Azmi *Journal of Inequalities and Applications* (2023) 2023:87
https://doi.org/10.1186/s13660-023-02999-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalized contraction mappings in double controlled metric type space and related fixed point theorems

Fatima M. Azmi[1*]

*Correspondence: fazmi@psu.edu.sa
[1]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia

Abstract

In this article, we introduce two new types of generalized contraction mappings in double controlled metric type spaces: Θ-double controlled contraction mapping and Ćirić-Reich-Rus-type-Θ-double controlled contraction mapping. For each contraction mapping, we establish the existence and uniqueness of the fixed point theorems on the complete double controlled metric type space and provide examples. We present an application of our results and demonstrate how our results generalize several existing fixed point theorems in the literature.

Keywords: Fixed point; Double controlled metric type space; Ćirić-Reich-Rus-type contraction; Generalized Contraction mapping; Θ- contraction; Zeros of high degree polynomials

1 Introduction

The fixed point theorem established the existence of a solution to an integral equation and was known as the Banach contraction principle [1]. It rapidly developed into a typical device for solving several existing problems in various areas of mathematics, including nonlinear analysis and its application. The fixed point theory advanced in two directions. One is to change the space under consideration, and the other is to change the contraction condition. Banach's original thesis dealt with the standard metric space [1]. Many researchers generalized the structure of the metric space by changing the space under consideration; for example, b-metric space was introduced by Bakhtin [2] as an extension of standard metric spaces. Later, Branciari [3] presented the notion of generalized metric (Branciari metric) spaces, where the triangle inequality is replaced with $d(a,b) \leq d(a,u) + d(u,v) + d(v,b)$; for all distinct pairwise points $a,b,u,v \in W$, the map d is called a Branciari distance. Numerous fixed point results were established on such spaces [4, 5]. Subsequently, the notion of extended b-metric spaces appeared [6], and fixed point results on these spaces were established in many articles [7, 8]. Later, a more general concept of controlled metric-type spaces was introduced [9]. A further generalization resulted in the concepts of double controlled metric type spaces [10] and double controlled metric-like spaces [11, 12]. Several authors have studied various generalizations of the Banach contraction mapping principle. One of the prominent generalizations of the Banach contraction was presented by Kannan [13], who introduced a new contraction now known as the Kannan contraction. By utilizing the Kannan contraction, fixed point results have been established on b-metric spaces [14] and extended b-metric spaces [15].

Some researchers weakened the contraction property of the operator using many other contraction mappings (e.g., [16–25]); for example, Ćirić [18] and Reich [20] independently proved the existence and uniqueness theorem on complete metric space, when the operator has a certain contraction property. The contraction mapping was later known as a Ćirić-Reich-Rus-type contraction. Jleli et al. ([26, 27]) introduced the concept of Θ-contraction to establish a generalization of the Banach fixed point theorem in the situation of Branciari metric spaces. Recently, Abdeljawad et al. [28] proposed the notion of Θ-Branciari contraction to establish a fixed point theorem in the extended Branciari b-distance space using the Ćirić-Reich-Rus-type contraction mapping.

In this article, by focusing on two contraction mappings, Ćirić-Reich-Rus-type and Θ-contraction, we introduce them on double controlled metric type spaces. Hence, we denote the first contraction mapping by Θ-double controlled contraction mapping, while the second is denoted by the Ćirić-Reich-Rus-type Θ-double controlled contraction mapping. Furthermore, we establish the existence and uniqueness of the fixed point theorems on complete double-controlled metric type spaces. We also present examples and demonstrate how our result generalizes many existing fixed point theorems. We end the article with an application of our results.

2 Preliminaries

Kamran et al. [6] defined the concept of extended b-metric spaces, and their work generalized many results [29–32].

Definition 2.1 ([6]) Consider a mapping $\omega : \mathcal{S} \times \mathcal{S} \to [1,\infty)$, where $\mathcal{S} \neq \emptyset$. The function $d_\omega : \mathcal{S} \times \mathcal{S} \to [0,\infty)$, is called an extended b-metric if for all $\hat{x},\hat{y},\hat{z} \in \mathcal{S}$,

1. $d_\omega(\hat{x},\hat{z}) = 0$ iff $\hat{x} = \hat{z}$;
2. $d_\omega(\hat{x},\hat{z}) = d_\omega(\hat{z},\hat{x})$, symmetric;
3. $d_\omega(\hat{x},\hat{z}) \leq \omega(\hat{x},\hat{z})[d_\omega(\hat{x},\hat{y}) + d_\omega(\hat{y},\hat{z})]$,

The pair $(\mathcal{S},d_\omega)$ is called an extended b-metric space.

Branciari introduced the concept of generalized metric spaces [3], while the extended Branciari b-distance space was introduced in [28] and defined as follows.

Definition 2.2 ([28]) Let S be a non-empty set, and let $\omega : S \times S \to [1,\infty)$ be a mapping. The function $d : S \times S \to [0,\infty)$, is called an extended Branciari b-distance if for all $x,y \in S$ and all $u,v \in S$ distinct from x, y, this holds:

1. $d(x,y) = 0$ iff $x = y$;
2. $d(x,y) = d(y,x)$, symmetric;
3. $d(x,y) \leq \omega(x,y)[d(x,u) + d(u,v) + d(v,y)]$,

The pair (S,d) is called an extended Branciari b-distance space.

An extension of the b-metric space into a controlled metric type space was introduced by Mlaiki et al. [9].

Definition 2.3 Let $\beta : Z_\beta \times Z_\beta \to [1,\infty)$ be a mapping, where $Z_\beta \neq \emptyset$. The function $d_\beta : Z_\beta \times Z_\beta \to [0,\infty)$, is called a controlled metric if, for all $\hat{x}, \hat{y}, \hat{z} \in Z_\beta$, the following conditions hold:

$(d1)$ $d_\beta(\hat{x},\hat{z}) = 0$ iff $\hat{x} = \hat{z}$;

$(d2)$ $d_\beta(\hat{x},\hat{z}) = d_\beta(\hat{z},\hat{x})$, symmetric;

$(d3)$ $d_\beta(\hat{x},\hat{z}) \leq \beta(\hat{x},\hat{y})d_\beta(\hat{x},\hat{y}) + \beta(\hat{y},\hat{z})d_\beta(\hat{y},\hat{z})$,

then, the pair (Z_β, d_β) is called a controlled metric type space.

Then a more general concept of a b-metric type space called a double controlled metric type space was introduced in [10] as follows.

Definition 2.4 ([10]) Consider two noncomparable functions $\beta, \mu : Z \times Z \to [1,\infty)$, defined on a non-empty set Z. The mapping $d_{\beta,\mu} : Z \times Z \to [0,\infty)$ is called a double controlled metric type space by β and μ if, for all $z_1, z_2, z_3 \in Z$, the following conditions hold:

$(Q1)$ $d_{\beta,\mu}(z_1,z_2) = 0$ iff $z_1 = z_2$;

$(Q2)$ $d_{\beta,\mu}(z_1,z_2) = d_{\beta,\mu}(z_2,z_1)$, symmetric;

$(Q3)$ $d_{\beta,\mu}(z_1,z_2) \leq \beta(z_1,z_3)d_{\beta,\mu}(z_1,z_3) + \mu(z_3,z_2)d_{\beta,\mu}(z_3,z_2)$.

The pair $(Z, d_{\beta,\mu})$ is called a double controlled metric type space.

Remark 2.5 The class of double controlled metric type spaces is larger than that of controlled metric type spaces, which, in turn, is larger than the class of extended b-metric spaces. Moreover, the class of extended b-metric spaces is larger than that of b-metric spaces. All these classes are larger than the class of standard metric spaces. Every extended b-metric space is a controlled metric type space and double controlled metric type space, but the converse is invalid. Also, every controlled metric type space is, in fact, a double controlled metric type space, but the converse is invalid (Fig. 1).

