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Conformable fractional Newton-type inequalities with respect to differentiable convex functions

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Abstract

The authors propose a new method of investigation of an integral identity according to conformable fractional operators. Moreover, some Newton-type inequalities are considered for differentiable convex functions by taking the modulus of the newly established equality. In addition, we prove several Newton-type inequalities with the aid of Hölder and power-mean inequalities. Furthermore, several new results are given by using special choices of the obtained inequalities. Finally, we give several inequalities of conformable fractional Newton-type for functions of bounded variation.

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1 Introduction

Simpson's second rule has the rule of the three-point Newton–Cotes quadrature. Computations for three steps with a quadratic kernel are usually called Newton-type results. In the literature, these results are called Newton-type inequalities. Newton-type inequalities have been investigated extensively by many mathematicians. For instance, in [14], Newton-type inequalities were considered for functions whose second derivatives are convex. Noor et al. established Newton-type inequalities associated with harmonic convex and p-harmonic convex functions in [28] and [29], respectively. In [25], Newton-type inequalities were proved by postquantum integrals. Moreover, several error estimates of the Newton-type quadrature formula by bounded variation and Lipschitzian mappings were presented in [12]. Furthermore, Newton-type inequalities were presented for quantum differentiable convex functions in [4]. The reader is referred to [18, 20, 27, 32] and the references therein for more information and unexplained subjects about Newton-type inequalities including convex differentiable functions.

Fractional calculus has increased in popularity in recent years because of its applications in a wide range of different domains of science. Due to the significance of fractional calculus, one can be considered different fractional integral operators. By using the Hermite–Hadamard-type and Simpson-type inequalities, the bounds of new formulas can be obtained. For example, in [31], Hermite–Hadamard-type and trapezoidal-type inequalities were established for the first time using the Riemann–Liouville fractional integrals. In [32], sundry Newton-type inequalities were established by using Riemann–Liouville fractional integrals for differentiable convex functions and the authors also acquired several Riemann–Liouville fractional Newton-type inequalities for functions of bounded variation. In addition, sundry Newton-type inequalities for the case of functions whose first derivatives in absolute value at certain powers are arithmetically harmonically convex were obtained in [11]. Furthermore, several Newton-type inequalities were given and some applications for the case of special cases of real functions were also presented in [14]. See [9, 10, 16, 19, 20, 35] for details and unexplained subjects.

Riemann–Liouville fractional integrals, conformable fractional integrals, and many types of fractional integrals have been considered with inequalities. Nowadays, it has piqued the curiosity of mathematicians, engineers, and physicists [6, 33]. In addition to this, fractional derivatives are also used to model a wide range of mathematical biology problems, as well as chemical processes and engineering problems [7, 13]. By using the derivative's fundamental limit formulation, a newly well-behaved fundamental fractional derivative known as the conformable derivative has proved in [23]. Several major requirements that cannot be applied by the Riemann–Liouville and Caputo definitions are applied by the conformable derivative. In [1] the author proved that the conformable approach in [23] cannot yield good results when compared to the Caputo definition for the case of specific functions. This flaw in the conformable definition was recovered by several extensions of the conformable approach [3, 8, 17, 26, 34].

This study takes the form of six sections, including the introduction. With the help of the ongoing studies and the above-mentioned papers, we investigated Newton-type inequalities involving conformable fractional operators. The fundamental definitions of fractional calculus and other relevant research in this discipline are given in Sect. 2. We will prove an integral equality in Sect. 3 that is critical in establishing the primary results of the presented paper. Moreover, sundry new Newton-type inequalities for conformable fractional integrals will be proven. In Sect. 4, several results will be given by using special choices of obtained inequalities. In Sect. 5, we will present some inequalities of conformable fractional Newton-type for functions of bounded variation. Finally, in Sect. 6, we will give several ideas for the further direction of research.

2 Preliminaries

Simpson-type inequalities are inequalities that are generated from Simpson's following rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_{\sigma}^{\delta} \mathcal{F}(x)\,dx \approx \frac{\delta-\sigma}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right].$$

ii. Newton–Cotes quadrature formula or Simpson's second formula (Simpson's 3/8 rule):

$$\int_{\sigma}^{\delta} \mathcal{F}(x)\,dx \approx \frac{\delta-\sigma}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right].$$

Definition 1 (See [30]) Let us consider that I is an interval of real numbers. Then, a function $\mathcal{F}: I \to \mathbb{R}$ is said to be convex, if

$$\mathcal{F}\big(\mu x + (1-\mu)y\big) \le \mu\mathcal{F}(x) + (1-\mu)\mathcal{F}(y)$$

is valid $\forall x, y \in I$ and $\forall \mu \in [0,1]$.

Definition 2 (See [15, 24]) Let $\mathcal{F} \in L_1[\sigma,\delta]$, $\sigma,\delta \in \mathbb{R}$ with $\sigma < \delta$. The Riemann–Liouville fractional integrals $J^{\beta}_{\sigma^+}\mathcal{F}$ and $J^{\beta}_{\delta^-}\mathcal{F}$ of order $\beta > 0$ are defined by

$$J^{\beta}_{\sigma^+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}(x-\mu)^{\beta-1}\mathcal{F}(\mu)\,d\mu, \quad x > \sigma \tag{1}$$

and

$$J^{\beta}_{\delta^-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}(\mu-x)^{\beta-1}\mathcal{F}(\mu)\,d\mu, \quad x < \delta, \tag{2}$$

respectively. Here, Γ denotes the Gamma function defined by

$$\Gamma(\beta) = \int_0^{\infty} e^{-u}u^{\beta-1}\,du.$$

Let us note that $J^{0}_{\sigma+}\mathcal{F}(x) = J^{0}_{\delta-}\mathcal{F}(x) = \mathcal{F}(x)$.

The fractional conformable integral operators were considered in [22]. These authors also derived sundry characteristics and relationships between these operators and some other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 3 (See [22]) Let $\beta > 0$ and $\alpha \in (0,1]$. For $\mathcal{F} \in L_1[\sigma,\delta]$, the fractional conformable integral operator the generalized fractional Riemann–Liouville integrals (FCIOs) ${}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}$ and ${}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta}\mathcal{F}$ are defined by

$${}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}\left(\frac{(x-\sigma)^{\alpha}-(\mu-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\mu-\sigma)^{1-\alpha}}\,d\mu, \quad x > \sigma \tag{3}$$

and

$${}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}\left(\frac{(\delta-x)^{\alpha}-(\delta-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\delta-\mu)^{1-\alpha}}\,d\mu, \quad x < \delta, \tag{4}$$

respectively.

Let us consider $\alpha = 1$ in equalities (3) and (4). Then, the fractional integral in (3) and (4) coincides with the Riemann–Liouville fractional integral in (1) and (2), respectively. See Refs. [2, 21] and the references therein for further information.

3 Main results

Throughout the paper, we assume that $\alpha \in (0,1]$, $\beta \in \mathbb{R}^+$.

