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A kind of bivariate Bernoulli-type multiquadric quasi-interpolation operator with higher approximation order

Ruifeng Wu[1,2,3*]

*Correspondence:
wuruifeng@jlufe.edu.cn
[1]School of Applied Mathematics, Jilin University of Finance and Economics, Changchun 130117, P.R. China
[2]GongQing Institute Of Science And Technology, Jiujiang 332020, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, a kind of bivariate Bernoulli-type multiquadric quasi-interpolation operator is studied by combining the known multiquadric quasi-interpolation operator with the generalized Taylor polynomial as the expansion in the bivariate Bernoulli polynomials. Some error bounds and convergence rates of the combined operators are studied. A selection of numerical examples is presented to compare the performances of the obtained scheme. Furthermore, our method can be applied to time-dependent differential equations. Its advantage is that the algorithm is very simple and easy to implement.

Mathematics Subject Classification: 41A05; 65D05; 65D15

Keywords: Bivariate Bernoulli polynomials; Multiquadric quasi-interpolation; Convergence rates; The generalized Taylor polynomial

1 Introduction

Although the interpolation method is a classical approximation method in numerical mathematics, the interpolation matrix quickly becomes ill-conditioned with the number of interpolation nodes increases. To overcome this problem, quasi-interpolation method was confirmed that it does not solve linear algebraic equations and can achieve the desired convergence order. For a set of nodes $\Xi = \{\mathbf{q}_1, \ldots, \mathbf{q}_n\} \subseteq \mathbb{R}^d$, real functional values $\{f(\mathbf{q}_j)\}_{j=1}^n$ and quasi-interpolation basis function $\Psi(\mathbf{q})$, the quasi-interpolation $(\mathcal{Q}f)(\mathbf{q})$ takes the following standard form by linear combination

$$(\mathcal{Q}f)(\mathbf{q}) = \sum_{j=1}^{n} f(\mathbf{q}_j)\Psi(\mathbf{q}), \quad \mathbf{q} \in \mathbb{R}^d.$$

For the nodes of the uniform grid with spacing h, Rabut [1] constructed the following quasi-interpolation operator based on the theory of principal shift-invariant spaces (see, e.g., [2])

$$(\mathcal{Q}f)(\mathbf{q}) = \sum_{j\in\mathcal{Z}^d} f(jh)\Psi\left(\frac{\mathbf{q}}{h} - j\right), \tag{1}$$

where the basis function Ψ of the quasi-interpolation is considered as a compactly supported or rapidly decaying function. In [3], by using a simple generator Ψ_0 and recursively defines generators $\Psi_1, \Psi_2, \ldots, \Psi_{m-1}$ with corresponding quasi-interpolants defined as (1) reproducing polynomials of degrees $3, 5, \ldots, 2m-1$ respectively, Bozzini et al. introduced a procedure in spaces of m-harmonic splines in $\mathbb{R}^d$. In [4, 5], the Strang-Fix condition is regarded as a sufficient and necessary condition for polynomial reproduction and convergence order of quasi-interpolation. By using the Strang-Fix condition for Ψ, several scholars, for example, Buhmann et al. [6], Dyn & Ron [7], Wu & Liu [8], and Wu & Xiong [9] constructed quasi-interpolants on the scattered data, which reproduce polynomials. We will discuss the bivariate multiquadric quasi-interpolation operator, which can reproduce higher degree polynomials in this paper.

Interpolation method based on radial basis function is an important tool in practical application. In 1971, Hardy [10] studied the multiquadrics as a kind of radial basis function. A review by Franke [11] suggested that the multiquadric interpolation is one of the best methods among some 29 interpolation methods in respect of accuracy, efficiency, and easy implementation. Micchelli [12] proved the existence of the solution of the associated multiquadric interpolation problem in 1986. Buhmann [13] investigated the accuracy of quasi-interpolation for infinite regular grid data in 1988.

In 1992, through shifts of the multiquadric basis function of first degree on finite scattered data, Beast and Powell [14] first studied a univariate quasi-interpolant. Wu and Schaback [15] introduced a modified univariate multiquadric quasi-interpolant reproducing polynomials of degree no more than 1. Recently, many works have been researched on this subject, see for example [16–22].

In 2005, by using the dimension-splitting technology, Ling [23] extended the univariate quasi-interpolant to bidimensional case. Feng et al. [24] and Wu et al. [25] improved the approximation order of the bivariate quasi-interpolant. In 2018, Feng et al. [26] constructed the quasi-interpolation scheme for arbitrary dimensional scattered data approximation. In 2019, to overcome numerical solution of high-dimensional shockwave equations, Zhang et al. [27] proposed the bivariate dimension-splitting multiquadric quasi-interpolant. In 2022, Li et al. [28] introduced the bivariate multiquadric quasi-interpolation(MQQI) on the gridded data for solving 2D sine-Gordon equations. Recently, some scholars have made progress in the interesting application [29–32]. Numerical examples verify the effectiveness and high accuracy of the method. However, the convergence orders of their bivariate multiquadric quasi-interpolation operators are low.

To increase the accuracy of the multiquadric quasi-interpolation method on the interesting applications, our paper is to present a kind of bivariate multiquadric quasi-interpolation operator with higher accuracy for gridded data. In the paper, we combine the multiquadric quasi-interpolant (see, e.g., [27]), with the generalized Taylor polynomial as the expansion in the bivariate Bernoulli polynomials.

The rest sections of this paper are organized as follows. In Sect. 2, we recall the generalized Taylor polynomial of degree (m, n) and give new results on the error of approximation that will be used later in the paper. In Sect. 3, we apply previous results to derive a kind of bivariate Bernoulli-type multiquadric quasi-interpolation operator, and get their convergence rates. In Sect. 4, we give some numerical tests to investigate that the constructed operators are able to compare the approximation capacity of our new operators, providing high accuracy. Finally, conclusions and future work are arranged in Sect. 5.

2 The generalized Taylor polynomial of degree (m,n)

The generalized Taylor polynomial is an expansion in Bernoulli polynomial of degree (m,n). We retrospect some results from [33] and [34]. Univariate Bernoulli polynomials are defined by means of the following relations [35]

$$\begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x), \quad n > 1, \\ \int_0^1 B_n(x)\,dx = 0, \quad n \geq 1. \end{cases} \tag{2}$$

For a given function $f \in C^m[a,b]$, $m \geq 1$, the following univariate Bernoulli-type interpolation formula is given by [34]:

$$f(x) = B_m[f;a,b;h](x) + R_m[f;a,b;h](x),$$

with

$$B_m[f;a,b](x) = f(a) + \sum_{i=1}^{m} S_i\left(\frac{x-a}{h}\right)\frac{h^{i-1}}{i!}\Delta_h f^{(i-1)}(a), \tag{3}$$

where $h = b - a$,

$$S_i\left(\frac{x-a}{h}\right) = B_i\left(\frac{x-a}{h}\right) - B_i, \tag{4}$$

$$\Delta_h f^{(i-1)}(a) = f^{(i-1)}(b) - f^{i-1)}(a), \quad 1 \leq i \leq m, \tag{5}$$

$B_i = B_i(0)$ denotes the Bernoulli numbers, and $R_m[f;a,b;h]$ denotes the remainder term.

Suppose that $I = [a,b] \times [c,d]$ is a rectangular domain in the plane $\mathbb{R}^2$. Let us denote by $\mathcal{C}^{(m,n)}(I)$ the space of functions $f : I \to \mathcal{R}^2$ with continuous partial derivatives

$$f^{(i,j)}(x,y) = \frac{\partial^{(i+j)}}{\partial x^i \partial y^j} f(x,y), \quad (x,y) \in I,$$

for all (i,j), $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$.

We denote $h = b - a$, $k = d - c$, and from the operators [34] it follows:

$$\begin{aligned} \Delta_{(h,0)} f(x,y) &= f(x+h,y) - f(x,y), \\ \Delta_{(0,k)} f(x,y) &= f(x,y+k) - f(x,y), \\ \Delta_{(h,k)} f(x,y) &= \Delta_{(h,0)}\Delta_{(0,k)} f(x,y) \\ &= \Delta_{(0,k)}\Delta_{(h,0)} f(x,y) \\ &= f(x,y) - f(x+h,k) + f(x+h,y+k) - f(x,y+k). \end{aligned}$$

For a given function $f \in \mathcal{C}^{(m,n)}(I)$, the polynomial approximation term $B_{m,n}[f;a,b;c,d;h,k](x,y)$ is the Bernoulli-type polynomial of degree (m,n) with variables x, y, obtained by the following formula [34]:

$$\begin{aligned} & B_{m,n}[f;a,b;c,d;h,k](x,y) \\ & \quad = f(a,c) + \sum_{i=1}^{m} \Delta_{(h,0)} f^{(i-1,0)}(a,c) \frac{h^{i-1}}{i!} S_i\left(\frac{x-a}{h}\right) \\ & \qquad + \sum_{j=1}^{n} \Delta_{(0,k)} f^{(0,j-1)}(a,c) \frac{k^{j-1}}{j!} S_j\left(\frac{y-c}{k}\right) \\ & \qquad + \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{(h,k)} f^{(i-1,j-1)}(a,c) \frac{h^{i-1}k^{j-1}}{i!j!} S_i\left(\frac{x-a}{h}\right) S_j\left(\frac{y-c}{k}\right), \end{aligned} \tag{6}$$

where S_k, $k > 1$ are given in (4).

As in the one-dimensional case [36], we also call the polynomial approximant $B_{m,n}[f;a,b;c,d;h,k](x,y)$ the generalized Taylor polynomial of degree (m,n). It can be derived from a nice property of this operator: its limit when $h \to 0$, $k \to 0$ is the well-known Taylor polynomial of degree (m,n) of f about point (a,c) [37]:

$$\lim_{h,k\to 0} B_{m,n}[f;a,b;c,d;h,k](x,y) = T_{m,n}\big[f;(a,c)\big](x,y),$$

where

$$T_{m,n}\big[f;(a,c)\big](x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{(x-a)^i(y-c)^j}{i!j!} f^{(i,j)}(a,c).$$

Moreover, the polynomial $B_{m,n}[f;a,b;c,d;h,k](x,y)$ satisfies the following interpolation conditions [34]:

$$\begin{aligned} & B_{m,n}[f;a,b;c,d;h,k](a,c) = f(a,c), \\ & \Delta_{(h,0)} B_{m,n}[f;a,b;c,d;h,k]^{(i,0)}(a,c) = \Delta_{(h,0)} f^{(i,0)}(a,c), \quad 0 \le i \le m-1, \\ & \Delta_{(0,k)} B_{m,n}[f;a,b;c,d;h,k]^{(0,j)}(a,c) = \Delta_{(0,k)} f^{(0,j)}(a,c), \quad 0 \le j \le n-1, \\ & \Delta_{(h,k)} B_{m,n}[f;a,b;c,d;h,k]^{(i,j)}(a,c) \\ & \quad = \Delta_{(h,k)} f^{(i,j)}(a,c), \quad 0 \le i \le m-1, 0 \le j \le n-1. \end{aligned} \tag{7}$$

Theorem 1 (See, [34]) *The degree of exactness of the operator $B_{m,n}[\cdot]$ is (m,n), i.e., for each bivariate polynomial $p \in \mathbb{P}^{(m,n)}$.*

According to [34], for a given function $f \in C^{(m,n)}(I)$, $I = [a,b] \times [c,d]$ $(a < b, c < d)$, we have the bivariate Bernoulli interpolation formula:

$$f(x,y) = B_{m,n}[f;a,b;c,d;h,k](x,y) + R_{m,n}[f;a,b;c,d;h,k](x,y), \tag{8}$$

where $R_{m,n}[f;a,b;c,d;h,k](x,y)$ is the remainder term.

Note that the generalized Taylor polynomial $B_{m,n}[f;a,b;c,d;h,k](x,y)$ can be extended in a natural way to the whole real plane. To study bounds for the remainder $R_{m,n}[f;a,b;c,d;h,$

$k](x,y)$ of the formula (8) even in points outside the rectangular domain $[a,b] \times [c,d]$, we take the operator: $f \to B_{m,n}[f;a,b;c,d;h,k]$ as acting on the space $\mathcal{C}^{(m,n)}([\overline{a},\overline{b}] \times [\overline{c},\overline{d}])$ with $\overline{a} < a$, $b < \overline{b}$, $\overline{c} < c$, and $d < \overline{d}$. Let us set

$$I_u = \begin{cases} [\overline{a},a], & u = 1, \\ [a,b], & u = 2, \\ [b,\overline{b}], & u = 3 \end{cases} \quad \text{and} \quad I_v = \begin{cases} [\overline{c},c], & v = 1, \\ [c,d], & v = 2, \\ [d,\overline{d}], & v = 3 \end{cases}$$

be the subinterval in x and y directions and define the subinterval

$$I_1^x = \begin{cases} [x,b], & x \in [\overline{a},a], \\ [a,b], & x \in [a,b], \\ [a,x], & x \in [b,\overline{b}] \end{cases} \quad \text{and} \quad I_2^y = \begin{cases} [y,d], & y \in [\overline{c},c], \\ [c,d], & y \in [c,d], \\ [c,y], & y \in [d,\overline{d}]. \end{cases}$$

By applying the Peano's theorem for bidimensional case (see, [38]), we have the following form of the remainder (8).