The following example shows a controlled metric type space that is not an extended b-metric space [9].

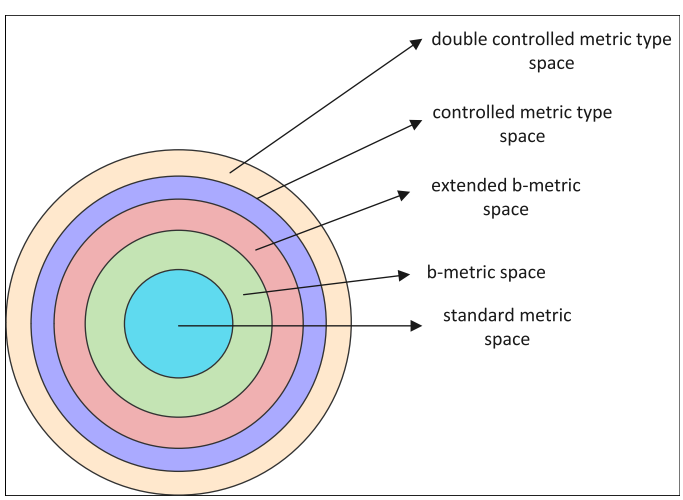

Figure 1 This figure displays the relation between different metric type spaces

Example 2.6 Let $Z = \mathbb{N}$, define $d_\alpha : Z \times Z \to [0,\infty)$, by

$$d_\alpha(u,v) = \begin{cases} 0, & \text{iff } u = v, \\ \frac{1}{u}, & \text{if } u \text{ is even and, } v \text{ is odd,} \\ \frac{1}{v} & \text{if } v \text{ is even and, } u \text{ is odd,} \\ 1 & \text{else.} \end{cases}$$

Then, (Z, d_α) is a controlled metric type space, with $\alpha : Z \times Z \to [1,\infty)$, defined by

$$\alpha(u,v) = \begin{cases} u & \text{if } u \text{ is even and, } v \text{ is odd,} \\ v & \text{if } v \text{ is even and, } u \text{ is odd,} \\ 1 & \text{else.} \end{cases}$$

The next example illustrates a double controlled metric type space, which is not a controlled metric type space [10].

Example 2.7 ([10]) Let $Z = [0,+\infty)$, define the mapping $d_{\beta,\mu} : Z \times Z \to [0,+\infty)$ by

$$d_{\beta,\mu}(u,v) = \begin{cases} 0 & \text{iff } u = v, \\ \frac{1}{u} & \text{if } u \geq 1 \text{ and } v \in [0,1), \\ \frac{1}{v} & \text{if } v \geq 1 \text{ and } u \in [0,1), \\ 1 & \text{otherwise.} \end{cases}$$

Let $\beta, \mu : Z \times Z \to [1,+\infty)$ be two functions defined by

$$\beta(u,v) = \begin{cases} u & \text{if } u, v \geq 1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\mu(u,v) = \begin{cases} 1 & \text{if } u, v \in [0,1), \\ \max\{u,v\} & \text{otherwise.} \end{cases}$$

To see $(Z, d_{\beta,\mu})$ is a double controlled metric type space. Observe that conditions $(Q1)$ and $(Q2)$ hold. To illustrate condition $(Q3)$ holds, note that if either $z = u$ or $z = v$, then $(Q3)$ holds. Thus, suppose $u \neq v$, which means $u \neq v \neq z$. We consider cases:

Case 1: if $u \geq 1$ and $v \in [0,1)$, or $v \geq 1$ and $u \in [0,1)$, then for any z, clearly condition $(Q3)$ is satisfied.

Case 2: if $u, v > 1$, and $z \geq 1$, then one can easily observe that

$$d_{\beta,\mu}(u,v) = 1 \leq u(1) + \max\{v,z\}(1).$$

While, if $z \in [0,1)$, then we obtain

$$d_{\beta,\mu}(u,v) = 1 \leq \frac{1}{u} + v\frac{1}{v}, \quad \text{hence } (Q3) \text{ holds.}$$

Case 3: if $u, v < 1$, and $z \geq 1$, we obtain

$$d_{\beta,\mu}(u,v) = 1 \leq \frac{1}{z} + z\frac{1}{z}.$$

While, if $z \in [0,1)$, then easily condition $(Q3)$ holds. Therefore, $(Z, d_{\beta,\mu})$ is a double controlled metric type space, which is not a controlled metric type space, by taking $\beta = \mu$, note

$$d_\beta\left(0, \frac{1}{2}\right) = 1 > \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \beta(0,4)d_\beta(0,4) + \beta\left(4, \frac{1}{2}\right)d_\beta\left(4, \frac{1}{2}\right).$$

Definition 2.8 Let $(Z, d_{\beta,\mu})$ be a double controlled metric type space, where $Z \neq \emptyset$, given any $\varepsilon > 0$, the open ball $B(y, \varepsilon)$ is defined as

$$B(y,\varepsilon) = \left\{w \in Z, d_{\beta,\mu}(y,w) < \varepsilon\right\}.$$

Let $\{y_n\}_{n\geq 0}$ be any sequence in Z. Then

(1) $\{y_n\}$ converges to some w in Z if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $d_{\beta,\mu}(y_n, w) < \varepsilon$ for all $n \geq N$, i.e. $\lim_{n\to\infty} d_{\beta,\mu}(y_n, w) = 0$.

(2) $\{y_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_{\beta,\mu}(y_m, y_n) < \varepsilon$, for all $m, n \geq N$.

(3) The space $(Z, d_{\beta,\mu})$ is called a complete double controlled metric type space if every Cauchy sequence in Z is convergent.

(4) A mapping $T : Z \to Z$ is said to be continuous at $y \in Z$ if for all $\varepsilon > 0$, there exists $\delta > 0$, such that $T(B(y,\delta)) \subseteq B(Ty, \varepsilon)$. Thus if $\{y_n\}_{n\geq 0}$ is any sequence which converges to u, i.e., $\lim_{n\to\infty} d_{\beta,\mu}(y_n, u) = 0$, then $\lim_{n\to\infty} d_{\beta,\mu}(Ty_n, Tu) = 0$.

Let (X, d) be a metric space, and let $T : X \to X$ be a mapping, then T is called a contraction [1] if there exists $r \in [0,1)$ such that this holds for all $x, y \in X$,

$$d(Tx, Ty) \leq rd(x,y).$$

Kannan [13] generalized the Banach contraction by introducing a new contraction on a metric space now known as the Kannan contraction. Thus, the mapping $T : X \to X$ is called the Kannan contraction if there exists $r \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq r\left[d(x, Tx) + d(y, Ty)\right],$$

for all $x, y \in X$. When (X, d) is complete, then every contraction and every Kannan contraction have a unique fixed point; see [1, 13]. Later many authors utilized Kannan-type contraction and established fixed point results on b-metric spaces [14], extended b-metric spaces [15], and more recently into double controlled dislocated quasi-metric type spaces [24].

The concept of Θ-contraction was proposed in [26] to extend some results on the fixed point theorem in the framework of the Branciari distance space. We recall the definition of the Θ set of functions.

Definition 2.9 ([26]) Let Θ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ obeying the following conditions:

(I) θ is nondecreasing;

(II) for each sequence $\{t_m\}$ of positive real numbers, this holds

$$\lim_{m\to\infty} t_m = 0^+ \quad \Longleftrightarrow \quad \lim_{m\to\infty} \theta(t_m) = 1;$$

(III) there exist k, with $0 < k < 1$, and $M \in (0,\infty]$, such that this holds

$$\lim_{t\to 0^+} \frac{\theta(t)-1}{t^k} = M.$$

It should be observed that Θ contains a large class of functions, for example, the functions $\theta_i : (0,\infty) \to (1,\infty)$, defined by $\theta_1(t) = e^{\sqrt{t}}$, and $\theta_2(t) = e^{\sqrt{te^t}}$ belongs to Θ.