Lemma 1 *Let $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ be a differentiable mapping on (σ,δ) with $\beta>0$ and $\alpha\in(0,1]$. If $\mathcal{F}'\in L[\sigma,\delta]$, then the following FCIOs identity holds*:

$$\begin{aligned}
&\frac{3^{\alpha\beta-1}\alpha^\beta}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{2\sigma+\delta}{3}}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+2\delta}{3}}\mathcal{F}(\delta)\right]\\
&\quad-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\\
&\quad=\frac{(\delta-\sigma)\alpha^\beta}{9}[I_1+I_2+I_3],
\end{aligned}\tag{5}$$

where $\Gamma(\beta)$ is an Euler Gamma function and

$$\begin{aligned}
I_1&=\int_0^1\left(\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{5}{8\alpha^\beta}\right)\mathcal{F}'\left(\mu\sigma+(1-\mu)\left(\frac{2\sigma+\delta}{3}\right)\right)d\mu,\\
I_2&=\int_0^1\left(\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right)\mathcal{F}'\left(\mu\left(\frac{2\sigma+\delta}{3}\right)+(1-\mu)\left(\frac{\sigma+2\delta}{3}\right)\right)d\mu,\\
I_3&=\int_0^1\left(\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right)\mathcal{F}'\left(\mu\left(\frac{\sigma+2\delta}{3}\right)+(1-\mu)\delta\right)d\mu.
\end{aligned}$$

Proof Using integration by parts and changing variables with $x=\mu\sigma+(1-\mu)(\frac{2\sigma+\delta}{3})$, we obtain

$$\begin{aligned}
I_1&=\int_0^1\left(\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{5}{8\alpha^\beta}\right)\mathcal{F}'\left(\mu\sigma+(1-\mu)\left(\frac{2\sigma+\delta}{3}\right)\right)d\mu\\
&=-\frac{3}{\delta-\sigma}\left(\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{5}{8\alpha^\beta}\right)\mathcal{F}\left(\mu\sigma+(1-\mu)\left(\frac{2\sigma+\delta}{3}\right)\right)\Bigg|_0^1\\
&\quad+\frac{3\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\mu\sigma+(1-\mu)\left(\frac{2\sigma+\delta}{3}\right)\right)d\mu\\
&=-\left[\frac{9}{8\alpha^\beta(\delta-\sigma)}\mathcal{F}(\sigma)+\frac{15}{8\alpha^\beta(\delta-\sigma)}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]\\
&\quad+\left(\frac{3}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_\sigma^{\frac{2\sigma+\delta}{3}}\left(\frac{(\frac{\delta-\sigma}{3})^\alpha-(x-\sigma)^\alpha}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(x-\sigma)^{1-\alpha}}dx\\
&=-\left[\frac{9}{8\alpha^\beta(\delta-\sigma)}\mathcal{F}(\sigma)+\frac{15}{8\alpha^\beta(\delta-\sigma)}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]\\
&\quad+\frac{3^{1+\alpha\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{1+\alpha\beta}}\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right].
\end{aligned}$$

Similar to the foregoing process, changing variables with $x=\mu(\frac{2\sigma+\delta}{3})+(1-\mu)(\frac{\sigma+2\delta}{3})$ and $x=\mu(\frac{\sigma+2\delta}{3})+(1-\mu)\delta$, we have

$$I_2=\int_0^1\left(\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right)\mathcal{F}'\left(\mu\left(\frac{2\sigma+\delta}{3}\right)+(1-\mu)\left(\frac{\sigma+2\delta}{3}\right)\right)d\mu$$

$$
\begin{aligned}
&= -\frac{3}{(\delta-\sigma)}\left(\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right)\mathcal{F}\left(\mu\left(\frac{2\sigma+\delta}{3}\right)+(1-\mu)\left(\frac{\sigma+2\delta}{3}\right)\right)\right|_0^1 \\
&\quad+\frac{3\beta}{(\delta-\sigma)}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\mu\left(\frac{2\sigma+\delta}{3}\right)\right. \\
&\quad\left.+(1-\mu)\left(\frac{\sigma+2\delta}{3}\right)\right)d\mu \\
&= -\left[\frac{3}{2\alpha^{\beta}(\delta-\sigma)}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\frac{3}{2\alpha^{\beta}(\delta-\sigma)}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right] \\
&\quad+\left(\frac{3}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_{\frac{2\sigma+\delta}{3}}^{\frac{\sigma+2\delta}{3}}\left(\frac{(\frac{\delta-\sigma}{3})^{\alpha}-(x-\frac{2\sigma+\delta}{3})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(x-\frac{2\sigma+\delta}{3})^{1-\alpha}}\,dx \\
&= -\left[\frac{3}{2\alpha^{\beta}(\delta-\sigma)}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\frac{3}{2\alpha^{\beta}(\delta-\sigma)}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right] \\
&\quad+\frac{3^{1+\alpha\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{1+\alpha\beta}}\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{2\sigma+\delta}{3}}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right)\mathcal{F}'\left(\mu\left(\frac{\sigma+2\delta}{3}\right)+(1-\mu)\delta\right)d\mu \\
&= -\frac{3}{\delta-\sigma}\left(\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right)\mathcal{F}\left(\mu\left(\frac{\sigma+2\delta}{3}\right)+(1-\mu)\delta\right)\right|_0^1 \\
&\quad+\frac{3\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\mu\left(\frac{\sigma+2\delta}{3}\right)+(1-\mu)\delta\right)d\mu \\
&= -\left[\frac{9}{8\alpha^{\beta}(\delta-\sigma)}\mathcal{F}(\delta)+\frac{15}{8\alpha^{\beta}(\delta-\sigma)}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right] \\
&\quad+\left(\frac{3}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_{\frac{\sigma+2\delta}{3}}^{\delta}\left(\frac{(\frac{\delta-\sigma}{3})^{\alpha}-(x-\frac{\sigma+2\delta}{3})^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(x-\frac{\sigma+2\delta}{3})^{1-\alpha}}\,dx \\
&= -\left[\frac{9}{8\alpha^{\beta}(\delta-\sigma)}\mathcal{F}(\delta)+\frac{15}{8\alpha^{\beta}(\delta-\sigma)}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right] \\
&\quad+\frac{3^{1+\alpha\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{1+\alpha\beta}}\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+2\delta}{3}}\mathcal{F}(\delta)\right].
\end{aligned}
$$

Finally, if we multiply $I_1+I_2+I_3$ by $\frac{(\delta-\sigma)\alpha^{\beta}}{9}$, then we have (5). This completes the proof of Lemma 1. □

