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On some inequalities for uniformly convex mapping with estimations to normal distributions

Saad Ihsan Butt[1], Yamin Sayyari[2], Praveen Agarwal[3,4], Juan J. Nieto[5*] and Muhammad Umar[1]

*Correspondence:
juanjose.nieto.roig@usc.es
[5]CITMAga, Department of Statistics, University of Santiago de Compostela, 15882, Santiago de Compostela, Spain
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce notable Jensen–Mercer inequality for a general class of convex functions, namely uniformly convex functions. We explore some interesting properties of such a class of functions along with some examples. As a result, we establish Hermite–Jensen–Mercer inequalities pertaining uniformly convex functions by considering the class of fractional integral operators. Moreover, we establish Mercer–Ostrowski inequalities for conformable integral operator via differentiable uniformly convex functions. Finally, we apply our inequalities to get estimations for normal probability distributions (Gaussian distributions).

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1 Introduction

A number of mathematical areas demonstrate the importance of convex functions. This theory offers a superb framework for initiating and creating numerical instruments to take on and research challenging mathematical topics. They are magical, especially in optimization theory, because of a variety of useful qualities. The theory of mathematical inequalities and convex functions have a beautiful relationship. Convexity arises in several related topics of basic optimization, namely information theory and inequalities theory. Interested readers can refer to [1–3].

The idea of the derivative operator from integer order n to arbitrary order is added in fractional calculus. Fractional integrals are effective tools for solving numerous issues from many scientific and engineering sectors in applied mathematics. Numerous mathematicians have been combining their efforts and developing fresh perspectives on fractional analysis over the past few years to add a fresh perspective and new elements to the fields of mathematical analysis and applied mathematics.

The following known fractional integrals are used throughout this paper (see, [4]).

$$ {}^{\beta}\mathcal{J}^{\alpha}_{\mu^+}\mathbb{F}(y_2) = \frac{1}{\Gamma(\beta)}\int_{\mu}^{y_2}\left(\frac{(y_2-\mu)^{\alpha}-(\gamma-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathbb{F}(\gamma)}{(\gamma-\mu)^{1-\alpha}}\,d\gamma \qquad (1) $$

and

$$ {}^{\beta}\mathcal{J}^{\alpha}_{\nu^-}\mathbb{F}(y_2) = \frac{1}{\Gamma(\beta)} \int_{y_2}^{\nu} \left(\frac{(\nu - y_2)^{\alpha} - (\nu - \gamma)^{\alpha}}{\alpha} \right)^{\beta - 1} \frac{\mathbb{F}(\gamma)}{(\nu - \gamma)^{1-\alpha}} \, d\gamma. \tag{2} $$

Note that if we choose $\alpha = 1$ in (1) and (2) then it reduces to classical Riemann–Liouville fractional integral operator. For some recent related results, see [5–7].

Integral inequalities have an important role in the expansion of all branches of mathematics. One of the most powerful of these integral inequalities is the Jensen–Mercer inequality, obtained for convex functions as follows:

Theorem 1.1 [8] *Let $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a convex mapping. Then the inequality*

$$ \mathbb{F}\left(\mu + \nu - \sum_{\jmath=1}^{n} p_{\jmath} y_{\jmath} \right) \le \mathbb{F}(\mu) + \mathbb{F}(\nu) - \sum_{\jmath=1}^{n} p_{\jmath} \mathbb{F}(y_{\jmath}) $$

holds for all $y_{\jmath} \in [\mu, \nu]$ and $p_{\jmath} \in [0, 1]$ with $\sum_{\jmath=1}^{n} p_{\jmath} = 1$.

In recent years, this Mercer variant of Jensen's inequality has been of supreme interest to many researchers. Many important extensions, refinements, improvements, and generalizations of Jensen–Mercer inequality were revealed in [9–12] along with some results in information theory [13]. It is not easy to formulate fractional variants of integral Jensen's inequality as there is still no breakthrough in achieving it. However, the variant of Hermite–Jensen–Mercer inequality introduced in [9] was recently presented by Sarikaya et al. in [14] for Riemann–Liouville fractional integral operators. However, their weighted fractional extensions and improvements were given by İşcan in [15]. Caputo fractional derivatives were given in [16, 17]. The conformable fractional integral operator was given in [18]. However, for ψ-Hilfer–Operator (with respect to monotone function) was studied in [19] and for Atangana–Baleanu fractional operator having non-singular kernel in [20].

We mentioned some Hermite–Jensen–Mercer inequalities and related results for conformable fractional integral operators of our interest as below:

Theorem 1.2 [18] *Let $\alpha, \beta > 0$ and $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a convex mapping. Then*

$$ \begin{aligned} \mathbb{F}\left(\mu + \nu - \frac{y_1 + y_2}{2} \right) &\le \frac{2^{\alpha\beta - 1} \alpha^{\beta} \Gamma(\beta + 1)}{(y_2 - y_1)^{\alpha\beta}} \times J\mathbb{F}(\alpha, \beta, y_1, y_2) \\ &\le \mathbb{F}(\mu) + \mathbb{F}(\nu) - \left(\frac{\mathbb{F}(y_1) + \mathbb{F}(y_2)}{2} \right) \end{aligned} $$

for all $y_1, y_2 \in [\mu, \nu]$, where

$$ J\mathbb{F}(\alpha, \beta, y_1, y_2) := {}^{\beta}\mathcal{J}^{\alpha}_{(\mu + \nu - \frac{y_1 + y_2}{2})^+} \mathbb{F}(\mu + \nu - y_1) + {}^{\beta}\mathcal{J}^{\alpha}_{(\mu + \nu - \frac{y_1 + y_2}{2})^-} \mathbb{F}(\mu + \nu - y_2). $$

Lemma 1.3 [18, Lemma 1] *Let $\alpha, \beta \in \mathbb{R}$, $y_1, y_2 \in [\mu, \nu]$ and $\mathbb{F} : [\mu, \nu] \longrightarrow \mathbb{R}$ be a differentiable mapping such that $\mathbb{F}' \in L[\mu, \nu]$. Then*

$$ \frac{2^{\alpha\beta - 1} \alpha^{\beta} \Gamma(\beta + 1)}{(y_2 - y_1)^{\alpha\beta}} J\mathbb{F}(\alpha, \beta, y_1, y_2) - \mathbb{F}\left(\mu + \nu - \frac{y_1 + y_2}{2} \right) $$

$$= \frac{y_2 - y_1}{4}\alpha^{\beta} \int_0^1 \left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta} \\ \times \left[\mathbb{F}'\left(\mu + \nu - \left(\frac{2-\gamma}{2}y_1 + \frac{\gamma}{2}y_2\right)\right) - \mathbb{F}'\left(\mu + \nu - \left(\frac{\gamma}{2}y_1 + \frac{2-\gamma}{2}y_2\right)\right)\right] d\gamma.$$

The organization of this article is in such a way that we first study some examples and important properties of uniformly convex functions. Then we introduce the variant of Jensen–Mercer inequality for them. As a result, we introduce several new generalized fractional variants of Hermite–Jensen–Mercer inequalities. Particular cases recapture several known results. Finally, we also first time introduced fractional Ostrowski–Mercer inequality.

2 Some results for uniformly convex functions

In this section, we start with the following important class of convex function:

Definition 2.1 ([21]) Let $\mathbb{F} : [\mu, \nu] \longrightarrow \mathbb{R}$ be a function. Then $\mathbb{F}$ is uniformly convex with modulus $\varphi : R_+ \longrightarrow [0, +\infty)$ if φ is increasing, vanishes only at 0, and

$$\mathbb{F}\big(\gamma y_1 + (1-\gamma)y_2\big) + \gamma(1-\gamma)\varphi\big(|y_1 - y_2|\big) \le \gamma\mathbb{F}(y_1) + (1-\gamma)\mathbb{F}(y_2)$$

for every $\gamma \in [0,1]$ and $y_1, y_2 \in [\mu, \nu]$.

The uniformly convex function is stronger than a convex function. Almost all convex functions on the finite interval $[\mu, \nu]$ can be considered as a uniformly convex functions. The algebraic properties of uniformly convex functions are given in the following references; see Bauschke [21, Page 144] and Zalinescu [22, Sect. 4].

We point out a few examples below:

(i) Let $\mathbb{F}(y_1) = {y_1}^2$. Since

$$\big(\gamma\mu + (1-\gamma)\nu\big)^2 + \gamma(1-\gamma)(\nu-\mu)^2 = \gamma\mu^2 + (1-\gamma)\nu^2 \le \gamma\mathbb{F}(\mu) + (1-\gamma)\mathbb{F}(\nu)$$

for all $\mu, \nu \in \mathbb{R}$ and all $\gamma \in [0,1]$, ${y_1}^2 : \mathbb{R} \to \mathbb{R}$ is uniformly convex with modulus $\varphi(y_1) = {y_1}^2$.

(ii) $e^{y_1} : (0, \infty) \to \mathbb{R}$ is uniformly convex with modulus $\varphi(y_1) = \frac{1}{2}{y_1}^2$;

(iii) $1/y_1 : (\mu, \nu) \to \mathbb{R}$ is uniformly convex with modulus $\varphi(y_1) = \frac{1}{\nu^3}{y_1}^2, \mu > 0$;

(iv) ${y_1}^4 : (\mu, \nu) \to \mathbb{R}$ is uniformly convex with modulus $\varphi(y_1) = 6\mu^2{y_1}^2, \mu > 0$.

Lemma 2.2 *Let $\mathbb{F} : I \to \mathbb{R}$ be a twice-differentiable function and*

$$m := \inf\big\{\mathbb{F}''(y_1) : y_1 \in I\big\} > 0.$$

Then $\mathbb{F}$ is uniformly convex with modulus $\varphi(r) = \frac{m}{2}r^2$.

Proof It is obvious that φ is increasing and vanishes only at 0. We consider two fixed points $y_1, y_2 \in I$ and define

$$\varphi(\alpha) := \alpha\mathbb{F}(y_1) + (1-\alpha)\mathbb{F}(y_2) - \mathbb{F}\big(\alpha y_1 + (1-\alpha)y_2\big) - \frac{m\alpha(1-\alpha)}{2}(y_1 - y_2)^2$$

for all $\alpha \in [0,1]$. Now, we show that $\varphi(\alpha) \geq 0$, for all $\alpha \in [0,1]$. Since $\varphi(0) = \varphi(1) = 0$ and

$$\frac{d^2\varphi}{d\alpha^2} = m(y_1 - y_2)^2 - (y_2 - y_1)^2\mathbb{F}''\big(\alpha y_1 + (1-\alpha)y_2\big) \leq 0,$$

$\mathbb{F}(\alpha) \geq 0$ for every $y_1, y_2 \in [\mu,\nu]$ and $\alpha \in [0,1]$. Hence,

$$\alpha\mathbb{F}(y_1) + (1-\alpha)\mathbb{F}(y_2) \geq \mathbb{F}\big(\alpha y_1 + (1-\alpha)y_2\big) + \frac{m\alpha(1-\alpha)}{2}(y_1 - y_2)^2.$$

Therefore, the proof is complete. □

Definition 2.3 [23] Let $\mathbb{F} : [\mu,\nu] \longrightarrow \mathbb{R}$ be a function. Then $\mathbb{F}$ is strongly convex with modulus $c > 0$ if

$$\mathbb{F}\big(\gamma y_1 + (1-\gamma)y_2\big) + c\gamma(1-\gamma)(y_1 - y_2)^2 \leq \gamma\mathbb{F}(y_1) + (1-\gamma)\mathbb{F}(y_2)$$

for every $\gamma \in [0,1]$ and $y_1, y_2 \in [\mu,\nu]$.

Lemma 2.4 *Let $\mathbb{F} : I \longrightarrow \mathbb{R}$ be a strongly convex function with modulus $c > 0$ on I, $\{y_\jmath\}_{\jmath=1}^n \subseteq [\mu,\nu]$ be a sequence and let π be a permutation on $\{1,\dots,n\}$ such that $y_{\pi(1)} \leq y_{\pi(2)} \leq \cdots \leq y_{\pi(n)}$. Then the inequality*

$$\mathbb{F}\left(\sum_{\jmath=1}^n p_\jmath y_\jmath\right) \leq \sum_{\jmath=1}^n p_\jmath \mathbb{F}(y_\jmath) - c\sum_{\jmath=1}^{n-1} p_{\pi(\jmath)}p_{\pi(\jmath+1)}(y_{\pi(\jmath+1)} - y_{\pi(\jmath)})^2 \tag{3}$$

holds for every convex combination $\sum_{\jmath=0}^n p_\jmath y_\jmath$ of points $y_\jmath \in I$.

Proof The result follows from Theorem 2.4 of [24] with $\varphi(r) = cr^2$. □

For the rest of the paper, we will use the following notations for classes of functions.

$$\mathbb{F} \in U\big(\varphi; [\mu,\nu]\big) = \mathbb{F} : [\mu,\nu] \subset (0,\infty) \to \mathbb{R}$$

be an uniformly convex mapping with modulus φ

and

$$\mathbb{F} \in S\big(c; [\mu,\nu]\big) = \mathbb{F} : [\mu,\nu] \subset (0,\infty) \to \mathbb{R}$$

be a strongly convex function with modulus c.