Theorem 2 *Let $f \in \mathcal{C}^{(m,n)}([\overline{a},\overline{b}] \times [\overline{c},\overline{d}])$ and $(x,y) \in [\overline{a},\overline{b}] \times [\overline{c},\overline{d}]$. Then for the remainder*

$$R_{m,n}[f;a,b;c,d;h,k](x,y) = f(x,y) - B_{m,n}[f;a,b;c,d;h,k](x,y). \tag{9}$$

Let $I_u \times I_v$ be a fixed rectangular partition of $[\overline{a},\overline{b}] \times [\overline{c},\overline{d}]$ and suppose that $(x,y) \in I_u \times I_v$, $u,v = 1,2,3$. Then

$$\begin{aligned} &R_{m,n}[f;a,b;c,d;h,k](x,y) \\ &\quad = \sum_{j<n} \int_{I_1^x} f^{(m,j)}(s,c) K_{m,j}(x,y,s)\,ds \\ &\qquad + \sum_{i<m} \int_{I_2^y} f^{(i,n)}(a,t) K_{i,n}(x,y,t)\,dt \\ &\qquad + \int\int_{I_1^x \times I_2^y} f^{(m,n)}(x,y,s,t) K_{m,n}(x,y,s,t)\,ds\,dt; \end{aligned} \tag{10}$$

where $K_{m,0}(x,y,s)K_{m,j}(x,y,s)$, $K_{0,n}(x,y,t)$, $K_{i,n}(x,y,t)$, and $K_{m,n}(x,y,s,t)$ are the Peano's kernels.

Proof On the one hand, in the polynomial approximation term (6) there are evaluations of derivatives of f up to the order $(m-1,n-1)$ in points of $[\overline{a},\overline{b}] \times [\overline{c},\overline{d}]$; on the other hand, the exactness of the polynomial approximant (6) on the space $\mathbb{P}^{(m,n)}$ supposes the exactness of the operator on the subspace $\mathbb{P}^{(m-1,n-1)}$. By using the Peano's kernel theorem, we have the following result:

$$\begin{aligned} &R_{m,n}[f;a,b;c,d;h,k](x,y) \\ &\quad = \sum_{j<n} \int_{\overline{a}}^{\overline{b}} f^{m,j}(s,c) K_{m,j}(x,y,s)\,ds \end{aligned}$$

$$+\sum_{i<m}\int_{\overline{c}}^{\overline{d}} f^{i,n}(a,t)K_{i,n}(x,y,t)\,dt+\int_{\overline{a}}^{\overline{b}}\int_{\overline{c}}^{\overline{d}} f^{m,n}(x,y,s,t)K_{m,n}(x,y,s,t)\,ds\,dt,$$

where the above kernel functions are obtained by the linear functional $f\to R_{m,n}[f;a,b;c,d;h,k](x,y)$ to $\frac{(x-s)_+^{m-1}}{(m-1)!}\frac{(y-c)^j}{j!}$, $\frac{(x-a)^i}{i!}\frac{(y-t)_+^{n-1}}{(n-1)!}$, and $\frac{(x-s)_+^{m-1}}{(m-1)!}\frac{(y-t)_+^{n-1}}{(n-1)!}$ considered as a function of x, y and $(\cdot)_+^k$ denotes the positive part of the kth power of the argument, i.e., $z_+^k=\max\{z^k,0\}$. If $x\in I_u\times I_v$, $u,v=1$, i.e., $x\in[\overline{a},a]$, $y\in[\overline{c},c]$, then we have

$$\begin{aligned}
&R_{m,n}[f;a,b;c,d;h,k](x,y)\\
&=\sum_{j<n}\biggl(\int_{\overline{a}}^{x} f^{(m,j)}(s,c)K_{m,j}(x,y,s)\,ds+\int_{x}^{a} f^{(m,j)}(s,)K_{m,j}(x,y,s)\,ds\\
&\quad+\int_{a}^{b} f^{(m,j)}(s,c)K_{m,j}(x,y,s)\,ds+\int_{b}^{\overline{b}} f^{(m,j)}(s,c)K_{m,j}(x,y,s)\,ds\biggr)\\
&\quad+\sum_{i<m}\biggl(\int_{\overline{c}}^{y} f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt+\int_{y}^{c} f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt\\
&\quad+\int_{c}^{d} f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt+\int_{d}^{\overline{d}} f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt\biggr)\\
&\quad+\Bigl(\int_{\overline{a}}^{x}\int_{\overline{c}}^{y} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{\overline{a}}^{x}\int_{y}^{c} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\\
&\quad+\int_{\overline{a}}^{x}\int_{c}^{d} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{\overline{a}}^{x}\int_{d}^{\overline{d}} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\\
&\quad+\int_{x}^{a}\int_{\overline{c}}^{y} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{x}^{a}\int_{y}^{c} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\\
&\quad+\int_{x}^{a}\int_{c}^{d} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{x}^{a}\int_{d}^{\overline{d}} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\\
&\quad+\int_{a}^{b}\int_{\overline{c}}^{y} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{a}^{b}\int_{y}^{c} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\\
&\quad+\int_{a}^{b}\int_{c}^{d} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{a}^{b}\int_{d}^{\overline{d}} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\\
&\quad+\int_{b}^{\overline{b}}\int_{\overline{c}}^{y} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{b}^{\overline{b}}\int_{y}^{c} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\\
&\quad+\int_{b}^{\overline{b}}\int_{c}^{d} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt+\int_{b}^{\overline{b}}\int_{d}^{\overline{d}} f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt\Bigr).
\end{aligned}$$

Note that if $\overline{a}<s<x$ or $\overline{c}<t<y$, then

$$\begin{aligned}
&K_{m,0}(x,y,s)\\
&=\frac{(x-s)^{m-1}}{(m-1)!}\\
&\quad-\left(\frac{(a-s)^{m-1}}{(m-1)!}+\sum_{p=1}^{m}\frac{1}{(m-p)!}\bigl((b-s)^{m-p}-(a-s)^{m-p}\bigr)\frac{h^{p-1}}{p!}S_p\left(\frac{x-a}{h}\right)\right)=0,
\end{aligned}$$

$$
\begin{aligned}
&K_{m,j}(x,y,s)\\
&\quad= \frac{(x-s)^{m-1}}{(m-1)!}\frac{(y-c)^j}{j!} - \Biggl(\sum_{q=1}^{j}\frac{(a-s)^{m-1}}{(m-1)!(j-q+1)!}\frac{k^j}{q!}S_q\biggl(\frac{y-c}{k}\biggr)\\
&\qquad+\sum_{q=1}^{j}\sum_{p=1}^{m}\frac{1}{(m-p)!(j-p+1)!}\bigl((b-s)^{m-p}-(a-s)^{m-p}\bigr)\\
&\qquad\times\frac{h^{p-1}}{p!}\frac{k^j}{q!}S_p\biggl(\frac{x-a}{h}\biggr)S_q\biggl(\frac{y-c}{k}\biggr)\Biggr) = 0, \quad j=1,2,\dots,n-1,\\
&K_{0,n}(x,y,t)\\
&\quad= \frac{(y-t)^{n-1}}{(n-1)!}\\
&\qquad-\Biggl(\frac{(c-t)^{n-1}}{(n-1)!}+\sum_{q=1}^{n}\frac{1}{(n-q)!}\bigl((d-t)^{n-q}-(c-t)^{n-q}\bigr)\frac{k^{q-1}}{q!}S_q\biggl(\frac{y-c}{k}\biggr)\Biggr) = 0,\\
&K_{i,n}(x,y,t)\\
&\quad= \frac{(x-a)^i}{i!}\frac{(y-t)^{n-1}}{(n-1)!} - \Biggl(\sum_{p=1}^{i}\frac{(c-t)^{n-1}}{(n-1)!(i-p+1)!}\frac{h^i}{p!}S_p\biggl(\frac{x-a}{h}\biggr)\\
&\qquad+\sum_{p=1}^{i}\sum_{q=1}^{n}\frac{1}{(n-q)!(i-q+1)!}\bigl((d-t)^{n-q}-(c-t)^{n-q}\bigr)\\
&\qquad\times\frac{h^i}{p!}\frac{k^{q-1}}{q!}S_p\biggl(\frac{x-a}{h}\biggr)S_q\biggl(\frac{y-c}{k}\biggr)\Biggr) = 0, \quad i=1,2,\dots,m-1,\\
&K_{m,n}(x,y,s,t)\\
&\quad= \frac{(x-s)^{m-1}}{(m-1)!}\frac{(y-t)^{n-1}}{(n-1)!} - \Biggl(\frac{(a-s)^{m-1}}{(m-1)!}\frac{(c-t)^{n-1}}{(n-1)!}\\
&\qquad+\sum_{p=1}^{m}\frac{(c-t)^{n-1}}{(m-p)!(n-1)!}\bigl((b-s)^{m-p}-(a-s)^{m-p}\bigr)\frac{h^{p-1}}{p!}S_p\biggl(\frac{x-a}{h}\biggr)\\
&\qquad+\sum_{q=1}^{n}\frac{(a-s)^{m-1}}{(m-1)!(n-q)!}\bigl((d-t)^{n-q}-(c-t)^{n-q}\bigr)\frac{k^{q-1}}{q!}S_q\biggl(\frac{y-c}{k}\biggr)\\
&\qquad+\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{1}{(m-p)!(n-q)!}\bigl((a-s)^{m-p}(c-t)^{n-q}-(b-s)^{m-p}(c-t)^{n-q}\\
&\qquad+(b-s)^{m-p}(d-t)^{n-q}-(a-s)^{m-p}(d-t)^{n-q}\bigr)\frac{h^{p-1}}{p!}\frac{k^{q-1}}{q!}S_p\biggl(\frac{x-a}{h}\biggr)\\
&\qquad\times S_q\biggl(\frac{y-c}{k}\biggr)\Biggr) = 0,
\end{aligned}
$$

because $\frac{(x-s)^i}{i!}\frac{(y-t)^j}{j!}$ is considered as a polynomial in x, y of degree (i,j), $i=0,1,\dots,m-1$, $j=0,1,\dots,n-1$ such that these must coincide with their generalized Taylor expansion (6). By definition of the positive part, these kernels are also zero in the interval $b < x < \overline{b}$ or

$d < y < \overline{d}$, so we give a proof of the first case (10) of Theorem 2. The remaining expressions of Theorem 2 can be proved by the analogous manners. □

Let us set

$$d[a,b](x) = \begin{cases} b-x, & x<a, \\ b-a, & a\le x\le b, \\ x-a, & b<x \end{cases} \quad \text{and} \quad d[c,d](y) = \begin{cases} d-y, & y<c, \\ d-c, & c\le y\le d, \\ y-c, & d<y, \end{cases} \tag{11}$$

i.e., $d[a,b](x)(d[c,d](y))$ is distance of $x(y)$ from the interval $[a,b]([c,d])$, plus $b-a(d-c)$; we set also

$$d^m[a,b](x) = \bigl(d[a,b](x)\bigr)^m, \qquad d^n[c,d](y) = \bigl(d[c,d](y)\bigr)^n.$$

By virtue of previous theorem, we obtain the desired bounds as follows.

Theorem 3 *Let $f \in \mathcal{C}^{(m,n)}([\overline{a},\overline{b}] \times [\overline{c},\overline{d}])$ and $(x,y) \in [\overline{a},\overline{b}] \times [\overline{c},\overline{d}]$. Then for the remainder* (9) *suppose that $(x,y) \in I_u \times I_v$, $u,v = 1,2,3$. Then*

$$\begin{aligned} &\bigl|R_{m,n}[f;a,b;c,d;h,k](x,y)\bigr| \\ &\quad\le F(m,n)\bigl(C_{m,n}^{y,d[c,d](y)} \cdot d^m[a,b](x) \\ &\qquad + C_{m,n}^{x,d[a,b](x)} \cdot d^n[c,d](y) + C_{m,n} \cdot d^m[a,b](x) \cdot d^n[c,d](y)\bigr); \end{aligned} \tag{12}$$

where

$$F(m,n) = \max_{\substack{0\le i\le m-1 \\ 0\le j\le n-1}} \{F_{m,j}, F_{i,n}, F_{m,n}\}, \qquad F_{i,j} = \sup_{(x,y)\in[\overline{a},\overline{b}]\times[\overline{c},\overline{d}]} \bigl|f^{(i,j)}(x,y)\bigr|, \tag{13}$$

$$\begin{aligned} C_{m,n}^{y,z} &= \frac{1}{m!}\Biggl(1 + \sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}\binom{p}{l_1}|B_{p-l_1}| \\ &\quad + \sum_{j=1}^{n-1}\sum_{q=1}^{j}\sum_{l_2=1}^{q}\Biggl(1 + \sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}|B_{p-l_1}|\Biggr)|B_{q-l_2}|\binom{j+1}{q}\frac{z^j}{(j+1)!}\Biggr), \end{aligned} \tag{14}$$

$$\begin{aligned} C_{m,n}^{x,z} &= \frac{1}{n!}\Biggl(1 + \sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}\binom{q}{l_2}|B_{q-l_2}| \\ &\quad + \sum_{i=1}^{m-1}\sum_{p=1}^{i}\sum_{l_1=1}^{p}\Biggl(1 + \sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}|B_{q-l_2}|\Biggr)|B_{p-l_1}|\binom{i+1}{p}\frac{z^i}{(i+1)!}\Biggr), \end{aligned} \tag{15}$$

$$\begin{aligned} C_{m,n} &= \frac{1}{m!n!}\Biggl(1 + \sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}\binom{p}{l_1}|B_{p-l_1}| + \sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}\binom{q}{l_2}|B_{q-l_2}| \\ &\quad + \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{l_1=1}^{p}\sum_{l_2=1}^{q}\binom{m}{p}\binom{n}{q}\binom{p}{l_1}\binom{q}{l_2}|B_{p-l_1}||B_{q-l_2}|\Biggr). \end{aligned} \tag{16}$$