Theorem 1 *Assume that all the assumptions of Lemma 1 hold. Moreover, let $|\mathcal{F}'|$ be a convex function on $[\sigma,\delta]$. Then, we have*

$$
\begin{aligned}
&\left|\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{2\sigma+\delta}{3}}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+2\delta}{3}}\mathcal{F}(\delta)\right]\right. \\
&\quad\left.-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\right| \\
&\quad\le\frac{(\delta-\sigma)\alpha^{\beta}}{27}\left[\left(2A_2(\alpha,\beta)+A_1(\alpha,\beta)+A_4(\alpha,\beta)+A_3(\alpha,\beta)+A_5(\alpha,\beta)\right)\left|\mathcal{F}'(\sigma)\right|\right.
\end{aligned}
$$

$$+\left(A_2(\alpha,\beta)-A_1(\alpha,\beta)+2A_4(\alpha,\beta)-A_3(\alpha,\beta)\right.$$
$$\left.+3A_6(\alpha,\beta)-A_5(\alpha,\beta)\right)\left|\mathcal{F}'(\delta)\right|\Big], \tag{6}$$

where

$$\begin{aligned}
A_1(\alpha,\beta) &= \int_0^1 \mu\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{5}{8\alpha^\beta}\right|d\mu \\
&= \frac{1}{\alpha^\beta}\left[\frac{5}{8}\left(C_1^2-\frac{1}{2}\right)+\frac{1}{\alpha}\left(2\mathcal{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{5}{8}\right)^{\frac{1}{\beta}}\right)\right.\right. \\
&\quad \left.\left.-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{5}{8}\right)^{\frac{1}{\beta}}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)+\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right)\right], \\
A_2(\alpha,\beta) &= \int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{5}{8\alpha^\beta}\right|d\mu \\
&= \frac{1}{\alpha^\beta}\left[\frac{5}{8}(2C_1-1)+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{5}{8}\right)^{\frac{1}{\beta}}\right)\right)\right], \\
A_3(\alpha,\beta) &= \int_0^1 \mu\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|d\mu \\
&= \frac{1}{\alpha^\beta}\left[\frac{1}{2}\left(C_2^2-\frac{1}{2}\right)+\frac{1}{\alpha}\left(2\mathcal{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right.\right. \\
&\quad \left.\left.-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)+\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right)\right], \\
A_4(\alpha,\beta) &= \int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|d\mu \\
&= \frac{1}{\alpha^\beta}\left[\frac{1}{2}(2C_2-1)+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right)\right], \\
A_5(\alpha,\beta) &= \int_0^1 \mu\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right|d\mu \\
&= \frac{1}{\alpha^\beta}\left[\frac{3}{8}\left(C_3^2-\frac{1}{2}\right)+\frac{1}{\alpha}\left(2\mathcal{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{3}{8}\right)^{\frac{1}{\beta}}\right)\right.\right. \\
&\quad \left.\left.-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{3}{8}\right)^{\frac{1}{\beta}}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)+\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
A_6(\alpha,\beta) &= \int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right|d\mu \\
&= \frac{1}{\alpha^\beta}\left[\frac{3}{8}(2C_3-1)+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{3}{8}\right)^{\frac{1}{\beta}}\right)\right)\right].
\end{aligned}$$

Here, $C_1 = 1-(1-(\frac{5}{8})^{\frac{1}{\beta}})^{\frac{1}{\alpha}}$, $C_2 = 1-(1-(\frac{1}{2})^{\frac{1}{\beta}})^{\frac{1}{\alpha}}$, $C_3 = 1-(1-(\frac{3}{8})^{\frac{1}{\beta}})^{\frac{1}{\alpha}}$, *the functions* $\mathfrak{B}(\cdot,\cdot)$ *and* $\mathcal{B}(\cdot,\cdot,\cdot)$ *are the Beta function and the incomplete Beta function defined as*

$$\begin{cases}\mathfrak{B}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,du,\\ \mathcal{B}(x,y,r) = \int_0^r u^{x-1}(1-u)^{y-1}\,du\end{cases}$$

for $x,y>0$ *and* $r\in[0,1]$.

Proof By Lemma 1, integrating by parts and the convexity of $|\mathcal{F}'|$, we obtain

$$\begin{aligned}
&\left|\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}_{\frac{2\sigma+\delta}{3}}^{\alpha}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+2\delta}{3}}^{\alpha}\mathcal{F}(\delta)\right]\right.\\
&\quad\left.-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\right|\\
&\quad\le\frac{(\delta-\sigma)\alpha^{\beta}}{9}\left[\left|\int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right)\mathcal{F}'\left(\mu\sigma+(1-\mu)\left(\frac{2\sigma+\delta}{3}\right)\right)d\mu\right|\right.\\
&\quad+\left|\int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right)\mathcal{F}'\left(\mu\left(\frac{2\sigma+\delta}{3}\right)+(1-\mu)\left(\frac{\sigma+2\delta}{3}\right)\right)d\mu\right|\\
&\quad\left.+\left|\int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right)\mathcal{F}'\left(\mu\left(\frac{\sigma+2\delta}{3}\right)+(1-\mu)\delta\right)d\mu\right|\right]\\
&\quad\le\frac{(\delta-\sigma)\alpha^{\beta}}{9}\left[\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\left(\frac{2+\mu}{3}\right)\sigma+\left(\frac{1-\mu}{3}\right)\delta\right)\right|d\mu\right.\\
&\quad+\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\left(\frac{1+\mu}{3}\right)\sigma+\left(\frac{2-\mu}{3}\right)\delta\right)\right|d\mu\\
&\quad\left.+\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{\mu}{3}\sigma+\left(\frac{3-\mu}{3}\right)\delta\right)\right|d\mu\right]\\
&\quad\le\frac{(\delta-\sigma)\alpha^{\beta}}{9}\left[\left|\mathcal{F}'(\sigma)\right|\int_0^1\left(\frac{2+\mu}{3}\right)\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right|d\mu\right.\\
&\quad+\left|\mathcal{F}'(\delta)\right|\int_0^1\left(\frac{1-\mu}{3}\right)\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right|d\mu\\
&\quad+\left|\mathcal{F}'(\sigma)\right|\int_0^1\left(\frac{1+\mu}{3}\right)\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|d\mu\\
&\quad+\left|\mathcal{F}'(\delta)\right|\int_0^1\left(\frac{2-\mu}{3}\right)\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|d\mu\\
&\quad+\left|\mathcal{F}'(\sigma)\right|\int_0^1\frac{\mu}{3}\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right|d\mu\\
&\quad\left.+\left|\mathcal{F}'(\delta)\right|\int_0^1\left(\frac{3-\mu}{3}\right)\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right|d\mu\right]\\
&\quad=\frac{(\delta-\sigma)\alpha^{\beta}}{27}\Big[\big(2A_2(\alpha,\beta)+A_1(\alpha,\beta)+A_4(\alpha,\beta)+A_3(\alpha,\beta)+A_5(\alpha,\beta)\big)\left|\mathcal{F}'(\sigma)\right|\\
&\quad+\big(A_2(\alpha,\beta)-A_1(\alpha,\beta)+2A_4(\alpha,\beta)-A_3(\alpha,\beta)+3A_6(\alpha,\beta)-A_5(\alpha,\beta)\big)\left|\mathcal{F}'(\delta)\right|\Big].
\end{aligned}$$