3 Jensen–Mercer inequalities for uniformly convex functions

Theorem 3.1 *Let $\mathbb{F} \in U(\varphi; [\mu,\nu])$ and $\mu < \varsigma < \nu$. Then following inequality is valid*

$$\mathbb{F}(\mu + \nu - \varsigma) + \mathbb{F}(\varsigma) + \frac{2(\nu - \varsigma)(\varsigma - \mu)}{(\nu - \mu)^2}\varphi(\nu - \mu) \leq \mathbb{F}(\mu) + \mathbb{F}(\nu). \tag{4}$$

Proof Let $\varsigma \in [\mu,\nu]$ be arbitrary and $\varsigma = \gamma\mu + (1-\gamma)\nu$. Then the following inequality for uniformly convex function holds

$$\mathbb{F}(\mu + \nu - \varsigma) = \mathbb{F}\big((1-\gamma)\mu + \gamma\nu\big) \leq (1-\gamma)\mathbb{F}(\mu) + \gamma\mathbb{F}(\nu) - \gamma(1-\gamma)\varphi(\nu - \mu)$$

$$
\begin{aligned}
&= \mathbb{F}(\mu) + \mathbb{F}(\nu) - \bigl[\gamma\mathbb{F}(\mu) + (1-\gamma)\mathbb{F}(\nu)\bigr] - \gamma(1-\gamma)\varphi(\nu-\mu) \\
&\le \mathbb{F}(\mu) + \mathbb{F}(\nu) - \mathbb{F}\bigl(\gamma\mu + (1-\gamma)\nu\bigr) - 2\gamma(1-\gamma)\varphi(\nu-\mu) \\
&= \mathbb{F}(\mu) + \mathbb{F}(\nu) - \mathbb{F}(\varsigma) - \frac{2(\nu-\varsigma)(\varsigma-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu).
\end{aligned}
$$

So, the proof is complete. □

Corollary 3.2 *Let $\mathbb{F} \in S(c;[\mu,\nu])$ and $\mu < \varsigma < \nu$. Then following inequality holds*

$$
\mathbb{F}(\mu+\nu-\varsigma) + \mathbb{F}(\varsigma) + 2c(\nu-\varsigma)(\varsigma-\mu) \le \mathbb{F}(\mu) + \mathbb{F}(\nu). \tag{5}
$$

Proof The result follows from Theorem 3.1 with $\varphi(r) = cr^2$. □

Theorem 3.3 *Let $\mathbb{F} \in U(\varphi;[\mu,\nu])$. Then Jensen–Mercer inequality for uniformly convex function holds*

$$
\begin{aligned}
&\mathbb{F}\bigl(\mu+\nu-\bigl(\gamma y_1 + (1-\gamma)y_2\bigr)\bigr) \\
&\quad\le \mathbb{F}(\mu) + \mathbb{F}(\nu) - \gamma\mathbb{F}(y_1) - (1-\gamma)\mathbb{F}(y_2) \\
&\qquad - \gamma(1-\gamma)\varphi\bigl(|y_1-y_2|\bigr) \\
&\qquad - \frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\bigl(\gamma(\nu-y_1)(y_1-\mu) + (1-\gamma)(\nu-y_2)(y_2-\mu)\bigr).
\end{aligned} \tag{6}
$$

Proof Let $y_1, y_2 \in [\mu,\nu]$.

$$
\begin{aligned}
&\mathbb{F}\bigl(\mu+\nu-\bigl(\gamma y_1 + (1-\gamma)y_2\bigr)\bigr) \\
&\quad= \mathbb{F}\bigl(\gamma(\mu+\nu-y_1) + (1-\gamma)(\mu+\nu-y_2)\bigr) \\
&\quad\le \gamma\mathbb{F}(\mu+\nu-y_1) + (1-\gamma)\mathbb{F}(\mu+\nu-y_2) - \gamma(1-\gamma)\varphi\bigl(|y_1-y_2|\bigr).
\end{aligned} \tag{7}
$$

With the use of Theorem 3.1, we have

$$
\begin{aligned}
&\gamma\mathbb{F}(\mu+\nu-y_1) + (1-\gamma)\mathbb{F}(\mu+\nu-y_2) \\
&\quad\le \gamma\left(\mathbb{F}(\mu) + \mathbb{F}(\nu) - \mathbb{F}(y_1) - \frac{2(\nu-y_1)(y_1-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu)\right) \\
&\qquad + (1-\gamma)\left(\mathbb{F}(\mu) + \mathbb{F}(\nu) - \mathbb{F}(y_2) - \frac{2(\nu-y_2)(y_2-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu)\right) \\
&\quad= \mathbb{F}(\mu) + \mathbb{F}(\nu) - \gamma\mathbb{F}(y_1) - (1-\gamma)\mathbb{F}(y_2) \\
&\qquad - \frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\bigl(\gamma(\nu-y_1)(y_1-\mu) + (1-\gamma)(\nu-y_2)(y_2-\mu)\bigr).
\end{aligned} \tag{8}
$$

A combination of (7) and (8), we have (6). □

Corollary 3.4 *Let $\mathbb{F} \in S(c;[\mu,\nu])$. Then, Jensen–Mercer inequality for strongly convex function holds*

$$
\mathbb{F}\bigl(\mu+\nu-\bigl(\gamma y_1 + (1-\gamma)y_2\bigr)\bigr)
$$

$$\begin{aligned}&\leq \mathbb{F}(\mu)+\mathbb{F}(\nu)-\gamma\mathbb{F}(y_1)-(1-\gamma)\mathbb{F}(y_2)\\&\quad -c\gamma(1-\gamma)(y_1-y_2)^2-2c\big(\gamma(\nu-y_1)(y_1-\mu)+(1-\gamma)(\nu-y_2)(y_2-\mu)\big)\end{aligned}\tag{9}$$

for all $y_1,y_2\in[\mu,\nu]$.

Proof The result follows from Theorem 3.3 with $\varphi(r)=cr^2$. □

Corollary 3.5 *Let* $\mathbb{F}\in U(\varphi;[\mu,\nu])$. *Then we have*

$$\frac{1}{\nu-\mu}\int_\mu^\nu \mathbb{F}(x)\,dx\leq \frac{\mathbb{F}(\mu)+\mathbb{F}(\nu)}{2}-\frac{1}{6}\varphi(\nu-\mu).$$

Proof Replacing y_1 by μ and y_2 by ν in Theorem 3.3, we get

$$\begin{aligned}&\mathbb{F}\big(\mu+\nu-(\gamma\mu+(1-\gamma)\nu)\big)\\&\quad\leq \mathbb{F}(\mu)+\mathbb{F}(\nu)-\gamma\mathbb{F}(\mu)-(1-\gamma)\mathbb{F}(\nu)-\gamma(1-\gamma)\varphi(\nu-\mu)\end{aligned}$$

for every $\gamma\in[\mu,\nu]$. Now, by integrating the above inequality w.r.t. γ over $[0,1]$, we obtain

$$\int_0^1\mathbb{F}\big(\mu+\nu-(\gamma\mu+(1-\gamma)\nu)\big)\,d\gamma\leq \mathbb{F}(\mu)+\mathbb{F}(\nu)-\frac{\mathbb{F}(\mu)+\mathbb{F}(\nu)}{2}-\frac{1}{6}\varphi(\nu-\mu),$$

which completes the proof. □

4 New fractional Hermite–Jensen–Mercer type inequalities

Theorem 4.1 *Let* $\mathbb{F}\in U(\varphi;[\mu,\nu])$. *Then midpoint Hermite–Jensen–Mercer type inequality for uniformly convex function*

$$\begin{aligned}&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)+D_1\varphi(\alpha,\beta,y_1,y_2)\\&\quad\leq \frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(y_2-y_1)^{\alpha\beta}}\times J\mathbb{F}(\alpha,\beta,y_1,y_2)\\&\quad\leq \mathbb{F}(\mu)+\mathbb{F}(\nu)-\left(\frac{\mathbb{F}(y_1)+\mathbb{F}(y_2)}{2}\right)-K_1\varphi(\alpha,\beta,y_1,y_2)\end{aligned}$$

holds for all $y_1,y_2\in[\mu,\nu]$ *where B is beta-function and,*

$$D_1\varphi(\alpha,\beta,y_1,y_2):=\frac{1}{8\beta}\int_0^1 u^{\beta-1}\varphi\big((1-u)^{\frac{1}{\alpha}}|y_1-y_2|\big)\,du$$

and

$$\begin{aligned}K_1\varphi(\alpha,\beta,y_1,y_2)&:=\frac{2\alpha^{-\beta}\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\big((\nu-y_1)(y_1-\mu)+(\nu-y_2)(y_2-\mu)\big)\\&\quad+\left(\frac{1}{2\beta\alpha^\beta}-\frac{1}{2\alpha^\beta}B\left(\beta,\frac{2}{\alpha}+1\right)\right)\varphi\big(|y_1-y_2|\big).\end{aligned}$$

Proof Since $\mathbb{F}$ is uniformly convex with modulus φ,

$$\mathbb{F}\left(\mu+\nu-\frac{x_1+x_2}{2}\right)=\mathbb{F}\left(\frac{2\mu+2\nu-x_1-x_2}{2}\right)\\ \le\frac{1}{2}\mathbb{F}(\mu+\nu-x_1)+\frac{1}{2}\mathbb{F}(\mu+\nu-x_2)-\frac{1}{4}\varphi\left(|x_2-x_1|\right)$$

for all $x_1,x_2\in[\mu,\nu]$.

Now, by using the change of variables $x_1=\frac{\gamma}{2}y_1+(1-\frac{\gamma}{2})y_2$ and $x_2=\frac{\gamma}{2}y_2+(1-\frac{\gamma}{2})y_1$ for $y_1,y_2\in[\mu,\nu]$ and $\gamma\in[0,1]$, we obtain

$$\begin{aligned}&2\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)\\&\quad\le\mathbb{F}\left(\mu+\nu-\left(\frac{\gamma}{2}y_1+\left(1-\frac{\gamma}{2}\right)y_2\right)\right)\\&\qquad+\mathbb{F}\left(\mu+\nu-\left(\left(1-\frac{\gamma}{2}\right)y_1+\frac{\gamma}{2}y_2\right)\right)-\frac{1}{4}\varphi\left((1-\gamma)|y_1-y_2|\right)\end{aligned}\tag{10}$$

Multiplying (10) by $(\frac{1-(1-\gamma)^\alpha}{\alpha})^{\beta-1}(1-\gamma)^{\alpha-1}:=\Gamma_{\alpha,\beta}(\gamma)$, integrating w.r.t. γ over $[0,1]$, and then combining the resulting inequality gives

$$\begin{aligned}&2\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\beta-1}(1-\gamma)^{\alpha-1}\\&\quad\le\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\beta-1}(1-\gamma)^{\alpha-1}\times\left[\mathbb{F}\left(\mu+\nu-\left(\frac{\gamma}{2}y_1+\left(1-\frac{\gamma}{2}\right)y_2\right)\right)\right.\\&\qquad\left.+\mathbb{F}\left(\mu+\nu-\left(\left(1-\frac{\gamma}{2}\right)y_1+\frac{\gamma}{2}y_2\right)\right)-\frac{1}{4}\varphi\left((1-\gamma)|y_1-y_2|\right)\right].\end{aligned}\tag{11}$$

On the other hand, we have

$$\begin{aligned}&\int_0^1\Gamma_{\alpha,\beta}(\gamma)\mathbb{F}\left(\mu+\nu-\left(\frac{\gamma}{2}y_1+\left(1-\frac{\gamma}{2}\right)y_2\right)\right)d\gamma\\&\quad=\left(\frac{2}{y_2-y_1}\right)^{\alpha\beta}\Gamma(\beta)^\beta J^\alpha_{(\mu+\nu-\frac{y_1+y_2}{2})^-}\mathbb{F}(\mu+\nu-y_2),\end{aligned}\tag{12}$$

$$\begin{aligned}&\int_0^1\Gamma_{\alpha,\beta}(\gamma)\mathbb{F}\left(\mu+\nu-\left(\left(1-\frac{\gamma}{2}\right)y_1+\frac{\gamma}{2}y_2\right)\right)d\gamma\\&\quad=\left(\frac{2}{y_2-y_1}\right)^{\alpha\beta}\Gamma(\beta)^\beta J^\alpha_{(\mu+\nu-\frac{y_1+y_2}{2})^+}\mathbb{F}(\mu+\nu-y_1),\end{aligned}\tag{13}$$

$$\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma=\frac{1}{\beta}\alpha^{-\beta}\tag{14}$$

and

$$\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\left((1-\gamma)|y_1-y_2|\right)d\gamma=\alpha^{-\beta}\int_0^1u^{\beta-1}\varphi\left((1-u)^{\frac{1}{\alpha}}|y_1-y_2|\right)du,\tag{15}$$

where $u=1-(1-\gamma)^\alpha$. The first inequality follows from (11), (12), (13) and (15).

