Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Inequalities for interval-valued Riemann diamond-alpha integrals

Martin Bohner[1], Linh Nguyen[2*], Baruch Schneider[3] and Tri Truong[3]

*Correspondence:
nguyenletoannhatlinh@tdtu.edu.vn
[2]Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Full list of author information is available at the end of the article

Abstract

We propose the concept of Riemann diamond-alpha integrals for time scales interval-valued functions. We first give the definition and some properties of the interval Riemann diamond-alpha integral that are naturally investigated as an extension of interval Riemann nabla and delta integrals. With the help of the interval Riemann diamond-alpha integral, we present interval variants of Jensen inequalities for convex and concave interval-valued functions on an arbitrary time scale. Moreover, diamond-alpha Hölder's and Minkowski's interval inequalities are proved. Also, several numerical examples are provided in order to illustrate our main results.

MSC: 26D15; 28B20; 34N05; 39A12

Keywords: Interval-valued functions; Time scales; Diamond-alpha integral; Jensen's inequalities; Hölder's inequalities; Minkowski's inequalities

1 Introduction

The description of deterministic real-world phenomena using mathematical models or computer models allows us to approach and study them effectively from a mathematical perspective. However, in some practical situations, modeled problems have appeared with uncertainties or vaguenesses due to uncertain data, imprecise measurement, etc. Depending on the characteristics of problems and the types of uncertainties, the corresponding modeled problems can be fuzzy, stochastic, or interval-valued. For instance, we can utilize the tools of interval analysis in situations where the values of the input data are uncertain, but we can determine or estimate the intervals to which these values belong. Interval analysis was pioneered by Ramon Moore (see [27]) in 1966. After that, there have been various studies on interval analysis in both theoretical and applied mathematics (see [11, 17, 23, 28, 34] and the references given in there). More recently, some well-known integral inequalities were generalized to the interval-valued case, such as Jensen's inequalities (see [12, 35]), Minkowski's inequality (see [30]), Chebyshev's inequality (see [36]), Opial's inequalities (see [14, 37]), and Wirtinger's inequalities (see [13]). These research works have provided fundamental tools used in mathematics as well as applied and engineering sciences.

Time scales were first introduced and studied by Stefan Hilger (1988) in his PhD thesis. This study constitutes a powerful and practical approach in attempting to unify standard concepts in discrete and continuous mathematics. Time scales theory was previously applied to numerous problems in applied and pure mathematics (see [5, 7, 16, 18, 22]). In 2006, Qin Sheng et al. (see [33]) introduced and studied a combined so-called diamond-alpha dynamic derivative as a linear combination of nabla and delta dynamic derivatives. Also, diamond-alpha integrals and their applications were studied in [4, 10, 24, 25]. In general, inequalities in classical calculus or time scales calculus play crucial roles in many areas of mathematical analysis. Hence, there have been numerous works in attempting to extend the classical inequalities to inequalities on time scales in some recent years, for instance, Jensen-type inequalities (see [3, 6, 26]), Ostrowski-type inequalities (see [21, 29]), and Hardy-type inequalities (see [2, 31]).

In recent decades, time scales calculus for interval or fuzzy contexts has been more and more attractive with various research works. First, using the so-called Hukuhara difference, Shihuang Hong proposed the Hukuhara–Hilger derivative of time scales multivalued functions to study multivalued dynamic equations on time scales (see [19]). Then, in order to offer tools for the study of interval dynamic equations, Vasile Lupulescu introduced generalized differentiability and Riemann delta integrability of dynamic interval-valued functions (see [23]). In 2019, Dafang Zhao et al. (see [38]) provided several time scales versions of interval integral inequalities. To do this, the authors proposed the concept of interval Darboux delta integral and interval Riemann delta integral for interval-valued functions. For further details about time scales calculus with uncertainties, we refer to [20, 32, 37] and references therein.

Motivated by the above observations, this paper aims to propose a new concept of interval Riemann diamond-α integral for a class of time scales interval-valued functions defined as a linear combination of interval Riemann nabla and delta integrals. With the help of this concept, we prove diamond-α Jensen's interval inequalities for convex and concave interval-valued functions. Also, interval versions of the diamond-α Hölder and Minkowski inequalities are presented. The set up of this paper is as follows. In Sect. 2, we first recall some basic properties from time scales calculus and interval analysis that will be used in the rest of the paper. The definition of interval Riemann diamond-α integrals for interval-valued functions and some of its essential properties are contained in Sect. 3. In Sect. 4, we present Jensen's interval inequalities, Hölder's interval inequality, and Minkowski's interval inequality for interval Riemann diamond-α integrals. Finally, in Sect. 5, several numerical examples are offered in order to illustrate our main findings.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{Z}$ the sets of real and integer numbers, respectively.

2.1 Time scales and diamond-α integrals

Definition 2.1 (See [9]) Any closed $\emptyset \neq \mathbb{T} \subset \mathbb{R}$ is called a *time scale.*

Definition 2.2 (See [9]) The so-called *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are given by

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} | s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Definition 2.3 (See [9]) A point $t \in \mathbb{T}$ is called *right-scattered*, *left-scattered*, *right-dense*, or *left-dense* if $\sigma(t) > t$, $\rho(t) < t$, $\sigma(t) = t$, or $\rho(t) = t$, respectively. A time scale is called *isolated* if all of its elements are both left-scattered and right-scattered.

In the sequel, we denote by $[a,b]_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$, and $(a,b)_{\mathbb{T}}$ the intersection with $\mathbb{T}$ of real intervals $[a,b]$, $[a,b)$, $(a,b]$, and (a,b), respectively.

Definition 2.4 (See [9]) Let $t \in \mathbb{T}$ and $\delta > 0$. A neighborhood of t is denoted by $U_{\mathbb{T}}(t,\delta)$ and defined by $U_{\mathbb{T}}(t,\delta) = (t-\delta, t+\delta)_{\mathbb{T}}$

Definition 2.5 Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$ and $a \in \mathbb{T}$. We say that $\{t_n\}_{n\in\mathbb{N}}$ convergences to a, denoted by $t_n \to a$, if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ with $t_n \in U_{\mathbb{T}}(a,\varepsilon)$ for all $n \geq N$.

Definition 2.6 (See [16]) $\phi : \mathbb{T} \to \mathbb{R}$ is called *ld-continuous* (or *rd-continuous*) if it is continuous in left-dense (right-dense) points in $\mathbb{T}$ and its right-sided (left-sided) limits exist as finite numbers in right-dense (left-dense) points of $\mathbb{T}$. The sets of all ld-continuous, rd-continuous, and continuous $\phi : \mathbb{T} \to \mathbb{R}$ are denoted by $\mathfrak{C}_{\mathrm{ld}}(\mathbb{T},\mathbb{R})$, $\mathfrak{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, and $\mathfrak{C}(\mathbb{T},\mathbb{R})$, respectively.

Definition 2.7 (See [24]) A *partition* of an interval $[a,b]_{\mathbb{T}}$ is an arbitrary, in increasing order arranged

$$P = \{a = t_0, t_1, \dots, t_n = b\} \subset [a,b]_{\mathbb{T}}.$$

The set of all of such partitions is denoted by $\mathcal{P} = \mathcal{P}((a,b)_{\mathbb{T}})$.

Lemma 2.1 (See [8]) *For each $\delta > 0$, there is $\{t_0,\dots,t_n\} = P \in \mathcal{P}$ with the property that, for all $i \in \{1,2,\dots,n\}$, either we have $t_i - t_{i-1} \leq \delta$ or otherwise $t_i - t_{i-1} > \delta$ and $\sigma(t_{i-1}) = t_i$.*

By $\mathcal{P}_\delta = \mathcal{P}_\delta((a,b)_{\mathbb{T}})$, we denote the collection of all partitions possessing the property described in Lemma 2.1.

Definition 2.8 (See [24]) Assume $0 \leq \alpha \leq 1$. Let $\phi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be bounded, and let $\{t_0,\dots,t_n\} = P \in \mathcal{P}$. For $1 \leq i \leq n$, we pick $\xi_i \in [t_{i-1},t_i)_{\mathbb{T}}$, $\eta_i \in (t_{i-1},t_i]_{\mathbb{T}}$, and put

$$S = \sum_{i=1}^{n} \big(\alpha\phi(\xi_i) + (1-\alpha)\phi(\eta_i)\big)(t_i - t_{i-1}).$$

The sum S is said to be a Riemann $\diamondsuit_\alpha$-sum of ϕ that corresponds to $P \in \mathcal{P}$. The function ϕ is called Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$ provided there is some $R \in \mathbb{R}$ so that for each $\varepsilon > 0$, there is $\delta > 0$ with the property that $P \in \mathcal{P}_\delta$ implies $|S - R| < \varepsilon$, independent of how ξ_i, η_i for $1 \leq i \leq n$ are chosen. Then, R is said to be the Riemann $\diamondsuit_\alpha$-integral of ϕ on $[a,b]$, and it is denoted by $\int_a^b \phi(s)\diamondsuit_\alpha s$.

The next result gives us a sufficient condition for the Riemann $\diamondsuit_\alpha$-integrability of a real-valued function on a time scale.

Theorem 2.1 *Let $a, b \in \mathbb{T}$, $\phi : \mathbb{T} \to \mathbb{R}$, and $0 \le \alpha \le 1$. Assume that ϕ is both Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$ and Riemann ∇-integrable on $(a,b]_{\mathbb{T}}$. Then, ϕ is Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b \phi(s)\Diamond_\alpha s = \alpha \int_a^b \phi(s)\Delta s + (1-\alpha)\int_a^b \phi(s)\nabla s.$$

Proof The proof can be found in [3]. □

Theorem 2.2 (See [24]) *Let $\phi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a real-valued function. If ϕ is continuous on $[a,b]_{\mathbb{T}}$, then ϕ is Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$.*

Proposition 2.1 (See [24]) *Assume $a, b, c \in \mathbb{T}$, $a < c < b$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and let $\phi, \psi : \mathbb{T} \to \mathbb{R}$ be continuous. Then, the following statements hold.*
- (i) $\int_a^b [\lambda_1\phi(s) + \lambda_2\psi(s)]\Diamond_\alpha s = \lambda_1 \int_a^b \phi(s)\Diamond_\alpha s + \lambda_2 \int_a^b \psi(s)\Diamond_\alpha s$.
- (ii) $\int_a^b \phi(s)\Diamond_\alpha s = \int_a^c \phi(s)\Diamond_\alpha s + \int_c^b \phi(s)\Diamond_\alpha s$.
- (iii) $\int_a^b \phi(s)\Diamond_\alpha s \ge 0$ *if* $\phi(s) \ge 0$ *for all* $s \in [a,b]_{\mathbb{T}}$.
- (iv) $\int_a^b \phi(s)\Diamond_\alpha s \ge \int_a^b \psi(s)\Diamond_\alpha s$ *if* $\phi(s) \ge \psi(s)$ *for all* $s \in [a,b]_{\mathbb{T}}$.

Proposition 2.2 (See [24]) *If $t \in \mathbb{T}$ and $\phi : \mathbb{T} \to \mathbb{R}$, then the following statements hold.*
- (i) *ϕ is Riemann $\Diamond_\alpha$-integrable on $[t, \sigma(t)]_{\mathbb{T}}$ and*

$$\int_t^{\sigma(t)} \phi(s)\Diamond_\alpha s = \mu(t)\bigl[\alpha\phi(t) + (1-\alpha)\phi\bigl(\sigma(t)\bigr)\bigr],$$

  where $\mu(t) := \sigma(t) - t$ for all $t \in \mathbb{T}$.
- (ii) *ϕ is Riemann $\Diamond_\alpha$-integrable on $[\rho(t), t]_{\mathbb{T}}$ and*

$$\int_{\rho(t)}^t \phi(s)\Diamond_\alpha s = \nu(t)\bigl[\alpha\phi\bigl(\rho(t)\bigr) + (1-\alpha)\phi(t)\bigr],$$

  where $\nu(t) := t - \rho(t)$ for all $t \in \mathbb{T}$.

For further details on the Riemann $\Diamond_\alpha$-integral of real-valued functions, we refer to [1, 3, 33] and the references therein.

2.2 Inequalities for diamond-α integrals

Next, we recall and prove the following results about diamond-α inequalities that play important roles in our analysis.

Proposition 2.3 (See [1, Theorem 2.2.5]) *Assume $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}}, (c,d))$ and $\phi \in \mathfrak{C}((c,d), \mathbb{R})$ is convex, then*

$$\phi\left(\frac{\int_a^b \varrho(s)\Diamond_\alpha s}{b-a}\right) \le \frac{1}{b-a}\int_a^b \phi\bigl(\varrho(s)\bigr)\Diamond_\alpha s. \tag{1}$$

If ϕ is strictly convex, then $\le$ in (1) may be replaced by $<$. If ϕ is concave, then $\le$ in (1) is reversed.

We call (1) the diamond-α Jensen inequality. The following proposition presents the extended diamond-α Jensen inequality.

Proposition 2.4 (See [1, Theorem 2.2.6]) *Assume $a,b\in\mathbb{T}$ and $c,d\in\mathbb{R}$. If $\varrho\in\mathfrak{C}([a,b]_{\mathbb{T}},(c,d))$, $\phi\in\mathfrak{C}((c,d),\mathbb{R})$ is convex, and $\theta\in\mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R})$ satisfies $\int_a^b|\theta(s)|\diamondsuit_\alpha s>0$, then*

$$\phi\left(\frac{\int_a^b|\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b|\theta(s)|\diamondsuit_\alpha s}\right)\le\frac{\int_a^b|\theta(s)|\phi(\varrho(s))\diamondsuit_\alpha s}{\int_a^b|\theta(s)|\diamondsuit_\alpha s}. \tag{2}$$

If ϕ is strictly convex, then $\le$ in (2) may be replaced by $<$. If ϕ is concave, then $\le$ in (2) is reversed.

