Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Stability of Cayley dynamic systems with impulsive effects

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Abstract

Linear dynamic systems with impulsive effects are considered. For such a system we define a new impulsive exponential matrix. Necessary and sufficient conditions for exponential stability and boundedness have been established. The fundamental tool is an impulsive exponential matrix for exponential stability.

Keywords: Dynamic systems; Impulsive exponential matrix; Cayley transform; Regressive matrix

1 Introduction

Researchers are totally aware of the importance of the theory of impulsive dynamic systems from a theoretical point of view, and their applications in control theory, dynamical games, and also in optimal control and mathematical programming [1, 2]. Wang et al. in [3], reviewed some recent developments in impulsive control theory. Lupulescu et al. [4] have addressed some very recent exciting results.

Wang et al. [3], described an impulsive differential equation by three components: a continuous-time differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which represents a set of jump events in which the impulse equation is active.

During the last decade of the last century, the time scale calculus was affirmed as a tool for an explanation of a lot of new phenomena in electricity, mechanics, biology, economics, etc. [5]. Time scale calculus confirmed a way to treat discrete versions of continuous scientific problems [6]. There are many papers and books discussing the theory and applications of time scales (see, for example, [6, 7]).

On the other hand, several stability results of impulsive systems on time scales were obtained in recent years. We refer to [8, 9]. The models of impulsive control systems have been constantly expended, including impulsive stochastic systems [10, 11], Impulsive neural networks [9, 12], impulsive chaotic systems [13] fractional order impulsive functional systems [14], stability of impulsive switched systems [15]. Besides to the traditional asymptotic (exponential) stability analysis [16], the dynamical properties of impulsive dynamical systems was also extended to the finite-time stability [17, 18], stability of two measures [19], input-to-state stability [20, 21].

1.1 Research gap and motivation

The impulsive dynamic systems can present a more accurate dynamical system compared to the non-impulsive dynamic systems where the sense of disturbance in nature is involved. On the other hand, dynamic systems on time scale calculus combine discrete and continuous phenomena. So, the hybridization of two different phenomena, where the disturbance in the mathematical model is concerned and the mixed domain (continuous and discrete) in the related parameters, claims to merge the notions of impulsive dynamic systems and the time scale theory.

Matrix exponential and the fundamental matrix of homogenous dynamic systems play a key role in the stability theory. However, as far as authors know, all these studies are performed on Hilger's exponential function. In the existing literature, we have noticed some articles like [4, 22] that addressed different solution properties of impulsive dynamic systems with Hilger's approach.

Recently, Cieslinski [23] has proposed a new definition of the exponential function on time scales, based on the Cayley transformation. Complex-valued Cayley transformation transforms the imaginary axis into the unit circle. Therefore, it is possible to define trigonometric and hyperbolic complex-valued functions on time scales in a standard way. The following functions maintain most of the qualitative properties of the analogous continuous functions. In special, Pythagorean trigonometric identities hold exactly on any time scale. Dynamic equations satisfied by Cayley-type functions have a natural resemblance to the corresponding differential equations, Cayley h-difference equations [24], and Cayley quantum equations [25]. The current work is an attempt to study the solution of linear impulsive dynamic systems by means of a Cayley-type fundamental matrix.

1.2 Novelties of the work

Motivated by the recent studies [22, 23], the current article claims its novelty from the following perspectives:

(i) We define fundamental matrices of linear impulsive dynamic systems on time scales that are based on Cayley's transformation. The impulsive transition matrix and its properties are key for the stability theory.
(ii) We establish the existence and uniqueness of the solution of impulsive dynamic systems.
(iii) Necessary and sufficient conditions for boundedness and exponential stability of the proposed solution are established in this article.

1.3 Structure of the paper

The outline of the paper is as follows. In Sect. 2, we review the definitions and properties of time-scale calculus and qualitative properties of dynamic systems on time scales. In Sect. 3, we develop the fundamental concepts of homogeneous impulsive dynamic systems on time scales. These properties are used to investigate linear nonhomogeneous dynamic systems. In Sect. 4, we get variations of the constants formula for linear dynamic systems. We investigate the stability and boundedness of solutions in Sect. 5. Finally, we conclude the paper with some brief comments.