To prove the second inequality, by (6),

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\left(\frac{\gamma}{2}y_1+\left(1-\frac{\gamma}{2}\right)y_2\right)\right)\\
&\quad\leq \mathbb{F}(\mu)+\mathbb{F}(\nu)-\frac{\gamma}{2}\mathbb{F}(y_1)-\left(1-\frac{\gamma}{2}\right)\mathbb{F}(y_2)\\
&\qquad-\frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\varphi\big(|y_1-y_2|\big)\\
&\qquad-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\left(\frac{\gamma}{2}(\nu-y_1)(y_1-\mu)+\left(1-\frac{\gamma}{2}\right)(\nu-y_2)(y_2-\mu)\right).
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\left(\left(1-\frac{\gamma}{2}\right)y_1+\frac{\gamma}{2}y_2\right)\right)\\
&\quad\leq \mathbb{F}(\mu)+\mathbb{F}(\nu)-\left(1-\frac{\gamma}{2}\right)\mathbb{F}(y_1)-\frac{\gamma}{2}\mathbb{F}(y_2)\\
&\qquad-\frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\varphi\big(|y_1-y_2|\big)\\
&\qquad-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\left(\left(1-\frac{\gamma}{2}\right)(\nu-y_1)(y_1-\mu)+\frac{\gamma}{2}(\nu-y_2)(y_2-\mu)\right).
\end{aligned}\tag{17}$$

Upon adding (16) and (17), we obtain

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\left(\frac{\gamma}{2}y_1+\left(1-\frac{\gamma}{2}\right)y_2\right)\right)+\mathbb{F}\left(\mu+\nu-\left(\left(1-\frac{\gamma}{2}\right)y_1+\frac{\gamma}{2}y_2\right)\right)\\
&\quad\leq 2\big[\mathbb{F}(\mu)+\mathbb{F}(\nu)\big]-\big[\mathbb{F}(y_1)+\mathbb{F}(y_2)\big]-\gamma\left(1-\frac{\gamma}{2}\right)\varphi\big(|y_1-y_2|\big)\\
&\qquad-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\big((\nu-y_1)(y_1-\mu)+(\nu-y_2)(y_2-\mu)\big).
\end{aligned}\tag{18}$$

Multiplying (18) by $\Gamma_{\alpha,\beta}(\gamma)$ and integrating the obtained inequality w.r.t. γ over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1\Gamma_{\alpha,\beta}(\gamma)\left\{\mathbb{F}\left(\mu+\nu-\left(\frac{\gamma}{2}y_1+\left(1-\frac{\gamma}{2}\right)y_2\right)\right)\right.\\
&\quad\left.+\mathbb{F}\left(\mu+\nu-\left(\left(1-\frac{\gamma}{2}\right)y_1+\frac{\gamma}{2}y_2\right)\right)\right\}d\gamma\leq\left\{2\big[\mathbb{F}(\mu)+\mathbb{F}(\nu)\big]-\big[\mathbb{F}(y_1)+\mathbb{F}(y_2)\big]\right.\\
&\quad\left.-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\big((\nu-y_1)(y_1-\mu)+(\nu-y_2)(y_2-\mu)\big)\right\}\times\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\quad-\varphi\big(|y_1-y_2|\big)\int_0^1\gamma\left(1-\frac{\gamma}{2}\right)\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\beta-1}(1-\gamma)^{\alpha-1}\,d\gamma.
\end{aligned}\tag{19}$$

Furthermore,

$$\int_0^1\gamma\left(1-\frac{\gamma}{2}\right)\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\beta-1}(1-\gamma)^{\alpha-1}\,d\gamma=\frac{1}{2\alpha^\beta}\int_0^1\big(1-t^{\frac{2}{\alpha}}\big)(1-t)^{\beta-1}\,dt$$

$$= \frac{1}{2\beta\alpha^{\beta}} - \frac{1}{2\alpha^{\beta}} B\left(\beta, \frac{2}{\alpha} + 1\right), \tag{20}$$

where $t = (1-\gamma)^{\alpha}$. The second inequality follows from (12), (13), (15), (19) and (20). □

Corollary 4.2 *If we set* $\alpha = 1$ *in Theorem* 4.1, *we get*

$$\begin{aligned} &\mathbb{F}\left(\mu + \nu - \frac{y_1 + y_2}{2}\right) + \frac{1}{8\beta}\int_0^1 u^{\beta-1}\varphi\big((1-u)|y_1 - y_2|\big)\,du \\ &\leq \frac{2^{\beta-1}\Gamma(\beta+1)}{(y_2 - y_1)^{\beta}}\left\{J^{\beta}_{(\mu+\nu-\frac{y_1+y_2}{2})^+}\mathbb{F}(\mu+\nu-y_1) + J^{\beta}_{(\mu+\nu-\frac{y_1+y_2}{2})^-}\mathbb{F}(\mu+\nu-y_2)\right\} \\ &\leq \mathbb{F}(\mu) + \mathbb{F}(\nu) - \left(\frac{\mathbb{F}(y_1) + \mathbb{F}(y_2)}{2}\right) - D\varphi(1, \beta, y_1, y_2). \end{aligned}$$

Corollary 4.3 *If we set* $\mu = y_1$, $\nu = y_2$ *and* $\alpha = 1$ *in Theorem* 4.1, *we get*

$$\begin{aligned} &\mathbb{F}\left(\frac{y_1 + y_2}{2}\right) + \frac{1}{8\beta}\int_0^1 u^{\beta-1}\varphi\big((1-u)|y_1 - y_2|\big)\,du \\ &\leq \frac{2^{\beta-1}\Gamma(\beta+1)}{(y_2 - y_1)^{\beta}}\left\{J^{\beta}_{(\frac{y_1+y_2}{2})^+}\mathbb{F}(y_2) + J^{\beta}_{(\frac{y_1+y_2}{2})^-}\mathbb{F}(y_1)\right\} \\ &\leq \frac{\mathbb{F}(y_1) + \mathbb{F}(y_2)}{2} - \left(\frac{1}{2\beta} - \frac{1}{2}B(\beta, 3)\right)\varphi\big(|y_1 - y_2|\big) \end{aligned}$$

Corollary 4.4 *If we set* $\mu = y_1$, $\nu = y_2$ *and* $\alpha = \beta = 1$ *in Theorem* 4.1, *we get*

$$\begin{aligned} &\mathbb{F}\left(\frac{y_1 + y_2}{2}\right) + \frac{1}{8}\int_0^1 \varphi\big((1-u)|y_1 - y_2|\big)\,du \\ &\leq \frac{1}{(y_2 - y_1)}\int_x^y \mathbb{F}(u)\,du \\ &\leq \frac{\mathbb{F}(y_1) + \mathbb{F}(y_2)}{2} - \frac{1}{3}\varphi\big(|y_1 - y_2|\big). \end{aligned}$$

Remark 4.5 If we set $\varphi = 0$ in Theorem 4.1, we get Theorem 2 of [18].

Theorem 4.6 *Let* $\mathbb{F} \in U(\varphi; [\mu, \nu])$. *Then, conformable Hermite–Jensen–Mercer type inequality for uniformly convex function*

$$\begin{aligned} &\mathbb{F}\left(\mu + \nu - \frac{y_1 + y_2}{2}\right) + D_2\varphi(\alpha, \beta, y_1, y_2) \\ &\leq \mathbb{F}(\mu) + \mathbb{F}(\nu) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y_2 - y_1)^{\alpha\beta}}\left\{{}^{\beta}J^{\alpha}_{y_1^+}\mathbb{F}(y_2) + {}^{\beta}J^{\alpha}_{y_2^-}\mathbb{F}(y_1)\right\} \\ &\leq \mathbb{F}(\mu) + \mathbb{F}(\nu) - \mathbb{F}\left(\frac{y_1 + y_2}{2}\right) - K_2\varphi(\alpha, \beta, y_1, y_2). \end{aligned}$$

holds for all $y_1, y_2 \in [\mu, \nu]$, *where*

$$K_2\varphi(\alpha, \beta, y_1, y_2) := \beta\alpha^{\beta}\int_0^1 \Gamma_{\alpha,\beta}(\gamma)\varphi\big(|2\gamma - 1|.|y_1 - y_2|\big)\,d\gamma$$

and

$$
\begin{aligned}
&D_2\varphi(\alpha,\beta,y_1,y_2)\\
&:= \frac{2\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\alpha^{-\beta}\big((\mu+\nu)(y_1+y_2)-2\mu\nu\big)\\
&\quad+\frac{1}{8}\alpha^{\beta}\int_0^1\int_0^1 \Gamma_{\alpha,\beta}(\gamma)u^{\beta-1}\varphi\big((1-u)^{\frac{1}{\alpha}}|2\gamma-1||y_1-y_2|\big)\Gamma_{\alpha,\beta}(\gamma)\,du\,d\gamma\\
&\quad+\frac{2({y_1}^2+{y_2}^2)\varphi(\nu-\mu)}{\alpha^{\beta}\beta(\nu-\mu)^2}\left(\frac{4}{\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)-\frac{6}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)+1\right)\\
&\quad+\left(\frac{1}{2}-\frac{\beta}{2}B\left(\beta,\frac{2}{\alpha}+1\right)\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\big(|2\gamma-1||y_1-y_2|\big)\,d\gamma.
\end{aligned}
$$

Proof It follows from Theorem 4.1 that

$$
\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\frac{x_1+x_2}{2}\right)+\frac{1}{8\beta}\int_0^1 u^{\beta-1}\varphi\big((1-u)^{\frac{1}{\alpha}}|x_1-x_2|\big)\,du\\
&\leq \mathbb{F}(\mu)+\mathbb{F}(\nu)-\left(\frac{\mathbb{F}(x_1)+\mathbb{F}(x_2)}{2}\right)\\
&\quad-\frac{2\alpha^{-\beta}\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\big((\nu-x_1)(x_1-\mu)+(\nu-x_2)(x_2-\mu)\big)\\
&\quad-\left(\frac{1}{2\beta\alpha^{\beta}}-\frac{1}{2\alpha^{\beta}}B\left(\beta,\frac{2}{\alpha}+1\right)\right)\varphi\big(|x_2-x_1|\big).
\end{aligned}
\tag{21}
$$

for all $x_1,x_2\in[\mu,\nu]$.

By changing the variables $x_1=\gamma y_1+(1-\gamma)y_2$ and $x_2=(1-\gamma)y_1+\gamma y_2$ for $y_1,y_2\in[\mu,\nu]$ and $\gamma\in[0,1]$ in (21), multiplying by $\Gamma_{\alpha,\beta}(\gamma)$ and then by using integration w.r.t. γ over $[0,1]$ leads to the conclusion that

$$
\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\quad+\frac{1}{8\beta}\int_0^1\int_0^1\Gamma_{\alpha,\beta}(\gamma)u^{\beta-1}\varphi\big((1-u)^{\frac{1}{\alpha}}|2\gamma-1||y_1-y_2|\big)\,du\,d\gamma\\
&\leq\big[\mathbb{F}(\mu)+\mathbb{F}(\nu)\big]\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\quad-\int_0^1\Gamma_{\alpha,\beta}(\gamma)\left(\frac{\mathbb{F}(\gamma y_1+(1-\gamma)y_2)+\mathbb{F}((1-\gamma)y_1+\gamma y_2)}{2}\right)d\gamma\\
&\quad-\frac{2\alpha^{-\beta}\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\big((\mu+\nu)(y_1+y_2)-2\mu\nu\big)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\\
&\quad-\frac{2\alpha^{-\beta}\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\big(({y_1}^2+{y_2}^2)(2\gamma^2-2\gamma+1)-4\gamma(1-\gamma)y_1y_2\big)\,d\gamma\\
&\quad-\left(\frac{1}{2\beta\alpha^{\beta}}-\frac{1}{2\alpha^{\beta}}B\left(\beta,\frac{2}{\alpha}+1\right)\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\big(|2\gamma-1||y_1-y_2|\big)\,d\gamma.
\end{aligned}
$$

Hence,

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)\frac{1}{\beta}\alpha^{-\beta} \\
&\quad+\frac{1}{8\beta}\int_0^1\int_0^1\Gamma_{\alpha,\beta}(\gamma)u^{\beta-1}\varphi\bigl((1-u)^{\frac{1}{\alpha}}|2\gamma-1||y_1-y_2|\bigr)\,du\,d\gamma \\
&\quad\leq\bigl[\mathbb{F}(\mu)+\mathbb{F}(\nu)\bigr]\frac{1}{\beta}\alpha^{-\beta} \\
&\quad-\int_0^1\Gamma_{\alpha,\beta}(\gamma)\left(\frac{\mathbb{F}(\gamma y_1+(1-\gamma)y_2)+\mathbb{F}((1-\gamma)y_1+\gamma y_2)}{2}\right)d\gamma \\
&\quad-\frac{2\alpha^{-\beta}\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\bigl((\mu+\nu)(y_1+y_2)-2\mu\nu\bigr)\frac{1}{\beta}\alpha^{-\beta} \\
&\quad-\frac{2\alpha^{-\beta}\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\bigl(\bigl({y_1}^2+{y_2}^2\bigr)\bigl(2\gamma^2-2\gamma+1\bigr)-4\gamma(1-\gamma)y_1y_2\bigr)\,d\gamma \\
&\quad-\left(\frac{1}{2\beta\alpha^{\beta}}-\frac{1}{2\alpha^{\beta}}B\left(\beta,\frac{2}{\alpha}+1\right)\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\bigl(|2\gamma-1||y_1-y_2|\bigr)\,d\gamma,
\end{aligned}$$

and thus,

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)+\frac{1}{8}\alpha^{\beta}\int_0^1\int_0^1\Gamma_{\alpha,\beta}(\gamma)u^{\beta-1}\varphi\bigl((1-u)^{\frac{1}{\alpha}}|2\gamma-1||y_1-y_2|\bigr)\,du\,d\gamma \\
&\quad\leq\mathbb{F}(\mu)+\mathbb{F}(\nu)-\beta\alpha^{\beta}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\left(\frac{\mathbb{F}(\gamma y_1+(1-\gamma)y_2)+\mathbb{F}((1-\gamma)y_1+\gamma y_2)}{2}\right)d\gamma \\
&\quad-\frac{2\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\alpha^{-\beta}\bigl((\mu+\nu)(y_1+y_2)-2\mu\nu\bigr) \\
&\quad-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\bigl(\bigl({y_1}^2+{y_2}^2\bigr)\bigl(2\gamma^2-2\gamma+1\bigr)-4\gamma(1-\gamma)y_1y_2\bigr)\,d\gamma \\
&\quad-\left(\frac{1}{2}-\frac{\beta}{2}B\left(\beta,\frac{2}{\alpha}+1\right)\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\bigl(|2\gamma-1||y_1-y_2|\bigr)\,d\gamma.
\end{aligned}$$