Proof Let $x \in I_k$, $k = 1$, i.e., $x \in [\overline{a}, a]$, $y \in [\overline{c}, c]$, then we have from the first case (10) of Theorem 2 that

$$\begin{aligned}
&R_{m,n}[f;a,b;c,d;h,k](x,y) \\
&= \left(\int_x^a f^{(m,0)}(s,c)K_{m,0}(x,y,s)\,ds + \int_a^b f^{(m,0)}(s,c)K_{m,0}(x,y,s)\,ds\right) \\
&\quad + \sum_{j=1}^{n-1}\left(\int_x^a f^{(m,j)}(s,c)K_{m,j}(x,y,s)\,ds + \int_a^b f^{(m,j)}(s,c)K_{m,j}(x,y,s)\,ds\right) \\
&\quad + \left(\int_y^c f^{(0,n)}(a,t)K_{0,n}(x,y,t)\,dt + \int_c^d f^{(0,n)}(a,t)K_{0,n}(x,y,t)\,dt\right) \\
&\quad + \sum_{i=1}^{m-1}\left(\int_y^c f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt + \int_c^d f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt\right) \\
&\quad + \int_x^a\int_y^c f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt + \int_x^a\int_c^d f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt \\
&\quad + \int_a^b\int_y^c f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt + \int_a^b\int_c^d f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt \\
&:= G_1 + G_2 + \sum_{j=1}^{n-1}\left(G_3^j + G_4^j\right) + G_5 + G_6 + \sum_{i=1}^{m-1}\left(G_7^i + G_8^i\right) + G_9 + G_{10} + G_{11} + G_{12}.
\end{aligned}$$

If $x < s < a$, then

$$\begin{aligned}
&K_{m,0}(x,y,s) \\
&= -\frac{(a-s)^{m-1}}{(m-1)!} - \sum_{p=1}^{m}\frac{h^{p-1}}{(m-p)!p!}\left((b-s)^{m-p} - (a-s)^{m-p}\right)S_p\left(\frac{x-a}{h}\right), \\
&K_{m,j}(x,y,s) \\
&= -\sum_{q=1}^{j}\frac{(a-s)^{m-1}}{(m-1)!(j-q+1)!}\frac{k^j}{q!}S_q\left(\frac{y-c}{k}\right) \\
&\quad - \sum_{q=1}^{j}\sum_{p=1}^{m}\frac{h^{p-1}k^j}{(m-p)!p!(j-q+1)!q!} \\
&\quad \times \left((b-s)^{m-p} - (a-s)^{m-p}\right)S_p\left(\frac{x-a}{h}\right)S_q\left(\frac{y-c}{k}\right), \quad j = 1,2,\ldots,n-1,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_x^a f^{(m,0)}(s,c)K_{m,0}(x,y,s)\,ds \\
&:= G_1 \\
&= -\frac{1}{(m-1)!}\int_x^a f^{(m,0)}(s,c)(a-s)^{m-1}\,ds - \sum_{p=1}^{m}\frac{1}{(m-p)!p!}h^{p-1}S_p\left(\frac{x-a}{h}\right)
\end{aligned}$$

$$\times \int_x^a f^{(m,0)}(s,c)\big((b-s)^{m-p}-(a-s)^{m-p}\big)\,ds,$$

$$\begin{aligned}
&\int_x^a f^{(m,j)}(s,c)K_{m,j}(x,y,s)\,ds \\
&:= G_3^j \\
&= -\sum_{q=1}^{j}\frac{k^j}{(m-1)!(j-q+1)!q!}S_q\left(\frac{y-c}{k}\right)\int_x^a f^{(m,j)}(s,c)(a-s)^{m-1}\,ds \\
&\quad -\sum_{q=1}^{j}\sum_{p=1}^{m}\frac{1}{(m-p)!(j-q+1)!p!q!}h^{p-1}k^jS_p\left(\frac{x-a}{h}\right)S_q\left(\frac{y-c}{k}\right) \\
&\quad \times \int_x^a f^{(m,j)}(s,c)\big((b-s)^{m-p}-(a-s)^{m-p}\big)\,ds, \quad j=1,2,\dots,n-1.
\end{aligned}$$

Note that the integrands are of type $h(s)f^{(m,j)}(s,c)$, $j=0,1,\dots,n-1$ with a $h(s)$ that does not change sign in $[x,a]$. By using the first mean value theorem for integrals [39], we find

$$\begin{aligned}
G_1 &= -\frac{1}{(m-1)!}f^{(m,0)}\big(\xi_{m,0}^{\langle x,a\rangle},c\big)\int_x^a (a-s)^{m-1}\,ds \\
&\quad -\sum_{p=1}^{m}\frac{h^{p-1}}{(m-p)!p!}S_p\left(\frac{x-a}{h}\right)f^{(m,0)}\big(\xi_{m,0,p}^{\langle x,a\rangle},c\big)\int_x^a \big((b-s)^{m-p}-(a-s)^{m-p}\big)\,ds, \\
G_3^j &= -\sum_{q=1}^{j}\frac{k^j}{(m-1)!(j-q+1)!q!}S_q\left(\frac{y-c}{k}\right)f^{(m,j)}\big(\xi_{m,j}^{\langle x,a\rangle},c\big)\int_x^a (a-s)^{m-1}\,ds \\
&\quad -\sum_{q=1}^{j}\sum_{p=1}^{m}\frac{h^{p-1}k^j}{(m-p)!(j-q+1)!p!q!}S_p\left(\frac{x-a}{h}\right)S_q\left(\frac{y-c}{k}\right)f^{(m,j)}\big(\xi_{m,j,p}^{\langle x,a\rangle},c\big) \\
&\quad \times \int_x^a \big((b-s)^{m-p}-(a-s)^{m-p}\big)\,ds, \quad j=1,2,\dots,n-1
\end{aligned}$$

for some $\xi_{m,j}^{\langle x,a\rangle}$, $\xi_{m,j,p}^{\langle x,a\rangle}\in[\overline{a},\overline{b}]$, $j=0,1,\dots,n-1$, $p=1,2,\dots,m$, so that we have after some calculations

$$\begin{aligned}
G_1 = {}& -f^{(m,0)}\big(\xi_{m,0}^{\langle x,a\rangle},c\big)\frac{(a-x)^m}{m!}-\frac{h^m}{m!}\sum_{p=1}^{m}\frac{m!}{(m-p+1)!p!}S_p\left(\frac{x-a}{h}\right) \\
&\times f^{(m,0)}\big(\xi_{m,0,p}^{\langle x,a\rangle},c\big)\left(-1+\sum_{u=0}^{m-p}\left(\frac{b-x}{h}\right)^{m-p-u}\left(\frac{a-x}{h}\right)^{u}\right),
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
G_3^j = {}& -\frac{1}{m!}\sum_{q=1}^{j}\frac{(j+1)!}{(j-q+1)!q!}\frac{k^j}{(j+1)!}S_q\left(\frac{y-c}{k}\right)f^{(m,j)}\big(\xi_{m,j}^{\langle x,a\rangle},c\big)(a-x)^m \\
&-\frac{h^m}{m!}\sum_{q=1}^{j}\sum_{p=1}^{m}\frac{m!(j+1)!}{(m-p+1)!(j-q+1)!p!q!}\frac{k^j}{(j+1)!}S_p\left(\frac{x-a}{h}\right)S_q\left(\frac{y-c}{k}\right)
\end{aligned} \tag{18}$$

$$\times f^{(m,j)}\big(\xi_{m,j,p}^{\langle x,a\rangle},c\big)\left(-1+\sum_{u=0}^{m-p}\left(\frac{b-x}{h}\right)^{m-p-u}\left(\frac{a-x}{h}\right)^{u}\right),\quad j=1,2,\ldots,n-1.$$

If $a < s < b$, then

$$\begin{aligned}K_{m,0}(x,y,s)&=-\sum_{p=1}^{m}\frac{h^{p-1}}{(m-p)!p!}(b-s)^{m-p}S_p\left(\frac{x-a}{h}\right),\\K_{m,j}(x,y,s)&=-\sum_{q=1}^{j}\sum_{p=1}^{m}\frac{h^{p-1}k^j}{(m-p)!p!(j-q+1)!q!}(b-s)^{m-p}S_p\left(\frac{x-a}{h}\right)\\&\quad\times S_q\left(\frac{y-c}{k}\right),\quad j=1,2,\ldots,n-1,\end{aligned}$$

so that

$$\begin{aligned}&\int_a^b f^{(m,0)}(s,c)K_{m,0}(x,y,s)\,ds\\&\quad:=G_2\\&\quad=-\sum_{p=1}^{m}\frac{h^{p-1}}{(m-p)!p!}S_p\left(\frac{x-a}{h}\right)\int_a^b f^{(m,0)}(s,c)(b-s)^{m-p}\,ds,\\&\int_a^b f^{(m,j)}(s,c)K_{m,j}(x,y,s)\,ds\\&\quad:=G_4^j\\&\quad=-\sum_{q=1}^{j}\sum_{p=1}^{m}\frac{h^{p-1}k^j}{(m-p)!p!(j-q+1)!q!}S_p\left(\frac{x-a}{h}\right)\\&\qquad\times S_q\left(\frac{y-c}{k}\right)\int_a^b f^{(m,j)}(s,c)(b-s)^{m-p}\,ds,\quad j=1,2,\ldots,n-1,\end{aligned}$$

and by using the first mean value theorem for integrals, we have after some calculations

$$G_2=-\frac{h^m}{m!}\sum_{p=1}^{m}\frac{m!}{(m-p+1)!p!}S_p\left(\frac{x-a}{h}\right)f^{(m,0)}\big(\xi_{m,0}^{\langle a,b\rangle},c\big),\tag{19}$$

and

$$\begin{aligned}G_4^j=&-\frac{h^m}{m!}\sum_{q=1}^{j}\sum_{p=1}^{m}\frac{m!(j+1)!}{(m-p+1)!(j-q+1)!p!q!}\frac{k^j}{(j+1)!}\\&\times S_p\left(\frac{x-a}{h}\right)S_q\left(\frac{y-c}{k}\right)f^{(m,j)}\big(\xi_{m,j,p}^{\langle a,b\rangle},c\big),\quad j=1,2,\ldots,n-1\end{aligned}\tag{20}$$

for some $\xi_{m,j}^{\langle a,b\rangle},\xi_{m,j,p}^{\langle a,b\rangle}\in[\overline{a},\overline{b}]$, $j=0,1,\ldots,n-1$, $p=1,2,\ldots,m$.