2 Preliminaries

Let $\mathbb{X}$ be a finite-dimensional Banach space with a norm $\|\cdot\|$. For simplicity, we shall denote all norms by the same symbol $\|\cdot\|$, without danger of confusion. For $M \in \mathbb{M}_n(\mathbb{R})$, let us denote its conjugate by M^T (its transpose). For an unbounded time scale $\mathbb{T}_+$ (a nonempty subset of real-line), consider the following jump operators:

$$x^\sigma = \sigma(x) := \inf\{s \in \mathbb{T} : s > x\};$$
$$x^\rho = \rho(x) := \sup\{s \in \mathbb{T} : s < x\},$$

and a real-valued function:

$$\mu : \mathbb{T} \to \mathbb{R}^+ : \mu(x) := x^\sigma - x.$$

From definitions of jump operators: for all $x \in \mathbb{T}$, $x^\sigma \geq x$ and $x^\rho \leq x$, (see, e.g., [6]).

Throughout this paper, we always assume that $\sup \mathbb{T} = \infty$ and for any $\tau \in \mathbb{T}$, let $\mathbb{T}_{(\tau)} := [\tau, \infty) \cap \mathbb{T}$ and $\mathbb{T}_+ := \mathbb{T}_{(0)}$.

A function f is said to be right-dense continuous (rd-continuous) if it is continuous at right-dense points and left-hand limits exist at left-dense points. The Δ-derivative of a function g is defined by

$$g^\Delta(x) = \lim_{\substack{s \to x \\ s \neq x^\sigma}} \frac{g(x^\sigma) - g(s)}{x^\sigma - s}.$$

Let us state the following basic spaces:

- $C_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X}) := \{g : \mathbb{T}_{(\tau)} \to \mathbb{X} : g \text{ is } rd\text{-continuous}\}$.
- $C^1_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X}) := \{g : \mathbb{T}_{(\tau)} \to \mathbb{X} : g\ \Delta\text{-derivative is in } C_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X})\}$.
- $\mathcal{R}(\mathbb{T}_{(\tau)}, \mathbb{X}) := \{g \in C_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X}) : (I_{\mathbb{X}} + \mu(x)g(x))^{-1} \text{ exists for all } x \in \mathbb{T}_{(\tau)}\}$.
- $\mathcal{R}^+(\mathbb{T}_{(\tau)}, \mathbb{R}) := \{g \in \mathcal{R}(\mathbb{T}_{(\tau)}, \mathbb{R}) : I_{\mathbb{X}} + \mu(x)g(x) > 0 \text{ for all } x \in \mathbb{T}_{(\tau)}\}$.

Here $\mathbb{X}$ can be $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ or $\mathbb{M}_n(\mathbb{R})$.

For $g \in \mathcal{R}(\mathbb{T}_+, \mathbb{C})$, Hilger defined the exponential function as follows:

$$e_g(x,s) = \exp\left(\int_s^x \frac{\operatorname{Log}(1 + \mu(r)g(r))}{\mu(r)} \Delta r\right),$$

where Log is the principal logarithm function.

If $g,f \in \mathcal{R}(\mathbb{T}_+, \mathbb{C})$, then the following properties hold:

- $e_f(x^\sigma, s) = (1 + \mu(x)f(x))e_f(x,s)$,
- $(e_f(x,s))^{-1} = e_{\ominus^\mu f}(x,s)$, where $\ominus^\mu f := \frac{-f}{1+\mu f}$,
- $e_f(x,s)e_f(s,r) = e_f(x,s)$,
- $e_g(x,s)e_f(x,s) = e_{f \oplus^\mu g}(x,s)$, where $f \oplus^\mu g := f + g + \mu fg$.

For each regressive matrix M, $e_M(t,s)$ is the only solution of IVP:

$$X^\Delta = MX, \quad X(s) = I_{\mathbb{M}_n(\mathbb{R})}$$

and $x(t) = e_M(t,s)x_0$, $t \geq s$, is the only solution of IVP:

$$x^\Delta = Mx, \quad x(s) = a_0$$

and $x(t) = e_{\ominus M^T}(t,s)x_0$, $t \geq s$ is the only solution of the adjoint IVP:

$$x^\Delta = -M^T x^\sigma, \quad x(s) = x_0.$$

3 Exponential function

3.1 Scalar case

Cieslinski [23], introduced an improved exponential function (or the Cayley-exponential function). To formulate a new definition, we need a new regressivity notion:

The function $f : \mathbb{T} \to \mathbb{C}$ is called regressive (Cieslinski definition) if $\mu(x)f(x) \neq \pm 2$ for any $x \in \mathbb{T}^k$. The set of all regressive functions $f : \mathbb{T} \to \mathbb{C}$ is denoted by $\mathcal{R}^C(\mathbb{T}_+, \mathbb{C})$.