3 Main results

Our work focuses on two types of contraction mappings: Ćirić-Reich-Rus-type and Θ-contraction. Following are two subsections for each type of contraction mappings on complete double controlled metric type spaces.

3.1 Θ-double controlled contraction mapping and fixed point theorems

With Θ as in Definition 2.9 and inspired by [26], we present the notion of Θ-double controlled contraction mapping and establish a fixed point theorem on complete double controlled metric type spaces.

Definition 3.1 Let $(Z, d_{\beta,\mu})$ be a double controlled metric type space. Let $T : Z \to Z$ be a self mapping. Then T is said to be Θ-double controlled contraction mapping if there exists a function $\theta \in \Theta$ and an $r \in (0,1)$ such that the following holds:

$$z, w \in Z, \quad d_{\beta,\mu}(Tz, Tw) \neq 0 \quad \Rightarrow \quad \theta\big(d_{\beta,\mu}(Tz, Tw)\big) \le \big[\theta\big(d_{\beta,\mu}(z,w)\big)\big]^r. \tag{3.1}$$

Lemma 3.2 *Let $(Z, d_{\beta,\mu})$ be a double controlled metric type space, and let $T : Z \to Z$ be a Θ-double controlled contraction mapping. Then T is continuous.*

Proof As T is Θ-double controlled contraction mapping, hence this holds

$$\theta\big(d_{\beta,\mu}(Tx, Ty)\big) \le \big[\theta\big(d_{\beta,\mu}(x,y)\big)\big]^r. \tag{3.2}$$

for some $\theta \in \Theta$ and $r \in (0,1)$, and for any $x, y \in Z$ with $Tx \neq Ty$. Applying ln on both sides of equation (3.2), we obtain

$$\ln\big[\theta\big(d_{\beta,\mu}(Tx, Ty)\big)\big] \le r \ln\big[\theta\big(d_{\beta,\mu}(x,y)\big)\big] \le \ln\big[\theta\big(d_{\beta,\mu}(x,y)\big)\big]. \tag{3.3}$$

As θ is nondecreasing, we obtain

$$d_{\beta,\mu}(Tx, Ty) \le d_{\beta,\mu}(x,y), \quad \forall x, y \in Z. \tag{3.4}$$

Hence T is continuous. □

Our first main result of the fixed point theorem.

Theorem 3.3 *Let $(Z, d_{\beta,\mu})$ be a complete double controlled metric type space with $\beta, \mu : Z \times Z \to [1, \infty)$ two noncomparable functions, defined on a non-empty set Z, and let $T : Z \to Z$ be a Θ-double controlled contraction self mapping satisfying the following*:

(1)

$$\sup_{m\geq 1} \lim_{n\to\infty} \beta(z_{n+1}, z_{n+2})\mu(z_{n+1}, z_m) < 1/p,$$

for some $p \in (0, 1)$, and

(2) *For any $z \in Z$, both $\lim_{n\to\infty} \beta(z, z_n)$ and $\lim_{n\to\infty} \mu(z_n, z)$ exist and are finite, where the sequence $\{z_n\}$ is defined as $z_n = T^n z_0$ for some $z_0 \in Z$.*

Then T has a unique fixed point in Z.

Proof Let z_0 be any arbitrary point in Z, then we have a sequence $\{z_n\}_{n\geq 0}$, with $T^n z_0 = z_n$, for all $n \in \mathbb{N}$.

If for some $m \in \mathbb{N}$, $T^m z_0 = T^{m+1} z_0$, then this implies that $T^m z_0$ is a fixed point of the mapping T. Thus, without loss of generality, we may assume that $z_n \neq z_{n+1}$, i.e., $d_{\beta,\mu}(T^n z_0, T^{n+1} z_0) > 0$, for all $n \in \mathbb{N}$.

Applying (3.1) recursively, we obtain

$$\begin{aligned} \theta\bigl(d_{\beta,\mu}(z_n, z_{n+1})\bigr) &= \theta\bigl(d_{\beta,\mu}(Tz_{n-1}, Tz_n)\bigr) \\ &\leq \bigl[\theta\bigl(d_{\beta,\mu}(z_{n-1}, z_n)\bigr)\bigr]^r \\ &\leq \bigl[\theta\bigl(d_{\beta,\mu}(z_{n-2}, z_{n-1})\bigr)\bigr]^{r^2} \\ &\leq \bigl[\theta\bigl(d_{\beta,\mu}(z_{n-3}, z_{n-2})\bigr)\bigr]^{r^3} \leq \cdots \leq \bigl[\theta\bigl(d_{\beta,\mu}(z_0, z_1)\bigr)\bigr]^{r^n}. \end{aligned} \tag{3.5}$$

As $\theta(t) > 1$, we have

$$1 < \theta\bigl(d_{\beta,\mu}(z_n, z_{n+1})\bigr) \leq \bigl[\theta\bigl(d_{\beta,\mu}(z_0, z_1)\bigr)\bigr]^{r^n}. \tag{3.6}$$

Since $0 < r < 1$, hence letting n tends to infinity in (3.6), we get

$$\lim_{n\to\infty} \theta\bigl(d_{\beta,\mu}(z_n, z_{n+1})\bigr) = 1.$$

Employing property (II) in Definition 2.9, we obtain

$$\lim_{n\to\infty} d_{\beta,\mu}(z_n, z_{n+1}) = 0. \tag{3.7}$$

In a similar method, one can show that

$$\lim_{n\to\infty} d_{\beta,\mu}(z_n, z_{n+2}) = 0. \tag{3.8}$$

By (III) of Definition 2.9, there exists $k \in (0, 1)$ and $M \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d_{\beta,\mu}(z_n, z_{n+1})) - 1}{[d_{\beta,\mu}(z_n, z_{n+1})]^k} = M. \tag{3.9}$$

Case 1: Let $0 < M < \infty$, and define $L = \frac{M}{2}$, from equation (3.9), there exists some $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ we obtain

$$\left| \frac{\theta(d_{\beta,\mu}(z_n, z_{n+1})) - 1}{[d_{\beta,\mu}(z_n, z_{n+1})]^k} - M \right| \leq L,$$

which implies that

$$L = M - L \leq \frac{\theta(d_{\beta,\mu}(z_n, z_{n+1})) - 1}{[d_{\beta,\mu}(z_n, z_{n+1})]^k}, \quad \forall n \geq n_0.$$

Hence, for all $n \geq n_0$, we have

$$n\big[d_{\beta,\mu}(z_n, z_{n+1})\big]^k \leq n\left[\frac{\theta(d_{\beta,\mu}(z_n, z_{n+1})) - 1}{L}\right].$$

By employing (3.5), we obtain

$$n\big[d_{\beta,\mu}(z_n, z_{n+1})\big]^k \leq n\left[\frac{[\theta(d_{\beta,\mu}(z_0, z_1))]^{r^n} - 1}{L}\right].$$

Letting $n \to \infty$, in the above inequality, we get

$$\lim_{n\to\infty} n\big[d_{\beta,\mu}(z_n, z_{n+1})\big]^k = 0. \tag{3.10}$$

Case 2: $M = \infty$, in this case, let $L > 0$ be any arbitrary number. Thus, by the definition of the limit, we can find some $n_1 \in \mathbb{N}$ such that

$$\frac{\theta(d_{\beta,\mu}(z_n, z_{n+1})) - 1}{[d_{\beta,\mu}(z_n, z_{n+1})]^k} \geq L, \quad \forall n \geq n_1,$$

which gives

$$n\big[d_{\beta,\mu}(z_n, z_{n+1})\big]^k \leq n\left[\frac{\theta(d_{\beta,\mu}(z_n, z_{n+1})) - 1}{L}\right].$$

Again employing (3.5) in the above inequality and then letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} n\big[d_{\beta,\mu}(z_n, z_{n+1})\big]^k = 0. \tag{3.11}$$

Thus, from equations (3.10) and (3.11), we deduce that for any $M \in (0, \infty]$ and $0 < k < 1$, there exists some $\hat{N} \in \mathbb{N}$, where $\hat{N} = \max\{n_0, n_1\}$ such that

$$d_{\beta,\mu}(z_n, z_{n+1}) \leq \frac{1}{n^{1/k}}, \quad \forall n \geq \hat{N}, \tag{3.12}$$

Next, we prove that T has a fixed point; we consider two cases.

