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E(\mathcal{G})$ for $i = 0, 1, \dots, n$. The vertices x and y are called connected if there is a path of some length from x to y. The graph $\mathcal{G}$ is called connected if there is a path between any two vertices of $\mathcal{G}$.

Let us define $[x]_{\mathcal{G}}^{l} = \{y \in X :$ there is a path directing from x to y having length $l\}$ for $l \in \mathbb{N}$ and a relation P on X by: $(xPy)_{\mathcal{G}}$ if and only if there is a directed path from x to y in $\mathcal{G}$. We write $w \in (xPy)_{\mathcal{G}}$ if w is contained in some directed path from x to y in $\mathcal{G}$. A sequence $\{x_n\}$ in X is called $\mathcal{G}$-termwise connected if for all $n \in \mathbb{N}$ we have $(x_n P x_{n+1})_{\mathcal{G}}$.

Definition 4 (Shukla et al. [24]) Let X be a nonempty set endowed with a graph $\mathcal{G}$ and $d_{\mathcal{G}} : X \times X \to \mathbb{R}$ be a function satisfying the following conditions:

(GM1) $d_{\mathcal{G}}(x,y) \geq 0$ for all $x, y \in X$;

(GM2) $d_{\mathcal{G}}(x,y) = 0$ if and only if $x = y$;

(GM3) $d_{\mathcal{G}}(x,y) = d_{\mathcal{G}}(y,x)$ for all $x, y \in X$;

(GM4) $(xPy)_{\mathcal{G}}$, $w \in (xPy)_{\mathcal{G}}$ implies $d_{\mathcal{G}}(x,y) \leq d_{\mathcal{G}}(x,w) + d_{\mathcal{G}}(w,y)$ for all $x, y, w \in X$.

Then the mapping $d_{\mathcal{G}}$ is called a graphical metric on X, and the pair $(X, d_{\mathcal{G}})$ is called a graphical metric space.

For examples and various interesting properties of a graphical metric space, the reader is referred to Shukla et al. [24] and Shukla and Künzi [22].

We next introduce the notion of graphical cone metric spaces.

3 Graphical cone metric spaces

Definition 5 Let X be a nonempty set endowed with a graph $\mathcal{G}$ and $\mathfrak{B}$ be a Banach algebra. Suppose that a mapping $d_{\mathcal{G}_c} : X \times X \to \mathfrak{B}$ satisfies the following conditions:

(GCM1) $d_{\mathcal{G}_c}(x,y) \succeq \theta$ for all $x,y \in X$;
(GCM2) $d_{\mathcal{G}_c}(x,y) = \theta$ if and only if $x = y$ for all $x,y \in X$;
(GCM3) $d_{\mathcal{G}_c}(x,y) = d_{\mathcal{G}_c}(y,x)$ for all $x,y \in X$;
(GCM4) $(xPy)_{\mathcal{G}}$, $w \in (xPy)_{\mathcal{G}}$ implies $d_{\mathcal{G}_c}(x,y) \preceq d_{\mathcal{G}_c}(x,w) + d_{\mathcal{G}_c}(w,y)$ for all $x,y,w \in X$.

Then the mapping $d_{\mathcal{G}_c}$ is called a graphical cone metric and the pair $(X, d_{\mathcal{G}_c})$ is called a graphical cone metric space over Banach algebra $\mathfrak{B}$.

Example 1 Every cone metric space (X,d) is a graphical cone metric space $(X, d_{\mathcal{G}_c})$, where $d_{\mathcal{G}_c} \equiv d$ and $\mathcal{G} = X \times X$ is the universal graph.

Example 2 Let $\mathfrak{B} = \mathbb{R}^2$ with the Euclidian norm and the coordinate-wise multiplication. Define $C = \{(a,b) \in \mathfrak{B} \colon a,b \geq 0\}$, then C is a solid cone in $\mathfrak{B}$. Let $X = [0,\infty)$ and $\mathcal{G}$ be defined by $V(\mathcal{G}) = X$ and $E(\mathcal{G}) = \Delta \cup \{(x,y) \in X \times X \colon 0 < y \leq x\}$. Define a function $d_{\mathcal{G}_c} \colon X \times X \to \mathbb{R}^2$ by

$$d_{\mathcal{G}_c}(x,y) = \begin{cases} (0,0), & \text{if } x = y; \\ \min\{x,y\}(1,a), & \text{if } x \neq y \text{ and } x,y \in X \setminus \{0\}; \\ \max\{x,y\}(1,b), & \text{otherwise}, \end{cases}$$

where $a,b > 0$ are fixed numbers. Then $(X, d_{\mathcal{G}_c})$ is a graphical cone metric space over Banach algebra $\mathfrak{B}$. Obviously, $(X, d_{\mathcal{G}_c})$ is not a cone metric space.

Example 3 Let $\mathfrak{B} = C^1_{\mathbb{R}}[0,1]$ with the norm $\|\cdot\|$ given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for all $f \in C^1_{\mathbb{R}}[0,1]$ and the multiplication be the point-wise multiplication. Define $C = \{f \in \mathfrak{B} \colon f(t) \geq 0 \text{ for all } t \in [0,1]\}$, then C is a solid cone in $\mathfrak{B}$. Let $X = [0,\infty)$, $\mathcal{G}$ be a graph given by $V(\mathcal{G}) = X$ and $E(\mathcal{G}) = \{(x,y) \in X \times X \colon x \leq y\}$ and define $d_{\mathcal{G}_c} \colon X \times X \to \mathfrak{B}$ by

$$d_{\mathcal{G}_c}(x,y)(t) = \begin{cases} 0, & \text{if } x = y; \\ e^{xyt}, & \text{otherwise}. \end{cases}$$

Then we show that $(X, d_{\mathcal{G}_c})$ is a graphical cone metric space over Banach algebra $\mathfrak{B}$. The properties (GCM1), (GCM2), and (GCM3) are obvious. For (GCM4), suppose $x,y,z \in X$ and $z \in (xPy)_{\mathcal{G}}$, i.e., $0 \leq x \leq z \leq y$. Then we must have $(xyt)^n \leq (xzt)^n + (zyt)^n$ for all $n \in \mathbb{N}$, hence for all $t \in [0,1]$ we have

$$\sum_{n=0}^{\infty} \frac{(xyt)^n}{n!} \leq \sum_{n=0}^{\infty} \frac{(xzt)^n + (yzt)^n}{n!}.$$

This shows that $d_{\mathcal{G}_c}(x,y) \preceq d_{\mathcal{G}_c}(x,z) + d_{\mathcal{G}_c}(z,y)$, hence (GCM4) holds. On the other hand, $(X, d_{\mathcal{G}_c})$ is not a cone metric space as $d_{\mathcal{G}_c}(1,2) \npreceq d_{\mathcal{G}_c}(1,0) + d_{\mathcal{G}_c}(0,2)$.

We next give some propositions with the help of which one can construct several more examples of graphical cone metric spaces.

Proposition 1 *Let $(X, d_{\mathcal{G}})$ be a graphical metric space and $\mathfrak{B}$ be a Banach algebra with cone C. Then $(X, d_{\mathcal{G}_c})$ is a graphical cone metric space over Banach algebra $\mathfrak{B}$, where $d_{\mathcal{G}_c} \colon X \times X \to \mathfrak{B}$ is defined by $d_{\mathcal{G}_c}(x,y) = d_{\mathcal{G}}(x,y) \cdot p$, where $p \in C \setminus \{\theta\}$ is a fixed vector.*

Proof It follows from the definition of cone C. □

Proposition 2 *Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space. Then $(X, d'_{\mathcal{G}_c})$ is a graphical cone metric space, where $d'_{\mathcal{G}_c} : X \times X \to \mathfrak{B}$ is defined by $d'_{\mathcal{G}_c}(x, y) = d_{\mathcal{G}_c}(x, y) \cdot p$, where $p \in C \setminus \{\theta\}$ is a fixed vector such that $\rho(e - p) < 1$.*

Proof It follows from the fact that p is invertible and $C^2 \subseteq C$. □

In the further discussion, unless specified otherwise, whenever we consider the Euclidean Banach algebra $\mathbb{R}^2$, it is assumed with cone $C = \{(a, b) \in \mathbb{R}^2 : a, b \geq 0\}$, the Euclidean norm, and the coordinate-wise multiplication.