For $f \in \mathcal{R}^C(\mathbb{T}_+, \mathbb{C})$, the Cayley-exponential function is defined by

$$E_f(x,s) := \exp\left(\int_s^x \frac{1}{\mu(r)} \operatorname{Log}\left(\frac{1 + \frac{1}{2}\mu(r)f(r)}{1 - \frac{1}{2}\mu(r)f(r)}\right)\Delta r\right), \quad \mu(r) > 0.$$

If $f, g \in \mathcal{R}^C(\mathbb{T}_+, \mathbb{C})$, then the following properties hold:

- $E_f(x^\sigma, s) = (\frac{1+\frac{1}{2}\mu(x)f(x)}{1-\frac{1}{2}\mu(x)f(x)})E_f(x,s)$,
- $(E_f(x,s))^{-1} = E_{-f}(x,s)$,
- $\overline{(E_f(x,s))} = E_{\bar{f}}(x,s)$,
- $E_f(x,s)E_f(s,r) = E_f(x,s)$,
- $E_f(x,s)E_g(x,s) = E_{f\oplus g}(x,s)$, where $f \oplus g := \frac{f+g}{1+\frac{1}{4}\mu^2 fg}$.

The function $f : \mathbb{T} \to \mathbb{R}$ is called positively regressive if for all $x \in \mathbb{T}^k$ we have $|f(x)\mu(x)| < 2$. The set of all positively regressive function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}^C_+(\mathbb{T}_+, \mathbb{C})$. $\mathcal{R}^C_+(\mathbb{T}_+, \mathbb{C})$ is a commutative group under $\oplus$. However, the set $\mathcal{R}^C(\mathbb{T}_+, \mathbb{C})$ is not closed under $\oplus$.

It is known that [23, Theorem 3.2], for real-valued function a on time scale $\mathbb{T}_+$, we have

$$a^\Delta(t) = \beta(t)a(t) \quad \Longleftrightarrow \quad a^\Delta(t) = \alpha(t)\langle a(t)\rangle,$$

where $\beta(t) = \frac{\alpha(t)}{1-\frac{1}{2}\mu(t)\alpha(t)}$ and $\langle a(t)\rangle := \frac{a(t)+a(\sigma(t))}{2}$.

3.2 Matrix case

Definition 3.1 (Regressivity) An $n \times n$ matrix-valued function M on a time scale $\mathbb{T}_+$ is called regressive, if

$$I \pm \frac{\mu(x)M(x)}{2} \quad \text{are invertible for all } x \in \mathbb{T}^k. \tag{1}$$

The class of all such regressive and rd-continuous matrix-valued functions is denoted by $\mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$.

In this part, we generalized Cayley exponential function for the following dynamic systems:

$$a^\Delta(t) = M(t)\langle a(t)\rangle, \tag{2}$$

where $M \in \mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$. Nonhomogenous system is stated as:

$$a^\Delta(t) = M(t)\langle a(t)\rangle + h(t) \tag{3}$$

with $h \in C_{rd}(\mathbb{T}_+, (\mathbb{R}^n))$.

An element ϕ of $C_{rd}^1(\mathbb{T}_+, (\mathbb{R}^n))$ is called a solution of IVP (3) on $\mathbb{T}_+$ such that $\phi^\Delta(t) = M(t)\langle\phi(t)\rangle + h(t)$ for all $t \in \mathbb{T}_+$.

Let us denote the transition matrix $E_M(x,\tau)$ of (2) at initial time $\tau \in \mathbb{T}_+$, and is the unique solution of the following matrix IVP:

$$A^\Delta(t) = M(t)\big\langle A(t)\big\rangle, \quad A(\tau) = I$$

and $a(t) = E_M(t,\tau)\eta$, $t \geq \tau$, satisfied:

$$a^\Delta(t) = M(t)\big\langle a(t)\big\rangle, \quad a(\tau) = \eta.$$

Definition 3.2 Assume M and $N \in \mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$. Then we define circle addition $\oplus$ of M and N by

$$(M \oplus N)(x) := (M + N)\left(I + \frac{1}{4}\mu^2 MN\right)^{-1} \quad \text{for all } x \in \mathbb{T}^k.$$

Example 3.3 Assume $M \in \mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ is constant $n \times n$-matrix. If $\mathbb{T} = \mathbb{R}$, then $E_M(x,x_0) = e^{M(x-x_0)}$.