This is the desired result of Theorem 1. □

Theorem 2 *Suppose that all the assumptions of Lemma 1 hold. Suppose also that $|\mathcal{F}'|^q$ is a convex function on $[\sigma,\delta]$, where $\frac{1}{p}+\frac{1}{q}=1$ with $p,q>1$. Then, we have*

$$\begin{aligned}
&\left|\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{2\sigma+\delta}{3}}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+2\delta}{3}}\mathcal{F}(\delta)\right]\right.\\
&\quad\left.-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\right|\\
&\quad\le\frac{(\delta-\sigma)\alpha^{\beta}}{9}\left[A_7^{\frac{1}{p}}(\alpha,\beta,p)\left(\frac{5|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right.\\
&\quad+A_8^{\frac{1}{p}}(\alpha,\beta,p)\left(\frac{|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{2}\right)^{\frac{1}{q}}\\
&\quad\left.+A_9^{\frac{1}{p}}(\alpha,\beta,p)\left(\frac{|\mathcal{F}'(\sigma)|^q+5|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right],
\end{aligned}\tag{7}$$

where

$$\begin{aligned}
A_7(\alpha,\beta,p)&=\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right|^p d\mu,\\
A_8(\alpha,\beta,p)&=\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^p d\mu
\end{aligned}$$

and

$$A_9(\alpha,\beta,p)=\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right|^p d\mu.$$

Proof If we consider Lemma 1, then we can readily obtain

$$\begin{aligned}
&\left|\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{2\sigma+\delta}{3}}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+2\delta}{3}}\mathcal{F}(\delta)\right]\right.\\
&\quad\left.-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\right|\\
&\quad\le\frac{(\delta-\sigma)\alpha^{\beta}}{9}\left[\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\left(\frac{2+\mu}{3}\right)\sigma+\left(\frac{1-\mu}{3}\right)\delta\right)\right|d\mu\right.\\
&\quad+\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\left(\frac{1+\mu}{3}\right)\sigma+\left(\frac{2-\mu}{3}\right)\delta\right)\right|d\mu\\
&\quad\left.+\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{3}{8\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{\mu}{3}\sigma+\left(\frac{3-\mu}{3}\right)\delta\right)\right|d\mu\right].
\end{aligned}\tag{8}$$

Now, we consider the integrals on the right side of (8). Using the convexity of $|\mathcal{F}'|^q$ and the well-known Hölder inequality, we have

$$\begin{aligned}
&\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\left(\frac{2+\mu}{3}\right)\sigma+\left(\frac{1-\mu}{3}\right)\delta\right)\right|d\mu\\
&\quad\le\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{5}{8\alpha^{\beta}}\right|^p d\mu\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times \left(\int_0^1 \left|\mathcal{F}'\left(\left(\frac{2+\mu}{3}\right)\sigma + \left(\frac{1-\mu}{3}\right)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}$$
$$\leq A_7^{\frac{1}{p}}(\alpha,\beta,p)\left(\int_0^1 \left(\frac{2+\mu}{3}\left|\mathcal{F}'(\sigma)\right|^q + \frac{1-\mu}{3}\left|\mathcal{F}'(\delta)\right|^q\right) d\mu\right)^{\frac{1}{q}}$$
$$= A_7^{\frac{1}{p}}(\alpha,\beta,p)\left[\frac{5|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{6}\right]^{\frac{1}{q}}. \tag{9}$$

In a similar manner, we readily obtain

$$\int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta - \frac{1}{2\alpha^\beta}\right|\left|\mathcal{F}'\left(\left(\frac{1+\mu}{3}\right)\sigma + \left(\frac{2-\mu}{3}\right)\delta\right)\right| d\mu$$
$$\leq \left(\int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta - \frac{1}{2\alpha^\beta}\right|^p d\mu\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left|\mathcal{F}'\left(\left(\frac{1+\mu}{3}\right)\sigma + \left(\frac{2-\mu}{3}\right)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}$$
$$\leq A_8^{\frac{1}{p}}(\alpha,\beta,p)\left(\int_0^1 \left(\frac{1+\mu}{3}\left|\mathcal{F}'(\sigma)\right|^q + \frac{2-\mu}{3}\left|\mathcal{F}'(\delta)\right|^q\right) d\mu\right)^{\frac{1}{q}}$$
$$= A_8^{\frac{1}{p}}(\alpha,\beta,p)\left[\frac{|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{2}\right]^{\frac{1}{q}} \tag{10}$$

and

$$\int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta - \frac{3}{8\alpha^\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{3}\sigma + \left(\frac{3-\mu}{3}\right)\delta\right)\right| d\mu$$
$$\leq \left(\int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta - \frac{3}{8\alpha^\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1 \left|\mathcal{F}'\left(\frac{\mu}{3}\sigma + \left(\frac{3-\mu}{3}\right)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}$$
$$\leq A_9^{\frac{1}{p}}(\alpha,\beta,p)\left(\int_0^1 \left(\frac{\mu}{3}\left|\mathcal{F}'(\sigma)\right|^q + \frac{3-\mu}{3}\left|\mathcal{F}'(\delta)\right|^q\right) d\mu\right)^{\frac{1}{q}}$$
$$= A_9^{\frac{1}{p}}(\alpha,\beta,p)\left[\frac{|\mathcal{F}'(\sigma)|^q + 5|\mathcal{F}'(\delta)|^q}{6}\right]^{\frac{1}{q}}. \tag{11}$$

If we insert from (9)–(11) into (8), then we have

$$\left|\frac{3^{\alpha\beta-1}\alpha^\beta}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^\beta_+\mathcal{J}^\alpha_\sigma\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + {}^\beta_+\mathcal{J}^\alpha_{\frac{2\sigma+\delta}{3}}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + {}^\beta_+\mathcal{J}^\alpha_{\frac{\sigma+2\delta}{3}}\mathcal{F}(\delta)\right]\right.$$
$$\left. - \frac{1}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right]\right|$$
$$\leq \frac{(\delta-\sigma)\alpha^\beta}{9}\left[A_7^{\frac{1}{p}}(\alpha,\beta,p)\left(\frac{5|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right.$$
$$\left. + A_8^{\frac{1}{p}}(\alpha,\beta,p)\left(\frac{|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{2}\right)^{\frac{1}{q}} + A_9^{\frac{1}{p}}(\alpha,\beta,p)\left(\frac{|\mathcal{F}'(\sigma)|^q + 5|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right].$$