Similarly, as the discussion of G_1, G_3^j, G_2, G_4^j, we have for some $\eta_{i,n}^{\langle y,c\rangle}$, $\eta_{i,n,q}^{\langle y,c\rangle}$, $\eta_{i,n}^{\langle c,d\rangle}$, $\eta_{i,n,q}^{\langle c,d\rangle}\in[\overline{c},\overline{d}]$, $i=0,1,\ldots,m-1$, $q=1,2,\ldots,n$, respectively:

$$\int_y^c f^{(0,n)}(a,t)K_{0,n}(x,y,t)\,dt$$

$$\begin{aligned}
&:= G_5 \\
&= -f^{(0,n)}\bigl(a,\eta_{0,n}^{\langle y,c\rangle}\bigr)\frac{(c-y)^n}{n!} - \frac{k^n}{n!}\sum_{q=1}^{n}\frac{n!}{(n-q+1)!q!}S_q\left(\frac{y-c}{k}\right) \\
&\quad\times f^{(0,n)}\bigl(a,\eta_{0,n,q}^{\langle y,c\rangle}\bigr)\left(-1+\sum_{\nu=0}^{n-q}\left(\frac{d-y}{k}\right)^{n-q-\nu}\left(\frac{c-y}{k}\right)^{\nu}\right),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\int_y^c f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt \\
&:= G_7^i \\
&= -\frac{1}{n!}\sum_{p=1}^{i}\frac{(i+1)!}{(i-p+1)!p!}\frac{h^i}{(i+1)!}S_p\left(\frac{x-a}{h}\right)f^{(i,n)}\bigl(a,\eta_{i,n}^{\langle y,c\rangle}\bigr)(c-y)^n \\
&\quad-\frac{k^n}{n!}\sum_{p=1}^{i}\sum_{q=1}^{n}\frac{n!(i+1)!}{(n-q+1)!(i-p+1)!p!q!}\frac{h^i}{(i+1)!}S_p\left(\frac{x-a}{h}\right)S_q\left(\frac{y-c}{k}\right) \\
&\quad\times f^{(i,n)}\bigl(a,\eta_{i,n,q}^{\langle y,c\rangle}\bigr)\left(-1+\sum_{\nu=0}^{n-q}\left(\frac{d-y}{k}\right)^{n-q-\nu}\left(\frac{c-y}{k}\right)^{\nu}\right), \quad i=1,2,\dots,m-1,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\int_c^d f^{(0,n)}(a,t)K_{0,n}(x,y,t)\,dt \\
&:= G_6 \\
&= -\frac{k^n}{n!}\sum_{q=1}^{n}\frac{n!}{(n-q+1)!q!}S_q\left(\frac{y-c}{k}\right)f^{(0,n)}\bigl(a,\eta_{0,n}^{\langle c,d\rangle}\bigr),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\int_c^d f^{(i,n)}(a,t)K_{i,n}(x,y,t)\,dt \\
&:= G_8^i \\
&= -\frac{k^n}{n!}\sum_{p=1}^{i}\sum_{q=1}^{n}\frac{n!(i+1)!}{(n-q+1)!(i-p+1)!p!q!}\frac{h^i}{(i+1)!}S_p\left(\frac{x-a}{h}\right)S_q\left(\frac{y-c}{k}\right) \\
&\quad\times f^{(i,n)}\bigl(a,\eta_{i,n,q}^{\langle c,d\rangle}\bigr), \quad i=1,2,\dots,m-1.
\end{aligned} \tag{24}$$

If $x < s < a$, $y < t < c$, then

$$\begin{aligned}
&K_{m,n}(x,y,s,t) \\
&= -\frac{(a-s)^{m-1}}{(m-1)!}\frac{(c-t)^{n-1}}{(n-1)!} \\
&\quad-\sum_{p=1}^{m}\frac{(c-t)^{n-1}h^{p-1}}{(m-p)!p!(n-1)!}\bigl((b-s)^{m-p}-(a-s)^{m-p}\bigr)S_p\left(\frac{x-a}{h}\right) \\
&\quad-\sum_{q=1}^{n}\frac{(a-s)^{m-1}k^{q-1}}{(m-1)!(n-q)!q!}\bigl((d-t)^{n-q}-(c-t)^{n-q}\bigr)S_q\left(\frac{y-c}{k}\right)
\end{aligned}$$

$$
- \sum_{p=1}^{m} \sum_{q=1}^{n} \frac{h^{p-1}k^{q-1}}{(m-p)!p!(n-q)!q!} \big((b-s)^{m-p} - (a-s)^{m-p}\big)\big((d-t)^{n-q} - (c-t)^{n-q}\big)
\times S_p\left(\frac{x-a}{h}\right) S_q\left(\frac{y-c}{k}\right),
$$

so that

$$
\begin{aligned}
&\int_x^a \int_y^c f^{(m,n)}(s,t) K_{m,n}(x,y,s,t)\, ds\, dt \\
&:= G_9 \\
&= -\frac{1}{(m-1)!(n-1)!} \int_x^a \int_y^c f^{(m,n)}(s,t)(a-s)^{m-1}(c-t)^{n-1}\, ds\, dt \\
&\quad - \frac{1}{(n-1)!} \sum_{p=1}^{m} \frac{h^{p-1}}{(m-p)!p!} S_p\left(\frac{x-a}{h}\right) \int_x^a \int_y^c f^{(m,n)}(s,t)(c-t)^{n-1} \\
&\quad \times \big((b-s)^{m-p} - (a-s)^{m-p}\big)\, ds\, dt \\
&\quad - \frac{1}{(m-1)!} \sum_{q=1}^{n} \frac{k^{q-1}}{(n-q)!q!} S_q\left(\frac{y-c}{k}\right) \int_x^a \int_y^c f^{(m,n)}(s,t)(a-s)^{m-1} \\
&\quad \times \big((d-t)^{n-q} - (c-t)^{n-q}\big)\, ds\, dt \\
&\quad - \sum_{p=1}^{m} \sum_{q=1}^{n} \frac{h^{p-1}k^{q-1}}{(m-p)!(n-q)!p!q!} S_p\left(\frac{x-a}{h}\right) S_q\left(\frac{y-c}{k}\right) \\
&\quad \times \int_x^a \int_y^c f^{(m,n)}(s,t)\big((b-s)^{m-p} - (a-s)^{m-p}\big)\big((d-t)^{n-q} - (c-t)^{n-q}\big)\, ds\, dt.
\end{aligned}
$$

Note that the integrands are of type $g(s)h(t)f^{(m,n)}(s,t)$, with a $g(s)h(t)$ that does not change sign in $[x,a] \times [y,c]$. By applying the first mean value theorem for integrals [39], we have

$$
\begin{aligned}
G_9 = &-\frac{1}{(m-1)!(n-1)!} f^{(m,n)}\big(\xi^{\langle a,c\rangle}, \eta^{\langle a,c\rangle}\big) \int_x^a \int_y^c (a-s)^{m-1}(c-t)^{n-1}\, ds\, dt \\
&- \frac{1}{(n-1)!} \sum_{p=1}^{m} \frac{h^{p-1}}{(m-p)!p!} S_p\left(\frac{x-a}{h}\right) f^{(m,n)}\big(\xi_{m,p}^{\langle a,c\rangle}, \eta_{m,p}^{\langle a,c\rangle}\big) \\
&\times \int_x^a \int_y^c (c-t)^{n-1}\big((b-s)^{m-p} - (a-s)^{m-p}\big)\, ds\, dt \\
&- \frac{1}{(m-1)!} \sum_{q=1}^{n} \frac{k^{q-1}}{(n-q)!q!} S_q\left(\frac{y-c}{k}\right) f^{(m,n)}\big(\xi_{n,q}^{\langle c,a\rangle}, \eta_{n,q}^{\langle c,a\rangle}\big) \\
&\times \int_x^a \int_y^c (a-s)^{m-1}\big((d-t)^{n-q} - (c-t)^{n-q}\big)\, ds\, dt \\
&- \sum_{p=1}^{m} \sum_{q=1}^{n} \frac{h^{p-1}k^{q-1}}{(m-p)!(n-q)!p!q!} S_p\left(\frac{x-a}{h}\right) S_q\left(\frac{y-c}{k}\right) f^{(m,n)}\big(\xi_{m,n,p,q}^{\langle a,c\rangle}, \eta_{m,n,p,q}^{\langle a,c\rangle}\big) \\
&\times \int_x^a \int_y^c \big((b-s)^{m-p} - (a-s)^{m-p}\big)\big((d-t)^{n-q} - (c-t)^{n-q}\big)\, ds\, dt
\end{aligned}
$$

for some points $(\xi^{\langle a,c\rangle},\eta^{\langle a,c\rangle})$, $(\xi^{\langle a,c\rangle}_{m,p},\eta^{\langle a,c\rangle}_{m,p})$, $(\xi^{\langle c,a\rangle}_{n,q},\eta^{\langle c,a\rangle}_{n,q})$, $(\xi^{\langle a,c\rangle}_{m,n,p,q},\eta^{\langle a,c\rangle}_{m,n,p,q})\in[\overline{a},\overline{b}]\times[\overline{c},\overline{d}]$, $p=1,2,\dots,m$, $q=1,2,\dots,n$, so that we have after some calculations

$$\begin{aligned}
G_9 = {}& -f^{(m,n)}\big(\xi^{\langle a,c\rangle},\eta^{\langle a,c\rangle}\big)\frac{(a-x)^m}{m!}\frac{(c-y)^n}{n!} \\
& -\frac{1}{n!m!}h^m\sum_{p=1}^{m}\frac{m!}{(m-p+1)!p!}S_p\left(\frac{x-a}{h}\right)f^{(m,n)}\big(\xi^{\langle a,c\rangle}_{m,p},\eta^{\langle a,c\rangle}_{m,p}\big) \\
& \times\left(-1+\sum_{u=0}^{m-p}\left(\frac{b-x}{h}\right)^{m-p-u}\left(\frac{a-x}{h}\right)^{u}\right)(c-y)^n \\
& -\frac{1}{m!n!}k^n\sum_{q=1}^{n}\frac{n!}{(n-q+1)!q!}S_q\left(\frac{y-c}{k}\right)f^{(m,n)}\big(\xi^{\langle c,a\rangle}_{n,q},\eta^{\langle c,a\rangle}_{n,q}\big) \\
& \times\left(-1+\sum_{v=0}^{n-q}\left(\frac{d-y}{k}\right)^{n-q-v}\left(\frac{c-y}{k}\right)^{v}\right)(a-x)^m \\
& -\frac{1}{m!n!}h^mk^n\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{m!n!S_p(\frac{x-a}{h})S_q(\frac{y-c}{k})}{(m-p+1)!(n-q+1)!p!q!}f^{(m,n)}\big(\xi^{\langle a,c\rangle}_{m,n,p,q},\eta^{\langle a,c\rangle}_{m,n,p,q}\big) \\
& \times\left(-1+\sum_{u=0}^{m-p}\left(\frac{b-x}{h}\right)^{m-p-u}\left(\frac{a-x}{h}\right)^{u}\right)\left(-1+\sum_{v=0}^{n-q}\left(\frac{d-y}{k}\right)^{n-q-v}\left(\frac{c-y}{k}\right)^{v}\right).
\end{aligned}\tag{25}$$

Similarly, as the discussion of G_9, we have also for some $(\xi^{\langle a,d\rangle}_{n,q},\eta^{\langle a,d\rangle}_{n,q})$, $(\xi^{\langle a,d\rangle}_{m,n,p,q},\eta^{\langle a,d\rangle}_{m,n,p,q})$, $(\xi^{\langle b,c\rangle}_{m,p},\eta^{\langle b,c\rangle}_{m,p})$, $(\xi^{\langle b,c\rangle}_{m,n,p,q},\eta^{\langle b,c\rangle}_{m,n,p,q})$, $(\xi^{\langle a,d\rangle}_{m,n,p,q},\eta^{\langle a,d\rangle}_{m,n,p,q})$, $(\xi^{\langle a,d\rangle}_{m,n,p,q},\eta^{\langle a,d\rangle}_{m,n,p,q})\in[\overline{a},\overline{b}]\times[\overline{c},\overline{d}]$, $p=1,2,\dots,m$, $q=1,2,\dots,n$ that

$$\begin{aligned}
&\int_x^a\int_c^d f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt \\
&\quad:= G_{10} \\
&\quad= -\frac{k^n}{m!n!}\sum_{q=1}^{n}\frac{n!}{(n-q+1)!q!}S_q\left(\frac{y-c}{k}\right)f^{(m,n)}\big(\xi^{\langle a,d\rangle}_{n,q},\eta^{\langle a,d\rangle}_{n,q}\big)(x-a)^m \\
&\qquad -\frac{h^mk^n}{m!n!}\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{m!n!S_p(\frac{x-a}{h})S_q(\frac{y-c}{k})}{(m-p+1)!(n-q+1)!p!q!}f^{(m,n)}\big(\xi^{\langle a,d\rangle}_{m,n,p,q},\eta^{\langle a,d\rangle}_{m,n,p,q}\big) \\
&\qquad \times\left(-1+\sum_{u=0}^{m-p}\left(\frac{b-x}{h}\right)^{m-p-u}\left(\frac{a-x}{h}\right)^{u}\right),
\end{aligned}\tag{26}$$

$$\begin{aligned}
&\int_a^b\int_y^c f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt \\
&\quad:= G_{11} \\
&\quad= -\frac{h^m}{m!n!}\sum_{p=1}^{m}\frac{m!S_p(\frac{x-a}{h})}{(m-p+1)!p!}f^{(m,n)}\big(\xi^{\langle b,c\rangle}_{m,p},\eta^{\langle b,c\rangle}_{m,p}\big)(c-y)^n \\
&\qquad -\frac{h^mk^n}{m!n!}\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{m!n!S_p(\frac{x-a}{h})S_q(\frac{y-c}{k})}{(m-p+1)!(n-q+1)!p!q!}f^{(m,n)}\big(\xi^{\langle b,c\rangle}_{m,n,p,q},\eta^{\langle b,c\rangle}_{m,n,p,q}\big)
\end{aligned}\tag{27}$$

$$\times \left(-1+\sum_{\nu=0}^{n-q}\left(\frac{d-y}{k}\right)^{n-q-\nu}\left(\frac{c-y}{k}\right)^{\nu}\right),$$

and

$$\begin{aligned}
&\int_a^b\int_c^d f^{(m,n)}(s,t)K_{m,n}(x,y,s,t)\,ds\,dt \\
&:= G_{12} \\
&= -\frac{h^m k^n}{m!n!}\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{m!n!S_p(\frac{x-a}{h})S_q(\frac{y-c}{k})}{(m-p+1)!(n-q+1)!p!q!}f^{(m,n)}\left(\xi_{m,n,p,q}^{\langle a,d\rangle},\eta_{m,n,p,q}^{\langle a,d\rangle}\right).
\end{aligned} \tag{28}$$