Proposition 2.5 (See [1, Theorem 2.3.11]) *Let $a,b\in\mathbb{T}$. Assume that $\theta,\phi,\psi\in\mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R})$ such that $\int_a^b\theta(s)\psi^q(s)\diamondsuit_\alpha s>0$. If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then*

$$\int_a^b|\theta(s)||\phi(s)\psi(s)|\diamondsuit_\alpha s\le\left(\int_a^b|\theta(s)||\phi(s)|^p\diamondsuit_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b|\theta(s)||\psi(s)|^q\diamondsuit_\alpha s\right)^{\frac{1}{q}}. \tag{3}$$

The inequality (3) is known as a diamond-α Hölder inequality on time scales. For $\alpha=0$ and $\alpha=1$, the inequality (3) becomes the nabla and delta Hölder inequalities, respectively. In the particular case $\theta(s)=1$ for all $s\in[a,b]_{\mathbb{T}}$ and $\phi,\psi\in\mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R}^+)$, the inequality (3) of Proposition 2.5 can be rewritten as

$$\int_a^b\phi(s)\psi(s)\diamondsuit_\alpha s\le\left(\int_a^b\phi^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b\psi^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}.$$

The following proposition gives us the reversed diamond-α Hölder inequality on time scales.

Proposition 2.6 *Let $a,b\in\mathbb{T}$. Assume that $\phi,\psi\in\mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R}^+)$ such that $0<k\le\phi^p/\psi^q\le K<\infty$. If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then*

$$\left(\int_a^b\phi^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b\psi^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}\le\left(\frac{K}{k}\right)^{\frac{1}{pq}}\int_a^b\phi(s)\psi(s)\diamondsuit_\alpha s. \tag{4}$$

Proof For $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, from the fact $0<k\le\phi^p/\psi^q$, we obtain $\phi\ge k^{\frac{1}{p}}\psi^{\frac{q}{p}}$ for all $\phi,\psi\in\mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R}^+)$. By applying Proposition 2.1(iv), we obtain

$$\int_a^b\phi(s)\psi(s)\diamondsuit_\alpha s\ge k^{\frac{1}{p}}\int_a^b\psi(s)\psi^{\frac{q}{p}}(s)\diamondsuit_\alpha s=k^{\frac{1}{p}}\int_a^b\psi^q(s)\diamondsuit_\alpha s,$$

which implies that

$$\left(\int_a^b\phi(s)\psi(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}\ge k^{\frac{1}{pq}}\left(\int_a^b\psi^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}. \tag{5}$$

Similarly, since $\phi^p/\psi^q \leq K$, we have $\phi^p = \phi\phi^{\frac{p}{q}} \leq K^{\frac{1}{q}}\phi\psi$. It follows from Proposition 2.1(iv) that

$$\left(\int_a^b \phi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} \leq K^{\frac{1}{pq}}\left(\int_a^b \phi(s)\psi(s)\Diamond_\alpha s\right)^{\frac{1}{p}}. \tag{6}$$

Combining the inequalities (5) and (6), we therefore derive

$$\left(\int_a^b \phi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b \psi^q(s)\Diamond_\alpha s\right)^{\frac{1}{q}} \leq \left(\frac{K}{k}\right)^{\frac{1}{pq}}\int_a^b \phi(s)\psi(s)\Diamond_\alpha s.$$

The proof is finished. □

The next proposition provides the diamond-α Minkowski inequality.

Proposition 2.7 *If $a, b \in \mathbb{T}$, $\theta, \phi, \psi \in \mathfrak{C}([a,b]_{\mathbb{T}}, \mathbb{R})$, and $p > 1$, then*

$$\begin{aligned}
&\left(\int_a^b |\theta(s)||\phi(s)+\psi(s)|^p\Diamond_\alpha s\right)^{\frac{1}{p}} \\
&\quad\leq \left(\int_a^b |\theta(s)||\phi(s)|^p\Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b |\theta(s)||\psi(s)|^p\Diamond_\alpha s\right)^{\frac{1}{p}}.
\end{aligned} \tag{7}$$

Proof For $p > 1$, from the triangle inequality, we get

$$\begin{aligned}
\int_a^b |\theta(s)||\phi(s)+\psi(s)|^p\Diamond_\alpha s &= \int_a^b |\theta(s)||\phi(s)+\psi(s)||\phi(s)+\psi(s)|^{p-1}\Diamond_\alpha s \\
&\leq \int_a^b |\theta(s)||\phi(s)||\phi(s)+\psi(s)|^{p-1}\Diamond_\alpha s \\
&\quad + \int_a^b |\theta(s)||\psi(s)||\phi(s)+\psi(s)|^{p-1}\Diamond_\alpha s.
\end{aligned}$$

By Hölder's inequality (3) in Proposition 2.5, we get

$$\begin{aligned}
&\int_a^b |\theta(s)||\phi(s)||\phi(s)+\psi(s)|^{p-1}\Diamond_\alpha s \\
&\quad\leq \left(\int_a^b |\theta(s)||\phi(s)|^p\Diamond_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b |\theta(s)|\left(|\phi(s)+\psi(s)|^{p-1}\right)^{\frac{p}{p-1}}\Diamond_\alpha s\right)^{1-\frac{1}{p}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b |\theta(s)||\psi(s)||\phi(s)+\psi(s)|^{p-1}\Diamond_\alpha s \\
&\quad\leq \left(\int_a^b |\theta(s)||\psi(s)|^p\Diamond_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b |\theta(s)|\left(|\phi(s)+\psi(s)|^{p-1}\right)^{\frac{p}{p-1}}\Diamond_\alpha s\right)^{1-\frac{1}{p}}.
\end{aligned}$$

Therefore, we obtain

$$\int_a^b |\theta(s)||\phi(s)+\psi(s)|^p\Diamond_\alpha s$$

$$\leq \left(\int_a^b |\theta(s)| \left(|\phi(s)+\psi(s)|^{p-1}\right)^{\frac{p}{p-1}} \Diamond_\alpha s\right)^{1-\frac{1}{p}}$$
$$\times \left[\left(\int_a^b |\theta(s)||\phi(s)|^p \Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b |\theta(s)||\psi(s)|^p \Diamond_\alpha s\right)^{\frac{1}{p}}\right],$$

and hence we arrive at the inequality (7). The proof is finished. □

Proposition 2.8 *Let $a, b \in \mathbb{T}$. Assume that $\phi, \psi \in \mathfrak{C}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ such that $0 < k \leq \phi/\psi \leq K < \infty$. If $p > 1$, then*

$$\left(\int_a^b \phi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \psi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} \leq \left(\frac{K}{k}\right)^{\frac{p-1}{p^2}} \left(\int_a^b [\phi(s)+\psi(s)]^p \Diamond_\alpha s\right)^{\frac{1}{p}}. \tag{8}$$

Proof For $p > 1$, using Proposition 2.6, we have

$$\int_a^b \phi(s)[\phi(s)+\psi(s)]^{p-1} \Diamond_\alpha s$$
$$\geq \left(\frac{k}{K}\right)^{\frac{p-1}{p^2}} \left(\int_a^b \phi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} \left(\int_a^b \left([\phi(s)+\psi(s)]^{p-1}\right)^{\frac{p}{p-1}} \Diamond_\alpha s\right)^{1-\frac{1}{p}}$$

and

$$\int_a^b \psi(s)[\phi(s)+\psi(s)]^{p-1} \Diamond_\alpha s$$
$$\geq \left(\frac{k}{K}\right)^{\frac{p-1}{p^2}} \left(\int_a^b \psi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} \left(\int_a^b \left([\phi(s)+\psi(s)]^{p-1}\right)^{\frac{p}{p-1}} \Diamond_\alpha s\right)^{1-\frac{1}{p}}.$$

It follows that

$$\int_a^b [\phi(s)+\psi(s)]^p \Diamond_\alpha s$$
$$= \int_a^b \phi(s)[\phi(s)+\psi(s)]^{p-1} \Diamond_\alpha s + \int_a^b \psi(s)[\phi(s)+\psi(s)]^{p-1} \Diamond_\alpha s$$
$$\geq \left(\frac{k}{K}\right)^{\frac{p-1}{p^2}} \left(\int_a^b \left([\phi(s)+\psi(s)]^{p-1}\right)^{\frac{p}{p-1}} \Diamond_\alpha s\right)^{1-\frac{1}{p}}$$
$$\times \left[\left(\int_a^b \phi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \psi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}}\right].$$

Therefore,

$$\left(\int_a^b [\phi(s)+\psi(s)]^p \Diamond_\alpha s\right)^{\frac{1}{p}} \geq \left(\frac{k}{K}\right)^{\frac{p-1}{p^2}} \left[\left(\int_a^b \phi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \psi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}}\right].$$

The proof is finished. □

2.3 Interval arithmetic and interval-valued functions on time scales

Further, we denote by $\mathcal{I} = \{I = [\underline{I},\overline{I}] | \underline{I},\overline{I} \in \mathbb{R} \text{ and } \underline{I} \leq \overline{I}\}$ the class of all non-empty compact intervals of real numbers. The interval $I \in \mathcal{I}$ is said to be *positive* (or *negative*) if $\underline{I} > 0$ (or $\overline{I} < 0$). The set of all positive intervals and negative intervals are denoted by $\mathcal{I}^+$ and $\mathcal{I}^-$, respectively.

The next definition gives us some arithmetic operations in $\mathcal{I}$ used in the rest of this paper.

Definition 2.9 (See [27]) Let $I = [\underline{I},\overline{I}]$, $J = [\underline{J},\overline{J}] \in \mathcal{I}$ and $\lambda \in \mathbb{R}$. We define

(i) Addition: $I \oplus J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$.

(ii) Scalar multiplication:

$$\lambda \cdot I = \begin{cases} [\lambda\underline{I}, \lambda\overline{I}], & \text{if } \lambda \geq 0, \\ [\lambda\overline{I}, \lambda\underline{I}], & \text{if } \lambda < 0. \end{cases}$$

(iii) Multiplication: $I \cdot J = [\min\{\underline{I}\underline{J}, \underline{I}\overline{J}, \overline{I}\underline{J}, \overline{I}\overline{J}\}, \max\{\underline{I}\underline{J}, \underline{I}\overline{J}, \overline{I}\underline{J}, \overline{I}\overline{J}\}]$.

(iv) Power:

$$I^n = \begin{cases} [\underline{I}^n, \overline{I}^n], & \text{if } I \in \mathcal{I}^+ \text{ or } n \text{ is odd}, \\ [\underline{I}^n, \overline{I}^n], & \text{if } I \in \mathcal{I}^- \text{ or } n \text{ is even}, \\ [0, |I|^n], & \text{if } 0 \in I \text{ or } n \text{ is even}. \end{cases}$$

(v) Inclusion: $I \subseteq J$ if and only if $\underline{J} \leq \underline{I}$ and $\overline{I} \leq \overline{J}$.

Definition 2.10 (See [34]) Let $I = [\underline{I},\overline{I}]$, $J = [\underline{J},\overline{J}]$ be intervals in $\mathcal{I}$. The *generalized Hukuhara difference* (gH-difference for short) of I and J is defined by

$$I \ominus_{\mathrm{gH}} J = \left[\min\{\underline{I} - \underline{J}, \overline{I} - \overline{J}\}, \max\{\underline{I} - \underline{J}, \overline{I} - \overline{J}\}\right].$$

The gH-difference is also represented by the form

$$I \ominus_{\mathrm{gH}} J = \begin{cases} [\underline{I} - \underline{J}, \overline{I} - \overline{J}] & \text{if } \ell(I) \geq \ell(J), \\ [\overline{I} - \overline{J}, \underline{I} - \underline{J}] & \text{if } \ell(I) < \ell(J), \end{cases}$$

where $\ell(I) = \overline{I} - \underline{I}$ is said to be the length of $I = [\underline{I},\overline{I}] \in \mathcal{I}$.

In addition to the mentioned algebraic operations, the set of intervals $\mathcal{I}$ is also a complete metric space with the Hausdorff distance $\mathfrak{D}$ defined by $\mathfrak{D} : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+ \cup \{0\}$ with

$$\mathfrak{D}(I,J) = \max\left\{|\underline{I} - \underline{J}|, |\overline{I} - \overline{J}|\right\}.$$

We collect some well-known and important properties of the Hausdorff metric.

Proposition 2.9 (See [34]) *Let $I, J, M, N \in \mathcal{I}$ and $\lambda \in \mathbb{R}$. Then, the following assertions are true.*

(i) $\mathfrak{D}(I \oplus M, J \oplus M) = \mathfrak{D}(I,J)$,

(ii) $\mathfrak{D}(\lambda \cdot I, \lambda \cdot J) = |\lambda|\mathfrak{D}(I,J)$,
(iii) $\mathfrak{D}(I \oplus J, M \oplus N) \leq \mathfrak{D}(I,M) + \mathfrak{D}(J,N)$.

In order to establish continuity of time scales interval-valued functions, we now give the following definition.

Definition 2.11 Let $\Phi : \mathbb{T} \to \mathcal{I}$ be an interval-valued function on a time scale $\mathbb{T}$, and let $t_0 \in \mathbb{T}$. An interval Λ is called the $\mathbb{T}$-limit of Φ as t tends to t_0, denoted by $\lim_{t\to t_0} \Phi(t) = \Lambda$, if, for any $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{T} \setminus \{\sigma(t_0), \rho(t_0)\}$ with $t_n \to t_0$, we have $\lim_{n\to\infty} \mathfrak{D}(\Phi(t_n), \Lambda) = 0$.

Definition 2.12 Assume $\Phi : \mathbb{T} \to \mathcal{I}$ and $t_0 \in \mathbb{T}$. An interval Λ is called the *left-sided* (or *right-sided*) $\mathbb{T}$-limit of Φ as t tends to t_0, denoted by $\lim_{t\to t_0^-} \Phi(t) = \Lambda$ (or $\lim_{t\to t_0^+} \Phi(t) = \Lambda$), if, for any $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{T} \setminus \{\sigma(t_0), \rho(t_0)\}$, $t_n \leq t_0$ (or $t_n \geq t_0$) with $t_n \to t_0$, we have $\lim_{n\to\infty} \mathfrak{D}(\Phi(t_n), \Lambda) = 0$.