This completes the proof of Theorem 2. □

Theorem 3 *Let us consider that all the assumptions of Lemma 1 hold. If $|\mathcal{F}'|^q$ is convex on $[\sigma,\delta]$, where $q \geq 1$, then we have the following Newton-type inequality*

$$\begin{aligned}
&\Bigg|\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\bigg[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathcal{F}\bigg(\frac{2\sigma+\delta}{3}\bigg)+{}_{+}^{\beta}\mathcal{J}_{\frac{2\sigma+\delta}{3}}^{\alpha}\mathcal{F}\bigg(\frac{\sigma+2\delta}{3}\bigg)+{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+2\delta}{3}}^{\alpha}\mathcal{F}(\delta)\bigg] \\
&\quad-\frac{1}{8}\bigg[\mathcal{F}(\sigma)+3\mathcal{F}\bigg(\frac{2\sigma+\delta}{3}\bigg)+3\mathcal{F}\bigg(\frac{\sigma+2\delta}{3}\bigg)+\mathcal{F}(\delta)\bigg]\Bigg| \\
&\quad\leq\frac{(\delta-\sigma)\alpha^{\beta}}{9}\bigg[A_2^{1-\frac{1}{q}}(\alpha,\beta)\bigg(\bigg(\frac{2A_2(\alpha,\beta)+A_1(\alpha,\beta)}{3}\bigg)\big|\mathcal{F}'(\sigma)\big|^q \\
&\quad+\bigg(\frac{A_2(\alpha,\beta)-A_1(\alpha,\beta)}{3}\bigg)\big|\mathcal{F}'(\delta)\big|^q\bigg)^{\frac{1}{q}} \\
&\quad+A_4^{1-\frac{1}{q}}(\alpha,\beta)\bigg(\bigg(\frac{A_4(\alpha,\beta)+A_3(\alpha,\beta)}{3}\bigg)\big|\mathcal{F}'(\sigma)\big|^q \\
&\quad+\bigg(\frac{2A_4(\alpha,\beta)-A_3(\alpha,\beta)}{3}\bigg)\big|\mathcal{F}'(\delta)\big|^q\bigg)^{\frac{1}{q}} \\
&\quad+A_6^{1-\frac{1}{q}}(\alpha,\beta)\bigg(\frac{A_5(\alpha,\beta)}{3}\big|\mathcal{F}'(\sigma)\big|^q+\bigg(\frac{3A_6(\alpha,\beta)-A_5(\alpha,\beta)}{3}\bigg)\big|\mathcal{F}'(\delta)\big|^q\bigg)^{\frac{1}{q}}\bigg], \qquad (12)
\end{aligned}$$

where $A_1(\alpha,\beta)$, $A_2(\alpha,\beta)$, $A_3(\alpha,\beta)$, $A_4(\alpha,\beta)$, $A_5(\alpha,\beta)$, and $A_6(\alpha,\beta)$ are defined in Theorem 1.

Proof If we consider the convexity of $|\mathcal{F}'|^q$ and power-mean inequality, then we obtain

$$\begin{aligned}
&\int_0^1\bigg|\bigg(\frac{1-(1-\mu)^{\alpha}}{\alpha}\bigg)^{\beta}-\frac{5}{8\alpha^{\beta}}\bigg|\bigg|\mathcal{F}'\bigg(\bigg(\frac{2+\mu}{3}\bigg)\sigma+\bigg(\frac{1-\mu}{3}\bigg)\delta\bigg)\bigg|\,d\mu \\
&\quad\leq\bigg(\int_0^1\bigg|\bigg(\frac{1-(1-\mu)^{\alpha}}{\alpha}\bigg)^{\beta}-\frac{5}{8\alpha^{\beta}}\bigg|\,d\mu\bigg)^{1-\frac{1}{q}} \\
&\quad\times\bigg(\int_0^1\bigg|\bigg(\frac{1-(1-\mu)^{\alpha}}{\alpha}\bigg)^{\beta}-\frac{5}{8\alpha^{\beta}}\bigg|\bigg|\mathcal{F}'\bigg(\bigg(\frac{2+\mu}{3}\bigg)\sigma+\bigg(\frac{1-\mu}{3}\bigg)\delta\bigg)\bigg|^q\,d\mu\bigg)^{\frac{1}{q}} \\
&\quad=A_2^{1-\frac{1}{q}}(\alpha,\beta)\bigg(\int_0^1\bigg|\bigg(\frac{1-(1-\mu)^{\alpha}}{\alpha}\bigg)^{\beta}-\frac{5}{8\alpha^{\beta}}\bigg| \\
&\quad\times\bigg|\mathcal{F}'\bigg(\bigg(\frac{2+\mu}{3}\bigg)\sigma+\bigg(\frac{1-\mu}{3}\bigg)\delta\bigg)\bigg|^q\,d\mu\bigg)^{\frac{1}{q}} \\
&\quad\leq A_2^{1-\frac{1}{q}}(\alpha,\beta)\bigg(\int_0^1\bigg|\bigg(\frac{1-(1-\mu)^{\alpha}}{\alpha}\bigg)^{\beta}-\frac{5}{8\alpha^{\beta}}\bigg| \\
&\quad\times\bigg[\frac{2+\mu}{3}\big|\mathcal{F}'(\sigma)\big|^q+\frac{1-\mu}{3}\big|\mathcal{F}'(\delta)\big|^q\bigg]\,d\mu\bigg)^{\frac{1}{q}} \\
&\quad=A_2^{1-\frac{1}{q}}(\alpha,\beta)\bigg(\bigg(\frac{2A_2(\alpha,\beta)+A_1(\alpha,\beta)}{3}\bigg)\big|\mathcal{F}'(\sigma)\big|^q \\
&\quad+\bigg(\frac{A_2(\alpha,\beta)-A_1(\alpha,\beta)}{3}\bigg)\big|\mathcal{F}'(\delta)\big|^q\bigg)^{\frac{1}{q}}. \qquad (13)
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
&\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|\left|\mathcal{F}'\left(\left(\frac{1+\mu}{3}\right)\sigma+\left(\frac{2-\mu}{3}\right)\delta\right)\right|d\mu\\
&\le\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|d\mu\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|\left|\mathcal{F}'\left(\left(\frac{1+\mu}{3}\right)\sigma+\left(\frac{2-\mu}{3}\right)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&=A_4^{1-\frac{1}{q}}(\alpha,\beta)\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|\left|\mathcal{F}'\left(\left(\frac{1+\mu}{3}\right)\sigma+\left(\frac{2-\mu}{3}\right)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&\le A_4^{1-\frac{1}{q}}(\alpha,\beta)\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|\right.\\
&\quad\left.\times\left[\frac{1+\mu}{3}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\mu}{3}\left|\mathcal{F}'(\delta)\right|^q\right]d\mu\right)^{\frac{1}{q}}\\
&=A_4^{1-\frac{1}{q}}(\alpha,\beta)\left(\left(\frac{A_4(\alpha,\beta)+A_3(\alpha,\beta)}{3}\right)\left|\mathcal{F}'(\sigma)\right|^q\right.\\
&\quad\left.+\left(\frac{2A_4(\alpha,\beta)-A_3(\alpha,\beta)}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
&\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{3}\sigma+\left(\frac{3-\mu}{3}\right)\delta\right)\right|d\mu\\
&\le\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right|d\mu\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{3}\sigma+\left(\frac{3-\mu}{3}\right)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&=A_6^{1-\frac{1}{q}}(\alpha,\beta)\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{3}\sigma+\left(\frac{3-\mu}{3}\right)\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&\le A_6^{1-\frac{1}{q}}(\alpha,\beta)\left(\int_0^1\left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta-\frac{3}{8\alpha^\beta}\right|\left[\frac{\mu}{3}\left|\mathcal{F}'(\sigma)\right|^q+\frac{3-\mu}{3}\left|\mathcal{F}'(\delta)\right|^q\right]d\mu\right)^{\frac{1}{q}}\\
&=A_6^{1-\frac{1}{q}}(\alpha,\beta)\left(\frac{A_5(\alpha,\beta)}{3}\left|\mathcal{F}'(\sigma)\right|^q+\left(\frac{3A_6(\alpha,\beta)-A_5(\alpha,\beta)}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}\tag{15}$$