We finally obtain the following desire estimation for the error in terms of the above expression:

$$\begin{aligned}
&\left|R_{m,n}[f;a,b;c,d;h,k](x,y)\right| \\
&\quad\le \left(|G_1|+|G_2|\right)+\sum_{j=1}^{n-1}\left(\left|G_3^j\right|+\left|G_4^j\right|\right)+\left(|G_5|+|G_6|\right)+\sum_{i=1}^{m-1}\left(\left|G_7^i\right|+\left|G_8^i\right|\right) \\
&\qquad+\left(|G_9|+|G_{10}|\right)+\left(|G_{11}|+|G_{12}|\right).
\end{aligned}$$

In order to prove the bound (12), we apply the well-known identities

$$\begin{aligned}
&S_p(x)=B_p(x)-B_p=\sum_{l_1=1}^{p}\binom{p}{l_1}B_{p-l_1}x^{l_1}, \quad p=1,2,\ldots, \\
&S_q(y)=B_q(y)-B_q=\sum_{l_2=1}^{q}\binom{q}{l_2}B_{q-l_2}y^{l_2}, \quad q=1,2,\ldots.
\end{aligned} \tag{29}$$

For the term $(|G_1|+|G_2|)$, we have after some calculations

$$\begin{aligned}
&|G_1|+|G_2| \\
&\quad\le \frac{F(m,n)}{m!}\left((a-x)^m+h^m\sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}\binom{p}{l_1}|B_{p-l_1}|\left(\frac{b-x}{h}\right)^{m-(p-l_1)}\right) \\
&\quad\le \frac{F(m,n)}{m!}h^m\left(1+\sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}\binom{p}{l_1}|B_{p-l_1}|\right)\left(\frac{b-x}{h}\right)^m.
\end{aligned}$$

For the term $(|G_3^j|+|G_4^j|)$, $j=1,2,\ldots,n-1$, we have after some calculations

$$\begin{aligned}
&\left|G_3^j\right|+\left|G_4^j\right| \\
&\quad\le \frac{F(m,n)}{m!}\left(\sum_{q=1}^{j}\sum_{l_2=1}^{q}\binom{j+1}{q}\frac{k^j}{(j+1)!}|B_{q-l_2}|(a-x)^m\left(\frac{d-y}{k}\right)^{l_2}\right. \\
&\qquad+h^m\sum_{q=1}^{j}\sum_{p=1}^{m}\sum_{l_1=1}^{p}\sum_{l_2=1}^{q}\binom{m}{p}\binom{j+1}{q}\frac{k^j}{(j+1)!}|B_{p-l_1}||B_{q-l_2}|
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\frac{b-x}{h}\right)^{m-(p-l_1)}\left(\frac{d-y}{k}\right)^{l_2}\Biggr)\\
&\leq\frac{F(m,n)}{m!}h^m k^j\Biggl(\sum_{q=1}^{j}\sum_{l_2=1}^{q}\binom{j+1}{q}\frac{1}{(j+1)!}|B_{q-l_2}|\\
&\quad+\sum_{q=1}^{j}\sum_{p=1}^{m}\sum_{l_1=1}^{p}\sum_{l_2=1}^{q}\binom{m}{p}\binom{j+1}{q}\frac{1}{(j+1)!}|B_{p-l_1}||B_{q-l_2}|\Biggr)\left(\frac{b-x}{h}\right)^{m}\left(\frac{d-y}{k}\right)^{j}.
\end{aligned}
$$

For the term $(|G_5|+|G_6|)$, we have after some calculations

$$
\begin{aligned}
&|G_5|+|G_6|\\
&\quad\leq\frac{F(m,n)}{n!}\left((c-y)^n+k^n\sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}\binom{q}{l_2}|B_{q-l_2}|\left(\frac{d-y}{k}\right)^{n-(q-l_2)}\right)\\
&\quad\leq\frac{F(m,n)}{n!}k^n\left(1+\sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}\binom{q}{l_2}|B_{q-l_2}|\right)\left(\frac{d-y}{k}\right)^{n}.
\end{aligned}
$$

For the term $(|G_7^i|+|G_8^i|)$, $i=1,2,\dots,m-1$, we have after some calculations

$$
\begin{aligned}
&\left|G_7^i\right|+\left|G_8^i\right|\\
&\quad\leq\frac{F(m,n)}{n!}\Biggl(\sum_{p=1}^{i}\sum_{l_1=1}^{p}\binom{i+1}{p}\frac{h^i}{(i+1)!}|B_{p-l_1}|(c-y)^n\left(\frac{b-x}{h}\right)^{l_1}\\
&\qquad+k^n\sum_{p=1}^{i}\sum_{q=1}^{n}\sum_{l_2=1}^{q}\sum_{l_1=1}^{p}\binom{n}{q}\binom{i+1}{p}\frac{h^i}{(i+1)!}|B_{q-l_2}||B_{p-l_1}|\\
&\qquad\times\left(\frac{d-y}{k}\right)^{n-(q-l_2)}\left(\frac{b-x}{h}\right)^{l_1}\Biggr)\\
&\quad\leq\frac{F(m,n)}{n!}h^i k^n\Biggl(\sum_{p=1}^{i}\sum_{l_1=1}^{p}\binom{i+1}{p}\frac{1}{(i+1)!}|B_{p-l_1}|\\
&\qquad+\sum_{p=1}^{i}\sum_{q=1}^{n}\sum_{l_2=1}^{q}\sum_{l_1=1}^{p}\binom{n}{q}\binom{i+1}{p}\frac{1}{(i+1)!}|B_{q-l_2}||B_{p-l_1}|\Biggr)\left(\frac{d-y}{k}\right)^{n}\left(\frac{b-x}{h}\right)^{i}.
\end{aligned}
$$

For the term $(|G_9|+|G_{10}|)$, we have after some calculations

$$
\begin{aligned}
&|G_9|+|G_{10}|\\
&\quad\leq\frac{F(m,n)}{m!n!}(a-x)^m(c-y)^n+\frac{F(m,n)}{m!n!}h^m\sum_{p=1}^{m}\frac{m!}{(m-p+1)!p!}\left|S_p\left(\frac{x-a}{h}\right)\right|\\
&\qquad\times\left(-1+\sum_{u=0}^{m-p}\left(\frac{b-x}{h}\right)^{m-p-u}\left(\frac{a-x}{h}\right)^{u}\right)(c-y)^n\\
&\qquad+\frac{F(m,n)}{m!n!}k^n\sum_{q=1}^{n}\frac{n!}{(n-q)!q!}\left|S_q\left(\frac{y-c}{k}\right)\right|\left(\frac{d-y}{k}\right)^{n-q}(a-x)^m
\end{aligned}
$$

$$+\frac{F(m,n)}{m!n!}h^m k^n \sum_{p=1}^{m}\sum_{q=1}^{n}\frac{m!n!}{(m-p+1)!p!(n-q)!q!}\left|S_p\left(\frac{x-a}{h}\right)\right|\left|S_q\left(\frac{y-c}{k}\right)\right|$$
$$\times\left(-1+\sum_{u=0}^{m-p}\left(\frac{b-x}{h}\right)^{m-p-u}\left(\frac{a-x}{h}\right)^{u}\right)\left(\frac{d-y}{k}\right)^{n-q}.$$

For the term $(|G_{11}|+|G_{12}|)$, we have after some calculations

$$|G_{11}|+|G_{12}|\le\frac{F(m,n)}{m!n!}h^m\sum_{p=1}^{m}\frac{m!}{(m-p+1)!p!}\left|S_p\left(\frac{x-a}{h}\right)\right|(c-y)^n$$
$$+\frac{F(m,n)}{m!n!}h^m k^n\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{m!n!}{(m-p+1)!p!(n-q)!q!}$$
$$\times\left|S_p\left(\frac{x-a}{h}\right)\right|\left|S_q\left(\frac{y-c}{k}\right)\right|\left(\frac{d-y}{k}\right)^{n-q}.$$

By combining the estimates of $|G_9|+|G_{10}|$, $|G_{11}|+|G_{12}|$, we obtain after some calculations

$$|G_9|+|G_{10}|+|G_{11}|+|G_{12}|$$
$$\le\frac{F(m,n)}{m!n!}\left((a-x)^m(c-y)^n+h^m\sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}\binom{p}{l_1}|B_{p-l_1}|\left(\frac{b-x}{h}\right)^{m-(p-L_1)}\right.$$
$$\times(c-y)^n+k^n\sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}\binom{q}{l_2}|B_{q-l_2}|\left(\frac{d-y}{k}\right)^{n-(q-l_2)}(a-x)^m$$
$$+h^m k^n\sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{l_1=1}^{p}\sum_{l_2=1}^{q}\binom{m}{p}\binom{n}{q}\binom{p}{l_1}\binom{q}{l_2}|B_{p-l_1}||B_{q-l_2}|$$
$$\left.\times\left(\frac{b-x}{h}\right)^{m-(p-l_1)}\left(\frac{d-y}{k}\right)^{n-(q-l_2)}\right)$$
$$\le\frac{F(m,n)}{m!n!}h^m k^n\left(1+\sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}\binom{p}{l_1}|B_{p-l_1}|+\sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}\binom{q}{l_2}|B_{q-l_2}|\right.$$
$$\left.+\sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{l_1=1}^{p}\sum_{l_2=1}^{q}\binom{m}{p}\binom{n}{q}\binom{p}{l_1}\binom{q}{l_2}|B_{p-l_1}||B_{q-l_2}|\right)\left(\frac{b-x}{h}\right)^m\left(\frac{d-y}{k}\right)^n.$$

By combining the estimates of $|G_1|+|G_2|$, $|G_3^j|+|G_4^j|$, $|G_5|+|G_6|$, and $(|G_9|+|G_{10}|+|G_{11}|+|G_{12}|)$, we finish the proof in the first case of (12) in Theorem 3. The other expressions of the bounds may be proved in an analogous manner. □

Since the degree of exactness of the operator $B_{m,n}[f;a,b;c,d;h,k]$ is equal to (m,n), we can obtain the following desired bounds in an analogous manner.

Theorem 4 *Let* $f\in C^{(m+1,n+1)}([\overline{a},\overline{b}]\times[\overline{c},\overline{d}])$ *and* $(x,y)\in[\overline{a},\overline{b}]\times[\overline{c},\overline{d}]$. *For the remainder* (9) *suppose that* $(x,y)\in I_u\times I_v$, $u,v=1,2,3$. *Then*

$$\left|R_{m,n}[f;a,b;c,d;h,k](x,y)\right|$$

$$\leq F(m+1,n+1)\big(C_{m+1,n+1}^{y,d[c,d](y)} \cdot d^{m+1}[a,b](x) + C_{m+1,n+1}^{x,d[a,b](x)} \cdot d^{n}[c,d](y) + C_{m+1,n+1} \cdot d^{m+1}[a,b](x) \cdot d^{n+1}[c,d](y)\big);$$

where $F(m,n)$, $C_{m,n}^{y,z}$, $C_{m,n}^{x,z}$, $C_{m,n}$ *are defined in* (13), (14), (15), (16), *respectively.*

3 The bivariate Bernoulli-type multiquadric quasi-interpolation operators

3.1 A kind of bivariate Bernoulli-type multiquadric quasi-interpolation operator with higher approximation order

In this section, we introduce a well-known univariate quasi-interpolation operator. On this basis, we develop the bivariate multiquadric quasi-interpolation.

Given data $\{x_j,f_j\}, f_j = f(x_j)$, the univariate multiquadric quasi-interpolation operator $\mathcal{L}_B$ is constructed by Beatson and Powell in [14] as follows:

$$(\mathcal{L}_B f)(x) = f(x_0)\psi_0(x) + \sum_{l=1}^{N_1-1} f(x_l)\psi_l(x) + f(x_{N_1})\psi_{N_1}(x), \quad x \in I, \tag{30}$$

where

$$\begin{aligned}\psi_0(x) &= \frac{1}{2}c_1^2\int_{-\infty}^{x_0}\frac{1}{[(x-t)^2+c_1^2]^{3/2}}\,dt + \frac{1}{2}c_1^2\int_{x_0}^{x_1}\frac{(x_1-t)/(x_1-x_0)}{[(x-t)^2+c_1^2]^{3/2}}\,dt \\ &= \frac{1}{2} + \frac{\phi_1(x)-\phi_0(x)}{2(x_1-x_0)},\end{aligned} \tag{31}$$

$$\begin{aligned}\psi_{N_1}(x) &= \frac{1}{2}c_1^2\int_{x_{N_1}}^{\infty}\frac{1}{[(x-t)^2+c_1^2]^{3/2}}\,dt \\ &\quad + \frac{1}{2}c_1^2\int_{x_{N_1-1}}^{x_{N_1}}\frac{(t-x_{N_1-1})/(x_{N_1}-x_{N_1-1})}{[(x-t)^2+c_1^2]^{3/2}}\,dt \\ &= \frac{1}{2} - \frac{\phi_{N_1}(x)-\phi_{N_1-1}(x)}{2(x_{N_1}-x_{N_1-1})},\end{aligned} \tag{32}$$

and

$$\begin{aligned}\psi_l(x) &= \frac{1}{2}c_1^2\int_{x_{l-1}}^{x_{l+1}}\frac{B_l(t)}{[(x-t)^2+c_1^2]^{3/2}}\,dt \\ &= \frac{\phi_{l+1}(x)-\phi_l(x)}{2(x_{l+1}-x_l)} - \frac{\phi_i(x)-\phi_{l-1}(x)}{2(x_l-x_{l-1})}\end{aligned} \tag{33}$$

for $l = 1,2,\dots,N_1-1$, where $\{B_l(t) : t \in \mathbb{R}\}$ is the hat function that has the nodes $\{x_{l-1},x_l,x_{l+1}\}$, that is identically zero outside the interval $x_{l-1} \leq t \leq x_{l+1}$, and that satisfies the normalization condition $B_l(x_l) = 1$. $\phi_l(x) = \sqrt{(x-x_l)^2+c_1^2}$ is named multiquadric function, c_1 is shape parameter. The operator $\mathcal{L}_B$ reproduces constants.