Case 1: Suppose $T^n z_0 = z_n = z_m = T^m z_0$ for some $m \neq n \in \mathbb{N}$. Assume $m > n$, we obtain $T^{m-n}(z_n) = z_n$. Denote z_n by x, and $q = m - n$, then we have $T^q x = x$, which means x is a periodic point of T. Hence

$$d_{\beta,\mu}(x, Tx) = d_{\beta,\mu}\big(T^q x, T^{q+1} x\big).$$

Following the above argument, one can easily show that $d_{\beta,\mu}(x, Tx) = 0$, i.e., $Tx = x$, implying that x is a fixed point of T.

Case 2: Suppose that $z_n = T^n z_0 \neq T^m z_0 = z_m$ for all $m, n \in \mathbb{N}$ and assume $m > n$.

Claim: $\{z_n\}$ is a Cauchy sequence in Z.

For all $m, n \in \hat{N}$ with $m > n$, and by (Q3) of Definition 2.4, we have:

$$\begin{aligned} d_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1}) d_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m) d_{\beta,\mu}(z_{n+1}, z_m) \\ &\leq \beta(z_n, z_{n+1}) d_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\big[\beta(z_{n+1}, z_{n+2}) d_{\beta,\mu}(z_{n+1}, z_{n+2}) \\ &\quad + \mu(z_{n+2}, z_m) d_{\beta,\mu}(z_{n+2}, z_m)\big] \\ &= \beta(z_n, z_{n+1}) d_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\beta(z_{n+1}, z_{n+2}) d_{\beta,\mu}(z_{n+1}, z_{n+2}) \\ &\quad + \mu(z_{n+1}, z_m)\mu(z_{n+2}, z_m) d_{\beta,\mu}(z_{n+2}, z_m) \\ &\leq \beta(z_n, z_{n+1}) d_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\beta(z_{n+1}, z_{n+2}) d_{\beta,\mu}(z_{n+1}, z_{n+2}) \\ &\quad + \mu(z_{n+1}, z_m)\mu(z_{n+2}, z_m)\big[\beta(z_{n+2}, z_{n+3}) d_{\beta,\mu}(z_{n+2}, z_{n+3}) \\ &\quad + \mu(z_{n+3}, z_m) d_{\beta,\mu}(z_{n+3}, z_m)\big] \end{aligned}$$

$$\vdots$$

$$\begin{aligned} d_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1}) d_{\beta,\mu}(z_n, z_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1}) d_{\beta,\mu}(z_i, z_{i+1}) \\ &\quad + \prod_{l=n+1}^{m-1}\mu(z_l, z_m) d_{\beta,\mu}(z_{m-1}, z_m), \\ d_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1}) d_{\beta,\mu}(z_n, z_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1}) d_{\beta,\mu}(z_i, z_{i+1}) \\ &\quad + \prod_{l=n+1}^{m-1}\mu(z_l, z_m)\beta(z_{m-1}, z_m) d_{\beta,\mu}(z_{m-1}, z_m), \\ d_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1}) d_{\beta,\mu}(z_n, z_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1}) d_{\beta,\mu}(z_i, z_{i+1}). \end{aligned}$$

Hence

$$d_{\beta,\mu}(z_n, z_m) \leq \sum_{i=n}^{m-1} d_{\beta,\mu}(z_i, z_{i+1})\left[\prod_{j=1}^{i}\mu(z_j, z_m)\right]\beta(z_i, z_{i+1}). \tag{3.13}$$

Note that the series

$$\sum_{n=1}^{\infty} d_{\beta,\mu}(z_n, z_{n+1})\left[\prod_{i=1}^{n}\mu(z_i, z_m)\right]\beta(z_n, z_{n+1}),$$

converges, by employing (3.12) and using conditions (1) and (2), we obtain

$$\sum_{n=1}^{\infty} d_{\beta,\mu}(z_n, z_{n+1})\left[\prod_{i=1}^{n}\mu(z_i, z_m)\right]\beta(z_n, z_{n+1}) \leq \sum_{n=1}^{\infty}\frac{1}{n^{1/k}}\prod_{i=1}^{n}\mu(z_i, z_m)\beta(z_n, z_{n+1})$$

$$< \frac{1}{p}\sum_{n=1}^{\infty}\frac{1}{n^{1/k}},$$

which is convergent as $\frac{1}{k} > 1$. Let S and S_n be defined as

$$S = \sum_{n=1}^{\infty} d_{\beta,\mu}(z_n, z_{n+1})\left[\prod_{i=1}^{n}\mu(z_i, z_m)\right]\beta(z_n, z_{n+1})$$

and

$$S_n = \sum_{j=1}^{n} d_{\beta,\mu}(z_j, z_{j+1})\left[\prod_{i=1}^{j}\mu(z_i, z_m)\right]\beta(z_j, z_{j+1}).$$

Therefore, (3.13) can be written as

$$d_{\beta,\mu}(z_n, z_m) \le S_{m-1} - S_{n-1}$$

letting $n, m \to \infty$, and by incorporating equations (3.7) and (3.8), and the fact the above series is convergent, we obtain

$$\lim_{n,m\to\infty} d_{\beta,\mu}(z_n, z_m) = 0. \tag{3.14}$$

Thus, $\{z_n\}$ is a Cauchy sequence in a complete double controlled metric type space $(Z, d_{\beta,\mu})$; therefore, it converges to some $\hat{x} \in Z$, i.e., $\lim_{n\to\infty} d_{\beta,\mu}(z_n, \hat{x}) = 0$.

Next, we show $\hat{x}$ is a fixed point of T, i.e., $T\hat{x} = \hat{x}$. By Lemma 3.2, T is continuous; thus, $\lim_{n\to\infty} d_{\beta,\mu}(Tz_n, T\hat{x}) = 0$, hence we have $Tz_n = z_{n+1} \to T\hat{x}$. Using the triangular property of Definition 2.4, we have

$$d_{\beta,\mu}(\hat{x}, T\hat{x}) \le \beta(\hat{x}, z_n)d_{\beta,\mu}(\hat{x}, z_n) + \mu(z_n, T\hat{x})d_{\beta,\mu}(z_n, T\hat{x}). \tag{3.15}$$

Taking the limit as n tends to infinity in the above inequality and using condition (2), we obtain $d_{\beta,\mu}(\hat{x}, T\hat{x}) = 0$; that is, $T\hat{x} = \hat{x}$.

To prove the uniqueness of the fixed point, assume that T has two fixed points $\hat{x}, \hat{y}$ such that $\hat{x} \ne \hat{y}$,

$$\begin{aligned}\theta\big(d_{\beta,\mu}(\hat{x}, \hat{y})\big) &= \theta\big(d_{\beta,\mu}(T\hat{x}, T\hat{y})\big)\\ &\le \big[\theta\big(d_{\beta,\mu}(\hat{x}, \hat{y})\big)\big]^r\\ &< \theta\big(d_{\beta,\mu}(\hat{x}, \hat{y})\big),\end{aligned}$$

which is a contradiction, hence $\hat{x} = \hat{y}$, so T has a unique fixed point. □

Next, we illustrate Theorem 3.3 by the following example.