If $\Phi : \mathbb{T} \to \mathcal{I}$ and $\Lambda \in \mathcal{I}$, then clearly $\lim_{t\to t_0} \Phi(t) = \Lambda$ iff $\lim_{t\to t_0^-} \Phi(t) = \lim_{t\to t_0^+} \Phi(t) = \Lambda$.

Theorem 2.3 *Let $\Phi : \mathbb{T} \to \mathcal{I}$, and let $t_0 \in \mathbb{T}$. The limit of Φ as $t \to t_0$, if it exists, is unique.*

Proof The proof can be obtained easily from the definition of $\mathbb{T}$-limit and the properties of the distance $\mathfrak{D}$. □

Theorem 2.4 *Let $\Phi : \mathbb{T} \to \mathcal{I}$ be an interval-valued function such that $\Phi(t) = [\underline{\Phi}(t), \overline{\Phi}(t)]$ for all $t \in \mathbb{T}$, and let $t_0 \in \mathbb{T}$. Then, existence of $\lim_{t\to t_0} \Phi(t)$ implies existence of both limits $\lim_{t\to t_0} \underline{\Phi}(t)$ and $\lim_{t\to t_0} \overline{\Phi}(t)$. Moreover,*

$$\lim_{t\to t_0} \Phi(t) = \Big[\lim_{t\to t_0} \underline{\Phi}(t), \lim_{t\to t_0} \overline{\Phi}(t)\Big].$$

Proof Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{T} \setminus \{\sigma(t_0), \rho(t_0)\}$ be such that $t_n \to t_0$. Assuming $\lim_{t\to t_0} \Phi(t) = \Lambda = [\Lambda_1, \Lambda_2] \in \mathcal{I}$, we obtain $\lim_{n\to\infty} \mathfrak{D}(\Phi(t_n), \Lambda) = 0$. From the way the Hausdorff distance is defined, we get

$$\lim_{n\to\infty} \left|\underline{\Phi}(t_n) - \Lambda_1\right| = \lim_{n\to\infty} \left|\overline{\Phi}(t_n) - \Lambda_2\right| = 0.$$

Therefore, $\lim_{t\to t_0} \underline{\Phi}(t) = \Lambda_1$ and $\lim_{t\to t_0} \overline{\Phi}(t) = \Lambda_2$. This shows the result. □

Definition 2.13 An interval-valued function $\Phi : \mathbb{T} \to \mathcal{I}$ is called *continuous* at $t_0 \in \mathbb{T}$ if $\lim_{t\to t_0} \Phi(t)$ exists and $\lim_{t\to t_0} \Phi(t) = \Phi(t_0)$. The function Φ is said to be *ld-continuous* (or *rd-continuous*) if it is continuous at left-dense (or right-dense) points in $\mathbb{T}$ and its right-sided (or left-sided) limits exist (and are finite) at right-dense (or left-dense) points in $\mathbb{T}$. The set of all ld-continuous, rd-continuous, and continuous functions $\Phi : \mathbb{T} \to \mathcal{I}$ are denoted by $\mathfrak{C}_{\mathrm{ld}}(\mathbb{T}, \mathcal{I})$, $\mathfrak{C}_{\mathrm{rd}}(\mathbb{T}, \mathcal{I})$, and $\mathfrak{C}(\mathbb{T}, \mathcal{I})$, respectively.

The following remark gives us the relationship between the continuity, ld-continuity, and rd-continuity of an interval-valued function.

Remark 2.1 Let $[\underline{\Phi}, \overline{\Phi}] = \Phi : \mathbb{T} \to \mathcal{I}$ and $t_0 \in \mathbb{T}$. Then Φ is continuous (ld-continuous or rd-continuous) at t_0 iff the real-valued functions $\underline{\Phi}$ and $\overline{\Phi}$ are continuous (ld-continuous or rd-continuous) at t_0. In addition, from [9, Theorem 1.60], it follows that if the real-valued functions $\underline{\Phi}$ and $\overline{\Phi}$ are continuous, then they are both ld-continuous and rd-continuous. Hence, $\mathfrak{C}(\mathbb{T},\mathcal{I}) \subset \mathfrak{C}_{\mathrm{ld}}(\mathbb{T},\mathcal{I}) \cap \mathfrak{C}_{\mathrm{rd}}(\mathbb{T},\mathcal{I})$.

3 Interval Riemann diamond-α integral for interval-valued functions

Our principal goal in this section is to propose a new integral definition for time scales interval-valued functions, called interval Riemann diamond-α integral. Moreover, some essential characteristics of this integral also are investigated. In what follows, we always assume $\alpha \in [0,1]$ unless we explicitly state some exceptions. First, we start with the concepts of interval Riemann nabla and delta integrals.

Definition 3.1 Let $\Phi : (a,b]_{\mathbb{T}} \to \mathcal{I}$ be bounded and $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}$. Choosing arbitrary points $\eta_i \in (t_{i-1}, t_i]_{\mathbb{T}}$, for $1 \le i \le n$, the sum

$$RS_{\nabla} = \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot \Phi(\eta_i)$$

is said to be the interval Riemann ∇-sum of Φ corresponding to P. Then Φ is called *interval Riemann ∇-integrable* on the region $(a,b]_{\mathbb{T}}$ if there is $I_{\nabla} \in \mathcal{I}$ so that for every $\varepsilon > 0$, there is $\delta > 0$ satisfying

$$\mathfrak{D}(RS_{\nabla}, I_{\nabla}) < \varepsilon$$

for each interval Riemann ∇-sum of Φ that corresponds to $P \in \mathcal{P}_{\delta}$, independent of the way $\eta_i \in (t_{i-1}, t_i]_{\mathbb{T}}$, for $1 \le i \le n$, is chosen. I_{∇} is said to be the *interval Riemann ∇-integral* of Φ on $(a,b]_{\mathbb{T}}$, and we write $I_{\nabla} = \int_a^b \Phi(s)\nabla s$.

Analogously, the interval Riemann delta integral of a time scales interval-valued function can be defined as follows.

Definition 3.2 (See [23]) Let $\Phi : [a,b)_{\mathbb{T}} \to \mathcal{I}$ be bounded and $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}$. Choosing arbitrary points $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$, for $1 \le i \le n$, the sum

$$RS_{\Delta} = \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot \Phi(\xi_i)$$

is said to be the interval Riemann Δ-sum of Φ corresponding to P. Then Φ is called *interval Riemann Δ-integrable* on the region $[a,b)_{\mathbb{T}}$ if there is $I_{\Delta} \in \mathcal{I}$ so that for every $\varepsilon > 0$, there is $\delta > 0$ satisfying

$$\mathfrak{D}(RS_{\Delta}, I_{\Delta}) < \varepsilon$$

for each Riemann Δ-sum of Φ that corresponds to $P \in \mathcal{P}_{\delta}$, independent of the way $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$, for $1 \le i \le n$, is chosen. I_{Δ} is said to be the *interval Riemann Δ-integral* of Φ on $[a,b)_{\mathbb{T}}$, and we write $I_{\Delta} = \int_a^b \Phi(s)\Delta s$.

Theorem 3.1 *Let* $[\underline{\Phi},\overline{\Phi}] = \Phi : \mathbb{T} \to \mathcal{I}$. *Then,* Φ *is interval Riemann* ∇*-integrable on* $(a,b]_{\mathbb{T}}$ *iff* $\underline{\Phi}$ *and* $\overline{\Phi}$ *are Riemann* ∇*-integrable on* $(a,b]_{\mathbb{T}}$. *In addition,*

$$\int_a^b \Phi(s)\nabla s = \left[\int_a^b \underline{\Phi}(s)\nabla s, \int_a^b \overline{\Phi}(s)\nabla s\right].$$

Proof The proof can be obtained easily using the technique from the proof of [23, Theorem 6]. □

More details about the interval Riemann Δ-integral can be found in [23].

Definition 3.3 Let Φ be a bounded interval-valued function on $[a,b]_{\mathbb{T}}$, and let $\{t_0,\dots,t_n\} = P \in \mathcal{P}$. For $1 \le i \le n$, we pick $\xi_i \in [t_{i-1},t_i)_{\mathbb{T}}$, $\eta_i \in (t_{i-1},t_i]_{\mathbb{T}}$, and put

$$RS_{\Diamond_\alpha} = \sum_{i=1}^{n}(t_i - t_{i-1}) \cdot \left[\alpha \cdot \Phi(\xi_i) \oplus (1-\alpha) \cdot \Phi(\eta_i)\right].$$

The sum $RS_{\Diamond_\alpha}$ is called the interval Riemann $\Diamond_\alpha$-sum of Φ that corresponds to $P \in \mathcal{P}$. Then Φ is called *interval Riemann* $\Diamond_\alpha$*-integrable* on $[a,b]_{\mathbb{T}}$ if there is $I_{\Diamond_\alpha} \in \mathcal{I}$ so that for each $\varepsilon > 0$, there is $\delta > 0$ satisfying

$$\mathfrak{D}(RS_{\Diamond_\alpha}, I_{\Diamond_\alpha}) < \varepsilon$$

for each interval Riemann $\Diamond_\alpha$-sum of Φ that corresponds to $P \in \mathcal{P}_\delta$ independent of the way to pick $\xi_i \in [t_{i-1},t_i)_{\mathbb{T}}$ and $\eta_i \in (t_{i-1},t_i]_{\mathbb{T}}$, for $1 \le i \le n$. $I_{\Diamond_\alpha}$ is said to be the *interval Riemann* $\Diamond_\alpha$*-integral* of Φ from a to b, and we write $\int_a^b \Phi(s)\Diamond_\alpha s$.

The next theorem presents a sufficient condition for the interval Riemann $\Diamond_\alpha$-integrability of an interval-valued function.

Theorem 3.2 *If* $\Phi : \mathbb{T} \to \mathcal{I}$ *is interval Riemann* ∇*-integrable on* $(a,b]_{\mathbb{T}}$ *and interval Riemann* Δ*-integrable on* $[a,b)_{\mathbb{T}}$, *then* Φ *is interval Riemann* $\Diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$. *Moreover,*

$$\int_a^b \Phi(s)\Diamond_\alpha s = \alpha \cdot \int_a^b \Phi(s)\Delta s \oplus (1-\alpha) \cdot \int_a^b \Phi(s)\nabla s.$$

Proof The proof is classical and directly follows from Definitions 3.1, 3.2, and 3.3. □

Theorem 3.3 *Assume* $\Phi : \mathbb{T} \to \mathcal{I}$ *is such that* $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ *for all* $s \in \mathbb{T}$. *Then,* Φ *is interval Riemann* $\Diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$ *if* $\underline{\Phi}$ *and* $\overline{\Phi}$ *are Riemann* $\Diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$. *Moreover,*

$$\int_a^b \Phi(s)\Diamond_\alpha s = \left[\int_a^b \underline{\Phi}(s)\Diamond_\alpha s, \int_a^b \overline{\Phi}(s)\Diamond_\alpha s\right].$$

Proof The proof is classical and follows straightforward from Definition 3.3 and the definition of the Hausdorff distance. □

Corollary 3.1 *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$, $a < b$. Assume that $\Phi \in \mathfrak{C}([a,b]_{\mathbb{T}}, \mathcal{I})$ such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ for all $s \in \mathbb{T}$. Then, the following statements hold.*

(i) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b \Phi(s)\Diamond_\alpha s = \int_a^b \Phi(s)\Delta s = \int_a^b \Phi(s)\nabla s = \int_a^b \Phi(s)\mathrm{d}s,$$

where $\int_a^b \Phi(s)\mathrm{d}s$ is the classical interval Riemann integral and $\int_a^b \Phi(s)\mathrm{d}s = [\int_a^b \underline{\Phi}(s)\mathrm{d}s, \int_a^b \overline{\Phi}(s)\mathrm{d}s]$.

(ii) *If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then*

$$\int_a^b \Phi(s)\Diamond_\alpha s = \Bigg[h \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}-1} \underline{\Phi}(kh) + \alpha\underline{\Phi}(a)h + (1-\alpha)\underline{\Phi}(b)h, \\ h \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}-1} \overline{\Phi}(kh) + \alpha\overline{\Phi}(a)h + (1-\alpha)\overline{\Phi}(b)h \Bigg].$$

(iii) *If $\mathbb{T} = \{t_i | t_i < t_{i+1}\}$ for $i \in \mathbb{N}_0$, and $m < n$, then*

$$\int_{t_m}^{t_n} \Phi(s)\Diamond_\alpha s = \Bigg[\sum_{i=m}^{n-1} (t_{i+1} - t_i)\big[\alpha\underline{\Phi}(t_i) + (1-\alpha)\underline{\Phi}(t_{i+1})\big], \\ \sum_{i=m}^{n-1} (t_{i+1} - t_i)\big[\alpha\overline{\Phi}(t_i) + (1-\alpha)\overline{\Phi}(t_{i+1})\big] \Bigg].$$

Example 3.1 Let us consider $\Lambda \in \mathcal{I}$ and $\Phi : \mathbb{T} \to \mathcal{I}$ such that $\Phi(s) = \Lambda$ for all $s \in \mathbb{T}$. Then, we have $\int_a^b \Phi(s)\Diamond_\alpha s = (b-a) \cdot \Lambda$.