Zhang [27] extended the univariate quasi-interpolation $\mathcal{Q}f(x) = \sum_i f_i\psi_i(x)$ to bivariate $(\mathcal{Q}f)(x,y) = \sum_i\sum_j f_{ij}\psi_i(x)\psi_j(y)$ using the dimension-splitting multiquadric basis function technique. However, it only obtain lower accuracy.

In this paper, based on the above dimension-splitting idea, we give a kind of bivariate Bernoulli-type quasi-interpolation operator for the multiquadric basis function for gridded data with higher approximation order as follows:

$$(\mathcal{L}_{B_{m,n}}f)(x,y) = \sum_{l=0}^{N_1}\sum_{r=0}^{N_2} \psi_l(x)\psi_r(y)B_{m,n}[f;x_l,x_{l+1};y_r,y_{r+1};h_l,k_r], \tag{34}$$

where $\{\psi_l(x)\}_{l=0}^{N_1}$ are given above, and $\{\psi_r(y)\}_{r=0}^{N_2}$ are represented as follows:

$$\begin{aligned}
&\phi_r(y) = \sqrt{(y-y_r)^2 + c_2^2},\\
&\psi_0(y) = \frac{1}{2} + \frac{\phi_1(y)-\phi_0(y)}{2(y_1-y_0)},\\
&\psi_r(y) = \frac{\phi_{r+1}(y)-\phi_r(y)}{2(y_{r+1}-y_r)} - \frac{\phi_r(y)-\phi_{r-1}(y)}{2(y_r-y_{r-1})}, \quad r = 1,2,\dots,N_2-1,\\
&\psi_{N_2}(y) = \frac{1}{2} - \frac{\phi_{N_2}(x)-\phi_{N_2-1}(x)}{2(x_{N_2}-x_{N_2-1})},
\end{aligned}$$

c_2 is a small constant. The polynomials $(B_{m,n}^{l,r}f)(x,y) := B_{m,n}[f;x_l,x_{l+1};y_r,y_{r+1};h_l,k_r]$, $l = 0,1,\dots,N_1$, $r = 0,1,\dots,N_2$, denote the bivariate Bernoulli operator in the rectangle with opposite vertices (x_l,y_r), (x_{l+1},y_{r+1}) and they are given by (6), having $h_l = x_{l+1}-x_l$, $k_r = y_{r+1}-y_r$, $l = 0,1,\dots,N_1$, $r = 0,1,\dots,N_2$, considering a fictive node $(x_{N_1+1},y_{N_2+1}) = (x_{N_1-1},y_{N_2-1})$.

The quasi-interpolation operator $\mathcal{L}_{B_{m,n}}$ has the following polynomial reproduction property.

Theorem 5 *The operator $\mathcal{L}_{B_{m,n}}$ has the degree of exactness (m,n).*

Proof The argument $\mathcal{L}_{Bm_n}p = p$ follows from the well-known property

$$\sum_{l=0}^{N_1}\sum_{r=0}^{N_2} \psi_l(x)\psi_r(y) = 1 \tag{35}$$

and

$$\bigl(B_{m,n}^{l,r}e_{p,q}\bigr)(x,y) = e_{p,q}(x,y) \quad \text{for } l = 0,1,\dots,N_1, r = 0,1,\dots,N_2,$$

where $e_{p,q}(x,y) \in \mathbb{P}^{(p,q)}$, with $0 \le p \le m$, $0 \le q \le n$. □

3.2 The convergence rates of the operators

Let $\Omega \subset \mathbb{R}^2$ be a bounded rectangle domain, which contains the point set $\{(x_l,y_r), l = 0,1,\dots,N_1, r = 0,1,\dots,N_2\}$. Let $x_0 < \cdots < x_{N_1}$, $y_0 < \cdots < y_{N_2}$. To study the convergence rates of the quasi-interpolation operator $\mathcal{L}_{B_{m,n}}$, we use the following notations:

$$\begin{aligned}
&I_{\rho_1}(x) = [x-\rho_1, x+\rho_1], \quad \rho_1 > 0,\\
&I_{\rho_2}(y) = [y-\rho_2, y+\rho_2], \quad \rho_2 > 0,\\
&\delta_1 = \inf\bigl\{\rho_1 > 0 : \forall x \in [x_0,x_{N_1}], I_{\rho_1}(x) \cap X \neq \emptyset\bigr\},\\
&\delta_2 = \inf\bigl\{\rho_2 > 0 : \forall y \in [y_0,y_{N_2}], I_{\rho_2}(y) \cap Y \neq \emptyset\bigr\},
\end{aligned}$$

$$\delta = \max\{\delta_1, \delta_2\},$$

$$M_x = \max_{x\in[x_0,x_{N_1}]} \#(I_{\delta_1} \cap X),$$

$$M_y = \max_{y\in[y_0,y_{N_2}]} \#(I_{\delta_2} \cap Y),$$

$$M = \max\{M_x, M_y\},$$

where $X = \{x_0, x_1, \dots, x_{N_1}\}$, $Y = \{y_0, y_1, \dots, y_{N_2}\}$ and $\#(\cdot)$ denotes the cardinality function. Thus, $2\delta_1 = \max_{1\le l\le N_1} |x_l - x_{l-1}|$, $2\delta_2 = \max_{1\le r\le N_2} |(y_r - y_{r-1})|$, and $M_x(M_y)$ defines the maximum number of points of $X(Y)$ contained in interval $I_{\delta_1}(x)$ $(I_{\delta_2}(y))$.

Theorem 6 *Assume that c_1 and c_2 satisfy*

$$c_1 \le D_1\delta_1^{r_1}, \qquad c_2 \le D_2\delta_2^{r_2},$$

where D_1, D_2 are positive constants, and r_1, r_2 are positive integers. If $f \in C^{(m,n)}(\Omega)$, then

$$\|L_{B_{m,n}}f - f\| \le CM^2F(m,n)\mathcal{E}_{m,n,r_1,r_2}(\delta), \tag{36}$$

where $\|\cdot\|$ denotes the sup norm in Ω,

$$\mathcal{E}_{m,n,r_1,r_2}(\delta) = \begin{cases} \delta|\ln\delta|, & m = 1, r_1 = 1 \text{ or } n = 1, r_2 = 1, \\ \delta, & m = 1, n = 1, r_1 > 1, r_2 > 1, \\ \delta^m, & m > 1, n > 1, m \le 2r_1 - 1, n \le 2r_2 - 1, m \le n, \\ \delta^m, & m > 1, n > 1, m \le 2r_1 - 1, n > 2r_2 - 1, m \le 2r_2 - 1, \\ \delta^n, & m > 1, n > 1, m \le 2r_1 - 1, n \le 2r_2 - 1, m > n, \\ \delta^n, & m > 1, n > 1, m > 2r_1 - 1, n \le 2r_2 - 1, n \le 2r_1 - 1, \\ \delta^{2r_1-1}, & m > 1, n > 1, m > 2r_1 - 1, n \le 2r_2 - 1, n > 2r_1 - 1, \\ \delta^{2r_1-1}, & m > 1, n > 1, m > 2r_1 - 1, n > 2r_2 - 1, r_1 \le r_2, \\ \delta^{2r_2-1}, & m > 1, n > 1, m \le 2r_1 - 1, n > 2r_2 - 1, m > 2r_2 - 1, \\ \delta^{2r_2-1}, & m > 1, n > 1, m > 2r_1 - 1, n > 2r_2 - 1, r_1 > r_2, \end{cases} \tag{37}$$

and C is a positive constant independent of x, y and X, Y.

Proof In relations (12), (14), and (15), we set also

$$C_{m,n}^{y,z} = C_0(m) + \sum_{j=1}^{n-1} C_j(m)z^j,$$

$$C_0(m) = \frac{1}{m!}\left(1 + \sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}\binom{p}{l_1}|B_{p-l_1}|\right),$$

$$C_j(m) = \frac{1}{m!}\sum_{q=1}^{j}\sum_{l_2=1}^{q}\left(1 + \sum_{p=1}^{m}\sum_{l_1=1}^{p}\binom{m}{p}|B_{p-l_1}|\right)|B_{q-l_2}|\binom{j+1}{q}\frac{1}{(j+1)!},$$

$$
\begin{aligned}
C_{m,n}^{x,z} &= C_0(n) + \sum_{i=1}^{m-1} C_i(n) z^i, \\
C_0(n) &= \frac{1}{n!}\left(1 + \sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}\binom{q}{l_2}|B_{q-l_2}|\right), \\
C_i(n) &= \frac{1}{n!}\sum_{p=1}^{i}\sum_{l_1=1}^{p}\left(1 + \sum_{q=1}^{n}\sum_{l_2=1}^{q}\binom{n}{q}|B_{q-l_2}|\right)|B_{p-l_1}|\binom{i+1}{p}\frac{1}{(i+1)!}.
\end{aligned}
$$

In view of (12), the following inequality holds:

$$
\begin{aligned}
&\left|(\mathcal{L}_{B_{m,n}}f)(x,y) - f(x,y)\right| \\
&\quad \le \left|\sum_{l=0}^{N_1}\sum_{r=0}^{N_2}\psi_l(x)\psi_r(y)B_{m,n}[f;x_l,x_{l+1};y_r,y_{r+1};h_l,k_r] - f(x,y)\right| \\
&\quad \le \sum_{l=0}^{N_1}\sum_{r=0}^{N_2}\psi_l(x)\psi_r(y)\left|B_{m,n}[f;x_l,x_{l+1};y_r,y_{r+1};h_l,k_r] - f(x,y)\right| \\
&\quad \le F(m,n)\Bigg(C_0(m)S_{r_1,m}(x) + \sum_{j=1}^{n-1}C_j(m)S_{r_1,r_2,m,j}(x,y) + C_0(n)S_{r_2,n}(y) \\
&\qquad + \sum_{i=1}^{m-1}C_i(n)S_{r_1,r_2,i,n}(x,y) + C(m,n)S_{r_1,r_2,m,n}(x,y)\Bigg) \\
&\quad \le \overline{C}(m,n)F(m,n)E_{r_1,r_2,m,n}(x,y),
\end{aligned}
$$

where

$$
\begin{aligned}
&S_{r_1,m}(x) = \sum_{l=0}^{N_1}\psi_l(x)d^m[x_l,x_{l+1}](x), \qquad S_{r_2,n}(y) = \sum_{r=0}^{N_2}\psi_r(y)d^n[y_r,y_{r+1}](y), \\
&S_{r_1,r_2,m,j}(x,y) \\
&\quad = \sum_{l=0}^{N_1}\sum_{r=0}^{N_2}\psi_l(x)\psi_r(y)d^m[x_l,x_{l+1}](x)d^j[y_r,y_{r+1}](y), \quad j = 1,2,\ldots,n-1, \\
&S_{r_1,r_2,i,n}(x,y) \\
&\quad = \sum_{l=0}^{N_1}\sum_{r=0}^{N_2}\psi_l(x)\psi_r(y)d^i[x_l,x_{l+1}](x)d^n[y_r,y_{r+1}](y), \quad i = 1,2,\ldots,m-1, \\
&S_{r_1,r_2,m,n}(x,y) = \sum_{l=0}^{N_1}\sum_{r=0}^{N_2}\psi_l(x)\psi_r(y)d^m[x_l,x_{l+1}](x)d^n[y_r,y_{r+1}](y), \\
&E_{r_1,r_2,m,n}(x,y) = S_{r_1,m}(x) + S_{r_2,n}(y) \\
&\qquad + \sum_{j=1}^{n-1}S_{r_1,r_2,m,j}(x,y) + \sum_{i=1}^{m-1}S_{r_1,r_2,i,n}(x,y) + S_{r_1,r_2,m,n}(x,y), \\
&\overline{C}(m,n) = \max\left\{C_0(m),C_1(m),\ldots,C_n(m),C_0(n),C_1(n),\ldots,C_m(n),C(m,n)\right\}.
\end{aligned}
$$

Let

$$n_1 = \left[\frac{x_{N_1} - x_0}{2\delta_1}\right] + 1, \qquad Q_{\delta_1}(u - \rho_1, u + \rho_1), \quad u \in [x_0, x_{N_1}], \rho_1 > 0,$$
$$n_2 = \left[\frac{y_{N_2} - y_0}{2\delta_2}\right] + 1, \qquad Q_{\delta_2}(u - \rho_2, u + \rho_2), \quad u \in [y_0, y_{N_2}], \rho_2 > 0,$$

and

$$T^x_{j_1} = Q_{\delta_1}(x - 2j_1\delta_1) \cup Q_{\delta_1}(x + 2j_1\delta_1), \quad j_1 = 0, 1, \ldots, n_1,$$
$$T^y_{j_2} = Q_{\delta_2}(y - 2j_2\delta_2) \cup Q_{\delta_2}(y + 2j_2\delta_2), \quad j_2 = 0, 1, \ldots, n_2,$$

where $[\cdot]$ denotes the integer part of the argument. Therefore, for each $l \in \{0, 1, \ldots, N_1\}$ ($r \in \{0, 1, \ldots, N_2\}$) there exists a unique $j_1 \in \{0, 1, \ldots, n_1\}$ ($j_2 \in \{0, 1, \ldots, n_2\}$) such that $x_l \in T^x_{j_1}$ ($y_r \in T^y_{j_2}$).