If we insert (13)–(15) into (8), then the proof of Theorem 3 is finished. □

4 Special cases

Remark 1 If we choose $\alpha = 1$ in (5), then the equality reduces to

$$\begin{aligned}
&\frac{3^{\beta-1}}{(\delta-\sigma)^\beta}\Gamma(\beta+1)\left[J_{\sigma^+}^{\beta}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+J_{(\frac{2\sigma+\delta}{3})^+}^{\beta}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+J_{(\frac{\sigma+2\delta}{3})^+}^{\beta}\mathcal{F}(\delta)\right]\\
&\quad-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]
\end{aligned}$$

$$= \frac{\delta-\sigma}{9}\left[I_1^* + I_2^* + I_3^*\right],$$

where

$$I_1^* = \int_0^1 \left(\mu^\beta - \frac{5}{8}\right)\mathcal{F}'\left(\mu\sigma + (1-\mu)\left(\frac{2\sigma+\delta}{3}\right)\right)d\mu,$$
$$I_2^* = \int_0^1 \left(\mu^\beta - \frac{1}{2}\right)\mathcal{F}'\left(\mu\left(\frac{2\sigma+\delta}{3}\right) + (1-\mu)\left(\frac{\sigma+2\delta}{3}\right)\right)d\mu,$$
$$I_3^* = \int_0^1 \left(\mu^\beta - \frac{3}{8}\right)\mathcal{F}'\left(\mu\left(\frac{\sigma+2\delta}{3}\right) + (1-\mu)\delta\right)d\mu.$$

This coincides with [32, Lemma 1].

Remark 2 If we select $\alpha = 1$ in (6), then we obtain

$$\begin{aligned}
&\left|\frac{3^{\beta-1}}{(\delta-\sigma)^\beta}\Gamma(\beta+1)\left[J_{\sigma^+}^\beta\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + J_{(\frac{2\sigma+\delta}{3})^+}^\beta\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + J_{(\frac{\sigma+2\delta}{3})^+}^\beta\mathcal{F}(\delta)\right]\right.\\
&\quad\left.-\frac{1}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right]\right|\\
&\quad\le \frac{\delta-\sigma}{27}\left[\left(2A_2^*(\beta) + A_1^*(\beta) + A_4^*(\beta) + A_3^*(\beta) + A_5^*(\beta)\right)\left|\mathcal{F}'(\sigma)\right|\right.\\
&\quad\left.+\left(A_2^*(\beta) - A_1^*(\beta) + 2A_4^*(\beta) - A_3^*(\beta) + 3A_6^*(\beta) - A_5^*(\beta)\right)\left|\mathcal{F}'(\delta)\right|\right].
\end{aligned}$$

Here,

$$A_1^*(\beta) = \int_0^1 \mu\left|\mu^\beta - \frac{5}{8}\right|d\mu = \frac{\beta}{\beta+2}\left(\frac{5}{8}\right)^{\frac{\beta+2}{\beta}} + \frac{1}{\beta+2} - \frac{5}{16},$$
$$A_2^*(\beta) = \int_0^1 \left|\mu^\beta - \frac{5}{8}\right|d\mu = \frac{2\beta}{\beta+1}\left(\frac{5}{8}\right)^{\frac{\beta+1}{\beta}} + \frac{1}{\beta+1} - \frac{5}{8},$$
$$A_3^*(\beta) = \int_0^1 \mu\left|\mu^\beta - \frac{1}{2}\right|d\mu = \frac{\beta}{\beta+2}\left(\frac{1}{2}\right)^{\frac{\beta+2}{\beta}} + \frac{1}{\beta+2} - \frac{1}{4},$$
$$A_4^*(\beta) = \int_0^1 \left|\mu^\beta - \frac{1}{2}\right|d\mu = \frac{2\beta}{\beta+1}\left(\frac{1}{2}\right)^{\frac{\beta+1}{\beta}} + \frac{1}{\beta+1} - \frac{1}{2},$$
$$A_5^*(\beta) = \int_0^1 \mu\left|\mu^\beta - \frac{3}{8}\right|d\mu = \frac{\beta}{\beta+2}\left(\frac{3}{8}\right)^{\frac{\beta+2}{\beta}} + \frac{1}{\beta+2} - \frac{3}{16},$$
$$A_6^*(\beta) = \int_0^1 \left|\mu^\beta - \frac{3}{8}\right|d\mu = \frac{2\beta}{\beta+1}\left(\frac{3}{8}\right)^{\frac{\beta+1}{\beta}} + \frac{1}{\beta+1} - \frac{3}{8},$$

which is established by Sitthiwirattham et al. [32, Theorem 4].

Remark 3 Consider $\alpha = \beta = 1$ in (6). Then, (6) becomes

$$\left|\frac{1}{\delta-\sigma}\int_\sigma^\delta \mathcal{F}(\mu)\,d\mu - \frac{1}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{25(\delta-\sigma)}{576}\left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right|\right),$$

which is given in [32, Remark 3].

Remark 4 Note that the inequality (7) for $\alpha = 1$ reduces to

$$\begin{aligned}
&\left|\frac{3^{\beta-1}}{(\delta-\sigma)^{\beta}}\Gamma(\beta+1)\left[J^{\beta}_{\sigma^+}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + J^{\beta}_{(\frac{2\sigma+\delta}{3})^+}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + J^{\beta}_{(\frac{\sigma+2\delta}{3})^+}\mathcal{F}(\delta)\right]\right.\\
&\qquad\left. - \frac{1}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right]\right|\\
&\qquad\leq \frac{\delta-\sigma}{9}\left[\left(A_7^*(\beta,p)\right)^{\frac{1}{p}}\left(\frac{5|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right.\\
&\qquad\left. + \left(A_8^*(\beta,p)\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{2}\right)^{\frac{1}{q}} + \left(A_9^*(\beta,p)\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}'(\sigma)|^q + 5|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\begin{aligned}
A_7^*(\beta,p) &= \int_0^1 \left|\mu^{\beta} - \frac{5}{8}\right|^p d\mu,\\
A_8^*(\beta,p) &= \int_0^1 \left|\mu^{\beta} - \frac{1}{2}\right|^p d\mu,\\
A_9^*(\beta,p) &= \int_0^1 \left|\mu^{\beta} - \frac{3}{8}\right|^p d\mu.
\end{aligned}$$

This is proved by Sitthiwirattham et al. [32].