The proof of the following propositions can be established directly from the definitions.

Proposition 3 *Let $(A, d_{\mathcal{G}_1})$ and $(B, d_{\mathcal{G}_2})$ be two graphical metric spaces. Then $(X, d_{\mathcal{G}_c})$ is a graphical cone metric space over the Euclidean Banach algebra $\mathfrak{B} = \mathbb{R}^2$, where $X = A \times B$, $d_{\mathcal{G}_c} : X \times X \to \mathfrak{B}$ is defined by*

$$d_{\mathcal{G}_c}\big((a_1, b_1), (a_2, b_2)\big) = \big(d_{\mathcal{G}_1}(a_1, a_2), d_{\mathcal{G}_2}(b_1, b_2)\big) \quad \textit{for all } (a_1, b_1), (a_2, b_2) \in X,$$

and $\mathcal{G}$ is the graph defined by

$$V(\mathcal{G}) = X, E(\mathcal{G}) = \big\{\big((a_1, b_1), (a_2, b_2)\big) \in X \times X : (a_1, a_2) \in E(\mathcal{G}_1), (b_1, b_2) \in E(\mathcal{G}_2)\big\}.$$

Proposition 4 *Let $(X, d_{\mathcal{G}_1})$ and $(X, d_{\mathcal{G}_2})$ be two graphical metric spaces. Let $\mathfrak{B} = \mathbb{R}^2$ be the Euclidean Banach algebra. Define a graph $\mathcal{G}$ by $V(\mathcal{G}) = X$ and $E(\mathcal{G}) = E(\mathcal{G}_1) \cap E(\mathcal{G}_2)$ and the function $d_{\mathcal{G}_c} : X \times X \to \mathfrak{B}$ by*

$$d_{\mathcal{G}_c}(x, y) = \big(d_{\mathcal{G}_1}(x, y), d_{\mathcal{G}_2}(x, y)\big) \quad \textit{for all } x, y \in X.$$

Then $(X, d_{\mathcal{G}_c})$ is a graphical cone metric space over Banach algebra $\mathfrak{B}$.

Suppose that $x \in X$ and $c \in C^\circ$. Then the open ball with center x and radius c is denoted by $B_{\mathcal{G}}(x, c)$, and it is defined by

$$B_{\mathcal{G}}(x, c) = \big\{y \in X : (xPy)_{\mathcal{G}}, d_{\mathcal{G}_c}(x, y) \ll c\big\}.$$

Since $\Delta \subseteq E(\mathcal{G})$, every open ball is nonempty. Moreover, we prove the following.

Theorem 2 *Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space over Banach algebra $\mathfrak{B}$, and let*

$$\mathfrak{U} = \big\{U \subseteq X : \textit{for all } x \in U \textit{ there is } c \in C^\circ \textit{ such that } B_{\mathcal{G}}(x, c) \subseteq U\big\}.$$

Then $\mathfrak{U}$ defines a topology on X.

Proof It is clear that $X, \emptyset \in \mathfrak{U}$, hence $\mathfrak{U} \neq \emptyset$. Suppose $U_1, U_2 \in \mathfrak{U}$. Let $x \in U_1 \cap U_2$, then there exist $c_1, c_2 \in C^\circ$ such that $B_{\mathcal{G}}(x, c_1) \subseteq U_1$ and $B_{\mathcal{G}}(x, c_2) \subseteq U_2$. Since $c_1, c_2 \in C^\circ$, there exists $c \in C^\circ$ such that $c \ll c_1$ and $c \ll c_2$, and so $B_{\mathcal{G}}(x, c) \subseteq B_{\mathcal{G}}(x, c_1) \cap B_{\mathcal{G}}(x, c_2) \subseteq U_1 \cap U_2$.

Hence, $U_1 \cap U_2 \in \mathfrak{U}$. Similarly, one can show that the union of any collection of elements of $\mathfrak{U}$ is again in $\mathfrak{U}$. □

It is easy to see that every open ball is an open set in the topology $\mathfrak{U}$ and the collection $\mathcal{B} = \{B_{\mathcal{G}}(x,c) : x \in X, c \in C^\circ\}$ is a basis for the topology defined by $\mathfrak{U}$. The topology defined by $\mathfrak{U}$ is called the topology induced by the graphical cone metric $d_{\mathcal{G}_c}$.

We next show that the topology $\mathfrak{U}$ is T_1.

Proposition 5 *Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space over Banach algebra $\mathfrak{B}$. Then the induced topology $\mathfrak{U}$ is T_1.*

Proof We shall show that the set $X \setminus \{x\}$ is open for all $x \in X$, i.e., $X \setminus \{x\} \in \mathfrak{U}$ for all $x \in X$. Then, let $x \in X$ and $y \in X \setminus \{x\}$, then $d_{\mathcal{G}_c}(x,y) \neq \theta$ and $d_{\mathcal{G}_c}(x,y) \in C$. Hence, there exists $c \in C^\circ$ such that $\frac{c - d_{\mathcal{G}_c}(x,y)}{n} \in C^\circ$ for all $n \in \mathbb{N}$. Let $c_n = \frac{c - d_{\mathcal{G}_c}(x,y)}{n}$ and consider the open balls $B_{\mathcal{G}}(y, c_n)$, $n \in \mathbb{N}$. We claim that there exists $n_0 \in \mathbb{N}$ such that $x \notin B_{\mathcal{G}}(y, c_{n_0})$. On the contrary, suppose that $x \in B_{\mathcal{G}}(y, c_n)$ for all $n \in \mathbb{N}$, Then we have $d_{\mathcal{G}_c}(x,y) \ll c_n$, i.e., $\frac{c - d_{\mathcal{G}_c}(x,y)}{n} - d_{\mathcal{G}_c}(x,y) \in C^\circ$ for all $n \in \mathbb{N}$. Since C is closed, we obtain $-d_{\mathcal{G}_c}(x,y) \in C$, and by the definition of C we have $d_{\mathcal{G}_c}(x,y) = \theta$. This contradiction shows that there exists $n_0 \in \mathbb{N}$ such that $x \notin B_{\mathcal{G}}(y, c_{n_0})$, and so $B_{\mathcal{G}}(y, c_{n_0}) \subseteq X \setminus \{x\}$. This proves the result. □

In the next example, we show that in general the topology $\mathfrak{U}$ is not T_2.

Example 4 Let $(X, d_{\mathcal{G}_c})$ be the graphical cone metric space as we have considered in Example 2. Consider the points $x = 1$, $y = 2$ in X. Note that $B_{\mathcal{G}}(x, c_1) \cap B_{\mathcal{G}}(y, c_2) \neq \emptyset$ for all $c_1, c_2 \in C^\circ$. Hence, the induced topology $\mathfrak{U}$ is not T_2.

We now define the convergent and Cauchy sequences in a graphical cone metric space and compare their properties in contrast with cone metric spaces.

Definition 6 Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space over a Banach algebra $\mathfrak{B}$, $x \in X$, and $\{x_n\}$ be a sequence in X. Then:

- The sequence $\{x_n\}$ is said to be convergent to $x \in X$ with respect to $d_{\mathcal{G}_c}$ if for every $c \in C^\circ$ there exists $n_0 \in \mathbb{N}$ such that $d_{\mathcal{G}_c}(x_n, x) \ll c$ for all $n > n_0$. In this case, x is called a limit of $\{x_n\}$ with respect to $d_{\mathcal{G}_c}$. We denote this fact by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
- The sequence $\{x_n\}$ is said to be convergent to $x \in X$ with respect to the induced topology $\mathfrak{U}$ if for every $c \in C^\circ$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_{\mathcal{G}}(x, c)$ for all $n > n_0$. In this case, x is called a limit of $\{x_n\}$ with respect to induced topology $\mathfrak{U}$.
- The sequence $\{x_n\}$ is said to be a Cauchy sequence if for every $c \in C^\circ$ there exists $n_0 \in \mathbb{N}$ such that $d_{\mathcal{G}_c}(x_n, x_m) \ll c$ for all $n, m > n_0$.

Example 5 Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space as we have considered in Example 2. Consider the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then, for every $x \in X$ and $c \in C^\circ$, we can obtain $n_0 \in \mathbb{N}$ such that $d_{\mathcal{G}_c}(x_n, x) \ll c$ for all $n > n_0$. Therefore, $\{x_n\}$ converges to x with respect to $d_{\mathcal{G}_c}$ for all $x \in X$.