Theorem 3.4 *If $M \in \mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ is a matrix-valued functions on $\mathbb{T}_+$, then*

1. $E_0(x,s) \equiv I$ and $E_M(x,x) \equiv I$
2. $E_M(\sigma(x),s) = (I - \frac{\mu M}{2})^{-1}(I + \frac{\mu M}{2})E_M(x,s)$;
3. $E_M(x,s) = E_M^{-1}(s,x)$;
4. $E_M^{-1}(x,s) = E_{-M^*}(x,s)$;
5. $E_M(x,s)E_M(s,r) = E_M(x,r)$.

Theorem 3.5 *Let $M \in \mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ and suppose that $f : \mathbb{T}_+ \to \mathbb{R}^n$ is rd-continuous. Let $x_0 \in \mathbb{T}_+$ and $a_0 \in \mathbb{R}^n$. Then the IVP*

$$\begin{aligned} a^\Delta(x) &= M\big\langle a(x)\big\rangle + \left(I - \frac{M}{2}\mu(x)\right)f(x), \\ a(x_0) &= a_0 \end{aligned} \tag{4}$$

has a unique solution, given by

$$a(x) = E_M(x,x_0)a_0 + \int_{x_0}^{x} E_M\big(x,\sigma(\tau)\big)f(\tau)\Delta\tau. \tag{5}$$

Proof First, a given by (5) is well defined and can be rewritten as

$$a(\cdot) = E_M(\cdot,x_0)\left\{a_0 + \int_{x_0}^{\cdot} E_M\big(x_0,\sigma(\tau)\big)f(\tau)\Delta\tau\right\}.$$

We use the product rule to differentiate a:

$$\begin{aligned}
a^{\Delta}(\cdot) &= M(\cdot)\langle E_M(\cdot,t_0)\rangle \left\{ a_0 + \int_{x_0}^{\cdot} E_M\big(x_0,(\tau^{\sigma})\big) f(\tau)\Delta\tau \right\} \\
&\quad + E_M\big(\cdot^{\sigma},x_0\big)E_M\big(x_0,\cdot^{\sigma}\big)f(\cdot) \\
&= M(\cdot)\langle E_M(\cdot,x_0)\rangle \left\{ a_0 + \int_{x_0}^{\cdot} E_M\big(x_0,\tau^{\sigma}\big) f(\tau)\Delta\tau \right\} + f(\cdot) \\
&= M(\cdot)\left(\frac{E_M(\cdot,x_0)+E_M(\cdot^{\sigma},x_0)}{2}\right) \\
&\quad \times \left\{ a_0 + \int_{x_0}^{\cdot} E_M\big(x_0,\tau^{\sigma}\big) f(\tau)\Delta\tau \right\} + f(\cdot) \\
&= \frac{M(\cdot)}{2}\left[E_M(\cdot,x_0)a_0 + \int_{x_0}^{\cdot} E_M\big(\cdot,\tau^{\sigma}\big) f(\tau)\Delta\tau \right] \\
&\quad + \frac{M(\cdot)}{2}\left[E_M\big(\cdot^{\sigma},x_0\big)a_0 + \int_{x_0}^{\cdot} E_M\big(\cdot^{\sigma},(\tau^{\sigma}\big) f(\tau)\Delta\tau \right] + f(\cdot).
\end{aligned} \tag{6}$$

We know that $\int_x^{x^{\sigma}} f(t)\Delta t = \mu(x)f(x)$. Therefore, we have

$$\int_x^{x^{\sigma}} E_M\big(x^{\sigma},\tau^{\sigma}\big)f(\tau)\Delta\tau = \mu(x)E_M\big(x^{\sigma},x^{\sigma}\big)f(x) = \mu(x)f(x).$$