When $j_1 \geq 2$ ($j_2 \geq 2$), the following inequalities hold:

$$(2j_1 - 1)\delta_1 \leq |x - x_l| \leq (2j_1 + 1)\delta_1,$$
$$\bigl(2(j_1 - 1) - 1\bigr)\delta_1 \leq |x - \xi_l| \leq \bigl(2(j_1 + 1) + 1\bigr)\delta_1, \quad \text{for } \xi_l \in [x_{l-1}, x_{l+1}],$$
$$(2j_2 - 1)\delta_2 \leq |y - y_r| \leq (2j_2 + 1)\delta_2,$$
$$\bigl(2(j_2 - 1) - 1\bigr)\delta_2 \leq |y - \eta_r| \leq \bigl(2(j_2 + 1) + 1\bigr)\delta_2, \quad \text{for } \eta_r \in [y_{r-1}, y_{r+1}],$$

and by (11)

$$d[x_l, x_{l+1}] \leq \bigl(2(j_1 + 1) + 1\bigr)\delta_1,$$
$$d[y_r, y_{r+1}] \leq \bigl(2(j_2 + 1) + 1\bigr)\delta_2.$$

It follows from the definition of $M_x(M_y)$ that

$$1 \leq \bigl(T^x_0 \cap X\bigr) \leq M_x, \qquad 1 \leq \bigl(T^x_{j_1} \cap X\bigr) \leq 2M_x, \quad j_1 = 1, 2, \ldots, n_1,$$
$$1 \leq \bigl(T^y_0 \cap Y\bigr) \leq M_y, \qquad 1 \leq \bigl(T^y_{j_2} \cap Y\bigr) \leq 2M_y, \quad j_2 = 1, 2, \ldots, n_2.$$

When $x_0 \in T^x_{j_1}$, ($j_1 \geq 2$), we have after some calculations

$$\begin{aligned}
\psi_0(x) &\leq \frac{1}{2}c_1^2 \int_{-\infty}^{x_0} \frac{1}{|x - t|^3}\,dt + \frac{1}{2}c_1^2 \frac{1}{[(x - \tau_0)^2 + c_1^2]^{3/2}} \int_{x_0}^{x_1} \frac{x_1 - t}{x_1 - x_0}\,dt \\
&\leq \frac{1}{4}c_1^2|x - x_0|^{-2} + \frac{1}{4}c_1^2(x_1 - x_0)|x - \tau_0|^{-3} \\
&\leq \frac{1}{4}c_1^2\bigl[(2j_1 - 1)^{-2}\delta_1{}^{-2} + 2(2j_1 - 3)^{-3}\delta_1{}^{-2}\bigr] \\
&\leq c_1^2\delta_1{}^{-2}(2j_1 - 3)^{-2},
\end{aligned}$$

where $\tau_0 \in [x_0, x_1]$. When $x_{N_1} \in T^x_{j_1}$, ($j_1 \geq 2$), we get in an analogous manner

$$\psi_{N_1}(x) \leq c_1^2\delta_1{}^{-2}(2j_1 - 3)^{-2}.$$

When $x_l(l = 1,\dots,N_1 - 1) \in T^x_{j_1}$, $(j_1 \geq 2)$, we also get

$$\begin{aligned}\psi_l(x) &\leq \frac{1}{2}c_1^2\frac{1}{[(x-\tau_l)^2+c_1^2]^{3/2}}\int_{x_{l-1}}^{x_{l+1}} B_l(t)\,dt \\ &\leq \frac{1}{4}c_1^2(x_{l+1}-x_{l-1})|x-\tau_l|^{-3} \\ &\leq c_1^2{\delta_1}^{-2}(2j_1-3)^{-2},\end{aligned}$$

where $\tau_i \in [x_{l-1}, x_{l+1}]$. Similarly, when $j_2 \geq 2$, we obtain

$$\psi_r(y) \leq c_2^2{\delta_2}^{-2}(2j_2-3)^{-2}, \quad r = 0,1,\dots,N_2.$$

Then, we can obtain

$$\begin{aligned}S_{r_1,m}(x) &= \sum_{l=0}^{N_1}\psi_l(x)d^m[x_l,x_{l+1}](x) \\ &\leq \sum_{x_l\in T_0^x,T_1^x}\psi_l(x)\dot{d}^m[x_l,x_{l+1}](x) + \sum_{j_1=2}^{n_1}\sum_{x_l\in T_{j_1}^x}\psi_l(x)\dot{d}^m[x_l,x_{l+1}](x) \\ &\leq M_x(3\delta_1)^m + 2M_x(5\delta_1)^m + 2M_x\sum_{j_1=2}^{n_1}c_1^2{\delta_1}^{-2}(2j_1-3)^{-2}\bigl((2j_1+3)\delta_1\bigr)^m \\ &\leq 2M_x(5\delta_1)^m + 2M_x(5\delta_1)^m + 2M_xc_1^2{\delta_1}^{-2}\sum_{j_1=2}^{n_1}(2j_1-3)^{-2}(2j_1+3)^m \\ &\leq 2M_x5^m\left(2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1}j_1^{m-2}\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}S_{r_2,n}(x) &= \sum_{r=0}^{N_2}\psi_r(y)d^n[y_r,y_{r+1}](y) \\ &\leq 2M_y5^n\left(2{\delta_2}^n + D_2^2{\delta_2}^{2r_2+n-2}\sum_{j_2=1}^{n_2}j_2^{n-2}\right),\end{aligned}$$

$$\begin{aligned}S_{r_1,r_2,m,j}(x,y) &= \left(\sum_{l=0}^{N_1}\psi_l(x)d^m[x_l,x_{l+1}](x)\right)\left(\sum_{r=0}^{N_2}\psi_r(y)d^j[y_r,y_{r+1}](y)\right) \\ &\leq 4M_xM_y5^m5^j \\ &\quad\times\left(2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1}j_1^{m-2}\right)\left(2{\delta_2}^j + D_2^2{\delta_2}^{2r_2+j-2}\sum_{j_2=1}^{n_2}j_2^{j-2}\right),\end{aligned}$$

$$\begin{aligned}S_{r_1,r_2,i,n}(x,y) &= \left(\sum_{l=0}^{N_1}\psi_l(x)d^i[x_l,x_{l+1}](x)\right)\left(\sum_{r=0}^{N_2}\psi_r(y)d^n[y_r,y_{r+1}](y)\right) \\ &\leq 4M_xM_y5^i5^n\end{aligned}$$

$$\times \left(2{\delta_1}^i + D_1^2{\delta_1}^{2r_1+i-2}\sum_{j_1=1}^{n_1} j_1^{i-2}\right)\left(2{\delta_2}^n + D_2^2{\delta_2}^{2r_2+n-2}\sum_{j_2=1}^{n_2} j_2^{n-2}\right),$$

$$\begin{aligned} S_{r_1,r_2,m,n}(x,y) &= \left(\sum_{l=0}^{N_1}\psi_l(x)d^m[x_l,x_{l+1}](x)\right)\left(\sum_{r=0}^{N_2}\psi_r(y)d^n[y_r,y_{r+1}](y)\right) \\ &\le 4M_xM_y5^m5^n \\ &\quad \times \left(2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1} j_1^{m-2}\right)\left(2{\delta_2}^n + D_2^2{\delta_2}^{2r_2+n-2}\sum_{j_2=1}^{n_2} j_2^{n-2}\right). \end{aligned}$$

Thus, we get

$$\begin{aligned} &E_{r_1,r_2,m,n}(x,y) \\ &\quad = S_{r_1,m}(x) + S_{r_2,n}(y) \\ &\qquad + \sum_{j=1}^{n-1} S_{r_1,r_2,m,j}(x,y) + \sum_{i=1}^{m-1} S_{r_1,r_2,i,n}(x,y) + S_{r_1,r_2,m,n}(x,y) \\ &\quad \le 4M^2 5^m5^n\left(2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1} j_1^{m-2}\right) \\ &\qquad + 4M^2 5^m5^n\left(2{\delta_2}^n + D_2^2{\delta_2}^{2r_2+n-2}\sum_{j_2=1}^{n_2} j_2^{n-2}\right) + 4M^2 5^m5^n \\ &\qquad \times \left(2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1} j_1^{m-2}\right)\sum_{j=1}^{n-1}\left(2{\delta_2}^j + D_2^2{\delta_2}^{2r_2+j-2}\sum_{j_2=1}^{n_2} j_2^{j-2}\right) \\ &\qquad + 4M^2 5^m5^n \\ &\qquad \times \left(2{\delta_2}^n + D_2^2{\delta_2}^{2r_2+n-2}\sum_{j_2=1}^{n_2} j_2^{n-2}\right)\sum_{i=1}^{m-1}\left(2{\delta_1}^i + D_1^2{\delta_1}^{2r_1+i-2}\sum_{j_1=1}^{n_1} j_1^{i-2}\right) \\ &\qquad + 4M^2 5^m5^n \\ &\qquad \times \left(2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1} j_1^{m-2}\right)\left(2{\delta_2}^n + D_2^2{\delta_2}^{2r_2+n-2}\sum_{j_2=1}^{n_2} j_2^{n-2}\right) \\ &\quad := 4M^2 5^m5^n(P_1+P_2+P_3+P_4+P_5). \end{aligned}$$

Case 1 ($m = 1$)

If $r_1 = 1$, then

$$\begin{aligned} 2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1} j_1^{m-2} &:= P_1 \\ &= \mathcal{O}\big(\delta_1|\ln\delta_1|\big) \\ &\le \mathcal{O}\big(\delta|\ln\delta|\big), \end{aligned}$$

$$\left(2{\delta_1}^m + D_1^2{\delta_1}^{2r_1+m-2}\sum_{j_1=1}^{n_1} j_1^{m-2}\right)\sum_{j=1}^{n-1}\left(2{\delta_2}^j + D_2^2{\delta_2}^{2r_2+j-2}\sum_{j_2=1}^{n_2} j_2^{j-2}\right)$$

$$:= P_3$$

$$= \begin{cases} \mathcal{O}(\delta_1|\ln\delta_1|)\mathcal{O}(\delta_2|\ln\delta_2|), & r_2 = 1, \\ \mathcal{O}(\delta_1|\ln\delta_1|)\mathcal{O}(\delta_2), & r_2 > 1, \end{cases}$$

$$\leq \begin{cases} \mathcal{O}(\delta|\ln\delta|)\mathcal{O}(\delta|\ln\delta|), & r_2 = 1, \\ \mathcal{O}(\delta|\ln\delta|)\mathcal{O}(\delta), & r_2 > 1, \end{cases}$$

$$\left(2{\delta_2}^n + D_2^2{\delta_2}^{2r_2+n-2}\sum_{j_2=1}^{n_2} j_2^{n-2}\right)\left(1 + \sum_{i=1}^{m}\left(2{\delta_1}^i + D_1^2{\delta_1}^{2r_1+i-2}\sum_{j_1=1}^{n_1} j_1^{i-2}\right)\right)$$

$$:= P_2 + P_4 + P_5$$

$$= \begin{cases} \mathcal{O}(\delta_2|\ln\delta_2|)\mathcal{O}(\delta_1|\ln\delta_1|), & n = 1, r_2 = 1, \\ \mathcal{O}(\delta_2)(1 + \mathcal{O}(\delta_1|\ln\delta_1|)), & n = 1, r_2 > 1, \\ \mathcal{O}(\delta_2^n)(1 + \mathcal{O}(\delta_1|\ln\delta_1|)), & n > 1, n \leq 2r_2 - 1, \\ \mathcal{O}(\delta_2^{2r_2-1})(1 + \mathcal{O}(\delta_1|\ln\delta_1|)), & n > 1, n > 2r_2 - 1, \end{cases}$$

$$\leq \begin{cases} \mathcal{O}(\delta|\ln\delta|)(1 + \mathcal{O}(\delta|\ln\delta|)), & n = 1, r_2 = 1, \\ \mathcal{O}(\delta)(1 + \mathcal{O}(\delta|\ln\delta|)), & n = 1, r_2 > 1, \\ \mathcal{O}(\delta^n)(1 + \mathcal{O}(\delta|\ln\delta|)), & n > 1, n \leq 2r_2 - 1, \\ \mathcal{O}(\delta^{2r_2-1})(1 + \mathcal{O}(\delta|\ln\delta|)), & n > 1, n > 2r_2 - 1. \end{cases}$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta|\ln\delta|\bigr).$$

Case 2 ($n = 1$)