Remark 2 In cone metric spaces, the limit of a convergent sequence is unique and the convergence with respect to induced topology and the convergence with respect to cone metric are equivalent. The above example shows that in a graphical cone metric space the limit of a convergent sequence may not be unique. In graphical cone metric spaces the convergence with respect to $\mathfrak{U}$ implies the convergence with respect to $d_{\mathcal{G}_c}$, but the above example shows that the convergence of a sequence with respect to $d_{\mathcal{G}_c}$ does not imply the convergence with respect to $\mathfrak{U}$. Indeed, $0 \in X$ is a limit of the sequence $\{x_n\}$ with respect to $d_{\mathcal{G}_c}$, but it is not a limit of $\{x_n\}$ with respect to the induced topology $\mathfrak{U}$. Indeed, $B_{\mathcal{G}}(0, c) = \{0\}$ for all $c \in C^\circ$.

In the remaining part of the paper, since our aim is to generalize and utilize the graphical cone metric analogue of metric fixed point results, we will consider only the convergence with respect to $d_{\mathcal{G}_c}$, and for the sake of convenience, we will write "convergence" instead of "convergence with respect to $d_{\mathcal{G}_c}$".

In cone metric spaces, every convergent sequence is a Cauchy sequence. But it is not the case for graphical cone metric spaces.

Example 6 Let $X = \mathbb{R}$ and $\mathfrak{B} = \mathbb{R}^2$ be the Euclidean Banach algebra. Let $A = (-\infty, 0)$, $B = (0, \infty)$, and $\overline{A} = (-\infty, 0]$, $\overline{B} = [0, \infty)$. Define a graph $\mathcal{G}$ by $V(\mathcal{G}) = X$ and $E(\mathcal{G}) = (A \times A) \cup (B \times B)$ and the function $d_{\mathcal{G}_c} : X \times X \to \mathfrak{B}$ by

$$d_{\mathcal{G}_c}(x, y) = \begin{cases} (0, 0), & \text{if } x = y; \\ |x - y|(1, 1), & \text{if } (x, y) \in (\overline{A} \times \overline{A}) \cup (\overline{B} \times \overline{B}); \\ (1, 1), & \text{otherwise.} \end{cases}$$

Then $(X, d_{\mathcal{G}_c})$ is a graphical cone metric space over Banach algebra $\mathfrak{B}$. Consider the sequence $\{x_n\}$ in X defined by $x_n = \frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$. Then it is easy to see that $x_n \to 0$ as $n \to \infty$. On the other hand, $\{x_n\}$ is not a Cauchy sequence. Indeed, $d_{\mathcal{G}_c}(x_n, x_{n+1}) = (1, 1)$ for all $n \in \mathbb{N}$, and so it is not possible to choose $n_0 \in \mathbb{N}$ such that for every given $c \in C^\circ$ we have $d_{\mathcal{G}_c}(x_n, x_{n+1}) \ll c$ for all $n > n_0$.

Definition 7 Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space over Banach algebra $\mathfrak{B}$. Then X is called complete if every Cauchy sequence in X is convergent to some $x \in X$. Let $\mathcal{H}$ be a graph such that $V(\mathcal{H}) \subseteq X$. Then X is called $\mathcal{H}$-complete if every $\mathcal{H}$-termwise connected Cauchy sequence in X is convergent to some $x \in X$.

Obviously, $\mathcal{H}$-completeness is a weaker assumption than the completeness of a graphical cone metric space.

In the next section, we state some fixed point results for a self-mapping of a graphical cone metric space satisfying some particular conditions.

4 Fixed point theorems

We first introduce the notion of $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contractions on graphical cone metric spaces.

Definition 8 Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space over Banach algebra $\mathfrak{B}$, $T : X \to X$ be a mapping, and $\hat{\mathcal{G}}$ be a subgraph of $\mathcal{G}$ such that $E(\hat{\mathcal{G}}) \supseteq \Delta$. Then T is called a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction with contractive vector α if:

(GCC1) T preserves the edges of $\hat{\mathcal{G}}$, i.e., for all $(z_1, z_2) \in E(\hat{\mathcal{G}})$, we have $(Tz_1, Tz_2) \in E(\hat{\mathcal{G}})$;
(GCC2) There exists $\alpha \in C$ such that $\rho(\alpha) < 1$ and $d_{\mathcal{G}_c}(Tz_1, Tz_2) \preceq \alpha d_{\mathcal{G}_c}(z_1, z_2)$ for all $z_1, z_2 \in X$ with $(z_1, z_2) \in E(\hat{\mathcal{G}})$.

The vector α is called the contractive vector of T. Note that we can treat $\hat{\mathcal{G}}$ as a weighted graph with vector weight, and the corresponding vector weight of each edge (z_1, z_2) in $E(\hat{\mathcal{G}})$ is given by $d_{\mathcal{G}_c}(z_1, z_2)$. Therefore, condition (GCC2) shows that a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction reduces the vector weight in the sense that $\rho(\alpha) < 1$. A sequence $\{z_n\}$ is said to be a T-Picard sequence (or Picard sequence generated by T) with the initial value $z_0 \in X$ if $z_n = Tz_{n-1}$ for all $n \in \mathbb{N}$.

In the rest of this paper, we assume $\hat{\mathcal{G}}$ to be a subgraph of $\mathcal{G}$ such that $E(\hat{\mathcal{G}}) \supseteq \Delta$.

Example 7 Since $E(\hat{\mathcal{G}}) \supseteq \Delta$, the constant mapping on any arbitrary graphical cone metric space is a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction with contractive vector $\alpha \in C$ such that $\rho(\alpha) < 1$.

Example 8 The identity mapping on any arbitrary graphical cone metric space can be converted into a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction by assuming $E(\hat{\mathcal{G}}) = \Delta$ and the contractive vector $\alpha \in C$ such that $\rho(\alpha) < 1$. Indeed, this fact is true for any arbitrary mapping defined on a graphical cone metric space.

The above example shows that any arbitrary mapping on a graphical cone metric space over Banach algebra $\mathfrak{B}$ can be converted into a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction. Obviously, the Picard sequences generated by such a mapping T may not converge to any point in space. As the convergence of T-Picard sequence is an important issue in existence theorems, in the next theorem, we ensure the convergence of T-Picard sequences generated by a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction under some particular conditions.

Theorem 3 *Let $(X, d_{\mathcal{G}_c})$ be a $\hat{\mathcal{G}}$-complete graphical cone metric space over Banach algebra $\mathfrak{B}$ and $T: X \to X$ be a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction. Suppose that the following conditions hold*:

(A) *There exists $z_0 \in X$ such that $Tz_0 \in [z_0]^l_{\hat{\mathcal{G}}}$ for some $l \in \mathbb{N}$;*
(B) *If a $\hat{\mathcal{G}}$-termwise connected T-Picard sequence $\{z_n\}$ converges in X, then there exist a limit $w \in X$ of $\{z_n\}$ and $n_0 \in \mathbb{N}$ such that $(z_n, w) \in E(\hat{\mathcal{G}})$ or $(w, z_n) \in E(\hat{\mathcal{G}})$ for all $n > n_0$.*

Then there exists $z^* \in X$ such that the T-Picard sequence $\{z_n\}$ with the initial value $z_0 \in X$ is $\hat{\mathcal{G}}$-termwise connected and converges to both z^* and Tz^*.