That is,

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right) \\
&\quad+\frac{1}{8}\alpha^{\beta}\int_0^1\int_0^1\Gamma_{\alpha,\beta}(\gamma)u^{\beta-1}\varphi\bigl((1-u)^{\frac{1}{\alpha}}|2\gamma-1||y_1-y_2|\bigr)\,du\,d\gamma \\
&\quad\leq\mathbb{F}(\mu)+\mathbb{F}(\nu)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y_2-y_1)^{\alpha\beta}}\bigl\{{}^{\beta}J^{\alpha}_{y_1+}\mathbb{F}(y_2)+{}^{\beta}J^{\alpha}_{y_2-}\mathbb{F}(y_1)\bigr\} \\
&\quad-\frac{2\varphi(\nu-\mu)}{\beta(\nu-\mu)^2}\alpha^{-\beta}\bigl((\mu+\nu)(y_1+y_2)-2\mu\nu\bigr) \\
&\quad-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\bigl(\bigl({y_1}^2+{y_2}^2\bigr)\bigl(2\gamma^2-2\gamma+1\bigr)-4\gamma(1-\gamma)y_1y_2\bigr)\,d\gamma \\
&\quad-\left(\frac{1}{2}-\frac{\beta}{2}B\left(\beta,\frac{2}{\alpha}+1\right)\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\bigl(|2\gamma-1||y_1-y_2|\bigr)\,d\gamma. \qquad (22)
\end{aligned}$$

Also, we have

$$\int_0^1 \bigl(1-(1-\gamma)^\alpha\bigr)^{\beta-1}(1-\gamma)^{\alpha-1}\,d\gamma = \frac{1}{\alpha\beta},$$
$$\int_0^1 \gamma\bigl(1-(1-\gamma)^\alpha\bigr)^{\beta-1}(1-\gamma)^{\alpha-1}\,d\gamma = -\frac{1}{\alpha^2\beta}B\bigl(\beta+1,\alpha^{-1}\bigr),$$
$$\int_0^1 \gamma^2\bigl(1-(1-\gamma)^\alpha\bigr)^{\beta-1}(1-\gamma)^{\alpha-1}\,d\gamma = -\frac{2}{\alpha^2\beta}\bigl[B\bigl(\beta+1,\alpha^{-1}\bigr)-B\bigl(\beta+1,2\alpha^{-1}\bigr)\bigr],$$
$$\begin{aligned}
&\int_0^1 \Gamma_{\alpha,\beta}(\gamma)\bigl(\bigl({y_1}^2+{y_2}^2\bigr)\bigl(2\gamma^2-2\gamma+1\bigr)-4\gamma(1-\gamma)y_1y_2\bigr)\,d\gamma\\
&\quad=\frac{{y_1}^2+{y_2}^2}{\alpha^\beta\beta}\left(-\frac{4}{\alpha}\left[B\left(\beta+1,\frac{1}{\alpha}\right)-B\left(\beta+1,\frac{2}{\alpha}\right)\right]-\frac{2}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)+1\right)\\
&\quad=\frac{{y_1}^2+{y_2}^2}{\alpha^\beta\beta}\left(\frac{4}{\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)-\frac{6}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)+1\right),
\end{aligned}$$

which completes the proof of the first inequality.

To prove the second inequality, from the uniformly convex of $\mathbb{F}$, for $\gamma \in [0,1]$ we obtain

$$\begin{aligned}
\mathbb{F}\left(\frac{y_1+y_2}{2}\right)&=\mathbb{F}\left(\frac{\gamma y_1+(1-\gamma)y_1+\gamma y_2+(1-\gamma)y_2}{2}\right)\\
&\le\frac{\mathbb{F}(\gamma y_1+(1-\gamma)y_2)+\mathbb{F}((1-\gamma)y_1+\gamma y_2)}{2}\\
&\quad-\frac{1}{4}\varphi\bigl(|2\gamma-1||y_1-y_2|\bigr).
\end{aligned}\tag{23}$$

Multiplying (23) by $\Gamma_{\alpha,\beta}(\gamma)$ and then by integrating the resulting inequality w.r.t. γ over $[0,1]$ gives

$$\begin{aligned}
&\mathbb{F}\left(\frac{y_1+y_2}{2}\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\quad\le\int_0^1\Gamma_{\alpha,\beta}(\gamma)\left\{\frac{\mathbb{F}(\gamma y_1+(1-\gamma)y_2)+\mathbb{F}((1-\gamma)y_1+\gamma y_2)}{2}\right\}d\gamma\\
&\qquad-\frac{1}{4}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\bigl(|2\gamma-1|.|y_1-y_2|\bigr)\,d\gamma,
\end{aligned}$$

that is,

$$\begin{aligned}
\mathbb{F}\left(\frac{y_1+y_2}{2}\right)&\le\frac{\alpha^\beta\Gamma(\beta+1)}{2(y_2-y_1)^{\alpha\beta}}\bigl\{{}^{\beta}J^{\alpha}_{y_1^+}\mathbb{F}(y_2)+{}^{\beta}J^{\alpha}_{y_2^-}\mathbb{F}(y_1)\bigr\}\\
&\quad-\beta\alpha^\beta\int_0^1\Gamma_{\alpha,\beta}(\gamma)\varphi\bigl(|2\gamma-1|.|y_1-y_2|\bigr)\,d\gamma.
\end{aligned}\tag{24}$$

Therefore,

$$\begin{aligned}
&\mathbb{F}(\mu)+\mathbb{F}(\nu)-\mathbb{F}\left(\frac{y_1+y_2}{2}\right)\\
&\quad\ge\mathbb{F}(\mu)+\mathbb{F}(\nu)-\frac{\alpha^\beta\Gamma(\beta+1)}{2(y_2-y_1)^{\alpha\beta}}\bigl\{{}^{\beta}J^{\alpha}_{y_1^+}\mathbb{F}(y_2)+{}^{\beta}J^{\alpha}_{y_2^-}\mathbb{F}(y_1)\bigr\}
\end{aligned}$$

$$+\beta\alpha^{\beta}\int_0^1 \Gamma_{\alpha,\beta}(\gamma)\varphi\big(|2\gamma-1|.|y_1-y_2|\big)\,d\gamma, \tag{25}$$

which completes the proof of the second inequality. □

Remark 4.7 If we set $\varphi(y_1)=0$ in Theorem 4.6, we get inequality (2.7) in Theorem 3 of [18].

Theorem 4.8 *Let $\mathbb{F}\in U(\varphi;[\mu,\nu])$. Then, Hemite–Jensen–Mercer type inequality for uniformly convex function*

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)+\frac{\beta}{4}\int_0^1 u^{\beta-1}\varphi\Big(\big(1+(1-u)^{\frac{1}{\alpha}}\big)|y_1-y_2|\Big)\,du\\
&\quad\le\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y_2-y_1)^{\alpha\beta}}\left\{{}^{\beta}J^{\alpha}_{(\mu+\nu-y_1)^-}\mathbb{F}(\mu+\nu-y_2)+{}^{\beta}J^{\alpha}_{(\mu+\nu-y_2)^+}\mathbb{F}(\mu+\nu-y_1)\right\}\\
&\quad\le\mathbb{F}(\mu)+\mathbb{F}(\nu)-\frac{\mathbb{F}(y_1)+\mathbb{F}(y_2)}{2}-\beta\varphi\big(|y_1-y_2|\big)\left[B\left(\frac{1}{\alpha}+1,\beta\right)-B\left(\frac{2}{\alpha}+1,\beta\right)\right]\\
&\qquad-\big[(\nu-y_1)(y_1-\mu)+(\nu-y_2)(y_2-\mu)\big]\frac{\varphi(\nu-\mu)}{(\nu-\mu)^2},
\end{aligned} \tag{26}$$

holds for all $y_1,y_2\in[\mu,\nu]$.

Proof To prove the inequality, we use the uniformly convex of $\mathbb{F}$ to get

$$\begin{aligned}
\mathbb{F}\left(\mu+\nu-\frac{x_1+x_2}{2}\right)&=\mathbb{F}\left(\frac{2\mu+2\nu-x_1-x_2}{2}\right)\\
&\le\frac{1}{2}\mathbb{F}(\mu+\nu-x_1)+\frac{1}{2}\mathbb{F}(\mu+\nu-x_2)-\frac{1}{4}\varphi\big(|x_2-x_1|\big)
\end{aligned} \tag{27}$$

for all $x_1,x_2\in[\mu,\nu]$.

Let $x_1=\gamma y_1+(1-\gamma)y_2$ and $x_2=\gamma y_2+(1-\gamma)y_1$. Then (27) leads to

$$\begin{aligned}
\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)&\le\frac{1}{2}\mathbb{F}\big(\mu+\nu-\big(\gamma y_1+(1-\gamma)y_2\big)\big)\\
&\quad+\frac{1}{2}\mathbb{F}\big(\mu+\nu-\big((1-\gamma)y_1+\gamma y_2\big)\big)-\frac{1}{4}\varphi\big(|2\gamma-1|.|y_1-y_2|\big).
\end{aligned} \tag{28}$$

Multiplying both sides of (28) by $\Gamma_{\alpha,\beta}(\gamma)$ and integrating the obtained inequality w.r.t. γ on $[0,1]$, we have

$$\begin{aligned}
&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\quad\le\frac{1}{2}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\big\{\mathbb{F}\big(\mu+\nu-\big(\gamma y_1+(1-\gamma)y_2\big)\big)+\mathbb{F}\big(\mu+\nu-\big((1-\gamma)y_1+\gamma y_2\big)\big)\big\}\,d\gamma\\
&\qquad-\frac{1}{4}\int_0^1\varphi\big(|2\gamma-1|.|y_1-y_2|\big)\Gamma_{\alpha,\beta}(\gamma)\,d\gamma.
\end{aligned} \tag{29}$$

Also,

$$\begin{aligned}&\int_0^1 \Gamma_{\alpha,\beta}(\gamma)\mathbb{F}\big(\mu+\nu-\big(\gamma y_1+(1-\gamma)y_2\big)\big)\,d\gamma\\&\quad=\frac{1}{\beta(y_2-y_1)^{\alpha\beta}}\Gamma(\beta+1)^{\beta}J^{\alpha}_{(\mu+\nu-y_1)^-}\mathbb{F}(\mu+\nu-y_2),\end{aligned}\tag{30}$$

$$\begin{aligned}&\int_0^1 \Gamma_{\alpha,\beta}(\gamma)\mathbb{F}\big(\mu+\nu-\big((1-\gamma)y_1+\gamma y_2\big)\big)\,d\gamma\\&\quad=\frac{1}{\beta(y_2-y_1)^{\alpha\beta}}\Gamma(\beta+1)^{\beta}J^{\alpha}_{(\mu+\nu-y_2)^+}\mathbb{F}(\mu+\nu-y_1),\end{aligned}\tag{31}$$

$$\int_0^1 \Gamma_{\alpha,\beta}(\gamma)\,d\gamma=\frac{1}{\beta}\alpha^{-\beta}\tag{32}$$

and

$$\begin{aligned}&\int_0^1 \Gamma_{\alpha,\beta}(\gamma)\varphi\big(\big(|1-2\gamma|\big)|y_1-y_2|\big)\,d\gamma\\&\quad=\alpha^{-\beta}\int_0^1 u^{\beta-1}\varphi\big(\big(1+(1-u)^{\frac{1}{\alpha}}\big)|y_1-y_2|\big)\,du,\end{aligned}\tag{33}$$

where $u=1-(1-\gamma)^{\alpha}$. By the use of (29), (30), (31) and (33), we get

$$\begin{aligned}&\mathbb{F}\left(\mu+\nu-\frac{y_1+y_2}{2}\right)+\frac{\beta}{4}\int_0^1 u^{\beta-1}\varphi\big(\big(1+(1-u)^{\frac{1}{\alpha}}\big)|y_1-y_2|\big)\,du\\&\quad\le\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y_2-y_1)^{\alpha\beta}}\big\{{}^{\beta}J^{\alpha}_{(\mu+\nu-y_1)^-}\mathbb{F}(\mu+\nu-y_2)+{}^{\beta}J^{\alpha}_{(\mu+\nu-y_2)^+}\mathbb{F}(\mu+\nu-y_1)\big\}.\end{aligned}\tag{34}$$

It follows from the uniformly convexity of $\mathbb{F}$ that

$$\begin{aligned}&\mathbb{F}\big(\gamma(\mu+\nu-y_1)+(1-\gamma)(\mu+\nu-y_2)\big)\\&\quad\le\gamma\mathbb{F}(\mu+\nu-y_1)+(1-\gamma)\mathbb{F}(\mu+\nu-y_2)-\gamma(1-\gamma)\varphi\big(|y_1-y_2|\big).\end{aligned}$$

and

$$\begin{aligned}&\mathbb{F}\big((1-\gamma)(\mu+\nu-y_1)+\gamma(\mu+\nu-y_2)\big)\\&\quad\le(1-\gamma)\mathbb{F}(\mu+\nu-y_1)+\gamma\mathbb{F}(\mu+\nu-y_2)-\gamma(1-\gamma)\varphi\big(|y_1-y_2|\big).\end{aligned}$$

Adding the above two inequalities and using Theorem 3.1 gives

$$\begin{aligned}&\mathbb{F}\big(\gamma(\mu+\nu-y_1)+(1-\gamma)(\mu+\nu-y_2)\big)\\&\qquad+\mathbb{F}\big((1-\gamma)(\mu+\nu-y_1)+\gamma(\mu+\nu-y_2)\big)\\&\quad\le\mathbb{F}(\mu+\nu-y_1)+\mathbb{F}(\mu+\nu-y_2)-2\gamma(1-\gamma)\varphi\big(|y_1-y_2|\big)\\&\quad\le2\big(\mathbb{F}(\mu)+\mathbb{F}(\nu)\big)-\big(\mathbb{F}(y_1)+\mathbb{F}(y_2)\big)-2\gamma(1-\gamma)\varphi\big(|y_1-y_2|\big)\\&\qquad-\frac{2(\nu-y_1)(y_1-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu)-\frac{2(\nu-y_2)(y_2-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu).\end{aligned}\tag{35}$$