Example 3.2 Let $\mathbb{T} = h\mathbb{Z}$, $h \in (0,1)$, and let $\Phi : [0,2]_{h\mathbb{Z}} \to \mathcal{I}$ be given by $\Phi(s) = [s^2 + s, e^s + 1]$ for all $s \in [0,2]_{h\mathbb{Z}}$. Then, from Corollary 3.1(ii), we have

$$\begin{aligned} \int_0^2 \Phi(s)\Diamond_\alpha s &= \Bigg[h \sum_{k=1}^{\frac{2}{h}-1} \big(k^2h^2 + kh\big) + 6(1-\alpha)h, h \sum_{k=1}^{\frac{2}{h}-1} \big(e^{kh} + 1\big) + 2\alpha h + \big(e^2 + 1\big)(1-\alpha)h \Bigg] \\ &= \Bigg[\frac{1}{3}\big(14 + h^2 + 9h - 18\alpha h\big), 2 + \big(e^2 - 1\big)(1-\alpha)h + \frac{e^2 - 1}{e^h - 1}h \Bigg]. \end{aligned}$$

Theorem 3.4 *Let $\Phi : \mathbb{T} \to \mathcal{I}$ be such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ for all $s \in \mathbb{T}$. Then, for $t \in \mathbb{T}$, Φ is interval Riemann $\Diamond_\alpha$-integrable on $[\rho(t), t]_{\mathbb{T}}$, and*

$$\int_{\rho(t)}^t \Phi(s)\Diamond_\alpha s = \nu(t) \cdot \big(\alpha \cdot \Phi\big(\rho(t)\big) \oplus (1-\alpha) \cdot \Phi(t)\big).$$

Proof Suppose that $\Phi : \mathbb{T} \to \mathcal{I}$ is such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ for all $s \in \mathbb{T}$. Then, $\Phi(\rho(t)) = [\underline{\Phi}(\rho(t)), \overline{\Phi}(\rho(t))]$. According to Proposition 2.2, it follows that $\underline{\Phi}$ and $\overline{\Phi}$ are Riemann $\Diamond_\alpha$-integrable on $[\rho(t), t]_{\mathbb{T}}$ and

$$\int_{\rho(t)}^t \underline{\Phi}(s)\Diamond_\alpha s = \nu(t)\big(\alpha\underline{\Phi}\big(\rho(t)\big) + (1-\alpha)\underline{\Phi}(t)\big)$$

and

$$\int_{\rho(t)}^{t} \overline{\Phi}(s)\Diamond_\alpha s = \nu(t)\big(\alpha\overline{\Phi}\big(\rho(t)\big) + (1-\alpha)\overline{\Phi}(t)\big).$$

From Theorem 3.3, we obtain

$$\begin{aligned}\int_{\rho(t)}^{t} \Phi(s)\Diamond_\alpha s &= \left[\int_{\rho(t)}^{t} \underline{\Phi}(s)\Diamond_\alpha s, \int_{\rho(t)}^{t} \overline{\Phi}(s)\Diamond_\alpha s\right] \\ &= \big[\nu(t)\big(\alpha\underline{\Phi}\big(\rho(t)\big) + (1-\alpha)\underline{\Phi}(t)\big), \nu(t)\big(\alpha\overline{\Phi}\big(\rho(t)\big) + (1-\alpha)\overline{\Phi}(t)\big)\big] \\ &= \nu(t)\cdot\big(\alpha\cdot\big[\underline{\Phi}\big(\rho(t)\big), \overline{\Phi}\big(\rho(t)\big)\big] \oplus (1-\alpha)\cdot\big[\underline{\Phi}(t), \overline{\Phi}(t)\big]\big),\end{aligned}$$

and hence $\int_{\rho(t)}^{t} \Phi(s)\Diamond_\alpha s = \nu(t)\cdot(\alpha\cdot\Phi(\rho(t)) \oplus (1-\alpha)\cdot\Phi(t))$. The proof is finished. □

The following theorem provides us the linearity of the interval Riemann diamond-α integral with addition and scalar product.

Theorem 3.5 *Assume that $\Phi, \Psi : \mathbb{T} \to \mathcal{I}$ are interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. Then, the following assertions hold.*

(i) *$\lambda\cdot\Phi$ is interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b \lambda\cdot\Phi(s)\Diamond_\alpha s = \lambda\cdot\int_a^b \Phi(s)\Diamond_\alpha s.$$

(ii) *$\Phi \oplus \Psi$ is interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b \big[\Phi(s) \oplus \Psi(s)\big]\Diamond_\alpha s = \int_a^b \Phi(s)\Diamond_\alpha s \oplus \int_a^b \Psi(s)\Diamond_\alpha s.$$

Proof The proof is classical and immediately follows from Definition 3.3. □

Theorem 3.6 *If $\Phi, \Psi : \mathbb{T} \to \mathcal{I}$ are interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$, then $\Phi \ominus_{\mathrm{gH}} \Psi$ is interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$ and, moreover,*

$$\int_a^b \Phi(s)\Diamond_\alpha s \ominus_{\mathrm{gH}} \int_a^b \Psi(s)\Diamond_\alpha s \subseteq \int_a^b \big[\Phi(s) \ominus_{\mathrm{gH}} \Psi(s)\big]\Diamond_\alpha s.$$

Proof By Theorem 3.3, $\Phi \ominus_{\mathrm{gH}} \Psi$ is interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. Put $\underline{\Upsilon} = \underline{\Phi} - \underline{\Psi}$ and $\overline{\Upsilon} = \overline{\Phi} - \overline{\Psi}$. From the inequalities

$$\begin{aligned}\int_a^b \min\big\{\underline{\Upsilon}(s), \overline{\Upsilon}(s)\big\}\Diamond_\alpha s &\le \min\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\} \\ &\le \max\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\} \\ &\le \int_a^b \max\big\{\underline{\Upsilon}(s), \overline{\Upsilon}(s)\big\}\Diamond_\alpha s,\end{aligned}$$

it follows that

$$\begin{aligned}&\left[\min\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\}, \max\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\}\right]\\ &\quad\subseteq \left[\int_a^b \min\{\underline{\Upsilon}(s), \overline{\Upsilon}(s)\}\Diamond_\alpha s, \int_a^b \max\{\underline{\Upsilon}(s), \overline{\Upsilon}(s)\}\Diamond_\alpha s\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int_a^b \Phi(s)\Diamond_\alpha s \ominus_{\mathrm{gH}} \int_a^b \Psi(s)\Diamond_\alpha s\\ &\quad= \left[\int_a^b \underline{\Phi}(s)\Diamond_\alpha s, \int_a^b \overline{\Phi}(s)\Diamond_\alpha s\right] \ominus_{\mathrm{gH}} \left[\int_a^b \underline{\Psi}(s)\Diamond_\alpha s, \int_a^b \overline{\Psi}(s)\Diamond_\alpha s\right]\\ &\quad= \left[\min\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\}, \max\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\}\right]\\ &\quad\subseteq \left[\int_a^b \min\{\underline{\Upsilon}(s), \overline{\Upsilon}(s)\}\Diamond_\alpha s, \int_a^b \max\{\underline{\Upsilon}(s), \overline{\Upsilon}(s)\}\Diamond_\alpha s\right]\\ &\quad= \int_a^b \left[\Phi(s) \ominus_{\mathrm{gH}} \Psi(s)\right]\Diamond_\alpha s.\end{aligned}$$

The proof is finished. □

Remark 3.1 With the assumptions as in Theorem 3.6 and adding the condition that $\ell(\Phi(s)) - \ell(\Psi(s))$ possesses a constant sign on $[a,b]_{\mathbb{T}}$, we have

$$\int_a^b \Phi(s)\Diamond_\alpha s \ominus_{\mathrm{gH}} \int_a^b \Psi(s)\Diamond_\alpha s = \int_a^b \left[\Phi(s) \ominus_{\mathrm{gH}} \Psi(s)\right]\Diamond_\alpha s. \tag{9}$$

Indeed, if $\ell(\Phi(s)) - \ell(\Psi(s)) \geq 0$ on $[a,b]_{\mathbb{T}}$, then $\underline{\Phi} - \underline{\Psi} \leq \overline{\Phi} - \overline{\Psi}$ on $[a,b]_{\mathbb{T}}$ and $\Phi \ominus_{\mathrm{gH}} \Psi = [\underline{\Upsilon}, \overline{\Upsilon}]$. Thus,

$$\int_a^b \left[\underline{\Phi}(s) - \underline{\Psi}(s)\right]\Diamond_\alpha s \leq \int_a^b \left[\overline{\Phi}(s) - \overline{\Psi}(s)\right]\Diamond_\alpha s.$$

Hence, we obtain

$$\begin{aligned}&\int_a^b \Phi(s)\Diamond_\alpha s \ominus_{\mathrm{gH}} \int_a^b \Psi(s)\Diamond_\alpha s\\ &\quad= \left[\int_a^b \underline{\Phi}(s)\Diamond_\alpha s, \int_a^b \overline{\Phi}(s)\Diamond_\alpha s\right] \ominus_{\mathrm{gH}} \left[\int_a^b \underline{\Psi}(s)\Diamond_\alpha s, \int_a^b \overline{\Psi}(s)\Diamond_\alpha s\right]\\ &\quad= \left[\min\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\}, \max\left\{\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right\}\right]\\ &\quad= \left[\int_a^b \underline{\Upsilon}(s)\Diamond_\alpha s, \int_a^b \overline{\Upsilon}(s)\Diamond_\alpha s\right]\\ &\quad= \int_a^b \left[\Phi(s) \ominus_{\mathrm{gH}} \Psi(s)\right]\Diamond_\alpha s,\end{aligned}$$

so (9) holds. For the second case, i.e., $\ell(\Phi(s)) - \ell(\Psi(s)) < 0$ on $[a,b]_{\mathbb{T}}$, using a similar argument, we can also obtain equality (9).

To illustrate Remark 3.1, let us consider the following example.

Example 3.3 Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $\Phi, \Psi : \mathbb{T} \to \mathcal{I}$ be such that

$$\Phi(s) = \left[s, s^2 + 1\right] \quad \text{and} \quad \Psi(s) = [1,2] \cdot s \quad \text{for all } s \in \mathbb{T}.$$

We remark that $\ell(\Phi(s)) - \ell(\Psi(s)) = (s-1)^2$ has a constant sign on $[1,4]_{\mathbb{T}}$. We have

$$\Phi(s) \ominus_{\mathrm{gH}} \Psi(s) = \left[0, s^2 - 2s + 1\right] \quad \text{for all } s \in [1,4]_{\mathbb{T}}$$

and

$$\int_1^4 \left[\Phi(s) \ominus_{\mathrm{gH}} \Psi(s)\right] \Diamond_\alpha s = \int_1^4 \left[0, s^2 - 2s + 1\right] \Diamond_\alpha s = [0, 19 - 17\alpha].$$

Moreover, we have

$$\int_1^4 \Phi(s) \Diamond_\alpha s = \int_1^4 \left[s, s^2 + 1\right] \Diamond_\alpha s = [10 - 5\alpha, 39 - 27\alpha]$$

and

$$\int_1^4 \Psi(s) \Diamond_\alpha s = \int_1^4 [1,2] \cdot s \Diamond_\alpha s = [1,2] \cdot (10 - 5\alpha).$$

Since

$$\ell\left(\int_1^4 \Phi(s) \Diamond_\alpha s\right) = 29 - 22\alpha > 10 - 5\alpha = \ell\left(\int_1^4 \Psi(s) \Diamond_\alpha s\right)$$

for all $\alpha \in [0,1]$, we obtain

$$\int_1^4 \Phi(s) \Diamond_\alpha s \ominus_{\mathrm{gH}} \int_1^4 \Psi(s) \Diamond_\alpha s = [0, 19 - 17\alpha],$$

and hence

$$\int_1^4 \Phi(s) \Diamond_\alpha s \ominus_{\mathrm{gH}} \int_1^4 \Psi(s) \Diamond_\alpha s = \int_1^4 \left[\Phi(s) \ominus_{\mathrm{gH}} \Psi(s)\right] \Diamond_\alpha s.$$

Theorem 3.7 *Suppose $\Phi, \Psi : \mathbb{T} \to \mathcal{I}$ satisfy $\Phi \subseteq \Psi$ on $[a,b]_{\mathbb{T}}$. If Φ and Ψ are interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b \Phi(s) \Diamond_\alpha s \subseteq \int_a^b \Psi(s) \Diamond_\alpha s.$$

Proof Suppose that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ and $\Psi(s) = [\underline{\Psi}(s), \overline{\Psi}(s)]$ for all $s \in \mathbb{T}$. From the assumption $\Phi \subseteq \Psi$ on $[a,b]_{\mathbb{T}}$, Definition 2.9 gives

$$\underline{\Psi}(s) \leq \underline{\Phi}(s) \quad \text{and} \quad \overline{\Phi}(s) \leq \overline{\Psi}(s) \quad \text{for all } s \in [a,b]_{\mathbb{T}}.$$

This yields

$$\int_a^b \underline{\Psi}(s)\Diamond_\alpha s \leq \int_a^b \underline{\Phi}(s)\Diamond_\alpha s \quad \text{and} \quad \int_a^b \overline{\Phi}(s)\Diamond_\alpha s \leq \int_a^b \overline{\Psi}(s)\Diamond_\alpha s.$$

Thus, $[\int_a^b \underline{\Phi}(s)\Diamond_\alpha s, \int_a^b \overline{\Phi}(s)\Diamond_\alpha s] \subseteq [\int_a^b \underline{\Psi}(s)\Diamond_\alpha s, \int_a^b \overline{\Psi}(s)\Diamond_\alpha s]$. By Theorem 3.3, we get

$$\int_a^b \Phi(s)\Diamond_\alpha s \subseteq \int_a^b \Psi(s)\Diamond_\alpha s.$$

This finishes the proof. □

The next example illustrates Theorem 3.7. All numbers in the following example are rounded to three decimal digits.