Proof Assume that $z_0 \in X$ is such that $Tz_0 \in [z_0]^l_{\hat{\mathcal{G}}}$ for some $l \in \mathbb{N}$, and $\{z_n\}$ is the T-Picard sequence with the initial value z_0. Then there is a path $\{y_k\}_{k=0}^l$ such that $z_0 = y_0$, $z_1 = Tz_0 = y_l$, and $(y_{k-1}, y_k) \in E(\hat{\mathcal{G}})$ for $k = 1, 2, 3, \ldots, l$. As T is a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction, it preserves the edges, and we have $(Ty_{k-1}, Ty_k) \in E(\hat{\mathcal{G}})$ for $k = 1, 2, 3, \ldots, l$. This yields a path $\{Ty_k\}_{k=0}^l$ from $Ty_0 = Tz_0 = z_1$ to $Ty_l = T(Tz_0) = T^2 z_0 = z_2$ having length l, and so $z_2 \in [z_1]^l_{\hat{\mathcal{G}}}$. Continuation of this process yields a path $\{T^n y_k\}_{k=0}^l$ from $T^n y_0 = T^n z_0 = z_n$ to $T^n y_l = T^n Tz_0 = z_{n+1}$ of length l, and hence $z_{n+1} \in [z_n]^l_{\hat{\mathcal{G}}}$ for all $n \in \mathbb{N}$. Thus, we obtain that $\{z_n\}$ is a $\hat{\mathcal{G}}$-termwise connected sequence. Since $(T^n y_{k-1}, T^n y_k) \in E(\hat{\mathcal{G}})$ for $k = 1, 2, 3, \ldots, l$ and for all $n \in \mathbb{N}$, by contractive condition (GCC2) we have

$$d_{\mathcal{G}_c}\big(T^n y_{k-1}, T^n y_k\big) \preceq \alpha d_{\mathcal{G}_c}\big(T^{n-1} y_{k-1}, T^{n-1} y_k\big) \preceq \cdots \preceq \alpha^n d_{\mathcal{G}_c}(y_{k-1}, y_k). \tag{1}$$

Since $\hat{\mathcal{G}}$ is a subgraph of $\mathcal{G}$ and $(T^n y_{k-1}, T^n y_k) \in E(\hat{\mathcal{G}})$ for $k = 1, 2, 3, \dots, l$, for all $n \in \mathbb{N}$, using (1) and (GCM4) we obtain

$$\begin{aligned} d_{\mathcal{G}_c}(z_n, z_{n+1}) &= d_{\mathcal{G}_c}\big(T^n z_0, T^{n+1} z_0\big) \\ &= d_{\mathcal{G}_c}\big(T^n y_0, T^n y_l\big) \\ &\preceq \sum_{k=1}^{l} d_{\mathcal{G}_c}\big(T^n y_{k-1}, T^n y_k\big) \\ &\preceq \sum_{k=1}^{l} \alpha^n d_{\mathcal{G}_c}(y_{k-1}, y_k) \\ &= \alpha^n \delta_l, \end{aligned}$$

where $\delta_l = \sum_{k=1}^{l} d_{\mathcal{G}_c}(y_{k-1}, y_k)$. As the sequence $\{z_n\}$ is a $\hat{\mathcal{G}}$-termwise connected sequence, for any pair n, m of positive integers with $m > n$ we obtain

$$\begin{aligned} d_{\mathcal{G}_c}(z_n, z_m) &\preceq \sum_{k=n}^{m-1} d_{\mathcal{G}_c}(z_k, z_{k+1}) \\ &= \sum_{k=n}^{m-1} \alpha^k \delta_l \\ &= \alpha^n \Bigg[\sum_{k=n}^{m-1} \alpha^{k-n}\Bigg] \delta_l \\ &\preceq \alpha^n (e - \alpha)^{-1} \delta_l. \end{aligned}$$

Since $\rho(\alpha) < 1$, by Lemma 1 the sequence $\{\alpha^n (e - \alpha)^{-1} \delta_l\}$ is a c-sequence. Hence, using the above inequality, for every $c \in \mathfrak{B}$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d_{\mathcal{G}_c}(z_n, z_m) \preceq \alpha^n (e - \alpha)^{-1} \delta_l \ll c \quad \text{for all } n > n_0.$$

Therefore, $\{z_n\}$ is a Cauchy sequence in X. Since X is $\hat{\mathcal{G}}$-complete, the sequence $\{z_n\}$ converges in X, and from condition (B) there exists $z^* \in X$, $n_1 \in \mathbb{N}$ such that $(z_n, z^*) \in E(\hat{\mathcal{G}})$ or $(z^*, z_n) \in E(\hat{\mathcal{G}})$ for all $n > n_1$ and $z_n \to z^*$ as $n \to \infty$.

Now, if $(z_n, z^*) \in E(\hat{\mathcal{G}})$ for all $n > n_1$, using (GCC2) we obtain

$$\begin{aligned} d_{\mathcal{G}_c}\big(z_{n+1}, Tz^*\big) &= d_{\mathcal{G}_c}\big(Tz_n, Tz^*\big) \\ &\preceq \alpha d_{\mathcal{G}_c}\big(z_n, z^*\big) \end{aligned}$$

for all $n > n_1$. Since $z_n \to z^*$ as $n \to \infty$, the sequence $\{d_{\mathcal{G}_c}(z_n, z^*)\}$ is a c-sequence, and by Lemma 1 the sequence $\{\alpha d_{\mathcal{G}_c}(z_n, z^*)\}$ is a c-sequence. By Remark 1 we obtain $\{d_{\mathcal{G}_c}(z_{n+1}, Tz^*)\}$ is a c-sequence, i.e., $z_n \to Tz^*$ as $n \to \infty$. If $(z^*, z_n) \in E(\hat{\mathcal{G}})$, then a similar result holds.

Thus, the sequence $\{z_n\}$ converges to both z^* and Tz^*. □

We have established the convergence of a Picard sequence generated by a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction under the conditions of the above theorem. Apart from the cone metric case, the limit of this convergent Picard sequence may not be a fixed point of $(\mathcal{G},\hat{\mathcal{G}})$-graphical contraction, as shown in the following example.

Example 9 Let $\mathfrak{B} = \mathbb{R}^2$ with the Euclidian norm and the coordinate-wise multiplication. Define $C = \{(a,b) \in \mathfrak{B} : a,b \geq 0\}$, then C is a solid cone in $\mathfrak{B}$. Let $X = \mathbb{R}$, $\mathcal{G}$ and $\hat{\mathcal{G}}$ be the graphs defined by $\mathcal{G} = \hat{\mathcal{G}}$ and $V(\mathcal{G}) = X$, and $E(\mathcal{G}) = \{(x,y) \in (X \setminus \mathbb{Q}) \times (X \setminus \mathbb{Q}) : 0 < y < x\} \cup \Delta$ and the function $d_{\mathcal{G}_c} : X \times X \to \mathfrak{B}$ by

$$d_{\mathcal{G}_c}(x,y) = \begin{cases} (0,0), & \text{if } x = y; \\ \min\{|x|,|y|\}(1,1), & \text{if } x,y \in X \setminus \{0\}, x \neq y; \\ \max\{|x|,|y|\}(1,1), & \text{otherwise.} \end{cases}$$

Then $d_{\mathcal{G}_c}$ is a graphical cone metric on X and $(X, d_{\mathcal{G}_c})$ is a $\hat{\mathcal{G}}$-complete graphical cone metric space over Banach algebra $\mathfrak{B}$. Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 1 + x, & \text{if } x \in \mathbb{Q}; \\ \frac{x}{2}, & \text{if } x \in X \setminus \mathbb{Q}. \end{cases}$$

Then T is a $(\mathcal{G},\hat{\mathcal{G}})$-graphical contraction with $\alpha = (\frac{1}{2},\frac{1}{2})$ and $\rho(\alpha) = \frac{1}{2}$. Also, for each $z_0 \in X \setminus \mathbb{Q}$, we have $(z_0, Tz_0) \in E(\hat{\mathcal{G}})$, i.e., $Tz_0 \in [z_0]^1_{\hat{\mathcal{G}}}$. Any $\hat{\mathcal{G}}$-termwise connected T-Picard sequence in X is a positive decreasing sequence that converges to 0 with respect to the usual metric of $\mathbb{R}$, therefore, has at least one limit $w \in X$ such that condition (B) of Theorem 3 holds true (indeed, every $x > 0$ is a limit of such a sequence with respect to $d_{\mathcal{G}_c}$). Note that, for each positive $z_0 \in X \setminus \mathbb{Q}$, the Picard sequence with the initial value z_0 converges to some z^* as well as to Tz^* although T has no fixed point.

To ensure the existence of a fixed point of a $(\mathcal{G},\hat{\mathcal{G}})$-graphical contraction, we introduce the following condition.