Example 3.4 Let $Z = \{0, 1, 2\}$. Consider the symmetric metric $d_{\beta,\mu} : Z \times Z \to [0, \infty)$, defined by

$$d_{\beta,\mu}(2, 2) = d_{\beta,\mu}(1, 1) = d_{\beta,\mu}(0, 0) = 0,$$

and

$$d_{\beta,\mu}(0,1)=1,\qquad d_{\beta,\mu}(0,2)=\frac{4}{5},\qquad d_{\beta,\mu}(1,2)=\frac{6}{25}.$$

The two symmetric functions $\beta,\mu : Z\times Z\to[1,\infty)$, are given by

$$\beta(2,2)=\beta(1,1)=\beta(0,0)=1,\qquad \beta(1,2)=\frac{8}{5},\qquad \beta(0,1)=\frac{6}{5},\qquad \beta(0,2)=\frac{151}{100}$$

and

$$\mu(2,2)=\mu(1,1)=\mu(0,0)=1,\qquad \mu(1,2)=\frac{30}{20},\qquad \mu(0,1)=\frac{6}{5},\qquad \mu(0,2)=\frac{7}{5}.$$

One can easily show that $(d_{\beta,\mu},Z)$ is a complete double controlled metric type space.

Consider the self mapping $T:Z\to Z$, defined by $T(0)=2$, and $T(1)=T(2)=1$, also let $\theta:(0,\infty)\to(1,\infty)$ be given by $\theta(t)=e^{\sqrt{t}}$, then θ satisfies the properties in Definition 2.9, with $k=1/2$.

First, we show that T is Θ-double controlled contraction mapping as in Definition 3.1 with $r=\frac{3}{5}\in(0,1)$.

For any $z,w\in Z$, such that $d_{\beta,\mu}(Tz,Tw)\neq 0$, we investigate if $\theta(d_{\beta,\mu}(Tz,Tw))\leq[\theta(d_{\beta,\mu}(z,w))]^{3/5}$. It is enough to explore these:

(i) $\theta\big(d_{\beta,\mu}(T0,T1)\big)=\theta\big(d_{\beta,\mu}(2,1)\big)=\theta\left(\frac{6}{25}\right)=e^{\sqrt{6/25}}\leq e^{3/5}=\big[\theta\big(d_{\beta,\mu}(0,1)\big)\big]^{3/5}$,

(ii) $$\theta\big(d_{\beta,\mu}(T0,T2)\big)=\theta\big(d_{\beta,\mu}(2,1)\big)=\theta\left(\frac{6}{25}\right)=e^{\sqrt{6/25}}\leq\big(e^{\sqrt{4/5}}\big)^{3/5}=\big[\theta\big(d_{\beta,\mu}(0,2)\big)\big]^{3/5}.$$

Next, we explore if conditions (1) and (2) of Theorem 3.3 are satisfied with $p=1/3\in(0,1)$. To show (1) holds, i.e.,

$$\sup_{m\geq1}\lim_{n\to\infty}\beta(z_{n+1},z_{n+2})\mu(z_{n+1},z_m)<1/p=3,\quad\text{for any } z\in Z,$$

take, for example, $z_0=0$, then $z_1=T(0)=2$ and $z_2=T(2)=1$, $z_3=T(1)=1$, $z_j=1$ for $j>3$, hence

(a) $$\beta(z_0,z_1)\mu(z_0,z)=\beta(0,2)\mu(0,z)=\begin{cases}1.50<3, & \text{if } z=0,\\ 1.81<3, & \text{if } z=1,\\ 2.11<3 & \text{if } z=2,\end{cases}$$

(b) $$\beta(z_1,z_2)\mu(z_1,z)=\beta(2,1)\mu(2,z)=\begin{cases}2.24<3, & \text{if } z=0,\\ 2.40<3, & \text{if } z=1,\\ 1.60<3 & \text{if } z=2,\end{cases}$$

(c) $$\beta(z_2,z_3)\mu(z_2,z)=\beta(1,1)\mu(1,z)=\begin{cases}1.2<3, & \text{if } z=0,\\ 1.0<3, & \text{if } z=1,\\ 1.5<3 & \text{if } z=2,\end{cases}$$

$$\text{(d)}\quad \beta(z_3,z_4)\mu(z_3,z)=\beta(1,1)\mu(1,z)=\begin{cases}1.2<5/3, & \text{if } z=0,\\ 1.0<5/3, & \text{if } z=1,\\ 1.5<5/3 & \text{if } z=2.\end{cases}$$

As $z_j = 1$ for $j > 3$, then $\beta(z_3,z_4)\mu(z_3,z)$ reduces to case (c). If we take $0 \neq z_0$, then similarly we get $\sup_{m\geq 1}\lim_{n\to\infty}\beta(z_{n+1},z_{n+2})\mu(z_{n+1},z_m) < 3$. Therefore, T is Θ-double controlled contraction mapping, satisfying conditions (1) and (2) of Theorem 3.3, so it has a unique fixed point $z = 1$.

By taking $\beta(z,w) = \mu(z,w)$ in Theorem 3.3 and modifying the two conditions (1) and (2) accordingly, we deduce the following corollary for the case of controlled metric type space.

Corollary 3.5 *Let (Z,d_β) be a complete controlled metric type space, and let $T : Z \to Z$ be a Θ-contraction self mapping. Then T has a unique fixed point in Z.*

In case $\beta(z,w) = \mu(z,w) = 1$, we get the following corollary.

Corollary 3.6 *Let (Z,d) be a complete metric space, and let $T : Z \to Z$ be a self mapping. Suppose there exists a function $\theta \in \Theta$ and an $r \in (0,1)$, such that the following holds:*

$$z,w\in Z,\quad d_{\beta,\mu}(Tz,Tw)\neq 0 \quad\Rightarrow\quad \theta\big(d_{\beta,\mu}(Tz,Tw)\big)\leq\big[\theta\big(d_{\beta,\mu}(z,w)\big)\big]^r.$$

Then T has a unique fixed point.

3.2 Ćirić-Reich-Rus-type Θ-double controlled contraction mapping and fixed point theorem

In 1971, Ćirić [18] generalized Banach's contraction principle theorem to a more general contraction. Then in 1974, Ćirić [19] generalized his own result [18] by introducing the quasi-contraction and obtained the fixed point theorem under the below condition.

Definition 3.7 ([19]) Let (Z,d) be a metric space. If $T : Z \to Z$ satisfies the quasi-contraction condition

$$d(Tx,Ty)\leq p\max\big\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big\}$$

for all $x,y \in Z$, and for some $p \in [0,1)$, then T has a unique fixed point in Z.

During the same time, Reich [20] stated the following result.

Definition 3.8 Let (Z,ρ) be a complete metric space. If $T : Z \to Z$ satisfies:

$$\rho(Tx,Ty)\leq\lambda\big[\rho(x,y)+\rho(x,Tx)+\rho(y,Ty)\big].$$

For all $x,y \in Z$ and $\lambda \in [0,1/3)$, then T has a unique fixed point.

It should be noted that Ćirić and Reich proved the result independently. That is why in the literature it is referred to as a Ćirić-Reich-Rus-type contraction. In [28], Abdeljawad et al. gave the following definition of the Ćirić-Reich-Rus-type contraction mapping, and they used it to prove the existence and uniqueness of the fixed point when the extended Branciari b-distance space is complete.

Definition 3.9 ([28]) Let (Z, d) be an extended Branciari b-distance space, then a mapping $T : Z \to Z$, is called a Ćirić-Reich-Rus-type contraction mapping, if there exists a function $\theta \in \Theta$, and $p \in (0, 1)$, such that

$$\theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(M(x, y)\big)\big]^p, \quad \text{for all } x, y \in Z,$$

where

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\}$$

with $\limsup_{n,m\to\infty} \omega(x_n, x_m) < 1/p$, here $x_n = T^n x_0$ for some $x_0 \in Z$.

In the following, inspired by Definition 3.9, we generalize the Ćirić-Reich-Rus-type contraction mapping in the setting of a double controlled metric type space. Therefore, we introduce the concept of Ćirić-Reich-Rus-type Θ-double controlled contraction mapping (for short CRR-Θ-double controlled contraction), with Θ as in Definition 2.9, and we establish the fixed point theorem.