Example 3.4 Let $h = 0.02$. We consider interval-valued functions $\Phi, \Psi : [1,2]_{h\mathbb{Z}} \to \mathcal{I}$ such that $\Phi(s) = (s-1)\cdot[1,2]$ and $\Psi(s) = [1-2s+s^2, 2\sqrt{s-1}]$ for all $s \in [1,2]_{h\mathbb{Z}}$. It is easy to see that $1-2s+s^2 \leq s-1$ and $2s-2 \leq 2\sqrt{s-1}$ for all $s \in [1,2]_{h\mathbb{Z}}$. Therefore, by Definition 2.9(v), we get $\Phi(s) \subseteq \Psi(s)$ for all $s \in [1,2]_{h\mathbb{Z}}$. Moreover, by Corollary 3.1(ii), we have

$$\begin{aligned}
\int_1^2 \Phi(s)\Diamond_\alpha s &= \int_1^2 [s-1, 2s-2]\Diamond_\alpha s \\
&= \left[\sum_{k=\frac{1}{h}+1}^{\frac{2}{h}-1} \left(kh^2 - h\right) + (1-\alpha)h, \sum_{k=\frac{1}{h}+1}^{\frac{2}{h}-1} \left(2kh^2 - 2h\right) + 2(1-\alpha)h\right] \\
&= \left[0.49 + 0.02(1-\alpha), 0.98 + 0.04(1-\alpha)\right]
\end{aligned}$$

and

$$\begin{aligned}
\int_1^2 \Psi(s)\Diamond_\alpha s &= \int_1^2 \left[1-2s+s^2, 2\sqrt{s-1}\right]\Diamond_\alpha s \\
&= \left[\sum_{k=\frac{1}{h}+1}^{\frac{2}{h}-1} \left(h - 2kh^2 + k^2h^3\right) + (1-\alpha)h, 2h\sum_{k=\frac{1}{h}+1}^{\frac{2}{h}-1} \sqrt{kh-1} + 2(1-\alpha)h\right] \\
&= \left[0.323 + 0.02(1-\alpha), 1.312 + 0.04(1-\alpha)\right].
\end{aligned}$$

It is obvious that

$$\left[0.49 + 0.02(1-\alpha), 0.98 + 0.04(1-\alpha)\right] \subseteq \left[0.323 + 0.02(1-\alpha), 1.312 + 0.04(1-\alpha)\right]$$

for all $\alpha \in [0,1]$. Thus, we can conclude that

$$\int_1^2 \Phi(s)\Diamond_\alpha s \subseteq \int_1^2 \Psi(s)\Diamond_\alpha s.$$

4 Diamond-α inequalities for interval-valued functions

In this section, we prove some Jensen-type inequalities for the class of convex and concave time scales interval-valued functions by using the interval Riemann diamond-α integral. Moreover, diamond-α Hölder-type and Minkowski-type inequalities for interval-valued functions are proved. To obtain the interval versions of the diamond-α Jensen inequalities, we need the concepts of convexity and concavity of interval-valued functions on time scales as follows.

A real-valued function $\phi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be *convex* on $[a,b]_{\mathbb{T}}$ if for all $s,t \in [a,b]_{\mathbb{T}}$, $\lambda \in [0,1]$ such that $\lambda s + (1-\lambda)t \in [a,b]_{\mathbb{T}}$, we have

$$\phi\big(\lambda s + (1-\lambda)t\big) \le \lambda\phi(s) + (1-\lambda)\phi(t),$$

while $\phi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is called *concave* on $[a,b]_{\mathbb{T}}$ if

$$\phi\big(\lambda s + (1-\lambda)t\big) \ge \lambda\phi(s) + (1-\lambda)\phi(t)$$

for all $s,t \in [a,b]_{\mathbb{T}}$, $\lambda \in [0,1]$ such that $\lambda s + (1-\lambda)t \in [a,b]_{\mathbb{T}}$. Note that if a real-valued function $\phi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is convex (concave) on $[a,b]_{\mathbb{T}}$, then it is continuous on $(a,b)_{\mathbb{T}}$ (see [15]).

Definition 4.1 (See [38]) An $\Phi : [a,b]_{\mathbb{T}} \to \mathcal{I}$ is said to be $\mathcal{I}$-*convex* on $[a,b]_{\mathbb{T}}$ if

$$\lambda \cdot \Phi(s) \oplus (1-\lambda) \cdot \Phi(t) \subseteq \Phi\big(\lambda s + (1-\lambda)t\big)$$

for all $s,t \in [a,b]_{\mathbb{T}}$, $\lambda \in [0,1]$ such that $\lambda s + (1-\lambda)t \in [a,b]_{\mathbb{T}}$.

Definition 4.2 (See [38]) An $\Phi : [a,b]_{\mathbb{T}} \to \mathcal{I}$ is said to be $\mathcal{I}$-*concave* on $[a,b]_{\mathbb{T}}$ if

$$\Phi\big(\lambda s + (1-\lambda)t\big) \subseteq \lambda \cdot \Phi(s) \oplus (1-\lambda) \cdot \Phi(t)$$

for all $s,t \in [a,b]_{\mathbb{T}}$, $\lambda \in [0,1]$ such that $\lambda s + (1-\lambda)t \in [a,b]_{\mathbb{T}}$.

The following theorem gives us the connection between $\mathcal{I}$-convexity (concavity) of an interval-valued function and classical convexity (concavity) of corresponding real-valued functions on time scales.

Theorem 4.1 (See [38]) *Let $[\underline{\Phi}, \overline{\Phi}] = \Phi : [a,b]_{\mathbb{T}} \to \mathcal{I}$. Then, the following assertions hold.*

(i) *Φ is $\mathcal{I}$-convex on $[a,b]_{\mathbb{T}}$ if and only if $\underline{\Phi}$ is convex on $[a,b]_{\mathbb{T}}$ and $\overline{\Phi}$ is concave on $[a,b]_{\mathbb{T}}$.*

(ii) *Φ is $\mathcal{I}$-concave on $[a,b]_{\mathbb{T}}$ if and only if $\underline{\Phi}$ is concave on $[a,b]_{\mathbb{T}}$ and $\overline{\Phi}$ is convex on $[a,b]_{\mathbb{T}}$.*

Theorem 4.2 *Let $\Phi : [a,b]_{\mathbb{T}} \to \mathcal{I}$ be such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ for all $s \in [a,b]_{\mathbb{T}}$. Assume that Φ is $\mathcal{I}$-convex or $\mathcal{I}$-concave on $[a,b]_{\mathbb{T}}$. Then, Φ is interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$.*

Proof Suppose that $[\underline{\Phi}, \overline{\Phi}] = \Phi : [a,b]_{\mathbb{T}} \to \mathcal{I}$ and Φ is $\mathcal{I}$-convex or $\mathcal{I}$-concave on $[a,b]_{\mathbb{T}}$. Then, Theorem 4.1 implies that $\underline{\Phi}$ and $\overline{\Phi}$ are continuous. According to Theorem 2.2 and Theorem 3.3, we conclude the interval Riemann diamond-α integrability of Φ on $[a,b]_{\mathbb{T}}$. □

Now, we present the diamond-α Jensen interval inequality on time scales via interval Riemann $\Diamond_\alpha$-integral.

Theorem 4.3 *Let $a,b \in \mathbb{T}$ and $c,d \in \mathbb{R}$. Assume that $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}}, (c,d))$ and $\Phi \in \mathfrak{C}((c,d), \mathcal{I})$ such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$. If Φ is an $\mathcal{I}$-convex interval-valued function, then*

$$\frac{1}{b-a} \cdot \int_a^b \Phi\big(\varrho(s)\big)\Diamond_\alpha s \subseteq \Phi\left(\frac{\int_a^b \varrho(s)\Diamond_\alpha s}{b-a}\right). \tag{10}$$

Proof Suppose that $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}}, (c,d))$ and $\Phi \in \mathfrak{C}((c,d), \mathcal{I})$ such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$. Then, it follows that $\Phi \circ \varrho \in \mathfrak{C}([a,b]_{\mathbb{T}}, \mathcal{I})$, and hence $\Phi \circ \varrho$ is interval Riemann $\Diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. From Theorem 3.3, it follows that

$$\int_a^b \Phi\big(\varrho(s)\big)\Diamond_\alpha s = \left[\int_a^b \underline{\Phi}\big(\varrho(s)\big)\Diamond_\alpha s, \int_a^b \overline{\Phi}\big(\varrho(s)\big)\Diamond_\alpha s\right].$$

Since Φ is $\mathcal{I}$-convex, from Theorem 4.1, we get that $\underline{\Phi}$ is convex, and $\overline{\Phi}$ is concave. According to Proposition 2.3, we have

$$\underline{\Phi}\left(\frac{\int_a^b \varrho(s)\Diamond_\alpha s}{b-a}\right) \le \frac{1}{b-a} \int_a^b \underline{\Phi}\big(\varrho(s)\big)\Diamond_\alpha s$$

and

$$\frac{1}{b-a} \int_a^b \overline{\Phi}\big(\varrho(s)\big)\Diamond_\alpha s \le \overline{\Phi}\left(\frac{\int_a^b \varrho(s)\Diamond_\alpha s}{b-a}\right).$$

Therefore, it follows from Definition 2.9(v) that

$$\begin{aligned}&\left[\frac{1}{b-a} \int_a^b \underline{\Phi}\big(\varrho(s)\big)\Diamond_\alpha s, \frac{1}{b-a} \int_a^b \overline{\Phi}\big(\varrho(s)\big)\Diamond_\alpha s\right]\\ &\quad\subseteq \left[\underline{\Phi}\left(\frac{\int_a^b \varrho(s)\Diamond_\alpha s}{b-a}\right), \overline{\Phi}\left(\frac{\int_a^b \varrho(s)\Diamond_\alpha s}{b-a}\right)\right]\end{aligned}$$

holds, i.e.,

$$\frac{1}{b-a} \cdot \int_a^b \Phi\big(\varrho(s)\big)\Diamond_\alpha s \subseteq \Phi\left(\frac{\int_a^b \varrho(s)\Diamond_\alpha s}{b-a}\right).$$

The proof is finished. □

In the following theorem, we present the generalized diamond-α Jensen interval inequality on time scales.

Theorem 4.4 *Assume $a,b \in \mathbb{T}$ and $c,d \in \mathbb{R}$. Assume $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}},(c,d))$, $\Phi \in \mathfrak{C}((c,d),\mathcal{I})$ is so that $\Phi(s) = [\underline{\Phi}(s),\overline{\Phi}(s)]$, and $\theta \in \mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R})$ satisfies*

$$\int_a^b |\theta(s)| \diamondsuit_\alpha s > 0.$$

If Φ is an $\mathcal{I}$-convex interval-valued function, then

$$\frac{\int_a^b |\theta(s)|\Phi(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s} \subseteq \Phi\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right). \tag{11}$$

Proof Suppose that $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}},(c,d))$ and $\Phi \in \mathfrak{C}((c,d),\mathcal{I})$ such that $\Phi(s) = [\underline{\Phi}(s),\overline{\Phi}(s)]$. Then, it follows that $\Phi \circ \varrho \in \mathfrak{C}([a,b]_{\mathbb{T}},\mathcal{I})$, and hence $\underline{\Phi} \circ \varrho$ and $\overline{\Phi} \circ \varrho$ are Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. Since $\theta \in \mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R})$, θ is Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$, which yields that $|\theta|(\underline{\Phi} \circ \varrho)$ and $|\theta|(\overline{\Phi} \circ \varrho)$ are Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. By Theorem 3.3, we get that $|\theta|(\Phi \circ \varrho)$ is interval Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$ and

$$\int_a^b |\theta(s)|\Phi(\varrho(s))\diamondsuit_\alpha s = \left[\int_a^b |\theta(s)|\underline{\Phi}(\varrho(s))\diamondsuit_\alpha s, \int_a^b |\theta(s)|\overline{\Phi}(\varrho(s))\diamondsuit_\alpha s\right].$$

Since Φ is $\mathcal{I}$-convex, it implies from Theorem 4.1 that $\underline{\Phi}$ is convex and $\overline{\Phi}$ is concave. According to Proposition 2.4, we obtain

$$\underline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right) \le \frac{\int_a^b |\theta(s)|\underline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}$$

and

$$\frac{\int_a^b |\theta(s)|\overline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s} \le \overline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right).$$

From Definition 2.9(v), we derive

$$\begin{aligned}
&\left[\frac{\int_a^b |\theta(s)|\underline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}, \frac{\int_a^b |\theta(s)|\overline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right] \\
&\quad \subseteq \left[\underline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right), \overline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right)\right],
\end{aligned}$$

i.e.,

$$\frac{\int_a^b |\theta(s)|\Phi(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s} \subseteq \Phi\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right).$$

The proof is finished. □

Remark 4.1 Note that if $\alpha = 0$, then the result in Theorem 4.4 provides us a nabla version for the generalized Jensen interval inequality. Moreover, if $\alpha = 1$, then we obtain a result similar to the one proved in [38, Theorem 9].

Next we present the reversed diamond-α Jensen interval inequality.