Definition 9 Let $(X, d_{\mathcal{G}_c})$ be a graphical cone metric space over Banach algebra $\mathfrak{B}$ and $T : X \to X$ be a mapping. We say that the quadruple $(X, d_{\mathcal{G}_c}, \hat{\mathcal{G}}, T)$ has property (N) if:

$$\text{whenever a } \hat{\mathcal{G}}\text{-termwise connected } T\text{-Picard sequence } \{z_n\} \text{ has two limits } z^* \text{ and } y^*, \\ \text{where } z^* \in X, y^* \in T(X), \text{then } z^* = y^*. \tag{N}$$

The set of all fixed points of T will be denoted by $\mathcal{F}(T)$. We use the notation

$$X_T^{\hat{\mathcal{G}}} = \{z \in X : (z, Tz) \in E(\hat{\mathcal{G}})\}.$$

Theorem 4 *If all the conditions of Theorem 3 are satisfied and, in addition, if the quadruple $(X, d_{\mathcal{G}_c}, \hat{\mathcal{G}}, T)$ has property (N), then T has a fixed point in X.*

Proof Theorem 3 ensures that the T-Picard sequence $\{z_n\}$ with the initial value z_0 converges to both z^* and Tz^*. Since $z^* \in X$ and $Tz^* \in T(X)$, by property (N), we must have $Tz^* = z^*$. Thus, z^* is a fixed point of T. □

Example 10 Let $\mathfrak{B} = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$, and multiplication is defined by $(x_1, x_2).(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$. Define $C = \{(x_1, x_2) \colon x_1, x_2 \geq 0\}$, then C is a solid cone in $\mathfrak{B}$. Let $X = [0,1] \times [0,1]$, $\mathcal{G}$ and $\hat{\mathcal{G}}$ be the graphs defined by $\mathcal{G} = \hat{\mathcal{G}}$ and $V(\mathcal{G}) = X$, and $E(\mathcal{G}) = \{((x_1, x_2), (y_1, y_2)) \colon x_1 \leq y_1, x_2 \leq y_2$, and $x_1, x_2, y_1, y_2 \in (0,1]\} \cup \Delta$ and the function $d_{\mathcal{G}_c} \colon X \times X \to \mathfrak{B}$ by $d_{\mathcal{G}_c}((x_1, x_2), (y_1, y_2)) = (\hat{d}(x_1, y_1), \hat{d}(x_2, y_2))$ and

$$\hat{d}(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = x_2; \\ \ln\{\frac{1}{x_1 x_2}\}, & \text{if } x_1, x_2 \in (0,1], x_1 \neq x_2; \\ 1, & \text{otherwise.} \end{cases}$$

Then $d_{\mathcal{G}_c}$ is a graphical cone metric on X and $(X, d_{\mathcal{G}_c})$ is a $\hat{\mathcal{G}}$-complete graphical cone metric space. Define a mapping $T \colon X \to X$ by

$$T(x_1, x_2) = \left(x_1^a, x_2^b\right) \quad \text{with } a, b < 1.$$

Then T is a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction with $\alpha = (\alpha_1, \alpha_2)$, where $a \leq \alpha_1 < 1$ and $b \leq \alpha_1$. Therefore all the conditions of Theorem 4 are satisfied, and so T must have at least one fixed point in X. Indeed, $\mathcal{F}(T) = \{(0,0), (0,1), (1,0), (1,1)\}$.

Remark 3 The importance of condition (N) in Theorem 4 can be justified by Example 9. Indeed, except condition (N), all the conditions of Theorem 4 are satisfied in Example 9, but T has no fixed point, and so, for the existence of a fixed point, condition (N) plays a crucial role.

Theorem 5 *Suppose that all the conditions of Theorem 4 are satisfied. In addition, suppose that $X_T^{\hat{\mathcal{G}}}$ is connected (as a subgraph of $\hat{\mathcal{G}}$), then T has a unique fixed point.*

Proof The existence of a fixed point z^* of T follows from Theorem 4. Suppose that $y^* \neq z^*$ is another fixed point of T. Since $E(\hat{\mathcal{G}})$ contains all the loops, $\mathcal{F}(T) \subseteq X_T^{\hat{\mathcal{G}}}$, i.e., $z^*, y^* \in X_T^{\hat{\mathcal{G}}}$.

Since $X_T^{\hat{\mathcal{G}}}$ is connected, there must be a path from z^* to y^*, i.e., $(z^* P y^*)_{\hat{\mathcal{G}}}$. Hence, there exists a sequence $\{z_k\}_{k=0}^r$, where $z_0 = z^*$ and $z_r = y^*$ with $(z_k, z_{k+1}) \in E(\hat{\mathcal{G}})$ for $k = 0, 1, 2, \ldots, r-1$.

Since T is a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction, by successive use of (GCC1) we have

$$\left(T^n z_k, T^n z_{k+1}\right) \in E(\hat{\mathcal{G}}) \quad \text{for } k = 0, 1, 2, \ldots, r-1 \text{ and for all } n \in \mathbb{N}.$$

Therefore, by (GCC2) we obtain

$$d_{\mathcal{G}_c}\left(T^n z_k, T^n z_{k+1}\right) \preceq \alpha d_{\mathcal{G}_c}\left(T^{n-1} z_k, T^{n-1} z_{k+1}\right) \preceq \cdots \preceq \alpha^n d_{\mathcal{G}_c}(z_k, z_{k+1})$$

for $k = 0, 1, 2, \ldots, r-1$ and for all $n \in \mathbb{N}$.

Using the above inequality and (GCM4), we have

$$d_{\mathcal{G}_c}\left(z^*, y^*\right) = d_{\mathcal{G}_c}\left(T^n z^*, T^n y^*\right) \preceq \sum_{k=0}^{r-1} d_{\mathcal{G}_c}\left(T^n z_k, T^n z_{k+1}\right) \preceq \alpha^n \sum_{k=0}^{r-1} d_{\mathcal{G}_c}(z_k, z_{k+1}).$$

Since $\rho(\alpha) < 1$, by Lemma 1 the sequence $\{\alpha^n \sum_{k=0}^{r-1} d_{\mathcal{G}_c}(z_k, z_{k+1})\}$ is a c-sequence. Hence, using Remark 1 in the above inequality, we get $d_{\mathcal{G}_c}(z^*, y^*) = \theta$, i.e., $z^* = y^*$. Thus, the fixed point of T is unique. □

Remark 4 In the above theorem, by replacing the condition "$X_T^{\hat{\mathcal{G}}}$ is connected" with the condition "$\mathcal{F}(T)$ is connected" the conclusion remains the same, i.e., the mapping T possesses a unique fixed point.

In the next example, we compute a fixed point of the mapping by iteration produced by the T-Picard sequence and illustrate its convergence behavior with different initial values.

Example 11 Let $(X, d_{\mathcal{G}_c})$, a, b, and α be the same as we have considered in Example 10. Suppose that the mapping $T: X \to X$ is defined by

$$T(x_1, x_2) = \begin{cases} (x_1^a, x_2^b), & \text{if } (x_1, x_2) \in (0,1] \times (0,1]; \\ (1,1), & \text{otherwise.} \end{cases}$$

Then, by Theorem 5 and Remark 4, the mapping has a unique fixed point, namely, $\mathcal{F}(T) = \{(1,1)\}$. In the Tables 1, 2, 3 and 4 the convergence behavior of the iteration produced by a T-Picard sequence with various initial values and two sets of values of a and b is shown. The Figures 1 and 2 below show the magnitude of errors decreasing in successive iterations ($\|\cdot\|_E$ denotes the Euclidean norm on $\mathbb{R}^2$).

We now state some consequences of our main results.