Multiplying (35) by $\Gamma_{\alpha,\beta}(\gamma)$ and then by using integration w.r.t. γ over $[0,1]$, we have

$$\begin{aligned}
&\big\{\mathbb{F}\big(\gamma(\mu+\nu-y_1)+(1-\gamma)(\mu+\nu-y_2)\big)\\
&\quad+\mathbb{F}\big((1-\gamma)(\mu+\nu-y_1)+\gamma(\mu+\nu-y_2)\big)\big\}\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\leq\big(2\big(\mathbb{F}(\mu)+\mathbb{F}(\nu)\big)-\big(\mathbb{F}(y_1)+\mathbb{F}(y_2)\big)\big)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\quad-2\varphi\big(|y_1-y_2|\big)\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma\\
&\quad-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\big[(\nu-y_1)(y_1-\mu)+(\nu-y_2)(y_2-\mu)\big]\int_0^1\Gamma_{\alpha,\beta}(\gamma)\,d\gamma.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^1\gamma\Gamma_{\alpha,\beta}(\gamma)\,d\gamma&=\alpha^{-\beta+1}\int_0^1\gamma\big(1-(1-\gamma)^\alpha\big)^{\beta-1}(1-\gamma)^\alpha\,d\gamma\\
&=\alpha^{-\beta+1}\int_0^1\big(1-(1-\gamma)^\alpha\big)^{\beta-1}(1-\gamma)^\alpha\,d\gamma\\
&\quad-\alpha^{-\beta+1}\int_0^1\big(1-(1-\gamma)^\alpha\big)^{\beta-1}(1-\gamma)^{\alpha+1}\,d\gamma\\
&=\alpha^{-\beta}\int_0^1 t^{\frac{1}{\alpha}}(1-t)^{\beta-1}\,d\gamma-\alpha^{-\beta}\int_0^1 t^{\frac{2}{\alpha}}(1-t)^{\beta-1}\,d\gamma\\
&=\alpha^{-\beta}\bigg[B\bigg(\frac{1}{\alpha}+1,\beta\bigg)-B\bigg(\frac{2}{\alpha}+1,\beta\bigg)\bigg],
\end{aligned}$$

where $t=(1-\gamma)^\alpha$ and B is beta-function,

$$\begin{aligned}
&\frac{\alpha^\beta\Gamma(\beta+1)}{2(y_2-y_1)^{\alpha\beta}}\big\{{}^\beta J^\alpha_{(\mu+\nu-y_1)^-}\mathbb{F}(\mu+\nu-y_2)+{}^\beta J^\alpha_{(\mu+\nu-y_2)^+}\mathbb{F}(\mu+\nu-y_1)\big\}\\
&\leq\mathbb{F}(\mu)+\mathbb{F}(\nu)-\frac{\mathbb{F}(y_1)+\mathbb{F}(y_2)}{2}-\beta\varphi\big(|y_1-y_2|\big)\bigg[B\bigg(\frac{1}{\alpha}+1,\beta\bigg)-B\bigg(\frac{2}{\alpha}+1,\beta\bigg)\bigg]\\
&\quad-\big[(\nu-y_1)(y_1-\mu)+(\nu-y_2)(y_2-\mu)\big]\frac{\varphi(\nu-\mu)}{(\nu-\mu)^2}.
\end{aligned}\tag{36}$$

Combining (34) and (36) leads to (26). □

Remark 4.9 If we set $\varphi(y_1)=0$ in Theorem 4.8, we get inequality (2.8) in Theorem 3 of [18].

5 New inequalities via differentiable uniformly convex function

Throughout this section, I is defined by

$$I:=\bigg|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(y_2-y_1)^{\alpha\beta}}\times J\mathbb{F}(\alpha,\beta,y_1,y_2)-\mathbb{F}\bigg(\mu+\nu-\frac{y_1+y_2}{2}\bigg)\bigg|,$$

and $B_n=B(\beta+1,\frac{n}{\alpha})$ for $n=1,2,3$.

Theorem 5.1 *Let $\alpha, \beta > 0$, $y_1 < y_2$ and $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a differentiable mapping such that $\mathbb{F}' \in L[\mu, \nu]$ and $|\mathbb{F}'|$ is an uniformly convex mapping with modulus φ. Then the inequality*

$$I \leq \frac{y_2 - y_1}{4\alpha}\bigg[2\big(|\mathbb{F}'(\mu)| + \big|\mathbb{F}'(\nu)\big|\big)B_1 + \big|\mathbb{F}'(y_2)\big|B_2 - \big|\mathbb{F}'(y_1)\big|B_1 - \frac{\varphi(y_2 - y_1)}{2}(B_1 - B_3) - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2}\big((\nu - y_1)(y_1 - \mu) + (\nu - y_2)(y_2 - \mu)\big)B_1\bigg]$$

holds for all $y_1, y_2 \in [\mu, \nu]$.

Proof It follows from Lemma 1.3 that

$$I = \left|\frac{y_2 - y_1}{4}\alpha^\beta \int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha}\right)^\beta \times \left[\mathbb{F}'\left(\mu + \nu - \left(\frac{2 - \gamma}{2}y_1 + \frac{\gamma}{2}y_2\right)\right) - \mathbb{F}'\left(\mu + \nu - \left(\frac{\gamma}{2}y_1 + \frac{2 - \gamma}{2}y_2\right)\right)\right] d\gamma\right|.$$

Hence,

$$\begin{aligned} I &\leq \frac{y_2 - y_1}{4}\alpha^\beta \int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha}\right)^\beta \\ &\quad \times \left[\left|\mathbb{F}'\left(\mu + \nu - \left(\frac{2 - \gamma}{2}y_1 + \frac{\gamma}{2}y_2\right)\right) - \mathbb{F}'\left(\mu + \nu - \left(\frac{\gamma}{2}y_1 + \frac{2 - \gamma}{2}y_2\right)\right)\right|\right] d\gamma \\ &\leq \frac{y_2 - y_1}{4}\alpha^\beta \left\{\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha}\right)^\beta \left|\mathbb{F}'\left(\mu + \nu - \left(\frac{2 - \gamma}{2}y_1 + \frac{\gamma}{2}y_2\right)\right)\right| d\gamma \right. \\ &\quad \left. + \int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha}\right)^\beta \left|\mathbb{F}'\left(\mu + \nu - \left(\frac{\gamma}{2}y_1 + \frac{2 - \gamma}{2}y_2\right)\right)\right| d\gamma\right\}. \end{aligned}$$

Since $|\mathbb{F}'|$ is uniformly convex with modulus φ, Theorem 3.3 asserts that

$$\begin{aligned} I &\leq \frac{y_2 - y_1}{4}\alpha^\beta \left\{\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha}\right)^\beta \left\{\big|\mathbb{F}'(\mu)\big| + \big|\mathbb{F}'(\nu)\big| - \frac{2 - \gamma}{2}\big|\mathbb{F}'(y_1)\big| - \frac{\gamma}{2}\big|\mathbb{F}'(y_2)\big| \right.\right. \\ &\quad - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2}\left(\frac{2 - \gamma}{2}(\nu - y_1)(y_1 - \mu) + \frac{\gamma}{2}(\nu - y_2)(y_2 - \mu)\right) \\ &\quad \left. - \frac{\gamma(2 - \gamma)}{4}\varphi(y_2 - y_1)\right\} d\gamma + \int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha}\right)^\beta \left\{\big|\mathbb{F}'(\mu)\big| + \big|\mathbb{F}'(\nu)\big| - \frac{\gamma}{2}\big|\mathbb{F}'(y_1)\big| \right. \\ &\quad - \frac{2 - \gamma}{2}\big|\mathbb{F}'(y_2)\big| - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2}\left(\frac{\gamma}{2}(\nu - y_1)(y_1 - \mu) + \frac{2 - \gamma}{2}(\nu - y_2)(y_2 - \mu)\right) \\ &\quad \left.\left. - \frac{\gamma(2 - \gamma)}{4}\varphi(y_2 - y_1)\right\} d\gamma\right\}. \end{aligned}$$

After some calculations, we get

$$\begin{aligned} I &\leq \frac{y_2 - y_1}{4}\alpha^\beta \left\{\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha}\right)^\beta \right. \\ &\quad \times \left\{\big|\mathbb{F}'(\mu)\big| + \big|\mathbb{F}'(\nu)\big| - \frac{2 - \gamma}{2}\big|\mathbb{F}'(y_1)\big| - \frac{\gamma}{2}\big|\mathbb{F}'(y_2)\big|\right\} d\gamma \end{aligned}$$

$$+ \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \left\{ |\mathbb{F}'(\mu)| + |\mathbb{F}'(\nu)| - \frac{\gamma}{2}|\mathbb{F}'(y_1)| - \frac{2-\gamma}{2}|\mathbb{F}'(y_2)| \right\} d\gamma \Bigg\}$$
$$- \frac{y_2 - y_1}{4}\alpha^\beta \left\{ \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \left\{ \frac{\gamma(2-\gamma)}{2}\varphi(y_2 - y_1) \right.\right.$$
$$\left.\left. + \frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\big((\nu - y_1)(y_1-\mu) + (\nu - y_2)(y_2 - \mu)\big) \right\} d\gamma \right\}.$$

Therefore,

$$I \le \frac{y_2 - y_1}{4\alpha}\bigg[2\big(|\mathbb{F}'(\mu)| + |\mathbb{F}'(\nu)|\big) B\left(\beta+1, \frac{1}{\alpha}\right)$$
$$+ |\mathbb{F}'(y_2)| B\left(\beta+1, \frac{2}{\alpha}\right) - |\mathbb{F}'(y_1)| B\left(\beta+1, \frac{1}{\alpha}\right)$$
$$- \frac{\varphi(y_2 - y_1)}{2}\left(B\left(\beta+1, \frac{1}{\alpha}\right) - B\left(\beta+1, \frac{3}{\alpha}\right)\right)$$
$$- \frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\big((\nu - y_1)(y_1-\mu) + (\nu - y_2)(y_2 - \mu)\big) B\left(\beta+1, \frac{1}{\alpha}\right)\bigg],$$

where we have used the facts that

$$\alpha \int_0^1 \big(1-(1-\gamma)^\alpha\big)^\beta \, d\gamma = B\big(\beta+1, \alpha^{-1}\big),$$
$$\alpha \int_0^1 \gamma\big(1-(1-\gamma)^\alpha\big)^\beta \, d\gamma = B\big(\beta+1, \alpha^{-1}\big) - B\big(\beta+1, 2\alpha^{-1}\big)$$

and

$$\alpha \int_0^1 \gamma(2-\gamma)\big(1-(1-\gamma)^\alpha\big)^\beta \, d\gamma = B\big(\beta+1, \alpha^{-1}\big) - B\big(\beta+1, 3\alpha^{-1}\big). \qquad \square$$

Theorem 5.2 *Let $\alpha, \beta > 0$, $y_1 < y_2$, $q > 1$, $p = \frac{q}{1-q}$ and $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a differentiable mapping such that $\mathbb{F}' \in L[\mu, \nu]$ and $|\mathbb{F}'|^q$ is an uniformly convex mapping with modulus φ. Then the inequality*

$$I \le \frac{y_2 - y_1}{4}\alpha^\beta \left(\frac{B_1}{\alpha^{\beta+1}}\right)^{\frac{1}{p}} \times \left[\left\{ \left(\frac{|\mathbb{F}'|^q(\mu) + |\mathbb{F}'|^q(\nu)}{\alpha^{\beta+1}} B_1\right) - \frac{|\mathbb{F}'|^q(y_1)}{2\alpha^{\beta+1}}(B_1 + B_2) \right.\right.$$
$$- \frac{|\mathbb{F}'|^q(y_2)}{2\alpha^{\beta+1}}(B_1 - B_2) - \frac{(\nu - y_1)(y_1 - \mu)}{(\nu-\mu)^2\alpha^{\beta+1}}\varphi(\nu-\mu)(B_1 + B_2)$$
$$- \frac{(\nu - y_2)(y_2 - \mu)}{(\nu-\mu)^2\alpha^{\beta+1}}\varphi(\nu-\mu)(B_1 - B_2)$$
$$\left. - \frac{\varphi(\nu-\mu)}{2(\nu-\mu)^2}\varphi(y_2 - y_1)\left(\frac{1}{\alpha^{\beta+1}}B_1 - \frac{1}{\alpha^{\beta+1}}B_3\right)\right\}^{\frac{1}{q}}$$
$$+ \left\{ \left(\frac{|\mathbb{F}'|^q(\mu) + |\mathbb{F}'|^q(\nu)}{\alpha^{\beta+1}} B_1\right) - \frac{|\mathbb{F}'|^q(y_2)}{2\alpha^{\beta+1}}(B_1 + B_2) \right.$$
$$- \frac{|\mathbb{F}'|^q(y_1)}{2\alpha^{\beta+1}}(B_1 - B_2) - \frac{(\nu - y_2)(y_2 - \mu)}{(\nu-\mu)^2\alpha^{\beta+1}}\varphi(\nu-\mu)(B_1 + B_2)$$

$$- \frac{(\nu - y_1)(y_1 - \mu)}{(\nu - \mu)^2 \alpha^{\beta+1}} \varphi(\nu - \mu)(B_1 - B_2)$$
$$- \frac{\varphi(\nu - \mu)}{2(\nu - \mu)^2} \varphi(y_2 - y_1) \left(\frac{1}{\alpha^{\beta+1}} B_1 - \frac{1}{\alpha^{\beta+1}} B_3 \right) \Bigg\}^{\frac{1}{q}} \Bigg],$$

holds for all $y_1, y_2 \in [\mu, \nu]$.