Theorem 4.5 *Let $a,b \in \mathbb{T}$ and $c,d \in \mathbb{R}$. Assume that $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}},(c,d))$, $\Phi \in \mathfrak{C}((c,d),\mathcal{I})$ such that $\Phi(s) = [\underline{\Phi}(s),\overline{\Phi}(s)]$, and $\theta \in \mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R})$ such that*

$$\int_a^b |\theta(s)| \diamondsuit_\alpha s > 0.$$

If Φ is an $\mathcal{I}$-concave interval-valued function, then

$$\Phi\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right) \supseteq \frac{\int_a^b |\theta(s)|\Phi(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}. \tag{12}$$

Proof Suppose that $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}},(c,d))$ and $\Phi \in \mathfrak{C}((c,d),\mathcal{I})$ such that $\Phi(s) = [\underline{\Phi}(s),\overline{\Phi}(s)]$. Then, it follows that $\Phi \circ \varrho \in \mathfrak{C}([a,b]_{\mathbb{T}},\mathcal{I})$, and hence $\underline{\Phi} \circ \varrho$ and $\overline{\Phi} \circ \varrho$ are Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. Since $\theta \in \mathfrak{C}([a,b]_{\mathbb{T}},\mathbb{R})$, θ is Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$, which yields that $|\theta|(\underline{\Phi} \circ \varrho)$ and $|\theta|(\overline{\Phi} \circ \varrho)$ are Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. By Theorem 3.3, this implies that $|\theta|(\Phi \circ \varrho)$ is interval Riemann $\diamondsuit_\alpha$-integrable on $[a,b]_{\mathbb{T}}$ and

$$\int_a^b |\theta(s)|\Phi(\varrho(s))\diamondsuit_\alpha s = \left[\int_a^b |\theta(s)|\underline{\Phi}(\varrho(s))\diamondsuit_\alpha s, \int_a^b |\theta(s)|\overline{\Phi}(\varrho(s))\diamondsuit_\alpha s\right].$$

Since Φ is $\mathcal{I}$-concave, Theorem 4.1 implies that $\underline{\Phi}$ is concave and $\overline{\Phi}$ is convex. According to Proposition 2.4, we obtain

$$\frac{\int_a^b |\theta(s)|\underline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s} \leq \underline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right)$$

and

$$\overline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right) \leq \frac{\int_a^b |\theta(s)|\overline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}.$$

From Definition 2.9(v), we derive

$$\begin{aligned}&\left[\underline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right), \overline{\Phi}\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right)\right] \\ &\quad\supseteq \left[\frac{\int_a^b |\theta(s)|\underline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}, \frac{\int_a^b |\theta(s)|\overline{\Phi}(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right],\end{aligned}$$

i.e.,

$$\Phi\left(\frac{\int_a^b |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}\right) \supseteq \frac{\int_a^b |\theta(s)|\Phi(\varrho(s))\diamondsuit_\alpha s}{\int_a^b |\theta(s)|\diamondsuit_\alpha s}.$$

The proof is finished. $\square$

Corollary 4.1 *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Assume that $\varrho \in \mathfrak{C}([a,b]_{\mathbb{T}}, (c,d))$, $\Phi \in \mathfrak{C}((c,d), \mathcal{I})$ such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$, and $\theta \in \mathfrak{C}([a,b]_{\mathbb{T}}, \mathbb{R})$ such that*

$$\int_a^b \left|\theta(s)\right| \Diamond_\alpha s > 0.$$

If Φ is both $\mathcal{I}$-convex and $\mathcal{I}$-concave, then

$$\Phi\left(\frac{\int_a^b |\theta(s)| \varrho(s) \Diamond_\alpha s}{\int_a^b |\theta(s)| \Diamond_\alpha s}\right) = \frac{\int_a^b |\theta(s)| \Phi(\varrho(s)) \Diamond_\alpha s}{\int_a^b |\theta(s)| \Diamond_\alpha s}. \tag{13}$$

Proof The proof follows directly from combining Theorem 4.4 and Theorem 4.5. □

The following result gives us a version of the diamond-α Hölder interval inequality.

Theorem 4.6 *Let $a, b \in \mathbb{T}$. Assume that $\Phi, \Psi \in \mathfrak{C}([a,b]_{\mathbb{T}}, \mathcal{I}^+)$ are such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ and $\Psi(s) = [\underline{\Psi}(s), \overline{\Psi}(s)]$ for all $s \in [a,b]_{\mathbb{T}}$. If $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ and $0 < k \leq \underline{\Phi}^p / \underline{\Psi}^q \leq K < \infty$, then*

$$\int_a^b \Phi(s) \cdot \Psi(s) \Diamond_\alpha s \subseteq \left[\left(\frac{k}{K}\right)^{\frac{1}{pq}}, 1\right] \cdot \left(\int_a^b \Phi^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} \cdot \left(\int_a^b \Psi^q(s) \Diamond_\alpha s\right)^{\frac{1}{q}}. \tag{14}$$

Proof From Definition 2.9(iii), we have

$$\Phi(s) \cdot \Psi(s) = [\min \mathcal{S}, \max \mathcal{S}], \tag{15}$$

where $\mathcal{S} = \{\underline{\Phi}(s)\underline{\Psi}(s), \underline{\Phi}(s)\overline{\Psi}(s), \overline{\Phi}(s)\underline{\Psi}(s), \overline{\Phi}(s)\overline{\Psi}(s)\}$. Since Φ and Ψ are positive interval-valued functions, it follows from (15) that

$$\Phi(s) \cdot \Psi(s) = \left[\underline{\Phi}(s)\underline{\Psi}(s), \overline{\Phi}(s)\overline{\Psi}(s)\right].$$

Therefore, we have

$$\int_a^b \Phi(s) \cdot \Psi(s) \Diamond_\alpha s = \left[\int_a^b \underline{\Phi}(s)\underline{\Psi}(s) \Diamond_\alpha s, \int_a^b \overline{\Phi}(s)\overline{\Psi}(s) \Diamond_\alpha s\right].$$

From the fact that $\underline{\Phi}$ and $\underline{\Psi}$ are positive and for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < k \leq \underline{\Phi}^p / \underline{\Psi}^q \leq K < \infty$, we derive from Proposition 2.6 that

$$\left(\int_a^b \underline{\Phi}^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} \left(\int_a^b \underline{\Psi}^q(s) \Diamond_\alpha s\right)^{\frac{1}{q}} \leq \left(\frac{K}{k}\right)^{\frac{1}{pq}} \int_a^b \underline{\Phi}(s)\underline{\Psi}(s) \Diamond_\alpha s. \tag{16}$$

Applying Proposition 2.5 for the two positive real-valued functions $\overline{\Phi}(s)$, $\overline{\Psi}(s)$, we obtain

$$\int_a^b \overline{\Phi}(s)\overline{\Psi}(s) \Diamond_\alpha s \leq \left(\int_a^b \overline{\Phi}^p(s) \Diamond_\alpha s\right)^{\frac{1}{p}} \left(\int_a^b \overline{\Psi}^q(s) \Diamond_\alpha s\right)^{\frac{1}{q}}. \tag{17}$$

Combining (17) and (16) with Definition 2.9(v), we get

$$\begin{aligned}
&\left[\int_a^b \underline{\Phi}(s)\underline{\Psi}(s)\diamondsuit_\alpha s, \int_a^b \overline{\Phi}(s)\overline{\Psi}(s)\diamondsuit_\alpha s\right] \\
&\subseteq \Bigg[\left(\frac{k}{K}\right)^{\frac{1}{pq}}\left(\int_a^b \underline{\Phi}^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b \underline{\Psi}^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}, \\
&\quad \left(\int_a^b \overline{\Phi}^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b \overline{\Psi}^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}\Bigg] \\
&= \left[\left(\frac{k}{K}\right)^{\frac{1}{pq}}, 1\right] \\
&\quad \cdot \left[\left(\int_a^b \underline{\Phi}^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b \underline{\Psi}^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}, \left(\int_a^b \overline{\Phi}^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}\left(\int_a^b \overline{\Psi}^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}\right] \\
&= \left[\left(\frac{k}{K}\right)^{\frac{1}{pq}}, 1\right] \cdot \left[\left(\int_a^b \underline{\Phi}^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}, \left(\int_a^b \overline{\Phi}^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}}\right] \\
&\quad \cdot \left[\left(\int_a^b \underline{\Psi}^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}, \left(\int_a^b \overline{\Psi}^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}\right],
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_a^b \Phi(s)\cdot\Psi(s)\diamondsuit_\alpha s \\
&\subseteq \left[\left(\frac{k}{K}\right)^{\frac{1}{pq}}, 1\right] \cdot \left[\int_a^b \underline{\Phi}^p(s)\diamondsuit_\alpha s, \int_a^b \overline{\Phi}^p(s)\diamondsuit_\alpha s\right]^{\frac{1}{p}} \cdot \left[\int_a^b \underline{\Psi}^q(s)\diamondsuit_\alpha s, \int_a^b \overline{\Psi}^q(s)\diamondsuit_\alpha s\right]^{\frac{1}{q}}.
\end{aligned}$$

On the other hand, from Definition 2.9(iv), it follows that

$$\Phi^p(s) = \left[\underline{\Phi}^p(s), \overline{\Phi}^p(s)\right] \quad \text{and} \quad \Psi^q(s) = \left[\underline{\Psi}^q(s), \overline{\Psi}^q(s)\right].$$

Therefore,

$$\int_a^b \Phi(s)\cdot\Psi(s)\diamondsuit_\alpha s \subseteq \left[\left(\frac{k}{K}\right)^{\frac{1}{pq}}, 1\right] \cdot \left(\int_a^b \Phi^p(s)\diamondsuit_\alpha s\right)^{\frac{1}{p}} \cdot \left(\int_a^b \Psi^q(s)\diamondsuit_\alpha s\right)^{\frac{1}{q}}.$$

The proof is finished. □

If the assumptions in Theorem 4.6 hold for the special case $p = q = 2$, then we get the diamond-α Cauchy–Schwarz interval inequality on time scales.

Corollary 4.2 *Let $a, b \in \mathbb{T}$. Assume that $\Phi, \Psi \in \mathfrak{C}([a,b]_\mathbb{T}, \mathcal{I}^+)$ are such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ and $\Psi(s) = [\underline{\Psi}(s), \overline{\Psi}(s)]$ for all $s \in [a,b]_\mathbb{T}$. Then, we have*

$$\int_a^b \Phi(s)\cdot\Psi(s)\diamondsuit_\alpha s \subseteq \left[\left(\frac{k}{K}\right)^{\frac{1}{4}}, 1\right] \cdot \left(\int_a^b \Phi^2(s)\diamondsuit_\alpha s\right)^{\frac{1}{2}} \cdot \left(\int_a^b \Psi^2(s)\diamondsuit_\alpha s\right)^{\frac{1}{2}},$$

where $\sqrt{k} \le \underline{\Phi}/\underline{\Psi} \le \sqrt{K} < \infty$.

The last theorem in this section is to represent the Minkowski interval inequality via Riemann $\Diamond_\alpha$-integral. The theorem is stated as follows.

Theorem 4.7 *Let $a,b \in \mathbb{T}$. Assume that $\Phi,\Psi \in \mathfrak{C}([a,b]_{\mathbb{T}},\mathcal{I}^+)$ are such that $\Phi(s) = [\underline{\Phi}(s),\overline{\Phi}(s)]$ and $\Psi(s) = [\underline{\Psi}(s),\overline{\Psi}(s)]$ for all $s \in [a,b]_{\mathbb{T}}$ and $0 < k \leq \overline{\Phi}/\overline{\Psi} \leq K < \infty$. If $p > 1$, then*

$$\begin{aligned}&\left(\int_a^b \Phi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} \oplus \left(\int_a^b \Psi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}}\\ &\quad\subseteq \left[1,\left(\frac{K}{k}\right)^{\frac{p-1}{p^2}}\right]\cdot\left(\int_a^b \big(\Phi(s)\oplus\Psi(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}}.\end{aligned} \tag{18}$$

Proof Since $\Phi = [\underline{\Phi},\overline{\Phi}]$ and $\Psi = [\underline{\Psi},\overline{\Psi}]$, it follows from Definition 2.9(iv) that

$$\big(\Phi(s)\oplus\Psi(s)\big)^p = \big[\big(\underline{\Phi}(s)+\underline{\Psi}(s)\big)^p,\big(\overline{\Phi}(s)+\overline{\Psi}(s)\big)^p\big].$$

According to Proposition 2.7, we have

$$\left(\int_a^b \big(\underline{\Phi}(s)+\underline{\Psi}(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}} \leq \left(\int_a^b \underline{\Phi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \underline{\Psi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}}. \tag{19}$$

For $p > 1$ with $0 < k \leq \overline{\Phi}/\overline{\Psi} \leq K < \infty$, Proposition 2.8 yields

$$\left(\int_a^b \overline{\Phi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \overline{\Psi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} \leq \left(\frac{K}{k}\right)^{\frac{p-1}{p^2}}\left(\int_a^b \big(\overline{\Phi}(s)+\overline{\Psi}(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}}. \tag{20}$$

Moreover, we have

$$\begin{aligned}&\left(\int_a^b \Phi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} \oplus \left(\int_a^b \Psi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}}\\ &\quad= \left[\int_a^b \underline{\Phi}^p(s)\Diamond_\alpha s,\int_a^b \overline{\Phi}^p(s)\Diamond_\alpha s\right]^{\frac{1}{p}} \oplus \left[\int_a^b \underline{\Psi}^p(s)\Diamond_\alpha s,\int_a^b \overline{\Psi}^p(s)\Diamond_\alpha s\right]^{\frac{1}{p}}\\ &\quad= \left[\left(\int_a^b \underline{\Phi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \underline{\Psi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}},\left(\int_a^b \overline{\Phi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \overline{\Psi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}}\right].\end{aligned}$$

Combining Definition 2.9(v) with (20) and (19), we derive

$$\begin{aligned}&\left[\left(\int_a^b \underline{\Phi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \underline{\Psi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}},\left(\int_a^b \overline{\Phi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} + \left(\int_a^b \overline{\Psi}^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}}\right]\\ &\quad\subseteq \left[\left(\int_a^b \big(\underline{\Phi}(s)+\underline{\Psi}(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}},\left(\frac{K}{k}\right)^{\frac{p-1}{p^2}}\left(\int_a^b \big(\overline{\Phi}(s)+\overline{\Psi}(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}}\right]\\ &\quad= \left[1,\left(\frac{K}{k}\right)^{\frac{p-1}{p^2}}\right]\cdot\left[\left(\int_a^b \big(\underline{\Phi}(s)+\underline{\Psi}(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}},\left(\int_a^b \big(\overline{\Phi}(s)+\overline{\Psi}(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}}\right]\\ &\quad= \left[1,\left(\frac{K}{k}\right)^{\frac{p-1}{p^2}}\right]\cdot\left(\int_a^b \big(\Phi(s)\oplus\Psi(s)\big)^p\Diamond_\alpha s\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\left(\int_a^b \Phi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} \oplus \left(\int_a^b \Psi^p(s)\Diamond_\alpha s\right)^{\frac{1}{p}} \subseteq \left[1, \left(\frac{K}{k}\right)^{\frac{p-1}{p^2}}\right] \cdot \left(\int_a^b \left(\Phi(s) \oplus \Psi(s)\right)^p \Diamond_\alpha s\right)^{\frac{1}{p}}.$$

The proof is finished. □

5 Illustrative computations

In this section, we analyze some examples to illustrate the main results presented in Sect. 4. First, we consider an example to illustrate Theorem 4.5 for the class of $\mathcal{I}$-concave interval-valued functions. Note that all numbers in this section are rounded to three decimal digits.