Table 1 Iteration of z_n for $a = b = 0.2$ with $(x)_\Delta = (x, x)$

$a = b = 0.2$	$z_0 = (0.75)_\Delta$	$z_0 = (0.8)_\Delta$	$z_0 = (0.85)_\Delta$	$z_0 = (0.9)_\Delta$
z_0	$(0.75000)_\Delta$	$(0.80000)_\Delta$	$(0.85000)_\Delta$	$(0.90000)_\Delta$
z_1	$(0.94409)_\Delta$	$(0.95635)_\Delta$	$(0.96802)_\Delta$	$(0.97915)_\Delta$
z_2	$(0.98856)_\Delta$	$(0.99111)_\Delta$	$(0.99352)_\Delta$	$(0.99579)_\Delta$
z_3	$(0.99770)_\Delta$	$(0.99822)_\Delta$	$(0.99870)_\Delta$	$(0.99916)_\Delta$
z_4	$(0.99954)_\Delta$	$(0.99964)_\Delta$	$(0.99974)_\Delta$	$(0.99983)_\Delta$
z_5	$(0.99991)_\Delta$	$(0.99993)_\Delta$	$(0.99995)_\Delta$	$(0.99997)_\Delta$
z_6	$(0.99998)_\Delta$	$(0.99999)_\Delta$	$(0.99999)_\Delta$	$(0.99999)_\Delta$
z_7	$(1.00000)_\Delta$	$(1.00000)_\Delta$	$(1.00000)_\Delta$	$(1.00000)_\Delta$
z_8	$(1.00000)_\Delta$	$(1.00000)_\Delta$	$(1.00000)_\Delta$	$(1.00000)_\Delta$

Table 2 Euclidean magnitude $M_{z_i} = \|e_i\|_E$ of error $e_i = d_{\mathcal{G}_c}((1)_\Delta, z_i)$ for $a = b = 0.2$

$a = b = 0.2$	$z_0 = (0.75)_\Delta$	$z_0 = (0.8)_\Delta$	$z_0 = (0.85)_\Delta$	$z_0 = (0.9)_\Delta$
M_{z_0}	0.40684	0.31557	0.22984	0.14900
M_{z_1}	0.08137	0.06311	0.04597	0.02980
M_{z_2}	0.01627	0.01262	0.00919	0.00596
M_{z_3}	0.00325	0.00252	0.00184	0.00119
M_{z_4}	0.00065	0.00050	0.00037	0.00024
M_{z_5}	0.00013	0.00010	0.00007	0.00005
M_{z_6}	0.00003	0.00002	0.00001	0.00001
M_{z_7}	0.00001	0.00000	0.00000	0.00000
M_{z_8}	0.00000	0.00000	0.00000	0.00000

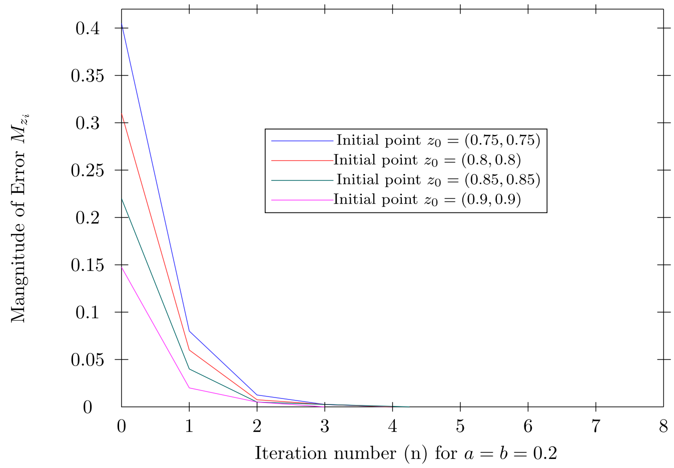

Figure 1 Convergence behavior of z_n for $a = b = 0.2$

Table 3 Iteration of z_n for $a = b = 0.1$ with $(x)_\Delta = (x, x)$

$a = b = 0.1$	$z_0 = (0.75)_\Delta$	$z_0 = (0.8)_\Delta$	$z_0 = (0.85)_\Delta$	$z_0 = (0.9)_\Delta$
z_0	$(0.75000)_\Delta$	$(0.80000)_\Delta$	$(0.85000)_\Delta$	$(0.90000)_\Delta$
z_1	$(0.97164)_\Delta$	$(0.97793)_\Delta$	$(0.98388)_\Delta$	$(0.98952)_\Delta$
z_2	$(0.99713)_\Delta$	$(0.99777)_\Delta$	$(0.99838)_\Delta$	$(0.99895)_\Delta$
z_3	$(0.99971)_\Delta$	$(0.99978)_\Delta$	$(0.99984)_\Delta$	$(0.99989)_\Delta$
z_4	$(0.99997)_\Delta$	$(0.99998)_\Delta$	$(0.99998)_\Delta$	$(0.99999)_\Delta$
z_5	$(1.00000)_\Delta$	$(1.00000)_\Delta$	$(1.00000)_\Delta$	$(1.00000)_\Delta$

Table 4 Euclidean magnitude $M_{z_i} = \|e_i\|_E$ of error $e_i = d_{\mathcal{G}_c}((1)_\Delta, z_i)$ for $a = b = 0.1$

$a = b = 0.1$	$z_0 = (0.75)_\Delta$	$z_0 = (0.8)_\Delta$	$z_0 = (0.85)_\Delta$	$z_0 = (0.9)_\Delta$
M_{z_0}	0.40684	0.31557	0.22984	0.14900
M_{z_1}	0.04068	0.03156	0.02298	0.01490
M_{z_2}	0.00407	0.00316	0.00230	0.00149
M_{z_3}	0.00041	0.00032	0.00023	0.00015
M_{z_4}	0.00004	0.00003	0.00002	0.00001
M_{z_5}	0.00000	0.00000	0.00000	0.00000

Remark 5 With $\mathfrak{B} = \mathbb{R}$ (with absolute value norm and ordinary multiplication) and $C = [0, \infty)$, all the results proved above reduce to the corresponding results of Shukla et al. [24].

Corollary 6 (Cone metric version of the Banach contraction principle) *Let (X, d) be a complete cone metric space over Banach algebra $\mathfrak{B}$ and $T: X \to X$ be a mapping. Suppose that there exists $\alpha \in C$ such that $\rho(\alpha) < 1$ and $d(Tx, Ty) \preceq \alpha d(x, y)$ for all $x, y \in X$. Then T has a unique fixed point in X.*

Proof Let $\mathcal{G} = \hat{\mathcal{G}}$ and $E(\mathcal{G}) = X \times X$, then the result follows from Theorem 5. □

We next give an improved cone metric version of the result of Ran and Reurings [17].

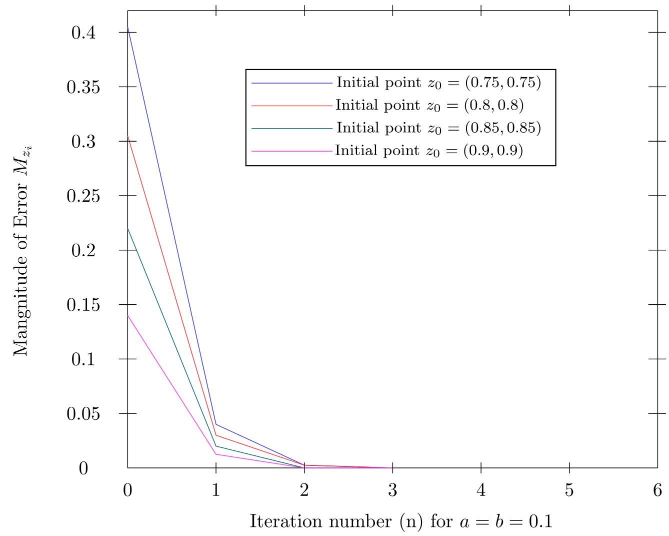

Figure 2 Convergence behavior of z_n for $a = b = 0.1$

Corollary 7 *Let $(X, \sqsubseteq)$ be a partially ordered set and d be a cone metric on X such that (X, d) is a complete cone metric space over Banach algebra $\mathfrak{B}$. Let $T : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$ and the following conditions hold*:

(a) *There exists $\alpha \in C$ such that $\rho(\alpha) < 1$ and*

$$d(Tx, Ty) \preceq \alpha d(x, y) \quad \text{for all } x, y \in X \text{ with } x \sqsubseteq y;$$

(b) *There exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;*

(c) *If $\{z_n\}$ is a sequence in X such that $z_n \sqsubseteq z_{n+1}$ for all $n \in \mathbb{N}$ and converges to $z \in X$, then there is $n_0 \in \mathbb{N}$ such that $z_n \sqsubseteq z$ or $z \sqsubseteq z_n$ for all $n > n_0$.*

Then T has a fixed point in X. Furthermore, if the set $\{x : x \sqsubseteq Tx\}$ is well ordered, then the fixed point of T is unique.