Proof Let $p = \frac{q}{1-q}$. It follows from Lemma 1.3 that

$$\begin{aligned} I &\leq \frac{y_2 - y_1}{4} \alpha^\beta \int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta \\ &\quad \times \left[\left| \mathbb{F}' \left(\mu + \nu - \left(\frac{2-\gamma}{2} y_1 + \frac{\gamma}{2} y_2 \right) \right) - \mathbb{F}' \left(\mu + \nu - \left(\frac{\gamma}{2} y_1 + \frac{2-\gamma}{2} y_2 \right) \right) \right| \right] d\gamma \\ &\leq \frac{y_2 - y_1}{4} \alpha^\beta \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta d\gamma \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta \left| \mathbb{F}' \left(\mu + \nu - \left(\frac{\gamma}{2} y_1 + \frac{2-\gamma}{2} y_2 \right) \right) \right|^q d\gamma \right)^{\frac{1}{q}} \\ &\quad + \frac{y_2 - y_1}{4} \alpha^\beta \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta d\gamma \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta \left| \mathbb{F}' \left(\mu + \nu - \left(\frac{2-\gamma}{2} y_1 + \frac{\gamma}{2} y_2 \right) \right) \right|^q d\gamma \right)^{\frac{1}{q}} \end{aligned}$$

Since $|\mathbb{F}'|^q$ is uniformly convex with modulus φ, Theorem 3.3 asserts that

$$\begin{aligned} I &\leq \frac{y_2 - y_1}{4} \alpha^\beta \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta d\gamma \right)^{\frac{1}{p}} \\ &\quad \times \Bigg[\Bigg(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta \Bigg[\left| \mathbb{F}' \right|^q (\mu) + \left| \mathbb{F}' \right|^q (\nu) \\ &\quad - \frac{2-\gamma}{2} \left| \mathbb{F}' \right|^q (y_1) - \frac{\gamma}{2} \left| \mathbb{F}' \right|^q (y_2) - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2} \left(\frac{2-\gamma}{2} \right)(\nu - y_1)(y_1 - \mu) \\ &\quad + \frac{\gamma}{2} (\nu - y_2)(y_2 - \mu) \Bigg) - \frac{\gamma(2-\gamma)}{4} \varphi(y_2 - y_1) \Bigg] d\gamma)^{\frac{1}{q}} \\ &\quad + \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^\beta \Bigg[\left| \mathbb{F}' \right|^q (\mu) + \left| \mathbb{F}' \right|^q (\nu) - \frac{\gamma}{2} \left| \mathbb{F}' \right|^q (y_1) - \frac{2-\gamma}{2} \left| \mathbb{F}' \right|^q (y_2) \right. \\ &\quad - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2} \left(\frac{\gamma}{2} (\nu - y_1)(y_1 - \mu) + \frac{2-\gamma}{2} (\nu - y_2)(y_2 - \mu) \right) \\ &\quad \left. - \frac{\gamma(2-\gamma)}{4} \varphi(y_2 - y_1) \Bigg] d\gamma \right)^{\frac{1}{q}} \Bigg] \end{aligned}$$

After some calculations, we get our desired result. □

Remark 5.3 Under the assumption of Theorem 5.2, we can conclude that:

(i) If we set $\varphi(y_1) = 0$ in Theorem 5.2, we get Theorem 5 of [18].

(ii) If we set $\varphi(y_1) = 0$, $\mu = y_1$ and $\nu = y_2$ in Theorem 5.2, we get Theorem 3 of [25].

(iii) If we set $\varphi(y_1) = 0$, $\beta = 1$ $\mu = y_1$ and $\nu = y_2$ in Theorem 5.2, we get Theorem 5 of [26].

Theorem 5.4 *Let $\alpha, \beta > 0$, $y_1 < y_2$, $q > 1$ and $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a differentiable mapping such that $\mathbb{F}' \in L[\mu, \nu]$ and $|\mathbb{F}'|^q$ is a uniformly convex mapping with modulus φ. Then the inequality*

$$\begin{aligned} I \leq{}& \frac{y_2 - y_1}{4} \alpha^\beta \left(\frac{1}{\alpha^{\beta p+1}} B\left(\beta p + 1, \frac{1}{\alpha} \right) \right)^{\frac{1}{p}} \times \left[\left(\left| \mathbb{F}'(\mu) \right|^q + \left| \mathbb{F}'(\nu) \right|^q - \frac{3}{4} \left| \mathbb{F}'(y_1) \right|^q \right.\right. \\ & - \frac{1}{4} \left| \mathbb{F}'(y_2) \right|^q - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2} \left(\frac{3}{4}(\nu - y_1)(y_1 - \mu) + \frac{1}{4}(\nu - y_2)(y_2 - \mu) \right) \\ & \left. - \frac{1}{6} \varphi(y_2 - y_1) \right)^{\frac{1}{q}} \\ & + \left(\left| \mathbb{F}'(\mu) \right|^q + \left| \mathbb{F}'(\nu) \right|^q - \frac{1}{4} \left| \mathbb{F}'(y_1) \right|^q - \frac{3}{4} \left| \mathbb{F}'(y_2) \right|^q - \frac{1}{6} \varphi(y_2 - y_1) \right. \\ & \left.\left. - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2} \left(\frac{1}{4}(\nu - y_1)(y_1 - \mu) + \frac{3}{4}(\nu - y_2)(y_2 - \mu) \right) \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof Let $p = \frac{q}{q-1}$. By using Lemma 1.3 and familiar Hölder integral inequality, we can write

$$\begin{aligned} I \leq{}& \frac{y_2 - y_1}{4} \alpha^\beta \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^{\beta p} d\gamma \right)^{\frac{1}{p}} \\ & \times \left\{ \left(\int_0^1 \left| \mathbb{F}'\left(\mu + \nu - \left(\frac{2-\gamma}{2} y_1 + \frac{\gamma}{2} y_2 \right) \right) \right|^q d\gamma \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left| \mathbb{F}'\left(\mu + \nu - \left(\frac{\gamma}{2} y_1 + \frac{2-\gamma}{2} y_2 \right) \right) \right|^q d\gamma \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{37}$$

By applying the uniform convexity of $|\mathbb{F}'|^q$ and Theorem 3.3, we have

$$\begin{aligned} & \left| \mathbb{F}'\left(\mu + \nu - \left(\frac{\gamma}{2} y_1 + \frac{2-\gamma}{2} y_2 \right) \right) \right|^q \\ & \quad \leq \left| \mathbb{F}'(\mu) \right|^q + \left| \mathbb{F}'(\nu) \right|^q - \frac{\gamma}{2} \left| \mathbb{F}'(y_1) \right|^q - \frac{2-\gamma}{2} \left| \mathbb{F}'(y_2) \right|^q \\ & \qquad - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2} \left(\frac{\gamma}{2}(\nu - y_1)(y_1 - \mu) + \frac{2-\gamma}{2}(\nu - y_2)(y_2 - \mu) \right) \\ & \qquad - \frac{\gamma(2-\gamma)}{4} \varphi(y_2 - y_1). \end{aligned} \tag{38}$$

It follows from (37) and (38) that

$$\begin{aligned} I \leq{}& \frac{y_2 - y_1}{4} \alpha^\beta \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha} \right)^{\beta p} d\gamma \right)^{\frac{1}{p}} \times \left[\left(\int_0^1 \left(\left| \mathbb{F}'(\mu) \right|^q + \left| \mathbb{F}'(\nu) \right|^q - \frac{\gamma}{2} \left| \mathbb{F}'(y_2) \right|^q \right.\right.\right. \\ & - \frac{2-\gamma}{2} \left| \mathbb{F}'(y_1) \right|^q - \frac{2\varphi(\nu - \mu)}{(\nu - \mu)^2} \left(\frac{2-\gamma}{2}(\nu - y_1)(y_1 - \mu) + \frac{\gamma}{2}(\nu - y_2)(y_2 - \mu) \right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{\gamma(2-\gamma)}{4}\varphi(y_2-y_1)\biggr)\,d\gamma\biggr)^{\frac{1}{q}} \\
&+ \biggl(\int_0^1 \biggl(\bigl|\mathbb{F}'(\mu)\bigr|^q + \bigl|\mathbb{F}'(\nu)\bigr|^q - \frac{\gamma}{2}\bigl|\mathbb{F}'(y_1)\bigr|^q - \frac{2-\gamma}{2}\bigl|\mathbb{F}'(y_2)\bigr|^q \\
&- \frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\biggl(\frac{\gamma}{2}(\nu-y_1)(y_1-\mu) + \frac{2-\gamma}{2}(\nu-y_2)(y_2-\mu)\biggr) \\
&- \frac{\gamma(2-\gamma)}{4}\varphi(y_2-y_1)\biggr)\,d\gamma\biggr)^{\frac{1}{q}}\biggr] \\
= {}& \frac{y_2-y_1}{4}\alpha^\beta\biggl(\frac{1}{\alpha^{\beta p+1}}B\biggl(\beta p+1,\frac{1}{\alpha}\biggr)\biggr)^{\frac{1}{p}} \times \biggl[\biggl(\bigl|\mathbb{F}'(\mu)\bigr|^q + \bigl|\mathbb{F}'(\nu)\bigr|^q - \frac{3}{4}\bigl|\mathbb{F}'(y_1)\bigr|^q \\
&- \frac{1}{4}\bigl|\mathbb{F}'(y_2)\bigr|^q - \frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\biggl(\frac{3}{4}(\nu-y_1)(y_1-\mu) + \frac{1}{4}(\nu-y_2)(y_2-\mu)\biggr) \\
&- \frac{1}{6}\varphi(y_2-y_1)\biggr)^{\frac{1}{q}} + \biggl(\bigl|\mathbb{F}'(\mu)\bigr|^q + \bigl|\mathbb{F}'(\nu)\bigr|^q - \frac{1}{4}\bigl|\mathbb{F}'(y_1)\bigr|^q - \frac{3}{4}\bigl|\mathbb{F}'(y_2)\bigr|^q \\
&- \frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\biggl(\frac{1}{4}(\nu-y_1)(y_1-\mu) + \frac{3}{4}(\nu-y_2)(y_2-\mu)\biggr) - \frac{1}{6}\varphi(y_2-y_1)\biggr)^{\frac{1}{q}}\biggr]. \qquad \square
\end{aligned}
$$

Corollary 5.5 *If we set $\alpha = \beta = 1$, $\mu = y_1$ and $\nu = y_2$ in Theorem 5.4, we get*

$$
\begin{aligned}
&\biggl|\frac{1}{(y_2-y_1)}\int_x^y \mathbb{F}(u)\,du - \mathbb{F}\biggl(\frac{y_1+y_2}{2}\biggr)\biggr| \le \frac{y_2-y_1}{4}\biggl(\frac{1}{p+1}\biggr)^{\frac{1}{p}} \\
&\times \biggl[\biggl(\frac{|\mathbb{F}'(y_1)|^q + 3|\mathbb{F}'(y_2)|^q}{4} - \frac{1}{6}\varphi(y_2-y_1)\biggr)^{\frac{1}{q}} \\
&+ \biggl(\frac{3|\mathbb{F}'(y_1)|^q + |\mathbb{F}'(y_2)|^q}{4} - \frac{1}{6}\varphi(y_2-y_1)\biggr)^{\frac{1}{q}}\biggr]
\end{aligned}
$$

Remark 5.6 If we set $\varphi(y_1) = 0$ in Theorem 5.4, we get Theorem 6 of [18].

Remark 5.7 If we set $\varphi(y_1) = 0$ and $\beta = 1$ in Theorem 5.4, we get Corollary 2 of [18].

Theorem 5.8 *Let $\alpha, \beta > 0$, $y_1 < y_2$, $q > 1$ and $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a differentiable mapping such that $\mathbb{F}' \in L[\mu, \nu]$ and $|\mathbb{F}'|^q$ is a uniformly convex mapping with modulus φ. Then the inequality*

$$
\begin{aligned}
I \le {}& \frac{y_2-y_1}{4\alpha}\biggl[\bigl(\bigl|\mathbb{F}'(\mu)\bigr|^q + \bigl|\mathbb{F}'(\nu)\bigr|^q\bigr)B_1 - \frac{|\mathbb{F}'(y_1)|^q}{2}(B_1+B_2) \\
&- \frac{|\mathbb{F}'(y_2)|^q}{2}(B_1-B_2) - \frac{(\nu-y_1)(y_1-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu)(B_1+B_2) \\
&- \frac{(\nu-y_2)(y_2-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu)(B_1-B_2) + \frac{\varphi(\nu-\mu)}{2(\nu-\mu)^2}\varphi(y_2-y_1)(B_1-B_3)\biggr]^{\frac{1}{q}} \\
&+ \frac{y_2-y_1}{4}\alpha^\beta\biggl[\bigl(\bigl|\mathbb{F}'(\mu)\bigr|^q + \bigl|\mathbb{F}'(\nu)\bigr|^q\bigr)B_1 - \frac{|\mathbb{F}'(y_2)|^q}{2}(B_1+B_2)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{|\mathbb{F}'(y_1)|^q}{2}(B_1-B_2)-\frac{(\nu-y_2)(y_2-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu)(B_1+B_2)\\
&-\frac{(\nu-y_1)(y_1-\mu)}{(\nu-\mu)^2}\varphi(\nu-\mu)(B_1-B_2)+\frac{\varphi(\nu-\mu)}{2(\nu-\mu)^2}\varphi(y_2-y_1)(B_1-B_3)\Bigg]^{\frac{1}{q}}.
\end{aligned}
$$