Example 5.1 Let $h = 0.01$ and $\varrho, \theta : [0,1]_{h\mathbb{Z}} \to \mathbb{R}$ be real-valued functions given by $\varrho(s) = \ln(s+1)$ and $\theta(s) = 1 + 2s^2$. We consider $\Phi : [0, \ln 2] \to \mathcal{I}$ such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$, where $\underline{\Phi}(s) = \cos s$ and $\overline{\Phi}(s) = e^s$ (Fig. 1a). Since $\underline{\Phi}$ is concave and $\overline{\Phi}$ is convex for all $s \in [0, \ln 2]$, from Theorem 4.1, it follows that Φ is $\mathcal{I}$-concave on $[0, \ln 2]$. According to Theorem 4.5, we have

$$\Phi\left(\frac{\int_0^1 |\theta(s)|\varrho(s)\Diamond_\alpha s}{\int_0^1 |\theta(s)|\Diamond_\alpha s}\right) \subseteq \frac{\int_0^1 |\theta(s)|\Phi(\varrho(s))\Diamond_\alpha s}{\int_0^1 |\theta(s)|\Diamond_\alpha s}. \tag{21}$$

Now we compute the integrals

$$\int_0^1 \left|\theta(s)\right|\Diamond_\alpha s = \int_0^1 \left(1 + 2s^2\right)\Diamond_\alpha s = \sum_{k=1}^{99} h\left(2k^2h^2 + 1\right) + \alpha h + 3(1-\alpha)h = 1.677 - 0.02\alpha$$

and

$$\begin{aligned}\int_0^1 \left|\theta(s)\right|\varrho(s)\Diamond_\alpha s &= \int_0^1 \left(1 + 2s^2\right)\ln(s+1)\Diamond_\alpha s \\ &= \sum_{k=1}^{99} h\left(1 + 2k^2h^2\right)\ln(kh+1) + 3h(1-\alpha)\ln 2 = 0.765 - 0.021\alpha.\end{aligned}$$

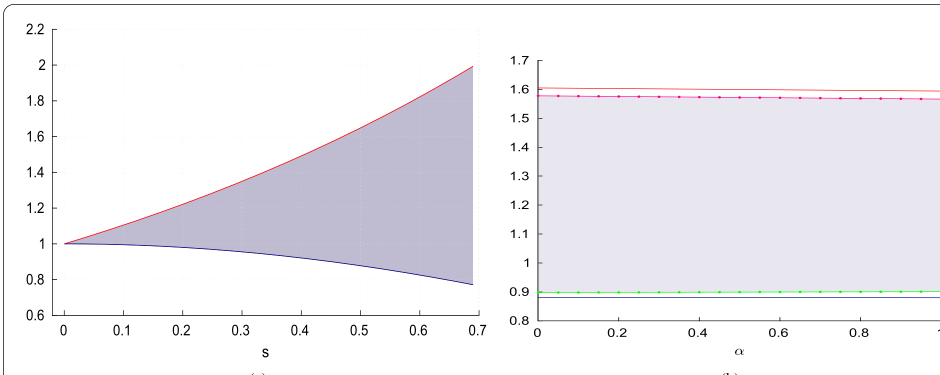

Figure 1 (**a**) Graph of $\Phi(s)$ (blue and red curves represent $\underline{\Phi}$ and $\overline{\Phi}$, respectively). (**b**) Illustration of Example 5.1 with different values of α (blue and red curves represent $\underline{\mathrm{RJ}}^\alpha$ and $\overline{\mathrm{RJ}}^\alpha$, respectively, and green and pink dotted curves are $\underline{\mathrm{LJ}}^\alpha$ and $\overline{\mathrm{LJ}}^\alpha$, respectively)

Therefore, we obtain

$$\Phi\left(\frac{\int_0^1 |\theta(s)|\varrho(s)\Diamond_\alpha s}{\int_a^b |\theta(s)|\Diamond_\alpha s}\right) = \left[\cos\left(\frac{0.765 - 0.021\alpha}{1.677 - 0.02\alpha}\right), \exp\left(\frac{0.765 - 0.021\alpha}{1.677 - 0.02\alpha}\right)\right].$$

Moreover,

$$\begin{aligned}\int_0^1 \left|\theta(s)\right|\Phi\bigl(\varrho(s)\bigr)\Diamond_\alpha s &= \int_0^1 \bigl(1 + 2s^2\bigr)\bigl[\cos\bigl(\ln(s+1)\bigr), s+1\bigr]\Diamond_\alpha s \\ &= [1.478 - 0.020\alpha, 2.692 - 0.05\alpha].\end{aligned}$$

Thus,

$$\frac{\int_0^1 |\theta(s)|\Phi(\varrho(s))\Diamond_\alpha s}{\int_0^1 |\theta(s)|\Diamond_\alpha s} = \left[\frac{1.478 - 0.02\alpha}{1.677 - 0.02\alpha}, \frac{2.692 - 0.05\alpha}{1.677 - 0.02\alpha}\right].$$

It is easy to see that

$$\begin{aligned}&\left[\cos\left(\frac{0.765 - 0.021\alpha}{1.677 - 0.02\alpha}\right), \exp\left(\frac{0.765 - 0.021\alpha}{1.677 - 0.02\alpha}\right)\right] \\ &\quad\subseteq \left[\frac{1.478 - 0.02\alpha}{1.677 - 0.02\alpha}, \frac{2.692 - 0.05\alpha}{1.677 - 0.02\alpha}\right]\end{aligned}$$

for all $\alpha \in [0,1]$. Hence inequality (21) holds. For brevity, we denote the values of the expressions on the left-hand side of (21) and the right-hand side of (21) corresponding to each $\alpha \in [0,1]$ by LJ^α and RJ^α, respectively. Figure 1b shows inequality (21) with different values of $\alpha \in [0,1]$.

The following example illustrates Corollary 4.1.

Example 5.2 Let $h > 0$ and $\varrho, \theta : [0,1]_{h\mathbb{Z}} \to \mathbb{R}$ be real-valued functions given by $\varrho(s) = s^3$ and $\theta(s) = 1 + 2s^2$. We consider an interval-valued function $\Phi : [0,1] \to \mathcal{I}$ given by $\Phi(s) = [2 - s, s + 3]$ for all $s \in [0,1]$. It is clear that Φ is both $\mathcal{I}$-convex and $\mathcal{I}$-concave on $[0,1]$. According to Corollary 4.1, we have

$$\frac{\int_0^1 |\theta(s)|\Phi(\varrho(s))\Diamond_\alpha s}{\int_0^1 |\theta(s)|\Diamond_\alpha s} = \Phi\left(\frac{\int_0^1 |\theta(s)|\varrho(s)\Diamond_\alpha s}{\int_0^1 |\theta(s)|\Diamond_\alpha s}\right).$$

Indeed, for all $\alpha \in [0,1]$, we have

$$\int_0^1 \left|\theta(s)\right|\Diamond_\alpha s = \int_0^1 \bigl(1 + 2s^2\bigr)\Diamond_\alpha s = \frac{1}{3}\bigl(-6h\alpha + h^2 + 3h + 5\bigr)$$

and

$$\int_0^1 \left|\theta(s)\right|\varrho(s)\Diamond_\alpha s = \int_0^1 s^3\bigl(1 + 2s^2\bigr)\Diamond_\alpha s = -\frac{1}{12}(h-1)^2\bigl(2h^2 + 4h - 7\bigr) + 3h(1-\alpha).$$

It follows that

$$\begin{aligned}
&\Phi\left(\frac{\int_0^1 |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_0^1 |\theta(s)|\diamondsuit_\alpha s}\right)\\
&\quad = \Phi\left(\frac{36h\alpha + 2h^4 - 13h^2 - 18h - 7}{4(6h\alpha - h^2 - 3h - 5)}\right)\\
&\quad = \left[\frac{12h\alpha - 2h^4 + 5h^2 - 6h - 33}{4(6h\alpha - h^2 - 3h - 5)}, \frac{108h\alpha + 2h^4 - 25h^2 - 54h - 67}{4(6h\alpha - h^2 - 3h - 5)}\right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^1 \bigl|\theta(s)\bigr|\Phi\bigl(\varrho(s)\bigr)\diamondsuit_\alpha s\\
&\quad = \int_0^1 \bigl(1 + 2s^2\bigr)\bigl[2 - s^3, s^3 + 3\bigr]\diamondsuit_\alpha s\\
&\quad = \frac{1}{12}\bigl[-12h\alpha + 2h^4 - 5h^2 + 6h + 33, -108h\alpha - 2h^4 + 25h^2 + 54h + 67\bigr]\\
&\quad = \left[\frac{1}{12}\bigl(2h^4 - 5h^2 - 30h + 33\bigr) + 2h\alpha + 3h(1 - \alpha),\right.\\
&\qquad \left.\frac{1}{12}\bigl(-2h^4 + 25h^2 - 90h + 67\bigr) + 3h\alpha + 12h(1 - \alpha)\right],
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{\int_0^1 |\theta(s)|\Phi(\varrho(s))\diamondsuit_\alpha s}{\int_0^1 |\theta(s)|\diamondsuit_\alpha s}\\
&\quad = \left[\frac{12h\alpha - 2h^4 + 5h^2 - 6h - 33}{4(6h\alpha - h^2 - 3h - 5)}, \frac{108h\alpha + 2h^4 - 25h^2 - 54h - 67}{4(6h\alpha - h^2 - 3h - 5)}\right]\\
&\quad = \Phi\left(\frac{\int_0^1 |\theta(s)|\varrho(s)\diamondsuit_\alpha s}{\int_0^1 |\theta(s)|\diamondsuit_\alpha s}\right).
\end{aligned}$$

The last two examples are presented to illustrate Theorem 4.6 and Theorem 4.7, respectively.

Example 5.3 Let $\alpha = 0.5$ and $\mathbb{T} = \{t_0, t_1, \ldots, t_8\} = \{1, \frac{21}{20}, \frac{10}{9}, \frac{56}{45}, \frac{3}{2}, \frac{79}{45}, \frac{17}{9}, \frac{39}{20}, 2\}$. Assume that $\Phi, \Psi \in \mathfrak{C}([1,2]_\mathbb{T}, \mathcal{I}^+)$ are such that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ and $\Psi(s) = [\underline{\Psi}(s), \overline{\Psi}(s)]$, where $\underline{\Phi}(s) = s$, $\overline{\Phi}(s) = s + 2$, $\underline{\Psi}(s) = \sqrt{s}$, $\overline{\Psi}(s) = \exp(\frac{s}{5}) + s$ for all $s \in \mathbb{T}$. It is clear that $1 \leq \underline{\Phi}^p/\underline{\Psi}^q \leq \sqrt[4]{2^9}$, with $p = 3$ and $\frac{1}{p} + \frac{1}{q} = 1$. According to Theorem 4.6, we have

$$\int_1^2 \Phi(s) \cdot \Psi(s)\diamondsuit_\alpha s \subseteq \left[\frac{1}{\sqrt{2}}, 1\right] \cdot \left(\int_1^2 \Phi^3(s)\diamondsuit_\alpha s\right)^{\frac{1}{3}} \cdot \left(\int_1^2 \Psi^{\frac{3}{2}}(s)\diamondsuit_\alpha s\right)^{\frac{2}{3}}. \tag{22}$$

Indeed, we have

$$\int_1^2 \Phi(s) \cdot \Psi(s)\diamondsuit_\alpha s = \int_1^2 \bigl[\underline{\Phi}(s)\underline{\Psi}(s), \overline{\Phi}(s)\overline{\Psi}(s)\bigr]\diamondsuit_\alpha s$$

$$= \left[\sum_{i=0}^{7} (s_{i+1} - s_i)\big[\alpha \underline{\Phi}(s_i)\underline{\Psi}(s_i) + (1-\alpha)\underline{\Phi}(s_{i+1})\underline{\Psi}(s_{i+1})\big], \right.$$

$$\left. \sum_{i=0}^{7} (s_{i+1} - s_i)\big[\alpha \overline{\Phi}(s_i)\overline{\Psi}(s_i) + (1-\alpha)\overline{\Phi}(s_{i+1})\overline{\Psi}(s_{i+1})\big] \right]$$

$$= [2.028 - 0.327\alpha, 10.749 - 1.304\alpha]$$

and

$$\int_1^2 \Phi^3(s)\Diamond_\alpha s = \int_1^2 \big[\underline{\Phi}^3(s), \overline{\Phi}^3(s)\big]\Diamond_\alpha s$$
$$= [4.396 - 1.233\alpha, 47.114 - 6.592\alpha].$$

Analogously, we obtain $\int_1^2 \Psi^{\frac{3}{2}}(s)\Diamond_\alpha s = [1.411 - 0.122\alpha, 5.137 - 0.575\alpha]$. Then, we have

$$\left(\int_1^2 \Phi^3(s)\Diamond_\alpha s\right)^{\frac{1}{3}} = \big[\sqrt[3]{4.396 - 1.233\alpha}, \sqrt[3]{47.114 - 6.592\alpha}\big]$$

and

$$\left(\int_1^2 \Psi^{\frac{3}{2}}(s)\Diamond_\alpha s\right)^{\frac{2}{3}} = \big[\sqrt[3]{(1.411 - 0.122\alpha)^2}, \sqrt[3]{(5.137 - 0.575\alpha)^2}\big].$$

Therefore, we have

$$\int_1^2 \Phi(s)\cdot\Psi(s)\Diamond_\alpha s = [2.028 - 0.327\alpha, 10.749 - 1.304\alpha]$$

$$\subseteq \left[\sqrt[3]{\frac{1}{\sqrt{2}}(4.396 - 1.233\alpha)(1.411 - 0.122\alpha)^2}, \right.$$

$$\left. \sqrt[3]{(47.114 - 6.592\alpha)(5.137 - 0.575\alpha)^2}\right]$$

$$= \left[\frac{1}{\sqrt{2}}, 1\right]\cdot\left(\int_1^2 \Phi^3(s)\Diamond_\alpha s\right)^{\frac{1}{3}}\cdot\left(\int_1^2 \Psi^{\frac{3}{2}}(s)\Diamond_\alpha s\right)^{\frac{2}{3}}.$$

We also denote the values of the expressions on the left-hand side of (22) and the right-hand side of (22) corresponding to each $\alpha \in [0,1]$ by LH^α and RH^α, respectively. Then, inequality (22) with different values of $\alpha \in [0,1]$ is shown in Fig. 2.