Proof Let the graph $\mathcal{G}$ and a subgraph $\hat{\mathcal{G}}$ be defined by $V(\mathcal{G}) = V(\hat{\mathcal{G}}) = X$, $E(\mathcal{G}) = X \times X$, $E(\hat{\mathcal{G}}) = \{(x, y) \in X \times X : x \sqsubseteq y\}$, then the result follows from Theorem 5. □

Let (X, d) be a cone metric space over Banach algebra $\mathfrak{B}$. Let M and N be two nonempty closed subsets of X, and let a mapping $T : M \cup N \to M \cup N$ satisfy the following conditions:

(a) $T(M) \subseteq N$ and $T(N) \subseteq M$;

(b) $d(Tx, Ty) \preceq \mu d(x, y)$ for all $x \in M$, $y \in N$, where $\mu \in C$ is such that $\rho(\mu) < 1$.

Then T is called a cyclic contraction (for cyclic contraction defined on metric spaces, see Kirk et al. [14]).

The following corollary is a cone metric version of the result of Kirk et al. [14].

Corollary 8 *Let (X, d) be a cone metric space over Banach algebra $\mathfrak{B}$, M and N be two nonempty closed subsets of X, and suppose that $T : M \cup N \to M \cup N$ is a cyclic contraction. Then T has a unique fixed point in $M \cap N$.*

Proof Define the graph $\mathcal{G}$ and a subgraph $\hat{\mathcal{G}}$ by $V(\mathcal{G}) = X$, $V(\hat{\mathcal{G}}) = M \cup N$, $E(\mathcal{G}) = X \times X$, and $E(\hat{\mathcal{G}}) = \Delta \cup \{(x,y)\colon (x,y) \in [M \times T(M)] \cup [N \times T(N)]\}$. Then the result follows from Theorem 5 (see also Shukla and Abbas [20] and Shukla et al. [24]). □

We define the cone metric analogue of ϵ-chainable metric spaces (see Edelstein [5]). Let (X,d) be a cone metric space over Banach algebra $\mathfrak{B}$ and $c \in C$ with $\theta \ll c$. Then (X,d) is called c-chainable if, given $x,y \in X$, there exist $k \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^k$ such that $x_0 = x$, $x_k = y$ and $d(x_i,x_{i-1}) \ll c$ for $i = 1,2,\dots,k$.

The following corollary is a cone metric version of the result of Edelstein [5].

Corollary 9 *Let (X,d) be a complete c-chainable cone metric space over Banach algebra $\mathfrak{B}$, and suppose that $T\colon X \to X$ is a mapping such that there exists $\alpha \in C$ with $\rho(\alpha) < 1$ and*

$$d(Tx,Ty) \preceq \alpha d(x,y) \quad \textit{for all } x,y \in X \textit{ with } d(x,y) \ll c.$$

Then T has a unique fixed point in X.

Proof Define the graph $\mathcal{G}$ and a subgraph $\hat{\mathcal{G}}$ by $V(\mathcal{G}) = V(\hat{\mathcal{G}}) = X$ and $E(\mathcal{G}) = X \times X$, $E(\hat{\mathcal{G}}) = \{(x,y) \in X \times X\colon d_{\mathcal{G}_c}(x,y) \ll c\}$. Then the result follows from Theorem 5. □

In the next section, we state an application of our results to a system of initial value problems.

5 Application to a system of initial value problems

In this section, we consider a pair of initial value problems, and by applying the fixed point results of the previous section, we show that the existence of a lower (or upper) solution of the pair of initial value problems ensures the existence of solution of the pair.

Let $a > 0$, $I = [0,a]$ and $X = C_{\mathbb{R}}(I) \times C_{\mathbb{R}}(I)$, where $C_{\mathbb{R}}(I)$ is the space of all continuous real-valued functions defined over the interval I. Let us consider the following pair of Cauchy initial value problems.

Suppose that $K_1,K_2\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $N\colon I \to \mathbb{R}$ are three continuous functions. We seek two differentiable functions u_1, u_2 on I that satisfy

$$\frac{du_i}{dt} = N(t)K_i\bigl(t,u_i(t),u_j(t)\bigr); \quad u_i(0) = 0, i,j = 1,2, i \neq j. \tag{2}$$

For $x(t) = (x_1(t),x_2(t))$, $y(t) = (y_1(t),y_2(t)) \in X$, we write $x(t) \sqsubseteq y(t)$ if $x_i(t) \leq y_i(t)$, $t \in I$, $i = 1,2$. For simplicity, we write x instead of $x(t)$ for $x(t) \in X$. Define $B \subset X$ by $B = \{(x_1,x_2) \in X\colon 0 \leq x_i(t) \leq 1, t \in I, i = 1,2\}$. For $\varrho > 0$, define $D\colon C_{\mathbb{R}}(I) \times C_{\mathbb{R}}(I) \to \mathbb{R}$ by

$$D(x_1,x_2) = \begin{cases} 0, & \text{if } x_1 = x_2; \\ \sup_{t\in I} |x_2(t) - x_1(t)|, & \text{if } (x_1,x_2) \in B, t \in I; \\ \varrho, & \text{otherwise.} \end{cases}$$

Define two graphs $\mathcal{G}$ and $\hat{\mathcal{G}}$ by $\mathcal{G} = \hat{\mathcal{G}}$, $V(\mathcal{G}) = X$ and

$$E(\mathcal{G}) = \Delta \cup \bigl\{(x,y) \in X \times X\colon x,y \in B, x \sqsubseteq y\bigr\}$$

and $d_{\mathcal{G}_c}: X \times X \to \mathbb{R}^2$ by $d_{\mathcal{G}_c}(x,y) = (D(x_1,y_1), D(x_2,y_2))$ for all $x = (x_1,x_2)$, $y = (y_1, y_2) \in X$.

Then $(X, d_{\mathcal{G}_c})$ is a $\hat{\mathcal{G}}$-complete graphical cone metric space over Banach algebra $\mathfrak{B} = \mathbb{R}^2$ with the norm $\|(u_1,u_2)\| = |u_1| + |u_2|$, the multiplication $(u_1,u_2).(v_1,v_2) = (u_1v_1, u_1v_2 + u_2v_1)$, and the cone $C = \{(u_1,u_2) \in \mathbb{R}^2 : u_1,u_2 \geq 0\}$.

Clearly system (2) is equivalent to the following system:

$$u_i(t) = \int_0^t N(s)K_i\bigl(s,u_i(s),u_j(s)\bigr)\,ds, \quad i,j = 1,2, i \neq j. \tag{3}$$

A pair $(\alpha_1,\alpha_2) \in X$ is a lower solution of (3) if

$$\alpha_i(t) \leq \int_0^t N(s)K_i\bigl(s,\alpha_i(s),\alpha_j(s)\bigr)\,ds \quad \text{for } i,j = 1,2, i \neq j \text{ and } t \in I.$$

Obviously, an upper solution of (2) satisfies the reverse inequality. We will show that the existence of lower solution of (3) ensures the existence of a solution of (2).

We consider the operator $T: X \to X$ such that

$$T(u_1,u_2) = \left(\int_0^t N(s)K_1(s,u_1,u_2)\,ds, \int_0^t N(s)K_2(s,u_2,u_1)\,ds\right)$$

and give sufficient conditions for the existence of a fixed point of T in X, i.e., a solution of system (2).

Theorem 10 *Suppose that the following conditions are satisfied*:

(a) *If* $s_1,s_2,s_3,s_4 \in [0,1]$, $s_1 \leq s_3$, $s_2 \leq s_4$, *then* $0 \leq K_i(t,s_1,s_2) \leq K_i(t,s_3,s_4)$ *and* $0 \leq N(t)$ *for* $t \in I$;

(b) *There exists* $\lambda \in (0,1)$ *such that for all* $x,y \in X$ *with* $(x,y) \in E(\mathcal{G})$ *we have for every* $t \in I$

$$K_i(t,y_i,y_j) - K_i(t,x_i,x_j) \leq y_i(t) - x_i(t), \quad i \neq j, i,j = 1,2,$$

and

$$\int_0^t N(s)\,ds \leq \lambda \quad \textit{for every } t \in I;$$

(c) $K_i(t,1,1) \leq 1/\lambda$ *for* $i = 1,2$, $t \in I$.

Then the existence of a lower solution of (3) *in* B *ensures the existence of a solution of* (2).