Definition 3.10 Let $(Z, d_{\beta,\mu})$ be a double controlled metric type space, and let $T : Z \to Z$ be a self mapping. Then T is a CRR-Θ-double controlled contraction mapping, if there exists a function $\theta \in \Theta$ such that θ is continuous, and $r \in (0, 1)$, so that the following holds:

$$\theta\big(d_{\beta,\mu}(Tz, Tw)\big) \leq \big[\theta\big(M(z, w)\big)\big]^r, \tag{3.16}$$

for all $z, w \in Z$, such that $d_{\beta,\mu}(Tz, Tw) \neq 0$, where

$$M(z, w) = \max\big\{d_{\beta,\mu}(z, w), d_{\beta,\mu}(z, Tz), d_{\beta,\mu}(w, Tw)\big\}.$$

Next, our second main result of the fixed point theorem.

Theorem 3.11 *Let $(Z, d_{\beta,\mu})$ be a complete double controlled metric type space, and let $\beta, \mu : Z \times Z \to [1, \infty)$ be two noncomparable functions, defined on a non-empty set Z. Let $T : Z \to Z$ be a CRR-Θ-double controlled contraction self mapping satisfying the following*:

$(M1)$

$$\sup_{m\geq 1} \lim_{n\to\infty} \beta(z_{n+1}, z_{n+2})\mu(z_{n+1}, z_m) < 1/p,$$

for some $p \in (0, 1)$, and

$(M2)$ *For any $z \in Z$, both $\lim_{n\to\infty} \beta(z, z_n)$ and $\lim_{n\to\infty} \mu(z_n, z)$ exist and are finite, where the sequence $\{z_n\}$ is defined as $z_n = T^n z_0$ for some $z_0 \in Z$.*

Then T has a unique fixed point in Z.

Proof Let z_0 be any arbitrary point in Z, then we have a sequence $\{z_n\}_{n\geq 0}$, with $T^n z_0 = z_n$, for all $n \in \mathbb{N}$.

If for some $q \in \mathbb{N}$, $T^q z_0 = T^{q+1} z_0$, then this implies that $T^q z_0$ is a fixed point of the mapping T. Thus, without loss of generality, we may assume that $d_{\beta,\mu}(z_n, z_{n+1}) = d_{\beta,\mu}(T^n z_0, T^{n+1} z_0) > 0$, for all $n \in \mathbb{N}$.

By Definition 3.10, we obtain

$$\theta\big(d_{\beta,\mu}(z_{n+1}, z_n)\big) \leq \big[\theta\big(M(z_n, z_{n-1})\big)\big]^r, \tag{3.17}$$

where

$$\begin{aligned} M(z_n, z_{n-1}) &= \max\big\{d_{\beta,\mu}(z_n, z_{n-1}), d_{\beta,\mu}(z_n, Tz_n), d_{\beta,\mu}(z_{n-1}, Tz_{n-1})\big\} \\ &= \max\big\{d_{\beta,\mu}(z_n, z_{n-1}), d_{\beta,\mu}(z_n, z_{n+1}), d_{\beta,\mu}(z_{n-1}, z_n)\big\} \\ &\leq \max\big\{d_{\beta,\mu}(z_n, z_{n-1}), d_{\beta,\mu}(z_n, z_{n+1})\big\}. \end{aligned}$$

In case $M(z_n, z_{n-1}) = d_{\beta,\mu}(z_n, z_{n+1})$, then the inequality in (3.17) turns into

$$\theta\big(d_{\beta,\mu}(z_{n+1}, z_n)\big) \leq \big[\theta\big(d_{\beta,\mu}(z_n, z_{n+1})\big)\big]^r,$$

which is a contradiction, since $0 < r < 1$. Therefore, we have

$$M(z_n, z_{n-1}) = d_{\beta,\mu}(z_n, z_{n-1}), \tag{3.18}$$

using (3.18), the inequality in (3.17) becomes

$$\begin{aligned} \theta\big(d_{\beta,\mu}(z_{n+1}, z_n)\big) &\leq \big[\theta\big(d_{\beta,\mu}(z_n, z_{n-1})\big)\big]^r \\ &\leq \big[\theta\big(d_{\beta,\mu}(z_{n-1}, z_{n-2})\big)\big]^{r^2} \leq \cdots \\ &\leq \big[\theta\big(d_{\beta,\mu}(z_0, z_1)\big)\big]^{r^n}. \end{aligned} \tag{3.19}$$

That is the same inequality as (3.5) in Theorem 3.3. Consequently, following the same steps of proof in Theorem 3.3 and utilizing conditions $(M1)$ and $(M2)$, we conclude that $\{z_n\}$ is a Cauchy sequence in a complete double controlled metric type space $(Z, d_{\beta,\mu})$. Therefore, it converges to some $\hat{z} \in Z$, i.e.,

$$\lim_{n\to\infty} d_{\beta,\mu}(z_n, \hat{z}) = 0. \tag{3.20}$$

We claim that $T(\hat{z}) = \hat{z}$. First, note that if there exists an integer N such that $z_N = \hat{z}$, then $\hat{z}$ is a fixed point since by (3.17), we have

$$\theta\big(d_{\beta,\mu}(Tz_N, z_N)\big) \leq \big[\theta\big(M(z_N, z_{N-1})\big)\big]^r.$$

Utilizing (3.18) and then repeating the steps as in (3.19), we obtain $\theta(d_{\beta,\mu}(Tz_N, z_N)) \leq [\theta(d_{\beta,\mu}(z_0, z_1))]^{r^N}$; applying the steps as in (3.6) and then taking the limit as N tends to infinity as in (3.7), we obtain $d_{\beta,\mu}(Tz_N, z_N) = 0$, i.e., $\hat{z} = z_N = Tz_N$ is a fixed point. Thus, without loss of generality assume that $z_n \neq \hat{z}$ for all n, to show $\hat{z}$ is a fixed point, observe that continuity of T together with (3.20) implies

$$\lim_{n\to\infty} d_{\beta,\mu}(z_{n+1}, T\hat{z}) = \lim_{n\to\infty} d_{\beta,\mu}(Tz_n, T\hat{z}) = 0. \tag{3.21}$$

Utilizing (3.20), we obtain

$$\lim_{n\to\infty} d_{\beta,\mu}(\hat{z}, z_{n+1}) = \lim_{n+1\to\infty} d_{\beta,\mu}(\hat{z}, z_{n+1}) = 0. \tag{3.22}$$

By (Q3) of Definition 2.4, we deduce

$$d_{\beta,\mu}(\hat{z}, T\hat{z}) \leq \beta(\hat{z}, z_{n+1}) d_{\beta,\mu}(\hat{z}, z_{n+1}) + \mu(z_{n+1}, T\hat{z}) d_{\beta,\mu}(z_{n+1}, T\hat{z}).$$

Upon taking the limit as n tends to infinity and using (3.21), (3.22), and ($M2$), we reach $d_{\beta,\mu}(\hat{z}, T\hat{z}) = 0$, hence $T\hat{z} = \hat{z}$, proving $\hat{z}$ is a fixed point.

Next, we show the uniqueness of the fixed point. Assume that T has two fixed points $\hat{z}$, $\hat{w}$ such that $\hat{z} \neq \hat{w}$, thus $d_{\beta,\mu}(\hat{z}, \hat{w}) = d_{\beta,\mu}(T\hat{z}, T\hat{w}) > 0$. Utilizing (3.16), we obtain

$$\begin{aligned}
\theta\left(d_{\beta,\mu}(\hat{z}, \hat{w})\right) &= \theta\left(d_{\beta,\mu}(T\hat{z}, T\hat{w})\right) \\
&\leq \left[\theta\left(M(\hat{z}, \hat{w})\right)\right]^r \\
&\leq \left[\theta\left(d_{\beta,\mu}(\hat{z}, \hat{w})\right)\right]^r < \left[\theta\left(d_{\beta,\mu}(\hat{z}, \hat{w})\right)\right]
\end{aligned}$$

which is a contradiction, hence $\hat{z} = \hat{w}$, so T has a unique fixed point. □

Next, we present a supporting example for Theorem 3.11 and also refer to [28].