For $r_2 = 1$. Similarly, as the discussion of Case 1 ($m = 1$), we have also

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta|\ln\delta|\bigr).$$

Case 3 ($m = 1$, $n = 1$)

If $r_1 > 1$, $r_2 > 1$, then

$$P_1 = \mathcal{O}(\delta_1) \leq \mathcal{O}(\delta); \qquad P_3 = \mathcal{O}(\delta_1)\mathcal{O}(\delta_2) \leq \mathcal{O}\bigl(\delta^2\bigr);$$

$$P_2 + P_4 + P_5 = \mathcal{O}(\delta_2)\bigl(1 + \mathcal{O}\bigl(\delta_1|\ln\delta_1|\bigr)\bigr) \leq \mathcal{O}(\delta)\bigl(1 + \mathcal{O}\bigl(\delta|\ln\delta|\bigr)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}(\delta).$$

Case 4 ($m > 1$, $n > 1$)

If $m \leq 2r_1 - 1$, $n \leq 2r_2 - 1$, $m \geq n$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^m\bigr) \leq \mathcal{O}\bigl(\delta^m\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^m\bigr)\mathcal{O}(\delta_2) \leq \mathcal{O}\bigl(\delta^m\bigr)\mathcal{O}(\delta),$$

$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^n\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \leq \mathcal{O}\bigl(\delta^n\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

If $m \le 2r_1 - 1$, $n > 2r_2 - 1$, $m \le 2r_2 - 1$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^m\bigr) \le \mathcal{O}\bigl(\delta^m\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^m\bigr)\mathcal{O}(\delta_2) \le \mathcal{O}\bigl(\delta^m\bigr)\mathcal{O}(\delta),$$
$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \le \mathcal{O}\bigl(\delta^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta^m\bigr).$$

If $m \le 2r_1 - 1$, $n \le 2r_2 - 1$, $m > n$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^m\bigr) \le \mathcal{O}\bigl(\delta^m\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^m\bigr)\mathcal{O}(\delta_2) \le \mathcal{O}\bigl(\delta^m\bigr)\mathcal{O}(\delta),$$
$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^n\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \le \mathcal{O}\bigl(\delta^n\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta^n\bigr).$$

If $m > 2r_1 - 1$, $n \le 2r_2 - 1$, $n \le 2r_1 - 1$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr) \le \mathcal{O}\bigl(\delta^{2r_1-1}\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr)\mathcal{O}(\delta_2) \le \mathcal{O}\bigl(\delta^{2r_1-1}\bigr)\mathcal{O}(\delta),$$
$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^n\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \le \mathcal{O}\bigl(\delta^n\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta^n\bigr).$$

If $m > 2r_1 - 1$, $n \le 2r_2 - 1$, $n > 2r_1 - 1$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr) \le \mathcal{O}\bigl(\delta^{2r_1-1}\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr)\mathcal{O}(\delta_2) \le \mathcal{O}\bigl(\delta^{2r_1-1}\bigr)\mathcal{O}(\delta),$$
$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^n\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \le \mathcal{O}\bigl(\delta^n\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta^{2r_1-1}\bigr).$$

If $m > 2r_1 - 1$, $n > 2r_2 - 1$, $r_1 \le r_2$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr) \le \mathcal{O}\bigl(\delta^{2r_1-1}\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr)\mathcal{O}(\delta_2) \le \mathcal{O}\bigl(\delta^{2r_1-1}\bigr)\mathcal{O}(\delta),$$
$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \le \mathcal{O}\bigl(\delta^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta^{2r_1-1}\bigr).$$

If $m \leq 2r_1 - 1$, $n > 2r_2 - 1$, $m > 2r_2 - 1$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^m\bigr) \leq \mathcal{O}\bigl(\delta^m\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^m\bigr)\mathcal{O}(\delta_2) \leq \mathcal{O}\bigl(\delta^m\bigr)\mathcal{O}(\delta),$$
$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \leq \mathcal{O}\bigl(\delta^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta^{2r_2-1}\bigr).$$

If $m > 2r_1 - 1$, $n > 2r_2 - 1$, $r_1 > r_2$, then

$$P_1 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr) \leq \mathcal{O}\bigl(\delta^{2r_1-1}\bigr); \qquad P_3 = \mathcal{O}\bigl(\delta_1^{2r_1-1}\bigr)\mathcal{O}(\delta_2) \leq \mathcal{O}\bigl(\delta^{2r_1-1}\bigr)\mathcal{O}(\delta),$$
$$P_2 + P_4 + P_5 = \mathcal{O}\bigl(\delta_2^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta_1)\bigr) \leq \mathcal{O}\bigl(\delta^{2r_2-1}\bigr)\bigl(1 + \mathcal{O}(\delta)\bigr).$$

So,

$$P_1 + P_2 + P_3 + P_4 + P_5 = \mathcal{O}\bigl(\delta^{2r_2-1}\bigr). \qquad \square$$

By using Theorem 6, we can obtain the following theorem in an analogous manner.

Theorem 7 *Assume that c_1 and c_2 satisfy*

$$c_1 \leq D_1\delta_1^{r_1}, \qquad c_2 \leq D_2\delta_2^{r_2},$$

where D_1, D_2 are positive constants, and r_1, r_2 are positive integers. If $f \in C^{(m+1,n+1)}(\Omega)$, then

$$\|L_{B_{m,n}}f - f\| \leq C'M^2F(m+1,n+1)\mathcal{E}'_{m+1,n+1,r_1,r_2}(\delta), \tag{38}$$

where

$$\mathcal{E}'_{m+1,n+1,r_1,r_2}(\delta) = \begin{cases} \delta^{m+1}, & m+1 \leq 2r_1 - 1, n+1 \leq 2r_2 - 1, m \leq n, \\ \delta^{m+1}, & m+1 \leq 2r_1 - 1, n+1 \leq 2r_2 - 1, m+1 \leq 2r_2 - 1, \\ \delta^{n+1}, & m+1 \leq 2r_1 - 1, n+1 \leq 2r_2 - 1, m > n, \\ \delta^{n+1}, & m+1 > 2r_1 - 1, n+1 \leq 2r_2 - 1, n+1 \leq 2r_1 - 1, \\ \delta^{2r_1-1}, & m+1 > 2r_1 - 1, n+1 \leq 2r_2 - 1, n+1 > 2r_1 - 1, \\ \delta^{2r_1-1}, & m+1 > 2r_1 - 1, n+1 > 2r_2 - 1, r_1 \leq r_2, \\ \delta^{2r_2-1}, & m+1 \leq 2r_1 - 1, n+1 > 2r_2 - 1, m+1 > 2r_2 - 1, \\ \delta^{2r_2-1}, & m+1 > 2r_1 - 1, n+1 > 2r_2 - 1, r_1 > r_2, \end{cases} \tag{39}$$

C' is a positive constant independent of x, y and X, Y.

Remark 1 To apply the newly constructed bivariate multiquadric quasi-interpolation operators to the interesting applications, the use of the Lagrange interpolation polynomials can avoid the use of the derivatives.

4 Numerical examples

To test the bivariate Bernoulli-type multiquadric quasi-interpolation operators, we consider the following test functions(see, e.g., [40]) on the computational domain $[0,1]\times[0,1]$:

$$\text{Gentle}\, f_1(x,y) = \frac{\exp[-\frac{81}{16}((x-0.5)^2+(y-0.5)^2)]}{3}, \tag{40}$$

$$\text{Sphere}\, f_2(x,y) = \frac{\sqrt{64-81((x-0.5)^2+(y-0.5)^2)}}{9} - 0.5, \tag{41}$$

$$\text{Saddle}\, f_3(x,y) = \frac{1.25+\cos(5.4y)}{6+6(3x-1)^2}. \tag{42}$$

For each function f_i, $i = 1,2,3$, we will compare the numerical results of our new operators $\mathcal{L}_{B_{m,n}}$ with other methods suitable for bivariate Bernoulli-type Shepard operators $S_{B_{m,n}}$ [41], bivariate Bernoulli-type thin-plate spline operators $Q_{B_{m,n}}$ [42] and the operators Q (see, [27] and [28]) on a finite field with $c_1 = (2\delta_1)^{r_1}$ and $c_2 = (2\delta_2)^{r_2}$. The operator $S_{B_{m,n}}$ is defined as follows

$$(S_{B_{m,n}}f)(x,y) = \sum_{i=0}^{N} A_{\mu,i}(x,y) B^i_{m,n}[f;x_i,x_{i+1};y_i,y_{i+1};h_i,k_i],$$

where

$$A_{\mu,i}(x,y) = \frac{(\sqrt{(x-x_i)^2+(y-y_i)^2})^{-\mu}}{\sum_{k=0}^{N}(\sqrt{(x-x_k)^2+(y-y_k)^2})^{-\mu}},$$

with the parameter $\mu > 0$, see [41] for details.

We consider also the combined thin-plate spline operators of Bernoulli type, and dimension-splitting multiquadric bivariate operator, denoted by $Q_{B_{m,n}}$, and Q, introduced in [42], and [27, 28], respectively. They are given by

$$\begin{aligned}(Q_{B_{m,n}}f)(x,y) &= \frac{1}{16\pi}\sum_{i=0}^{N}\psi\left(\frac{x-x_i}{h},\frac{y-y_i}{h}\right)B^i_{m,n}[f;x_i,x_{i+1};y_i,y_{i+1};h_i,k_i], \quad h\in(0,1),\\ (\mathcal{Q}f)(x,y) &= \sum_i\sum_j f_{ij}\psi_i(x)\psi_j(y).\end{aligned}$$

We use uniform grids of 121 nodes for the operators $Q_{B_{m,n}}$, $S_{B_{m,n}}$, $\mathcal{L}_{B_{m,n}}$, Q with a width $2\delta_1 = 2\delta_2 = 0.1$ as $(m,n) = (2,2),(3,3)$, $\mu = 4$, $h = \frac{1}{10}$, $r_1 = 3$ and $r_2 = 3$. In order to estimate the errors as accurately as possible, we compute the approximating functions at the points $(\frac{i}{21},\frac{j}{21})$, $(i = 1,2,\ldots,20; j = 1,2,\ldots,20)$. Tables 1-2 display mean and max errors for the different approximation operators above. From Table 1, one can observe that the error for the bivariate Bernoulli-type operator with multiquadrics $\mathcal{L}_{B_{m,n}}$ is lower than that for the bivariate Shepard-Bernoulli operator $S_{B_{m,n}}$, thin-plate spline operators $Q_{B_{m,n}}$ of Bernoulli type and dimension-splitting multiquadric bivariate operators Q as $(m,n) = (2,2)$. Table 2 shows that the smallest error is for the bivariate Bernoulli-type operator with multiquadrics $\mathcal{L}_{B_{m,n}}$ as $(m,n) = (3,3)$, too. We remark good approximation properties of the pro-

Table 1 The maximum and mean approximation error of the operators on $(m,n) = (2,2)$

	f_1		f_2		f_3	
	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$
$\mathcal{L}_{B_{2,2}} f$	0.000436	0.001481	0.000109	0.000648	0.000624	0.004172
$S_{B_{2,2}} f$ [41]	0.000499	0.002384	0.000150	0.000737	0.000694	0.005629
$Q_{B_{2,2}} f$ [42]	0.001299	0.004533	0.001702	0.005598	0.001650	0.007147
Qf [27]	0.001757	0.006318	0.002535	0.007270	0.001990	0.012885
Qf [28]	0.001757	0.006319	0.002537	0.006278	0.001990	0.012884

Table 2 The maximum and mean approximation error of the operators on $(m,n) = (3,3)$

	f_1		f_2		f_3	
	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$
$\mathcal{L}_{B_{3,3}} f$	0.000055	0.000361	0.000024	0.000102	0.000100	0.001169
$S_{B_{3,3}} f$ [41]	0.000099	0.000378	0.000028	0.000219	0.000155	0.001645
$Q_{B_{3,3}} f$ [42]	0.000821	0.002419	0.001440	0.007019	0.000907	0.010815
Qf [27]	0.001757	0.006318	0.002535	0.007270	0.001990	0.012885
Qf [28]	0.001757	0.006319	0.002537	0.006278	0.001990	0.012884

posed multiquadric quasi-interpolation operators $\mathcal{L}_{B_{m,n}}$ when compared with the other methods.

5 Conclusions

In this paper, a kind of bivariate Bernoulli-type multiquadric quasi-interpolation operator is constructed by combining the known multiquadric quasi-interpolation operator with the generalized Taylor polynomial as the expansion in the bivariate Bernoulli polynomials. A result on the convergence rates of the new operators is given. Numerical tests show that our method provides higher accuracy. Furthermore, the associated algorithm is easily implemented.

In our future work, we plan to apply it to solve partial differential equations, and good results may be obtained. Moreover, we could construct stochastic quasi-interpolation with Bernoulli polynomials.

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Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Ruifeng Wu wrote the main manuscript text and prepared all experiments. All authors reviewed the manuscript.

Author details
[1]School of Applied Mathematics, Jilin University of Finance and Economics, Changchun 130117, P.R. China. [2]GongQing Institute Of Science And Technology, Jiujiang 332020, P.R. China. [3]Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University, Changchun 130117, P.R. China.

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