Proof Notice that the operator T is well defined. Suppose that $x = (x_1,x_2)$, $y = (y_1,y_2) \in X$ with $(x,y) \in E(\mathcal{G})$, then we have $x,y \in B$ and $0 \leq x_i(t) \leq y_i(t) \leq 1$, $t \in I$, $i = 1,2$, and by conditions (a), (b), and (c), we get

$$\begin{aligned}(0,0) &\sqsubseteq T(x_1,x_2)\\ &= \left(\int_0^t N(s)K_1(s,x_1,x_2)\,ds, \int_0^t N(s)K_2(s,x_2,x_1)\,ds\right)\end{aligned}$$

$$\begin{aligned}
&\sqsubseteq \left(\int_0^t N(s)K_1(s,1,1)\,ds, \int_0^t N(s)K_2(s,1,1)\,ds\right)\\
&\sqsubseteq \left(\frac{1}{\lambda}\int_0^t N(s)\,ds, \frac{1}{\lambda}\int_0^t N(s)\,ds\right)\\
&\sqsubseteq (1,1)
\end{aligned}$$

and

$$\begin{aligned}
(0,0) &\sqsubseteq T(x_1,x_2)\\
&= \left(\int_0^t N(s)K_1(s,x_1,x_2)\,ds, \int_0^t N(s)K_2(s,x_2,x_1)\,ds\right)\\
&\sqsubseteq \left(\int_0^t N(s)K_1(s,y_1,y_2)\,ds, \int_0^t N(s)K_2(s,y_2,y_1)\,ds\right)\\
&= T(y_1,y_2).
\end{aligned}$$

And for $x = (x_1,x_2)$, $y = (y_1,y_2) \in X$ with $(x,y) \in E(\mathcal{G})$ and $t \in I$, we obtain

$$\begin{aligned}
d_{\mathcal{G}_c}\big(T(x_1,x_2), T(y_1,y_2)\big) &= d_{\mathcal{G}_c}\left(\left(\int_0^t N(s)K_1(s,x_1,x_2)\,ds, \int_0^t N(s)K_2(s,x_2,x_1)\,ds\right),\right.\\
&\quad \left.\left(\int_0^t N(s)K_1(s,y_1,y_2)\,ds, \int_0^t N(s)K_2(s,y_2,y_1)\,ds\right)\right)\\
&= \left(D\left(\int_0^t N(s)K_1(s,x_1,x_2)\,ds, \int_0^t N(s)K_1(s,y_1,y_2)\,ds\right),\right.\\
&\quad \left. D\left(\int_0^t N(s)K_2(s,x_2,x_1)\,ds, \int_0^t N(s)K_2(s,y_2,y_1)\,ds\right)\right).
\end{aligned}$$

Therefore, the definition of D yields

$$\begin{aligned}
d_{\mathcal{G}_c}\big(T(x_1,x_2), T(y_1,y_2)\big) &= \left(\sup_{t\in I}\left[\int_0^t N(s)K_1(s,y_1,y_2)\,ds - \int_0^t N(s)K_1(s,x_1,x_2)\,ds\right],\right.\\
&\quad \left.\sup_{t\in I}\left[\int_0^t N(s)K_2(s,y_2,y_1)\,ds - \int_0^t N(s)K_2(s,x_2,x_1)\,ds\right]\right)\\
&= \left(\sup_{t\in I}\int_0^t N(s)\big[K_1(s,y_1,y_2) - K_1(s,x_1,x_2)\big]\,ds,\right.\\
&\quad \left.\sup_{t\in I}\int_0^t N(s)\big[K_2(s,y_2,y_1) - K_2(s,x_2,x_1)\big]\,ds\right).
\end{aligned}$$

Using condition (b) in the above equality, we obtain

$$\begin{aligned}
&d_{\mathcal{G}_c}\big(T(x_1,x_2), T(y_1,y_2)\big)\\
&\quad\preceq \left(\sup_{t\in I}\int_0^t N(s)\big[y_1(s) - x_1(s)\big]\,ds, \sup_{t\in I}\int_0^t N(s)\big[y_2(s) - x_2(s)\big]\,ds\right)\\
&\quad\preceq \big(\lambda D\big(x_1(t),y_1(t)\big), \lambda D\big(x_2(t),y_2(t)\big)\big)\\
&\quad\preceq \left(\lambda D(x_1,y_1), \lambda D(x_2,y_2) + \frac{1}{2}D(x_1,y_1)\right)
\end{aligned}$$

$$= \left(\lambda, \frac{1}{2}\right)\big(D(x_1,y_1), D(x_2,y_2)\big)$$
$$= \left(\lambda, \frac{1}{2}\right) d_{\mathcal{G}_c}\big((x_1,x_2),(y_1,y_2)\big).$$

Note that $\rho(\lambda, \frac{1}{2}) = \lambda < 1$. Consequently, T is a $(\mathcal{G}, \hat{\mathcal{G}})$-graphical contraction and the existence of a lower solution of (3) in B, say (α_1, α_2), implies that property (A) of Theorem 3 holds true, that is, $T(\alpha_1,\alpha_2) \in [(\alpha_1,\alpha_2)]^1_{\hat{\mathcal{G}}}$. It is easy to see that property (B) of Theorem 3 holds true and the quadruple $(X, d_{\mathcal{G}_c}, \hat{\hat{\mathcal{G}}}, T)$ has property (N). Thus all the conditions of Theorem 4 are satisfied, and hence there exists a fixed point of T that is a solution of system (2). □

Remark 6 One can establish Theorem 10 in the case of existence of an upper solution by considering the following graph:

$$E(\mathcal{G}) = \Delta \cup \big\{(x,y) \in X \times X : x, y \in B, y \sqsubseteq x; t \in I\big\}$$

retaining the rest as above.

Next, we present a simple example that illustrates Theorem 10.

Example 12 Consider the following system of initial value problems: for $x, y \in C^1_{\mathbb{R}}[0,1]$ and $t \in I = [0,1]$,

$$\left.\begin{aligned} \frac{dx}{dt} &= tx + t, \quad x(0) = 0; \\ \frac{dy}{dt} &= ty + \frac{t}{2}, \quad y(0) = 0. \end{aligned}\right\} \tag{4}$$

Define the functions $N : I \to \mathbb{R}$, $K_1, K_2 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $N(t) = t$ and $K_1(t,x,y) = x + 1$, $K_2(t,y,x) = y + 1/2$ for all $t \in I$, $x, y \in \mathbb{R}$. Consider $X = C_{\mathbb{R}}(I) \times C_{\mathbb{R}}(I)$ and the operator $T : X \to X$ such that

$$\begin{aligned} T(x,y) &= \left(\int_0^t N(s)K_1(s,x,y)\,ds, \int_0^t N(s)K_2(s,y,x)\,ds\right) \\ &= \left(\int_0^t \big(sx(s) + s\big)\,ds, \int_0^t \left(sy(s) + \frac{s}{2}\right) ds\right). \end{aligned}$$

We observe that $z_0 = (0,0) \in X$ is a lower solution of (4) lying in B, and N, K_1, K_2 satisfy all the conditions of Theorem 10 with $a = 1$ and $\lambda = \frac{1}{2}$. Hence, by Theorem 10 system (4) has a solution. Furthermore, the T-Picard sequence $\{z_n\} = \{T^n z_0\}$ is given by

$$z_n = \left(\sum_{i=1}^n \frac{1}{i!}\frac{t^{2i}}{2^i}, \sum_{i=1}^n \frac{1}{2(i!)}\frac{t^{2i}}{2^i}\right), \quad n \in \mathbb{N},$$

and converges to the fixed point $(e^{t^2/2} - 1, \frac{e^{t^2/2}}{2} - 1) \in X$ of T and is the solution of system (4).

6 Conclusions

In this paper, we have unified the concepts of cone metric and graphical metric by introducing the notion of graphical cone metric, which yields a generalization of the usual metric. The topology induced by graphical cone metric is T_1 but not necessarily T_2, and this is the limitation of the topology induced by a graphical cone metric. Nevertheless, graphical cone metric spaces may have potential applications in the fields where non-T_2 spaces play a role, e.g., in the domain theory and in the foundation of semantics of computer languages. On the other hand, the fixed point theory for contraction mappings in graphical cone metric spaces, which generalizes the usual theory of fixed point, has been introduced. It is shown that the fixed point theory in this new setting has a wider scope and can be applied in finding the solutions of a particular type of a system of a pair of initial value problems. This approach can be generalized to the systems consisting of more than two initial value problems. Furthermore, this approach can be applied to the existence and uniqueness of solutions of systems of integral equations.

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