Example 3.12 Let $Z = \{S_n : n \in \mathbb{N}\}$, where

$$S_n = \frac{n(n+1)(n+2)}{3}.$$

Define the mapping $d_{\beta,\mu} : Z \times Z \to [0, +\infty)$ by $d_{\beta,\mu}(x, y) = |x - y|^2$, and let $\beta, \mu : Z \times Z \to [1, +\infty)$ be defined by

$$\beta(x, y) = \begin{cases} \frac{y}{x} & \text{if } y > x, \\ \frac{x}{y} & \text{otherwise}, \end{cases}$$

and $\mu(x, y) = \frac{1}{y} + 1$. Then $(Z, d_{\beta,\mu})$ is a complete double controlled metric type space. The mapping $T : Z \to Z$ is given by

$$T(S_n) = \begin{cases} S_1 & \text{if } n = 1, \\ S_{n-1} & \text{if } n \geq 2. \end{cases}$$

To show T is CRR-Θ-double controlled contraction mapping with $\theta(t) = e^t$, we need to show (3.16) holds, i.e.,

$$\theta\left(d_{\beta,\mu}(Tx, Ty)\right) \leq \left[\theta\left(M(x, y)\right)\right]^r, \quad \text{for some } r \in (0, 1).$$

Note that if the above equation holds, it yields, $e^{d_{\beta,\mu}(Tx,Ty)} \leq [e^{\theta(M(x,y))}]^r$. Applying ln on both sides, we get

$$d_{\beta,\mu}(Tx,Ty) \leq rM(x,y), \tag{3.23}$$

where $M(x,y) = \max\{d_{\beta,\mu}(x,y), d_{\beta,\mu}(x,Tx), d_{\beta,\mu}(y,Ty)\}$.

Hence to show T is CRR-Θ-double controlled contraction mapping, it is enough to show that (3.23) holds.

Case 1: $n = 1$ and $m > 2$, then

$$d_{\beta,\mu}(TS_1, TS_m) = d_{\beta,\mu}(S_1, S_{m-1}) = \left|\frac{m(m-1)(m+1)-6}{3}\right|^2.$$

Observe that

$$\begin{aligned} M(S_1,S_m) &= \max\{d_{\beta,\mu}(S_1,S_m), d_{\beta,\mu}(S_1,S_1), d_{\beta,\mu}(S_m,S_{m-1})\} \\ &= d_{\beta,\mu}(S_1,S_m). \end{aligned}$$

Since

$$d_{\beta,\mu}(S_1,S_m) = \left|\frac{m(m+1)(m+2)-6}{3}\right|^2 > d_{\beta,\mu}(S_m,S_{m-1}) = \left|\frac{m(m+1)3}{3}\right|^2.$$

Thus,

$$\begin{aligned} \frac{d_{\beta,\mu}(TS_1,TS_m)}{M(S_1,S_m)} &= \frac{d_{\beta,\mu}(TS_1,TS_m)}{d_{\beta,\mu}(S_1,S_m)} \\ &= \left|\frac{m(m-1)(m+1)-6}{m(m+1)(m+2)-6}\right|^2 < r. \end{aligned}$$

Case 2: For $m > n > 1$, we have

$$\begin{aligned} d_{\beta,\mu}(TS_n,TS_m) &= d_{\beta,\mu}(S_{n-1},S_{m-1}) \\ &= \left|\frac{n(n-1)(n+1)-m(m-1)(m+1)}{3}\right|^2 \\ &= \left|\frac{(n-m)(n^2+nm+m^2-1)}{3}\right|^2. \end{aligned}$$

While

$$\begin{aligned} M(S_n,S_m) &= \max\{d_{\beta,\mu}(S_n,S_m), d_{\beta,\mu}(S_n,S_{n-1}), d_{\beta,\mu}(S_m,S_{m-1})\} \\ &= d_{\beta,\mu}(S_n,S_m). \end{aligned}$$

Since

$$d_{\beta,\mu}(S_n,S_m) = \left|\frac{n(n+1)(n+2)-m(m+1)(m+2)}{3}\right|^2$$

$$= \left| \frac{(n-m)[(n^2+nm+m^2)+3(n+m)+2]}{3} \right|^2,$$

and

$$d_{\beta,\mu}(S_n, S_{n-1}) = \left| \frac{n(n+1)3}{3} \right|^2, \qquad d_{\beta,\mu}(S_m, S_{m-1}) = \left| \frac{m(m+1)3}{3} \right|^2.$$

Hence

$$\begin{aligned} \frac{d_{\beta,\mu}(TS_n, TS_m)}{M(S_n, S_m)} &= \left| \frac{(n-m)(n^2+nm+m^2-1)}{(n-m)[(n^2+nm+m^2)+3(n+m)+2]} \right|^2 \\ &= \left| \frac{n^2+nm+m^2-1}{n^2+nm+m^2+3(n+m)+2} \right|^2 < r, \quad \text{where } r \in (0,1). \end{aligned}$$

Thus, we have shown T is CRR-Θ-double controlled contraction mapping. Next, we show that conditions $(M1)$ and $(M2)$ of Theorem 3.11 hold. We form a sequence $\{z_n\}$ by taking $z_0 = S_n$, with $n > 2$. As $T(S_n) = S_{n-1}$, thus $z_1 = S_{n-1}$, $z_2 = S_{n-2}$, so $z_j = S_{n-j}$. Note that

$$\beta(S_{n+1}, S_n) = \frac{S_{n+1}}{S_n} = \frac{(n+1)(n+2)(n+3)}{n(n+1)(n+2)} = \frac{n+3}{n}$$

and

$$\mu(S_{n+1}, S_m) = \frac{1}{S_m} + 1 = \frac{3}{m(m+1)(m+2)} + 1.$$

Then,

$$\begin{aligned} \sup_{m\geq 1} \lim_{n\to\infty} \beta(S_{n+1}, S_n)\mu(S_{n+1}, S_m) &= \sup_{m\geq 1} \lim_{n\to\infty} \left(\frac{n+3}{n}\right)\left(\frac{3}{m(m+1)(m+2)} + 1\right) \\ &< 4 = 1/p, \end{aligned}$$

take $p = 1/4 \in (0,1)$. Clearly for any $x \in Z$, then $\lim_{n\to\infty} \beta(x, S_n)$ and $\lim_{n\to\infty} \mu(S_n, x)$ exist and are finite. Therefore, T satisfies the properties of Theorem 3.11, hence T has a unique fixed point S_1.

By taking $\beta = \mu$ in Theorem 3.11 and modifying the conditions $(M1)$ and $(M2)$ accordingly, we obtain the following immediate result for the case of controlled metric type space.

Corollary 3.13 *Let (Z, d_β) be a complete controlled metric type space, and let $T : Z \to Z$ be a CRR- Θ-double controlled contraction self mapping satisfying the following*:

$(C1)$

$$\sup_{m\geq 1} \lim_{n\to\infty} \beta(z_{n+1}, z_{n+2})\beta(z_{n+1}, z_m) < 1/p,$$

for some $p \in (0,1)$, and

$(C2)$ *For any $z \in Z$, then $\lim_{n\to\infty} \beta(z, z_n)$ and $\lim_{n\to\infty} \beta(z_n, z)$ exist and are finite, where the sequence $\{z_n\}$ is defined as $z_n = T^n z_0$ for some $z_0 \in Z$.*

Then T has a unique fixed point in Z.

In case $\beta(z,w) = \mu(z,w) = 1$ in Theorem 3.11, we get the following corollary.