Remark 5 Consider $\alpha = \beta = 1$ in (7). Then, (7) coincides with

$$\begin{aligned}
&\left|\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\mu)\,d\mu - \frac{1}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right]\right|\\
&\qquad\leq \frac{\delta-\sigma}{9}\left[\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}'(\sigma)|^q + 5|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right.\\
&\qquad + \left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{2}\right)^{\frac{1}{q}}\\
&\qquad\left. + \left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is given in [32, Remark 5].

Remark 6 For $\alpha = 1$, the inequality (12) becomes

$$\begin{aligned}
&\left|\frac{3^{\beta-1}}{(\delta-\sigma)^{\beta}}\Gamma(\beta+1)\left[J^{\beta}_{\sigma^+}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + J^{\beta}_{(\frac{2\sigma+\delta}{3})^+}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + J^{\beta}_{(\frac{\sigma+2\delta}{3})^+}\mathcal{F}(\delta)\right]\right.\\
&\qquad\left. - \frac{1}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right]\right|
\end{aligned}$$

$$\begin{aligned}&\leq \frac{\delta-\sigma}{9}\Bigg[\left(A_2^*(\beta)\right)^{1-\frac{1}{q}}\left(\left(\frac{2A_2^*(\beta)+A_1^*(\beta)}{3}\right)\left|\mathcal{F}'(\sigma)\right|^q+\left(\frac{A_2^*(\beta)-A_1^*(\beta)}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\\&+\left(A_4^*(\beta)\right)^{1-\frac{1}{q}}\left(\left(\frac{A_4^*(\beta)+A_3^*(\beta)}{3}\right)\left|\mathcal{F}'(\sigma)\right|^q+\left(\frac{2A_4^*(\beta)-A_3^*(\beta)}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\\&+\left(A_6^*(\beta)\right)^{1-\frac{1}{q}}\left(\frac{A_5^*(\beta)}{3}\left|\mathcal{F}'(\sigma)\right|^q+\left(\frac{3A_6^*(\beta)-A_5^*(\beta)}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

Here, $A_1^*(\beta), A_2^*(\beta), A_3^*(\beta), A_4^*(\beta), A_5^*(\beta), A_6^*(\beta)$ are defined in Remark 2. For the proof, we refer to [32, Theorem 5].

Remark 7 Consider $\alpha=\beta=1$ in (12). Then, (12) becomes

$$\begin{aligned}&\left|\frac{1}{\delta-\sigma}\int_\sigma^\delta \mathcal{F}(\mu)\,d\mu-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\right|\\&\leq\frac{\delta-\sigma}{36}\Bigg[\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left(\frac{251|\mathcal{F}'(\sigma)|^q+973|\mathcal{F}'(\delta)|^q}{1152}\right)^{\frac{1}{q}}\\&+\left(\frac{|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left(\frac{973|\mathcal{F}'(\sigma)|^q+251|\mathcal{F}'(\delta)|^q}{1152}\right)^{\frac{1}{q}}\Bigg],\end{aligned}$$

which is given in [32, Remark 4].

5 Fractional Newton-type inequality for functions of bounded variation

In this section, we establish a fractional Newton-type inequality for function of bounded variation.

Theorem 4 *Let us consider that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a function of bounded variation on $[\sigma,\delta]$. Then, we obtain the following Newton-type inequality for FCIOs*

$$\begin{aligned}&\Bigg|\frac{3^{\alpha\beta-1}\alpha^\beta}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{2\sigma+\delta}{3}}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+2\delta}{3}}\mathcal{F}(\delta)\right]\\&-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\Bigg|\leq\frac{5}{24}\bigvee_\sigma^\delta(\mathcal{F}).\end{aligned}$$

Here, $\bigvee_c^d(\mathcal{F})$ is the total variation of $\mathcal{F}$ on $[c,d]$.

Proof Consider

$$\Phi_\alpha^\beta(x)=\begin{cases}((\frac{\delta-\sigma}{3})^\alpha-(x-\sigma)^\alpha)^\beta-\frac{5}{8}(\frac{\delta-\sigma}{3})^{\alpha\beta}, & \sigma\leq x\leq\frac{2\sigma+\delta}{3},\\((\frac{\delta-\sigma}{3})^\alpha-(x-\frac{2\sigma+\delta}{3})^\alpha)^\beta-\frac{1}{2}(\frac{\delta-\sigma}{3})^{\alpha\beta} & \frac{2\sigma+\delta}{3}<x\leq\frac{\sigma+2\delta}{3},\\((\frac{\delta-\sigma}{3})^\alpha-(x-\frac{\sigma+2\delta}{3})^\alpha)^\beta-\frac{3}{8}(\frac{\delta-\sigma}{3})^{\alpha\beta} & \frac{\sigma+2\delta}{3}<x\leq\delta.\end{cases}$$

Then, this yields

$$\int_\sigma^\delta\Phi_\alpha^\beta(x)\,d\mathcal{F}(x)=\int_\sigma^{\frac{2\sigma+\delta}{3}}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^\alpha-(x-\sigma)^\alpha\right)^\beta-\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)$$

$$+\int_{\frac{2\sigma+\delta}{3}}^{\frac{\sigma+2\delta}{3}}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{2\sigma+\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{1}{2}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)$$
$$+\int_{\frac{\sigma+2\delta}{3}}^{\delta}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{\sigma+2\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{3}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x). \tag{16}$$

By using integration by parts, we obtain

$$\begin{aligned}
&\int_{\sigma}^{\frac{2\sigma+\delta}{3}}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-(x-\sigma)^{\alpha}\right)^{\beta}-\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)\\
&=\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-(x-\sigma)^{\alpha}\right)^{\beta}-\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)\mathcal{F}(x)\Big|_{\sigma}^{\frac{2\sigma+\delta}{3}}\\
&\quad+\alpha\beta\int_{\sigma}^{\frac{2\sigma+\delta}{3}}\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-(x-\sigma)^{\alpha}\right)^{\beta-1}(x-\sigma)^{\alpha-1}\mathcal{F}(x)\,dx\\
&=-\left[\frac{3}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\mathcal{F}(\sigma)+\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]\\
&\quad+\alpha^{\beta}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right].
\end{aligned} \tag{17}$$