It implies that

$$\begin{aligned}
a^{\Delta}(x) &= \frac{M(x)}{2}\left[E_M(x,x_0)a_0 + \int_{x_0}^{x} E_M\big(x_0,\tau^{\sigma}\big) f(\tau)\Delta\tau \right] \\
&\quad + \frac{M(x)}{2}\left[E_M\big(x^{\sigma},x_0\big)a_0 + \int_{x_0}^{x} E_M\big(x^{\sigma},\tau^{\sigma}\big) f(\tau)\Delta\tau \right] \\
&\quad + f(x) + \frac{M}{2}\mu(x)f(x) - \frac{M}{2}\mu(x)f(x),
\end{aligned}$$

we have

$$\begin{aligned}
a^{\Delta}(x) &= \frac{M(x)}{2}a(x) \\
&\quad + \frac{M(x)}{2}\left[E_M\big(x^{\sigma},x_0\big)a_0 + \int_{x_0}^{x} E_M\big(x^{\sigma},\tau^{\sigma}\big) f(\tau)\Delta\tau \right] \\
&\quad + f(x) + \frac{M(x)}{2}\mu(x)f(x) - \frac{M(x)}{2}\mu(x)f(x) \\
&= \frac{M(x)}{2}\big(a(x)\big) + \frac{M(x)}{2}a\big(x^{\sigma}\big) + \left(I - \frac{M(x)}{2}\mu(x)\right)f(x) \\
&= M(x)\langle a(x)\rangle + \left(I - \frac{M(x)}{2}\mu(x)\right)f(x).
\end{aligned}$$

Hence

$$a^{\Delta}(x) = M(x)\langle a(x)\rangle + \left(I - \frac{M(x)}{2}\mu(x)\right)f(x),$$
$$a(x_0) = a_0. \qquad \square$$

Theorem 3.6 (Putzer Algorithm) *Let $M \in \mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ be a constant $n \times n$-matrix. Suppose $x_0 \in \mathbb{T}_+$. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of M, then*

$$E_M(x, x_0) = \sum_{i=0}^{n-1} r_{i+1}(x) P_i, \tag{7}$$

where $r(x) := (r_1(x), r_2(x), \ldots, r_n(x))^T$ is the solution of IVP

$$r^{\Delta} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \ddots & \vdots \\ 0 & 1 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_n \end{pmatrix} \langle r\rangle, \quad r(x_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{8}$$

and the P matrices $P_0, P_1, \ldots, P_n$ are recursively defined by $P_0 = I$ and

$$P_{k+1} = (M - \lambda_{k+1} I)P_k \quad \textit{for } 0 \le k \le n-1.$$

Example 3.7 Let us consider the following dynamic system:

$$a^{\Delta}(x) = M\langle a(x)\rangle, \quad \text{where } M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ such that } b > a > 0 \tag{9}$$

on time scales $\mathbb{T}_+$ with step size function μ. We find the matrix exponential $E_M(x, x_0)$. M is regressive for $\mu \neq \frac{2}{a}, \frac{2}{b}$ that is $\det\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \frac{\mu}{2}\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right) = \frac{1}{2}a\mu + \frac{1}{2}b\mu + \frac{1}{4}ab\mu^2 + 1 = \frac{1}{4}(b\mu + 2)(a\mu + 2) \neq 0$.

Since by definition $\mu \ge 0$, therefore $\det\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \frac{\mu}{2}\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right) \neq 0$ and $\det\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \frac{\mu}{2}\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right) = \frac{1}{4}ab\mu^2 - \frac{1}{2}b\mu - \frac{1}{2}a\mu + 1 = \frac{1}{4}(b\mu - 2)(a\mu - 2) \neq 0$ for $\mu \neq \frac{2}{a}, \frac{2}{b}$.

Equation (9) is equivalently written as

$$a^{\Delta}(x) = \left(I - \frac{\mu}{2}M\right)^{-1} Ma(x).$$

It is easy to see that

$$E_M(x, x_0) = \begin{pmatrix} e_{\frac{2a}{2-a\mu}}(x, x_0) & 0 \\ 0 & e_{\frac{2b}{2-b\mu}}(x, x_0) \end{pmatrix}$$

and

$$a(x) = \begin{pmatrix} e_{\frac{2a}{2-a\mu}}(x,x_0) & 0 \\ 0 & e_{\frac{2b}{2-b\mu}}(x,x_0) \end{pmatrix} a_0.$$