Corollary 3.14 *Let (Z,d) be a complete metric space, let $T : Z \to Z$ be a self mapping. Suppose there exists a continuous function $\theta \in \Theta$ and an $r \in (0,1)$, such that the following holds*:

$$\theta\big(d_{\beta,\mu}(Tz,Tw)\big) \leq \big[\theta\big(M(z,w)\big)\big]^r,$$

for all $z,w \in Z$, such that $d_{\beta,\mu}(Tz,Tw) \neq 0$, where

$$M(z,w) = \max\big\{d_{\beta,\mu}(z,w), d_{\beta,\mu}(z,Tz), d_{\beta,\mu}(w,Tw)\big\}.$$

Then T has a unique fixed point.

Corollary 3.15 *Let $(Z,d_{\beta,\mu})$ be a complete double controlled metric type space, and let $T : Z \to Z$ be a self mapping. Suppose there exists $\gamma \in (0,1)$ such that the following holds*:

$(S1)$

$$d_{\beta,\mu}(Tz,Tw) \leq \gamma \max\big\{d_{\beta,\mu}(z,w), d_{\beta,\mu}(z,Tz), d_{\beta,\mu}(w,Tw)\big\}, \quad \textit{for all } z,w \in Z.$$

$(S2)$

$$\sup_{m\geq 1} \lim_{n\to\infty} \beta(z_{n+1},z_{n+2})\mu(z_{n+1},z_m) < 1/p,$$

for some $p \in (0,1)$, and

$(S3)$ *For any $z \in Z$, then $\lim_{n\to\infty}\beta(z,z_n)$ and $\lim_{n\to\infty}\mu(z_n,z)$ exist and are finite, where the sequence $\{z_n\}$ is defined as $z_n = T^n z_0$ for some $z_0 \in Z$.*

Then T has a unique fixed point in Z.

Proof Let $\theta : (0,\infty) \to (1,\infty)$ be defined by $\theta(t) = e^{\sqrt{t}}$, then $\theta \in \Theta$ as in Definition 2.9, and it is continuous. Property (S1) implies

$$\begin{aligned}\theta\big(d_{\beta,\mu}(Tz,Tw)\big) &= e^{\sqrt{d_{\beta,\mu}(Tz,Tw)}} \\ &\leq \big[e^{\sqrt{\max\{d_{\beta,\mu}(z,w),d_{\beta,\mu}(z,Tz),d_{\beta,\mu}(w,Tw)\}}}\big]^{\sqrt{\gamma}} \\ &= \big[\theta\big(M(z,w)\big)\big]^{\sqrt{\gamma}}.\end{aligned}$$

Let $r = \sqrt{\gamma} \in (0,1)$ Therefore, the existence and uniqueness of the fixed point follow from Theorem 3.11. □

4 Application

In the closing, we would like to illustrate the importance of our Theorem 3.3 in finding a unique real solution for an mth degree polynomial. There are many other methods for root finding problems, such as numerical methods, but utilizing the fixed point results, it becomes quite easy, as it is presented below.

Theorem 4.1 *For any natural number* $m \geq 3$, *the equation*

$$\xi^m - (m^4 - 1)\xi^{m+1} - m^4\xi + 1 = 0 \tag{4.1}$$

has a unique real solution in the interval $[-1, 1]$.

Proof If $|\xi| > 1$, then equation (4.1) does not have a solution, and therefore $|\xi| \leq 1$. Let $Z = [-1, 1]$, then for any $\xi, \nu \in Z$, let the metric be defined by $d_{\beta,\mu}(\xi, \nu) = |\xi - \nu|$.

The two functions $\beta, \mu : Z \times Z \to [1, \infty)$ are defined by

$$\beta(\xi, \nu) = \max\{\xi, \nu\} + 2, \quad \text{and} \quad \mu(\xi, \nu) = \max\{\xi, \nu\} + 1.$$

One can easily show that $(Z, d_{\beta,\mu})$ is a complete double controlled metric type space. Let $T : Z \to Z$ be a mapping defined by

$$T\xi = \frac{\xi^m + 1}{(m^4 - 1)\xi^m + m^4}, \tag{4.2}$$

and let $\theta : (0, \infty) \to (1, \infty)$ be defined by $\theta(t) = e^{\sqrt{t}}$, then $\theta \in \Theta$.

We will show that T is Θ-double controlled contraction mapping.

As $m \geq 3$, we can conclude that $m^4 \geq 81$. Therefore,

$$\begin{aligned}
d_{\beta,\mu}(T\xi, T\nu) &= \left| \frac{\xi^m + 1}{(m^4 - 1)\xi^m + m^4} - \frac{\nu^m + 1}{(m^4 - 1)\nu^m + m^4} \right| \\
&= \left| \frac{\xi^m - \nu^m}{((m^4 - 1)\xi^m + m^4)((m^4 - 1)\nu^m + m^4)} \right| \\
&\leq \frac{|\xi - \nu|}{m^4} \\
&\leq \frac{|\xi - \nu|}{81} \\
&= \frac{1}{81} d_{\beta,\mu}(\xi, \nu) \\
&\leq e^{-\tau} d_{\beta,\mu}(\xi, \nu), \quad \text{for some}, 3 < \tau \leq 5.
\end{aligned}$$

This yields

$$e^{\sqrt{d_{\beta,\mu}(T\xi,T\nu)}} \leq e^{\sqrt{e^{-\tau} d_{\beta,\mu}(\xi,\nu)}} = \left(e^{\sqrt{d_{\beta,\mu}(\xi,\nu)}}\right)^{\sqrt{e^{-\tau}}} = \left(e^{\sqrt{d_{\beta,\mu}(\xi,\nu)}}\right)^{r},$$

where $r = \sqrt{e^{-\tau}} \in (0, 1)$, since $3 < \tau \leq 5$. Hence

$$\theta\left(d_{\beta,\mu}(T\xi, T\nu)\right) \leq \left[\theta(d_{\beta,\mu}(\xi, \nu))\right]^r, \quad \text{for } \xi, \nu \in Z, \text{with } d_{\beta,\mu}(T\xi, T\nu) \neq 0.$$

Next, we show conditions (1) and (2) of Theorem 3.3 hold by taking $p = 1/4 \in (0, 1)$. Hence for any $z_0 \in Z$, we define the sequence $\{z_n\} \in Z$ by

$$z_n = T^n z_0 \leq \frac{2}{m^4}, \quad \text{for any } n. \tag{4.3}$$

Using the definitions of β, μ, and (4.3), we obtain

$$\sup_{m\geq 1}\lim_{n\to\infty}\beta(z_{n+1},z_{n+2})\mu(z_{n+1},z_m)\leq \frac{10}{m^4}+2<4.$$

Also, both $\lim_{n\to\infty}\beta(z,z_n)$ and $\lim_{n\to\infty}\mu(z_n,z)$ exist and are finite. Thus, all the conditions of Theorem 3.3 are satisfied, and therefore T has a unique fixed point in Z, which is a unique real solution of equation (4.1). □

5 Conclusion

Our space was a double controlled metric type space; there, we introduced two new types of generalized contraction mappings. In the first one, inspired by the work [26], we introduced Θ-double controlled contraction mapping, while in the second one, inspired by the work [28], we introduced a Ćirić-Reich-Rus-type Θ-double controlled contraction mapping. Under these mappings, we established the existence and uniqueness of the fixed point theorems on complete double controlled metric type spaces and presented some examples.

Karapinar introduced the notion of an interpolative Kannan-type contraction in [33]. Recently, Aydi et al. initiated the concept of ω-interpolative Ćirić-Reich-Rus-type contractions and established fixed point results [34, 35]. We propose some suggestions for future research directions, such as utilizing ω-interpolative Ćirić-Reich-Rus-type contractions on complete double controlled metric type space and exploring fixed point results.

Acknowledgements

The author would like to thank Prince Sultan University for paying the publication fees for this article through TAS Research LAB. Also, the author is grateful to the anonymous reviewers, their comments and remarks helped improve the paper.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

The author fully contributed to writing this article and reviewed the manuscript.

Received: 20 June 2022 Accepted: 12 June 2023 Published online: 22 June 2023

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