Example 5.4 Let $h = 0.02$ and $\Phi, \Psi : [0,2]_{h\mathbb{Z}} \to \mathcal{I}^+$ be so that $\Phi(s) = [\underline{\Phi}(s), \overline{\Phi}(s)]$ and $\Psi(s) = [\underline{\Psi}(s), \overline{\Psi}(s)]$, with $\underline{\Phi}(s) = \sqrt{1+s}$, $\overline{\Phi}(s) = s + \frac{2}{s+1}$, $\underline{\Psi}(s) = \sqrt{2s+1}$, and $\overline{\Psi}(s) = s + 2$ for all $s \in [0,2]_{h\mathbb{Z}}$. It is clear that $\frac{2}{3} \leq \overline{\Phi}(s)/\overline{\Psi}(s) \leq 1$ for all $s \in [0,2]_{h\mathbb{Z}}$. According to Theorem 4.7, we have

$$\left(\int_0^2 \Phi^2(s)\Diamond_\alpha s\right)^{\frac{1}{2}} \oplus \left(\int_0^2 \Psi^2(s)\Diamond_\alpha s\right)^{\frac{1}{2}} \subseteq \left[1, \sqrt[4]{\frac{3}{2}}\right]\cdot\left(\int_0^2 \big(\Phi(s)\oplus\Psi(s)\big)^2\Diamond_\alpha s\right)^{\frac{1}{2}}. \quad (23)$$

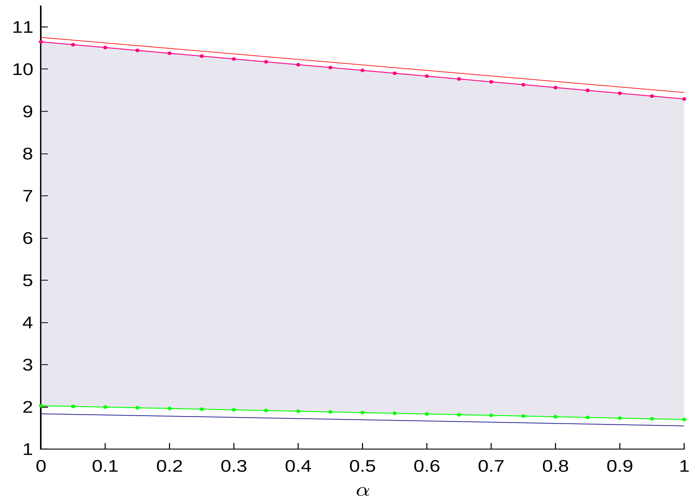

Figure 2 Illustration of Example 5.3 with different values of α (blue and red curves represent $\underline{RH}^{\alpha}$ and $\overline{RH}^{\alpha}$, respectively, and green and pink dotted curves are $\underline{LH}^{\alpha}$ and $\overline{LH}^{\alpha}$, respectively)

Indeed, we have

$$\begin{aligned}\int_0^2 (\Phi(s) \oplus \Psi(s))^2 \Diamond_\alpha s &= \int_0^2 \left[(\sqrt{1+s} + \sqrt{2s+1})^2, \left(2s + 2 + \frac{2}{s+1}\right)^2\right] \\ &= [19.901 - 0.235\alpha, 53.619 - 0.569\alpha].\end{aligned}$$

Hence, $(\int_0^2 (\Phi(s) \oplus \Psi(s))^2 \Diamond_\alpha s)^{\frac{1}{2}} = [\sqrt{19.901 - 0.235\alpha}, \sqrt{53.619 - 0.569\alpha}]$. On the other hand, we have

$$\int_0^2 \Phi^2(s) \Diamond_\alpha s = \int_0^2 \left[1 + s, \left(s + \frac{2}{s+1}\right)^2\right] \Diamond_\alpha s = [4.02 - 0.04\alpha, 8.97 - 0.062\alpha]$$

and

$$\int_0^2 \Psi^2(s) \Diamond_\alpha s = \int_0^2 \left[2s+1, (s+2)^2\right] \Diamond_\alpha s = [6.04 - 0.08\alpha, 18.787 - 0.24\alpha].$$

Then, we have

$$\begin{aligned}&\left(\int_0^2 \Phi^2(s) \Diamond_\alpha s\right)^{\frac{1}{2}} \oplus \left(\int_0^2 \Psi^2(s) \Diamond_\alpha s\right)^{\frac{1}{2}} \\ &\quad = [\sqrt{4.02 - 0.04\alpha} + \sqrt{6.04 - 0.08\alpha}, \sqrt{8.97 - 0.062\alpha} + \sqrt{18.787 - 0.24\alpha}] \\ &\quad \subseteq \left[1, \sqrt[4]{\frac{3}{2}}\right] \cdot [\sqrt{19.901 - 0.235\alpha}, \sqrt{53.619 - 0.569\alpha}] \\ &\quad = \left[1, \sqrt[4]{\frac{3}{2}}\right] \cdot \left(\int_0^2 (\Phi(s) \oplus \Psi(s))^2 \Diamond_\alpha s\right)^{\frac{1}{2}}.\end{aligned}$$

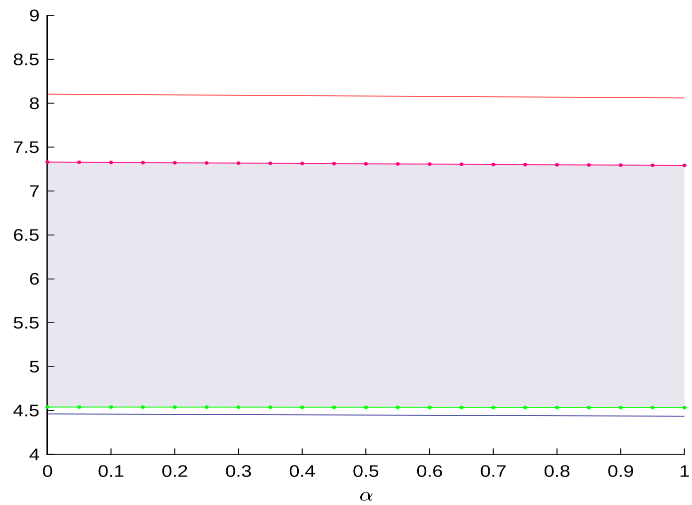

Figure 3 Illustration of Example 5.4 with different values of α (blue and red curves represent $\underline{\mathrm{RM}}^{\alpha}$ and $\overline{\mathrm{RM}}^{\alpha}$, respectively, and green and pink dotted curves are $\underline{\mathrm{LM}}^{\alpha}$ and $\overline{\mathrm{LM}}^{\alpha}$, respectively)

We also denote the values of the expressions on the left-hand side of (23) and the right-hand side of (23) corresponding to each $\alpha \in [0,1]$ by LM^{α} and RM^{α}, respectively. Then, the inequality (23) with different values of $\alpha \in [0,1]$ is shown in Fig. 3.

Acknowledgements
The fourth author was partially supported by project "Algebra, analysis, mathematical physics and number theory" under the grant SGS09/PřF/2023. The third author was supported by the Polish National Agency for Strategic Partnership under Grant No. BPI/PST/2021/1/00031/U/00001.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Tri Truong prepared the original manuscript. Martin Bohner, Linh Nguyen and Baruch Schneider reviewed and edited. Tri Truong finalized the manuscript.

Author details
[1]Missouri S&T, Rolla, MO 65409, USA. [2]Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. [3]Department of Mathematics, Faculty of science, University of Ostrava, Ostrava, 701 03, Czech Republic.

Received: 6 December 2022 Accepted: 30 May 2023 Published online: 19 June 2023

References

1. Agarwal, R.P., O'Regan, D., Saker, S.H.: Dynamic Inequalities on Time Scales. Springer, Cham (2014)
2. Agarwal, R.P., O'Regan, D., Saker, S.H.: Hardy Type Inequalities on Time Scales. Springer, Cham (2016)
3. Ammi, M.R.S., Ferreira, R.A.C., Torres, D.F.M.: Diamond-α Jensen's inequality on time scales. J. Inequal. Appl. **2008**, 1 (2008)
4. Ammi, M.R.S., Torres, D.F.M.: Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales. An. Univ. Craiova, Ser. Mat. Inform. **37**(1), 1–11 (2010)
5. Atici, F.M., Biles, D.C., Lebedinsky, A.: An application of time scales to economics. Math. Comput. Model. **43**(7–8), 718–726 (2006)
6. Barić, J., Bibi, R., Bohner, M., Nosheen, A., Pečarić, J.: Jensen Inequalities on Time Scales: Theory and Applications. Monographs in Inequalities, vol. 9. Element, Zagreb (2015)

7. Barilla, D., Bohner, M., Heidarkhani, S., Moradi, S.: Existence results for dynamic Sturm-Liouville boundary value problems via variational methods. Appl. Math. Comput. **409**, 125614 (2021)
8. Bohner, M., Guseinov, G.: Riemann and Lebesgue integration. In: Advances in Dynamic Equations on Time Scales, pp. 117–163. Birkhäuser Boston, Boston (2003)
9. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser Boston, Boston (2001)
10. Brito da Cruz, A.M.C., Martins, N., Torres, D.F.M.: The diamond integral on time scales. Bull. Malays. Math. Sci. Soc. **38**(4), 1453–1462 (2015)
11. Chalco-Cano, Y., Lodwick, W.A., Rufian-Lizana, A.: Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative. Fuzzy Optim. Decis. Mak. **12**(3), 305–322 (2013)
12. Costa, T.M.: Jensen's inequality type integral for fuzzy-interval-valued functions. Fuzzy Sets Syst. **327**, 31–47 (2017)
13. Costa, T.M., Chalco-Cano, Y., Román-Flores, H.: Wirtinger-type integral inequalities for interval-valued functions. Fuzzy Sets Syst. **396**, 102–114 (2020)
14. Costa, T.M., Román-Flores, H., Chalco-Cano, Y.: Opial-type inequalities for interval-valued functions. Fuzzy Sets Syst. **358**, 48–63 (2019)
15. Dinu, C.: Convex functions on time scales. An. Univ. Craiova, Ser. Mat. Inform. **33**, 87–96 (2008)
16. Georgiev, S.G.: Functional Dynamic Equations on Time Scales. Springer, Cham (2019)
17. Ghosh, D., Chauhan, R.S., Mesiar, R., Debnath, A.K.: Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions. Inf. Sci. **510**, 317–340 (2020)
18. Gu, J., Meng, F.: Some new nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales. Appl. Math. Comput. **245**, 235–242 (2014)
19. Hong, S.: Differentiability of multivalued functions on time scales and applications to multivalued dynamic equations. Nonlinear Anal. **71**(9), 3622–3637 (2009)
20. Khastan, A., Hejab, S.: First order linear fuzzy dynamic equations on time scales. Iran. J. Fuzzy Syst. **16**(2), 183–196 (2019)
21. Liu, W., Quôc, A.N., Chen, W.: On new Ostrowski type inequalities for double integrals on time scales. Dyn. Syst. Appl. **19**(1), 189–198 (2010)
22. Lungan, C., Lupulescu, V.: Random dynamical systems on time scales. Electron. J. Differ. Equ. **2012**, 86 (2012)
23. Lupulescu, V.: Hukuhara differentiability of interval-valued functions and interval differential equations on time scales. Inf. Sci. **248**, 50–67 (2013)
24. Malinowska, A.B., Torres, D.F.M.: On the diamond-alpha Riemann integral and mean value theorems on time scales. Dyn. Syst. Appl. **18**(3–4), 469–481 (2009)
25. Martynyuk, A.A.: Stability Theory for Dynamic Equations on Time Scales. Systems & Control: Foundations & Applications. Springer, Cham (2016)
26. Mikić, R., Pečarić, J.: Jensen-type inequalities on time scales for *n*-convex functions. Commun. Math. Anal. **21**(2), 46–67 (2018)
27. Moore, R.E.: Interval Analysis. Prentice Hall International, Englewood Cliffs (1966)
28. Moore, R.E., Kearfott, R.B., Cloud, M.J.: Introduction to Interval Analysis. SIAM, Philadelphia (2009)
29. Özkan, U.M., Yildirim, H.: Ostrowski type inequality for double integrals on time scales. Acta Appl. Math. **110**(1), 283–288 (2010)
30. Román-Flores, H., Chalco-Cano, Y., Lodwick, W.A.: Some integral inequalities for interval-valued functions. Comput. Appl. Math. **37**(2), 1306–1318 (2018)
31. Saker, S.H., O'Regan, D.: Hardy's type integral inequalities on time scales. Appl. Math. Inf. Sci. **9**(6), 2955–2962 (2015)
32. Shahidi, M., Khastan, A.: Linear fuzzy Volterra integral equations on time scales. Comput. Appl. Math. **39**(3), 1–23 (2020)
33. Sheng, Q., Fadag, M., Henderson, J., Davis, J.M.: An exploration of combined dynamic derivatives on time scales and their applications. Nonlinear Anal., Real World Appl. **7**(3), 395–413 (2006)
34. Stefanini, L., Bede, B.: Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. Nonlinear Anal. **71**(3–4), 1311–1328 (2009)
35. Zhang, D., Guo, C., Chen, D., Wang, G.: Jensen's inequalities for set-valued and fuzzy set-valued functions. Fuzzy Sets Syst. **404**, 178–204 (2021)
36. Zhao, D., An, T., Ye, G., Liu, W.: Chebyshev type inequalities for interval-valued functions. Fuzzy Sets Syst. **396**, 82–101 (2020)
37. Zhao, D., An, T., Ye, G., Liu, W.: Some generalizations of Opial type inequalities for interval-valued functions. Fuzzy Sets Syst. **436**, 128–151 (2022)
38. Zhao, D., Ye, G., Liu, W., Torres, D.F.M.: Some inequalities for interval-valued functions on time scales. Soft Comput. **23**, 6005–6015 (2019)