4 Impulsive exponential matrix

In this section, we assume the following IDS:

$$\begin{cases} a^{\Delta}(x) = M(x)\langle a(x)\rangle, & x \in \mathbb{T}_+,\ x \neq x_k, \\ a(x_k^+) = (I + B_k)a(x_k), & k = 1, 2, \ldots. \end{cases} \tag{10}$$

and

$$\begin{cases} a^{\Delta}(x) = M(x)\langle a(x)\rangle,\ x \in \mathbb{T}_{(\tau),} & x \neq x_k, \\ a(x_k^+) = (I + B_k)a(x_k), & k = 1, 2, \ldots, \\ a(\tau^+) = \eta, & \tau \geq 0. \end{cases} \tag{11}$$

with the following assumption:

(H): Assume $B_k \in M_n(\mathbb{R})$, $k = 1, 2, \ldots$, $M \in \mathcal{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$, $0 = x_0 < x_1 < x_2 < \cdots < x_k < \cdots$, with $\lim_{k\to\infty} x_k = \infty$, and $x_k^{\sigma} = x_k$.

For solution of (11), let us define:

$$\Omega := \left\{ \begin{array}{c} a : \mathbb{T}_+ \to \mathbb{R}^n;\ a \in C\big((x_k, x_{k+1}, \mathbb{R}^n\big),\ k = 0, 1, 2, \ldots, a\big(x_k^+\big) \\ \text{and } a\big(x_k^-\big) \text{ exists with } a\big(x_k^-\big) = a(x_k),\ k = 1, 2, \ldots, \end{array} \right\}$$

and

$$\Omega^1 := \big\{a \in \Omega;\ a \in C^1\big((x_k, x_{k+1}),\ \mathbb{R}^n\big),\ k = 0, 1, 2, \ldots\big\}.$$

A function $a \in \Omega^1$ is called the solution of (10), if it satisfies $a^{\Delta}(x) = M(x)\langle a(x)\rangle$ everywhere on $\mathbb{T}_{(\tau)} \setminus \{\tau, x_{k(\tau)}, x_{k(\tau)+1}, \ldots\}$ such that $k(\tau) := \min\{\ k = 0, 1, 2, \ldots; \tau < x_k\}$ and for each $i = k(\tau), k(\tau) + 1, \ldots$ satisfies the initial $a(\tau^+) = \eta$ and impulsive $a(x_i^+) = a(x_i) + B_i a(x_i)$ conditions.

Theorem 4.1 *If* (H) *holds, then any solution of the IVP* (10) *satisfies*

$$a(x) = a(\tau) + \int_{\tau}^{x} M(s)\big\langle a(s)\big\rangle\Delta s + \sum_{\tau<x_j<x} B_j a(x_j), \quad x \in \mathbb{T}_+, \tag{12}$$

and vice versa.

Proof By a similar discussion as in the proof of [26, Theorem 3.1], we can obtain this result. □

By using [26, Lemma 3.1] and [27, Lemma 3.1], we can obtain the following impulsive dynamic inequality:

Lemma 4.2 *Let $\tau \in \mathbb{T}_+, a \in \mathcal{R}(\mathbb{T}_+, \mathbb{R}), p \in \mathcal{R}_+^C(\mathbb{T}_+, \mathbb{R})$, and $c_i, d_i \in \mathbb{R}_+$, $i = 1, 2, \ldots$. If*

$$\begin{cases} a^{\Delta}(x) \leq p(x)\langle a(x)\rangle + b(x), & x \in \mathbb{T}_{(\tau)}, x \neq x_i, \\ a(x_i^+) \leq c_i a(x_i) + d_i, & i = 1, 2, \ldots, \end{cases}$$

then

$$a(x) \leq a(\tau) \prod_{\tau < x_i < x} c_i E_p(x, \tau) + \sum_{\tau < x_i < x} \left(\prod_{x_i < x_j < x} c_j E_p(x, x_i) \right) d_i + \int_{\tau}^{x} \prod_{s < x_i < x} c_i \big\langle E_{-p}(s, x) \big\rangle b(s) \Delta s.$$