Proof Let $p=\frac{q}{q-1}$. Following similar step like in the proof of the previous theorem, by using (38) and Lemma 1.3, we get

$$
\begin{aligned}
I\leq{}&\frac{y_2-y_1}{4}\alpha^{\beta}\Bigg\{\left(\int_0^1 1\,d\gamma\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\right.\\
&\times\left.\left|\mathbb{F}'\left(\mu+\nu-\left(\frac{2-\gamma}{2}y_1+\frac{\gamma}{2}y_2\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 1\,d\gamma\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\left|\mathbb{F}'\left(\mu+\nu-\left(\frac{\gamma}{2}y_1+\frac{2-\gamma}{2}y_2\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\Bigg\}\\
\leq{}&\frac{y_2-y_1}{4}\alpha^{\beta}\Bigg\{\left(\int_0^1\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\right.\\
&\times\left(\left|\mathbb{F}'(\mu)\right|^q+\left|\mathbb{F}'(\nu)\right|^q-\frac{2-\gamma}{2}\left|\mathbb{F}'(y_1)\right|^q-\frac{\gamma}{2}\left|\mathbb{F}'(y_2)\right|^q\right.\\
&-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\left(\frac{2-\gamma}{2}(\nu-y_1)(y_1-\mu)+\frac{\gamma}{2}(\nu-y_2)(y_2-\mu)\right)\\
&\left.\left.-\frac{\gamma(2-\gamma)}{4}\varphi(y_2-y_1)\right)d\gamma\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\beta}\left(\left|\mathbb{F}'(\mu)\right|^q+\left|\mathbb{F}'(\nu)\right|^q-\frac{\gamma}{2}\left|\mathbb{F}'(y_1)\right|^q-\frac{2-\gamma}{2}\left|\mathbb{F}'(y_2)\right|^q\right.\right.\\
&-\frac{2\varphi(\nu-\mu)}{(\nu-\mu)^2}\left(\frac{\gamma}{2}(\nu-y_1)(y_1-\mu)+\frac{2-\gamma}{2}(\nu-y_2)(y_2-\mu)\right)\\
&\left.\left.-\frac{\gamma(2-\gamma)}{4}\varphi(y_2-y_1)|^q\right)d\gamma\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

After some calculations, we get our desired result. □

Corollary 5.9 *If we set $\alpha=\beta=1$, $\mu=y_1$ and $\nu=y_2$ in Theorem 5.8, we get*

$$
\begin{aligned}
&\left|\frac{1}{(y_2-y_1)}\int_{y_1}^{y_2}\mathbb{F}(u)\,du-\mathbb{F}\left(\frac{y_1+y_2}{2}\right)\right|\\
&\quad\leq\frac{y_2-y_1}{4}\left(\frac{|\mathbb{F}'(y_1)|^q}{6}+\frac{|\mathbb{F}'(y_2)|^q}{3}+\frac{(5\varphi(y_2-y_1))^2}{24(y_2-y_1)^2}\right)^{\frac{1}{q}}\\
&\qquad+\frac{y_2-y_1}{4}\left(\frac{|\mathbb{F}'(y_2)|^q}{6}+\frac{|\mathbb{F}'(y_1)|^q}{3}+\frac{5(\varphi(y_2-y_1))^2}{24(y_2-y_1)^2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Remark 5.10 If we set $\varphi(y_1)=0$ in Theorem 5.8, we get Theorem 7 of [18].

6 New Ostrowski–Mercer type inequalities for uniformly convex functions

Let $\mathbb{F}:[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $\mathbb{F}'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|\mathbb{F}'(y_1)|\le M$, then the following inequality (see [27], page 468):

$$\left|\mathbb{F}(y_1)-\frac{1}{b-a}\int_a^b \mathbb{F}(y_2)\,dy_2\right|\le \frac{M}{b-a}\left[\frac{(y_1-a)^2+(b-y_1)^2}{2}\right] \tag{39}$$

holds. This result is known in the literature as the Ostrowski inequality. For recent results and generalizations concerning Ostrowski inequality, see [28, 29] and the references therein.

In this section, Mercer–Ostrowski inequalities for the conformable integral operator are obtained for uniformly convex functions. For this purpose, we give a new conformable integral operator identity that will serve as an auxiliary result to produce subsequent results for improvements.

Lemma 6.1 *Suppose that the mapping $\mathbb{F}:I=[a,b]\to\Re$ is differentiable on (a,b) with $b>a$. If $\mathbb{F}'\in L_1[a,b]$ then for all $y_1,\mu,\nu_2\in[a,b]$ and $\alpha,\beta>0$, the following identity*

$$\begin{aligned}
&\alpha^\beta(y_1-\mu)^2\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1+a-(\gamma\mu+(1-\gamma)y_1)\big)\,d\gamma\\
&\quad-\alpha^\beta(\nu-y_1)^2\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1+b-(\gamma\nu+(1-\gamma)y_1)\big)\,d\gamma\\
&\quad=(y_1-\mu)\mathbb{F}(y_1+a-\mu)+(\nu-y_1)\mathbb{F}(y_1+b-\nu)\\
&\quad\quad-\frac{\alpha^\beta\Gamma(\beta+1)}{(\nu-y_1)^{\alpha\beta-1}}\left\{{}^\beta J^\alpha_{(y_1+a-\mu)^-}\mathbb{F}(a)+{}^\beta J^\alpha_{(y_1+b-\nu)^+}\mathbb{F}(b)\right\}:=\mathbb{L}
\end{aligned} \tag{40}$$

Proof Let

$$I=\alpha^\beta(y_1-\mu)^2 I_1-\alpha^\beta(\nu-y_1)^2 I_2, \tag{41}$$

$$\begin{aligned}
I_1&=\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1+a-(\gamma\mu+(1-\gamma)y_1)\big)\,d\gamma\\
&=\frac{\mathbb{F}(y_1+a-\mu)}{\alpha^\beta}\\
&\quad-\frac{\beta}{(y_1-\mu)}\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\beta-1}(1-\gamma)^{\alpha-1}\mathbb{F}\big(y_1+a-(\gamma\mu+(1-\gamma)y_1)\big)\,d\gamma\\
&=\frac{\mathbb{F}(y_1+a-\mu)}{\alpha^\beta(y_1-\mu)}-\frac{\gamma(\beta+1)}{(y_1-\mu)^{\alpha\beta+1}}{}^b\beta J^\alpha_{(y_1+a-\mu)^-}\mathbb{F}(a)
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2&=\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1+b-(\gamma\nu+(1-\gamma)y_1)\big)\,d\gamma\\
&=\frac{\mathbb{F}(y_1+b-\nu)}{\alpha^\beta(\nu-\mu)}-\frac{\Gamma(\beta+1)}{(\nu-y_1)^{\alpha\beta+1}}{}^\beta J^\alpha_{(y_1+b-\nu)^+}\mathbb{F}(b)
\end{aligned}$$

Substitute the values of I_1 and I_2 in (41), we get the required result. □

Corollary 6.2 *If we set* $\alpha = 1$ *in lemma* 6.1

$$(y_1-\mu)^2\int_0^1 \gamma^{\beta}\mathbb{F}'\big(y_1+a-\big(\gamma\mu+(1-\gamma)y_1\big)\big)\,d\gamma$$
$$-(\nu-y_1)^2\int_0^1 \gamma^{\beta}\mathbb{F}'\big(y_1+b-\big(\gamma\nu+(1-\gamma)y_1\big)\big)\,d\gamma$$
$$=(y_1-\mu)\mathbb{F}(y_1+a-\mu)+(\nu-y_1)\mathbb{F}(y_1+b-\nu)$$
$$-\frac{\Gamma(\beta+1)}{(\nu-y_1)^{\beta-1}}\big\{{}^{\beta}J_{(y_1+a-\mu)^-}\mathbb{F}(a)+{}^{\beta}J_{(y_1+b-\nu)^+}\mathbb{F}(b)\big\}$$

Remark 6.3 If we set $\mu = a$, $\nu = b$ and $\alpha = \beta = 1$ in Lemma 6.1, then it reduces to Lemma 1 in [28].

Theorem 6.4 *Let* $\alpha,\beta > 0$, $a < b$ *and* $\mathbb{F}:[\mu,\nu]\to\mathbb{R}$ *be a differentiable mapping such that* $\mathbb{F}' \in L[a,b]$ *and* $|\mathbb{F}'|$ *is a uniformly convex mapping with modulus* φ. *Then the inequality holds*

$$|\mathbb{L}| \le \frac{(y_1-\mu)^2}{\alpha}\bigg\{\big[\big|\mathbb{F}'(y_1)\big|+\big|\mathbb{F}'(a)\big|\big]B_1-\big|\mathbb{F}'(\mu)\big|[B_1-B_2]-\big|\mathbb{F}'(y_1)\big|B_2$$
$$-\varphi(y_1-\mu)[B_2-B_3]-\frac{2\varphi(a-y_1)}{(a-y_1)^2}(a-\mu)(\mu-y_1)[B_1-B_2]\bigg\}$$
$$+\frac{(\nu-y_1)^2}{\alpha}\bigg\{\big[\big|\mathbb{F}'(y_1)\big|+\big|\mathbb{F}'(b)\big|\big]B_1-\big|\mathbb{F}'(\nu)\big|[B_1-B_2]-\big|\mathbb{F}'(y_1)\big|B_2$$
$$-\varphi(y_1-\nu)[B_2-B_3]-\frac{2\varphi(b-y_1)}{(b-y_1)^2}(b-\nu)(\nu-y_1)[B_1-B_2]\bigg\}.$$

Proof It follows from Lemma 6.1 that

$$|\mathbb{L}| = \bigg|(y_1-\mu)^2\alpha^{\beta}\int_0^1\bigg(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\bigg)^{\beta}\mathbb{F}'\big(y_1+a-\big(\gamma\mu+(1-\gamma)y_1\big)\big)\,d\gamma$$
$$-(\nu-y_1)^2\alpha^{\beta}\int_0^1\bigg(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\bigg)^{\beta}\mathbb{F}'\big(y_1+b-\big(\gamma\nu+(1-\gamma)y_1\big)\big)\,d\gamma\bigg|$$
$$\le (y_1-\mu)^2\alpha^{\beta}\int_0^1\bigg(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\bigg)^{\beta}\big|\mathbb{F}'\big(y_1+a-\big(\gamma\mu+(1-\gamma)y_1\big)\big)\big|\,d\gamma$$
$$-(\nu-y_1)^2\alpha^{\beta}\int_0^1\bigg(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\bigg)^{\beta}\big|\mathbb{F}'\big(y_1+b-\big(\gamma\nu+(1-\gamma)y_1\big)\big)\big|\,d\gamma$$

Since $|\mathbb{F}'|$ is uniformly convex with modulus φ,

$$|\mathbb{L}| \le (y_1-\mu)^2\alpha^{\beta}\int_0^1\bigg(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\bigg)^{\beta}\bigg\{\big|\mathbb{F}'(y_1)\big|+\big|\mathbb{F}'(a)\big|-\gamma\big|\mathbb{F}'(\mu)\big|-(1-\gamma)\big|\mathbb{F}'(y_1)\big|$$
$$-\frac{2\varphi(a-y_1)}{(a-y_1)^2}\big(\gamma(a-\mu)(\mu-y_1)+(1-\gamma)(a-y_1)(y_1-y_1)\big)$$
$$-\gamma(1-\gamma)\varphi(y_1-\mu)\bigg\}\,d\gamma$$

$$+ (\nu - y_1)^2\alpha^\beta \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \Big\{ |\mathbb{F}'(y_1)| + |\mathbb{F}'(b)| - \gamma|\mathbb{F}'(\nu)| - (1-\gamma)|\mathbb{F}'(y_1)|$$
$$- \frac{2\varphi(b-y_1)}{(b-y_1)^2}\big(\gamma(b-\nu)(\nu-y_1) + (1-\gamma)(b-y_1)(y_1-y_1)\big)$$
$$- \gamma(1-\gamma)\varphi(y_1-\nu)\Big\}\, d\gamma.$$

After some calculations, we get our desired result. □

Corollary 6.5 *Let $\alpha, \beta > 0$, $a < b$ and $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a differentiable mapping such that $\mathbb{F}' \in L[\mu, \nu]$ and $|\mathbb{F}'|$ is a strongly convex mapping with modulus c. Then the inequality holds*

$$|\mathbb{L}| \le \frac{(y_1-\mu)^2}{\alpha}\big\{\big[|\mathbb{F}'(y_1)| + |\mathbb{F}'(a)|\big]B_1 - |\mathbb{F}'(\mu)|[B_1-B_2] - |\mathbb{F}'(y_1)|B_2$$
$$- c(y_1-\mu)^2[B_2-B_3] - 2c(a-\mu)(\mu-y_1)[B_1-B_2]\big\}$$
$$+ \frac{(\nu-y_1)^2}{\alpha}\big\{\big[|\mathbb{F}'(y_1)| + |\mathbb{F}'(b)|\big]B_1$$
$$- |\mathbb{F}'(\nu)|[B_1-B_2] - |\mathbb{F}'(y_1)|B_2 - c(y_1-\nu)^2[B_2-B_3]$$
$$- 2c(b-\nu)(\nu-y_1)[B_1-B_2]\big\}.$$