In a similar manner, we obtain

$$\begin{aligned}
&\int_{\frac{2\sigma+\delta}{3}}^{\frac{\sigma+2\delta}{3}}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{2\sigma+\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{1}{2}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)\\
&=-\left[\frac{1}{2}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\frac{1}{2}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]\\
&\quad+\alpha^{\beta}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{2\sigma+\delta}{3}}^{\alpha}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\int_{\frac{\sigma+2\delta}{3}}^{\delta}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{\sigma+2\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{3}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)\\
&=-\left[\frac{3}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\mathcal{F}(\delta)+\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\\
&\quad+\alpha^{\beta}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+2\delta}{3}}^{\alpha}\mathcal{F}(\delta)\right].
\end{aligned} \tag{19}$$

By inserting the equalities (17)–(19) into (16), we have

$$\begin{aligned}
&\int_{\sigma}^{\delta}\Phi_{\alpha}^{\beta}(x)\,d\mathcal{F}(x)\\
&=\alpha^{\beta}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}_{\frac{2\sigma+\delta}{3}}^{\alpha}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+2\delta}{3}}^{\alpha}\mathcal{F}(\delta)\right]\\
&\quad-\frac{1}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right].
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+_{+}^{\beta}\mathcal{J}_{\frac{2\sigma+\delta}{3}}^{\alpha}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+_{+}^{\beta}\mathcal{J}_{\frac{\sigma+2\delta}{3}}^{\alpha}\mathcal{F}(\delta)\right]\\
&\quad-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\\
&\quad=\frac{3^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\int_{\sigma}^{\delta}\Phi_{\alpha}^{\beta}(x)\,d\mathcal{F}(x).
\end{aligned}\tag{20}$$

If we take modules of equality (20), then we readily obtain

$$\begin{aligned}
&\left|\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+_{+}^{\beta}\mathcal{J}_{\frac{2\sigma+\delta}{3}}^{\alpha}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+_{+}^{\beta}\mathcal{J}_{\frac{\sigma+2\delta}{3}}^{\alpha}\mathcal{F}(\delta)\right]\right.\\
&\quad\left.-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\right|\\
&\quad\le\frac{3^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\left|\int_{\sigma}^{\delta}\Phi_{\alpha}^{\beta}(x)d\mathcal{F}(x)\right|.
\end{aligned}\tag{21}$$

It is well known that if $\mathcal{F},g:[\sigma,\delta]\to\mathbb{R}$ are such that g is continuous on $[\sigma,\delta]$ and $\mathcal{F}$ is of bounded variation on $[\sigma,\delta]$, then $\int_{\sigma}^{\delta}g(\mu)\,d\mathcal{F}(\mu)$ exists and

$$\left|\int_{\sigma}^{\delta}g(\mu)\,d\mathcal{F}(\mu)\right|\le\sup_{\mu\in[\sigma,\delta]}\left|g(\mu)\right|\bigvee_{\sigma}^{\delta}(\mathcal{F}).\tag{22}$$

By using the inequality (22), we obtain

$$\begin{aligned}
&\left|\int_{\sigma}^{\delta}\Phi_{\alpha}^{\beta}(x)\,d\mathcal{F}(x)\right|\\
&\le\left|\int_{\sigma}^{\frac{2\sigma+\delta}{3}}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-(x-\sigma)^{\alpha}\right)^{\beta}-\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)\right|\\
&\quad+\left|\int_{\frac{2\sigma+\delta}{3}}^{\frac{\sigma+2\delta}{3}}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{2\sigma+\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{1}{2}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)\right|\\
&\quad+\left|\int_{\frac{\sigma+2\delta}{3}}^{\delta}\left(\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{\sigma+2\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{3}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right)d\mathcal{F}(x)\right|\\
&\le\sup_{x\in[\sigma,\frac{2\sigma+\delta}{3}]}\left|\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-(x-\sigma)^{\alpha}\right)^{\beta}-\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right|\bigvee_{\sigma}^{\frac{2\sigma+\delta}{3}}(\mathcal{F})\\
&\quad+\sup_{x\in[\frac{2\sigma+\delta}{3},\frac{\sigma+2\delta}{3}]}\left|\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{2\sigma+\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{1}{2}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right|\bigvee_{\frac{2\sigma+\delta}{3}}^{\frac{\sigma+2\delta}{3}}(\mathcal{F})\\
&\quad+\sup_{x\in[\frac{\sigma+2\delta}{3},\delta]}\left|\left(\left(\frac{\delta-\sigma}{3}\right)^{\alpha}-\left(x-\frac{\sigma+2\delta}{3}\right)^{\alpha}\right)^{\beta}-\frac{3}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\right|\bigvee_{\frac{\sigma+2\delta}{3}}^{\delta}(\mathcal{F})
\end{aligned}$$

$$= \left[\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\bigvee_{\sigma}^{\frac{2\sigma+\delta}{3}}(\mathcal{F})+\frac{1}{2}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\bigvee_{\frac{2\sigma+\delta}{3}}^{\frac{\sigma+2\delta}{3}}(\mathcal{F})+\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\bigvee_{\frac{\sigma+2\delta}{3}}^{\delta}(\mathcal{F})\right]$$
$$\leq \frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\bigvee_{\sigma}^{\delta}(\mathcal{F}).$$

Then, similar to the foregoing process, we readily have

$$\left|\int_{\sigma}^{\delta}\Psi_{\alpha}(x)\,d\mathcal{F}(x)\right|\leq\frac{5}{8}\left(\frac{\delta-\sigma}{3}\right)^{\alpha\beta}\bigvee_{\sigma}^{\delta}(\mathcal{F}). \tag{23}$$

If we substitute the inequality (23) in (21), then the following inequality holds:

$$\left|\frac{3^{\alpha\beta-1}\alpha^{\beta}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\sigma}^{\alpha}\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}_{\frac{2\sigma+\delta}{3}}^{\alpha}\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+2\delta}{3}}^{\alpha}\mathcal{F}(\delta)\right]\right.$$
$$\left.-\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]\right|\leq\frac{5}{24}\bigvee_{\sigma}^{\delta}(\mathcal{F}),$$

which is the desired result of Theorem 4. □

Remark 8 If we assign $\alpha = 1$ in Theorem 4, then we obtain the following inequality

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\mu)\,d\mu\right|$$
$$\leq\frac{5}{24}\bigvee_{\sigma}^{\delta}(\mathcal{F}),$$

which is established by Alomari in [5].

6 Summary and concluding remarks

Several new versions of Newton-type inequalities are considered for the case of differentiable convex functions by using conformable fractional integrals. To be more precise, several Newton-type inequalities for differentiable convex functions are constructed by using the Hölder and power-mean inequalities. Furthermore, more results are presented by using special choices of the obtained inequalities. Finally, we establish some inequalities of conformable fractional Newton type for functions of bounded variation.

In the future work, the ideas and strategies for our results related to Newton-type inequalities by conformable fractional integrals may open up new ways for mathematicians in this area. Moreover, one can try to generalize our results by utilizing a different version of convex function classes or another type of fractional integral operators. Finally, one can obtain these type of inequalities by conformable fractional integrals for convex functions by using quantum calculus.

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