Lemma 4.3 *Let $\tau \in \mathbb{T}_+, a, b \in \mathcal{R}^C(\mathbb{T}_+, \mathbb{R}), p \in \mathcal{R}_+^C(\mathbb{T}_+, \mathbb{R})$ and $c, b_i \in \mathbb{R}_+$, $i = 1, 2, 3, \ldots$. If*

$$a(x) \leq c + \int_{\tau}^{x} p(s) \big\langle a(s) \big\rangle \Delta s + \sum_{\tau < x_i < x} b_i a(x_i), \quad x \in \mathbb{T}_{(\tau)} \tag{13}$$

then

$$a(x) \leq c \prod_{\tau < x_i < x} (1 + b_i) E_p(x, \tau), \quad x \geq \tau. \tag{14}$$

Proof Let

$$v(x) := c + \int_{\tau}^{x} p(s) \big\langle a(s) \big\rangle \Delta s + \sum_{\tau < x_i < x} b_i a(x_i), \quad x \geq \tau$$

then

$$\begin{cases} v^{\Delta}(x) = p(x)\langle a(x)\rangle, & x \neq x_i, \ v(\tau) = c, \\ v(x_i^+) = v(x_i) + b_i a(x_i), & i = 1, 2, 3, \ldots. \end{cases}$$

Since $\langle a(x)\rangle \leq \langle v(x)\rangle$, $x \geq \tau$, we then have

$$\begin{cases} v^{\Delta}(x) \leq p(x)\langle v(x)\rangle, & x \neq x_i, \ v(\tau) = c, \\ v(x_i^+) = v(x_i) + b_i v(x_i), & i = 1, 2, 3, \ldots. \end{cases}$$

Lemma 4.2 yields

$$v(x) \leq c \prod_{\tau < x_k < x} (1 + b_k) E_p(x, \tau), \quad x \geq \tau,$$

which implies (14). □

Theorem 4.4 *If (H) holds, then any solution of (11) satisfies the following estimate*

$$\|a(x)\| \leq \|a(\tau)\| \prod_{\tau < x_k \leq x} \big(1 + \|B_k\|\big) \exp\left(\int_{\tau}^{x} \frac{\|M(s)\|}{1 - \frac{\mu(s)\|M(s)\|}{2}} \Delta s \right), \quad \mu(s) > 0 \tag{15}$$

for $\tau, x \in \mathbb{T}_+$ with $x \geq \tau$.

Proof From (12), we obtain that

$$\begin{aligned} \|a(x)\| &\leq \|a(\tau)\| + \int_\tau^x \|M(s)\|\langle\|a(s)\|\rangle\Delta s \\ &\quad + \sum_{\tau<x_j\leq x} \|B_j\|\,\|a(x_j)\|, \quad x \geq \tau. \end{aligned}$$

Lemma 4.3, yields

$$\|a(x)\| \leq \|a(\tau)\| \prod_{\tau<x_k\leq x} \bigl(1+\|B_k\|\bigr)E_{\|M(\cdot)\|}(x,\tau), \quad x \geq \tau.$$

Since for any $a \geq 0$,

$$\lim_{u\to\mu(s)} \frac{\ln(\frac{1+\frac{au}{2}}{1-\frac{au}{2}})}{u} = \begin{cases} a & \text{if } \mu(s)=0, \\ \frac{\ln(\frac{1+\frac{a\mu(s)}{2}}{1-\frac{a\mu(s)}{2}})}{\mu(s)} \leq \frac{a}{1-\frac{a\mu(s)}{2}} & \text{if } \mu(s)>0, \end{cases}$$

then

$$\begin{aligned} E_{\|M(\cdot)\|}(x,\tau) &= \exp\biggl(\int_\tau^x \lim_{u\to\mu(s)} \frac{\ln(\frac{1+\frac{\|M(s)\|\mu(s)}{2}}{1-\frac{\|M(s)\|\mu(s)}{2}})}{\mu(s)}\Delta s\biggr) \\ &\leq \exp\biggl(\int_\tau^x \frac{\|M(s)\|}{1-\frac{\mu(s)\|M(s)\|}{2}}\Delta s\Delta s\biggr), \quad x \geq \tau. \end{aligned}$$

Thus, we obtain (15). □

Remark 4.5 Estimate (15) is μ dependent. However, in Hilger's exponential case, it is independent of μ, see, for example, [22, Theorem 3.2].