Proof The result follows from Theorem 6.4 with $\varphi(r) = cr^2$. □

Theorem 6.6 *Let $\alpha, \beta, k > 0$, $a < b$ and $\mathbb{F} : [\mu, \nu] \to \mathbb{R}$ be a differentiable mapping such that $\mathbb{F}' \in L[a, b]$ and $|\mathbb{F}'|$ is a uniformly convex mapping with modulus φ. Then the inequality holds*

$$|\mathbb{L}| \le \left(\frac{B_1}{\alpha^{\beta+1}}\right)^{\frac{1}{p}} \frac{(y_1-\mu)^2}{\alpha}\Big\{\big[|\mathbb{F}'(y_1)|^q + |\mathbb{F}'(a)|^q\big]B_1 - |\mathbb{F}'(\mu)|^q[B_1-B_2] - |\mathbb{F}'(y_1)|^q B_2$$
$$- \varphi(y_1-\mu)[B_2-B_3] - \frac{2\varphi(a-y_1)}{(a-y_1)^2}(a-\mu)(\mu-y_1)[B_1-B_2]\Big\}^{\frac{1}{q}}$$
$$+ \left(\frac{B_1}{\alpha^{\beta+1}}\right)^{\frac{1}{p}} \frac{(\nu-y_1)^2}{\alpha}\Big\{\big[|\mathbb{F}'(y_1)|^q + |\mathbb{F}'(b)|^q\big]B_1 - |\mathbb{F}'(\nu)|^q[B_1-B_2] - |\mathbb{F}'(y_1)|^q B_2$$
$$- \varphi(y_1-\nu)[B_2-B_3] - \frac{2\varphi(b-y_1)}{(b-y_1)^2}(b-\nu)(\nu-y_1)[B_1-B_2]\Big\}^{\frac{1}{q}}.$$

Proof It follows from Lemma 6.1 that

$$|\mathbb{L}| = \bigg|(y_1-\mu)^2\alpha^\beta \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1 + a - (\gamma\mu + (1-\gamma)y_1)\big)\, d\gamma$$
$$- (\nu-y_1)^2\alpha^\beta \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1 + b - (\gamma\nu + (1-\gamma)y_1)\big)\, d\gamma\bigg|$$
$$\le (y_1-\mu)^2\alpha^\beta \left(\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta d\gamma\right)^{\frac{1}{p}}$$

$$\times \left[\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \left|\mathbb{F}'\big(y_1+a-\big(\gamma\mu+(1-\gamma)y_1\big)\big)\right|^q d\gamma\right]^{\frac{1}{q}}$$
$$+ (\nu-y_1)^2\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)^{\frac{1}{p}}$$
$$\times \left[\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \left|\mathbb{F}'\big(y_1+b-\big(\gamma\nu+(1-\gamma)y_1\big)\big)\right|^q d\gamma\right]^{\frac{1}{q}}$$

Since $|\mathbb{F}'|$ is uniformly convex with modulus φ,

$$|\mathbb{L}| \le (y_1-\mu)^2\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)^{\frac{1}{p}}$$
$$\times \left[\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \left\{\left|\mathbb{F}'(y_1)\right|^q + \left|\mathbb{F}'(a)\right|^q - \gamma\left|\mathbb{F}'(\mu)\right|^q - (1-\gamma)\left|\mathbb{F}'(y_1)\right|^q\right.\right.$$
$$- \frac{2\varphi(a-y_1)}{(a-y_1)^2}\big(\gamma(a-\mu)(\mu-y_1)+(1-\gamma)(a-y_1)(y_1-y_1)\big)$$
$$\left.\left. - \gamma(1-\gamma)\varphi(y_1-\mu)\right\} d\gamma\right]^{\frac{1}{q}}$$
$$+ (\nu-y_1)^2\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta\right)^{\frac{1}{p}}$$
$$\times \left[\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \left\{\left|\mathbb{F}'(y_1)\right|^q + \left|\mathbb{F}'(b)\right| - \gamma\left|\mathbb{F}'(\nu)\right|^q - (1-\gamma)\left|\mathbb{F}'(y_1)\right|^q\right.\right.$$
$$- \frac{2\varphi(b-y_1)}{(b-y_1)^2}\big(\gamma(b-\nu)(\nu-y_1)+(1-\gamma)(b-y_1)(y_1-y_1)\big)$$
$$\left.\left. - \gamma(1-\gamma)\varphi(y_1-\nu)\right\} d\gamma\right]^{\frac{1}{q}}.$$

After some calculations, we get the required result. □

Theorem 6.7 *Let $\alpha,\beta>0$, $a<b$ and $\mathbb{F}:[\mu,\nu]\to\mathbb{R}$ be a differentiable mapping such that $\mathbb{F}'\in L[a,b]$ and $|\mathbb{F}'|$ is a uniformly convex mapping with modulus φ. Then the inequality holds*

$$|\mathbb{L}| \le \left(\frac{B(p\beta+1,\frac{1}{\alpha})}{\alpha^{p\beta+1}}\right)^{\frac{1}{p}} \frac{(y_1-\mu)^2}{\alpha}\left\{\left|\mathbb{F}'(y_1)\right|^q + \left|\mathbb{F}'(a)\right|^q\right.$$
$$\left. - \frac{|\mathbb{F}'(\mu)|^q-|\mathbb{F}'(y_1)|^q}{2} - \frac{\varphi(y_1-\mu)}{6} - \frac{\varphi(a-y_1)}{(a-y_1)^2}(a-\mu)(\mu-y_1)\right\}^{\frac{1}{q}}$$
$$+ \left(\frac{B(p\beta+1,\frac{1}{\alpha})}{\alpha^{p\beta+1}}\right)^{\frac{1}{p}} \frac{(\nu-y_1)^2}{\alpha}\left\{\left|\mathbb{F}'(y_1)\right|^q + \left|\mathbb{F}'(b)\right|^q\right.$$
$$\left. - \frac{|\mathbb{F}'(\nu)|^q-|\mathbb{F}'(y_1)|^q}{2} - \frac{\varphi(y_1-\nu)}{6} - \frac{\varphi(b-y_1)}{(b-y_1)^2}(b-\nu)(\nu-y_1)\right\}^{\frac{1}{q}}.$$

Proof From Lemma 6.1 and applying Hölder inequality, we have

$$\begin{aligned}
|\mathbb{L}| &= \bigg|(y_1-\mu)^2\alpha^\beta \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1+a-\big(\gamma\mu+(1-\gamma)y_1\big)\big)\,d\gamma \\
&\quad - (\nu-y_1)^2\alpha^\beta \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \mathbb{F}'\big(y_1+b-\big(\gamma\nu+(1-\gamma)y_1\big)\big)\,d\gamma\bigg| \\
&\le (y_1-\mu)^2\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{p\beta} d\gamma\right)^{\frac{1}{p}} \left(\big|\mathbb{F}'\big(y_1+a-\big(\gamma\mu+(1-\gamma)y_1\big)\big)\big|^q\,d\gamma\right)^{\frac{1}{q}} \\
&\quad + (\nu-y_1)^2\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\beta}\right)^{\frac{1}{p}} \left[\big|\mathbb{F}'\big(y_1+b-\big(\gamma\nu+(1-\gamma)y_1\big)\big)\big|^q\,d\gamma\right]^{\frac{1}{q}}
\end{aligned}$$

Since $|\mathbb{F}'|$ is uniformly convex with modulus φ,

$$\begin{aligned}
I &\le (y_1-\mu)^2\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{p\beta}\right)^{\frac{1}{p}}\bigg[\int_0^1\Big\{\big|\mathbb{F}'(y_1)\big|^q+\big|\mathbb{F}'(a)\big|^q-\gamma\big|\mathbb{F}'(\mu)\big|^q \\
&\quad -(1-\gamma)\big|\mathbb{F}'(y_1)\big|^q-\frac{2\varphi(a-y_1)}{(a-y_1)^2}\big(\gamma(a-\mu)(\mu-y_1)+(1-\gamma)(a-y_1)(y_1-y_1)\big) \\
&\quad -\gamma(1-\gamma)\varphi(y_1-\mu)\Big\}\,d\gamma\bigg]^{\frac{1}{q}} + (\nu-y_1)^2\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{p\beta}\right)^{\frac{1}{p}} \\
&\quad \times\bigg[\int_0^1\Big\{\big|\mathbb{F}'(y_1)\big|^q+\big|\mathbb{F}'(b)\big|-\gamma\big|\mathbb{F}'(\nu)\big|^q-(1-\gamma)\big|\mathbb{F}'(y_1)\big|^q-\gamma(1-\gamma)\varphi(y_1-\nu) \\
&\quad -\frac{2\varphi(b-y_1)}{(b-y_1)^2}\big(\gamma(b-\nu)(\nu-y_1)+(1-\gamma)(b-y_1)(y_1-y_1)\big)\Big\}\,d\gamma\bigg]^{\frac{1}{q}}.
\end{aligned}$$

After some calculations, we get the required result. □

7 Applications

A random variable X is said to have a normal distribution [30], with σ (the standard deviation) and then translated by μ (the mean value): $\mathbb{F}(y_1)=\frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2}(\frac{y_1-\mu}{\sigma})^2\}$, $-\infty<y_1<\infty$. A random variable X normal-distributed with parameters μ and σ will be denoted by $X\sim N(\mu,\sigma)$.

The normal distribution, often known as the Gaussian distribution, is a symmetric probability distribution about the mean. This shows that data near to the mean occur more frequently than data distant from the mean. Like every probability distribution, the normal distribution describes the distribution of values of a variable. It is the most important probability distribution in statistics because it properly captures the distribution of values for numerous natural events. Commonly, traits that are the result of several different unique processes are described using normal distributions. For instance, the normal distribution may be shown for IQ scores, blood pressure, heights, and measurement inaccuracy.

In this section, we try to estimate the normal probability distribution with the help of inequalities.

Proposition 7.1 *Let $\nu>\mu>0$ and X has normal distribution with $X\sim N(\frac{\mu+\nu}{2},\frac{\nu}{\sqrt{2}})$. Then*

$$p(\mu\le X\le\nu)\le\frac{\nu-\mu}{6\mu\nu^2\sqrt{\pi}}\exp\left(-\left(\frac{\nu-\mu}{2\nu}\right)^2\right)\big(\mu^2+\nu^2+4\mu\nu\big).$$

Proof By the use of (Lemma 2.1 in [24]) the function $\mathbb{F}(x) = -\log(x) \in U(\frac{1}{2\nu^2}(.)^2; [\mu, \nu])$. Set $\mathbb{F}(x) = -\log(x)$ and $\varphi(r) = \frac{1}{2\nu^2} r^2$ in Theorem 3.1, we have

$$\frac{(\nu-\gamma)(\gamma-\mu)}{\nu^2} \leq \log\left(\frac{\gamma(\mu+\nu-\gamma)}{\mu\nu}\right)$$

for all $\gamma \in [\mu, \nu]$. Therefore,

$$\exp\left(\frac{(\nu-\gamma)(\gamma-\mu)}{\nu^2}\right) \leq \frac{\gamma(\mu+\nu-\gamma)}{\mu\nu}$$

or

$$\exp\left(-\frac{1}{2}\left(\frac{\gamma - \frac{\mu+\nu}{2}}{\frac{1}{\sqrt{2}}\nu}\right)^2\right) \leq \frac{\gamma(\mu+\nu-\gamma)}{\mu\nu} \times \exp\left(\frac{\mu}{\nu} - \frac{(\mu+\nu)^2}{4\nu^2}\right) \tag{42}$$

for all $\gamma \in [\mu, \nu]$. Multiplying (42) by $\frac{1}{\nu\sqrt{\pi}}$ and integrating the obtained inequality w.r.t. γ over $[\mu, \nu]$, we get

$$p(\mu \leq X \leq \nu) \leq \frac{\nu-\mu}{6\mu\nu^2\sqrt{\pi}} \exp\left(\frac{-(\nu-\mu)^2}{4\nu^2}\right)\left(\mu^2+\nu^2+4\mu\nu\right). \qquad \square$$

Proposition 7.2 *Let* $\mu > 0$, $k \geq 1$ *and* X *has normal distribution with* $X \sim N(\frac{k+1}{2}\mu, \frac{k\mu}{\sqrt{2}})$. *Then*

$$p(\mu \leq X \leq k\mu) \leq \frac{(k-1)(5k^2+1)}{6k^2\sqrt{\pi}} \exp\left(-\frac{(k-1)^2}{4k^2}\right).$$

Proof Setting $k := \frac{\nu}{\mu}$ in Proposition 7.1. $\square$

Proposition 7.3 *Let* $\frac{\sqrt{2}}{2} \leq a < b$. *Then the inequality*

$$\begin{aligned}(b-a)e^{-(\frac{a+b}{2})^2} + \frac{(2a^2-1)(b-a)^3}{24}e^{-b^2} &\leq \int_a^b e^{-u^2}\,du \\ &\leq \frac{e^{-a^2}+e^{-b^2}}{2}(b-a) - \frac{(2a^2-1)(b-a)^3}{3}e^{-b^2}\end{aligned}$$

holds.

Proof Applying Lemma 2.2 and Corollary 4.4 with $\mathbb{F}(u) = e^{-u^2}$, $y_1 = x = a$, $y_2 = y = b$ and $\varphi(r) = (2a^2-1)e^{-b^2}r^2$. $\square$

8 Conclusion

This paper is devoted to the study of inequalities for uniformly convex functions along with their properties. We give some examples of such convexity and gave the new concept of Jensen–Mercer inequality for it in a classical sense. In the later part, we employed our main inequality to get new fractional inequalities for uniformly convex functions. We used generalized conformable fractional integrals and configure Hermite–Jensen–Mercer inequalities for them. Some new extensions of fractional Hermite–Mercer type inequalities for differentiable uniformly convex functions are also presented. Finally, we employed our newly obtained results to explore new fractional variants of Ostrowski–Mercer type inequalities. It is pertinent to mention that by special substitution, we got all such inequalities for strongly convex functions. Also, we pointed out some particular cases of fractional integral inequalities.

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Declarations

Competing interests
The authors declare no competing interests.

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Author details
[1]Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan. [2]Department of Mathematics, Sirjan University Of Technology, Sirjan, Iran. [3]Department of Mathematics, Anand International College of Engineering, Jaipur, India. [4]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE. [5]CITMAga, Department of Statistics, University of Santiago de Compostela, 15882, Santiago de Compostela, Spain.

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