Let us define generalized exponential matrix $G_M(x,y)$, $0 \leq y \leq x$, for impulsive effects $\{B_i, x_i\}_{i=1}^\infty$:

$$G_M(x,y) = \begin{cases} E_M(x,x_i^+)[\prod_{y<x_k\leq x}(I+B_k)E_M(x_k,x_{k-1}^+)](I+B_j)E_M(x_j,y), \\ \quad \text{for } x_{j-1} \leq y < x_j < \cdots < x_i < x < x_{i+1}; \\ E_M(x,x_i^+)(I+B_i)E_M(x_i,y), \quad \text{for } x_{i-1} \leq y \leq x_i < x < x_{i+1}; \\ E_M(x,y), \quad \text{for } x_{i-1} \leq y \leq x \leq x_i. \end{cases} \tag{16}$$

Remark 4.6 By definition, we can obtain the following equality:

$$\begin{aligned} G_M(x,y) &= E_M\bigl(x,x_k^+\bigr)(I+B_k)E_M\bigl(x_k,x_{k-1}^+\bigr) \\ &\quad \times \biggl[\prod_{s<x_j<x_k} (I+B_j)E_M\bigl(x_j,x_{j-1}^+\bigr)\biggr](I+B_i)E_M(x_i,y). \end{aligned}$$

It follows that

$$G_M(x,y) = E_M\bigl(x,x_k^+\bigr)(I+B_k)G_M(x_k,y)$$

for $x_{i-1} \leq y < x_i < \cdots < x_k \leq x < x_{k+1}$.

For the following result, let us set $X_M(t) := G_M(t,0)$, $t \in \mathbb{T}_+$.

Theorem 4.7 *If* (H) *holds, then* $G_M(t,y)$ *has the following properties*:

1. $G_M(t,y) = X_M(t)X_M^{-1}(y)$, $0 \leq y \leq t$;
2. $G_M(t,t) = I$, $t \geq 0$;
3. If $(I + B_i)^{-1}$ exists for each i, then $G_M(t,y) = G_M^{-1}(y,t)$, $0 \leq y \leq t$;
4. $G_M(\sigma(x),s) = (I - \frac{\mu}{2}M)^{-1}(I + \frac{\mu}{2}M)G_M(x,s)$, $0 \leq s \leq x$;
5. $G_M(t,y)G_M(y,r) = G_M(t,r)$, $0 \leq r \leq y \leq t$.

Proof

1. Let

$$Y(t) := X_M(t)X_M^{-1}(y), \quad 0 \leq y \leq t.$$

Then we have

$$\begin{aligned} Y^\Delta(t) &= X_M^\Delta(t)X_M^{-1}(y) \\ &= M(t)\langle X_M(t)\rangle X_M^{-1}(y) \\ &= M(t)\langle X_M(t)X_M^{-1}(y)\rangle \\ &= M(t)\langle Y(t)\rangle, \quad t \neq x_k. \end{aligned}$$

Also

$$Y(y) = X_M(y)X_M^{-1}(y) = I,$$

and

$$\begin{aligned} Y(x_k^+) - Y(x_k) &= X_M(x_k^+)X_M^{-1}(y) - X_M(x_k)X_M^{-1}(y) \\ &= \left[X_M(x_k^+) - X_M(x_k)\right]X_M^{-1}(y) \\ &= B_k X_M(x_k)X_M^{-1}(y) \\ &= B_k Y(x_k) \quad \text{for each } x_k \geq y. \end{aligned}$$

Therefore, $Y(t) = X_M(t)X_M^{-1}(y)$ solves the IVP (19), which has exactly one solution. Therefore,

$$G_M(t,y) = X_M(t)X_M^{-1}(y), \quad 0 \leq y \leq t.$$

2. From part 1: $G_M(t,t) = X_M(t)X_M^{-1}(t) = I$.
3. By using part 1, we have $G_M(t,y) = X_M(t)X_M^{-1}(y) = (X_M(y)X_M^{-1}(t))^{-1} = G_M^{-1}(y,t)$.

4. Well-known relation implies that

$$\begin{aligned} G_M\big(\sigma(t),y\big) &= G_M(t,y)+\mu(t)G_M^{\Delta}(t,y) \\ &= G_M(t,y)+\mu(t)M(t)\big\langle G_M(t,y)\big\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} G_M\big(\sigma(t),y\big) &= G_M(t,y)+\mu(t)M(t)\big\langle G_M(t,y)\big\rangle; \\ &= G_M(t,y)+\mu(t)M(t)\left(\frac{G_M(t,y)+G_M(\sigma(t),y)}{2}\right) \\ &= G_M(t,y)+\frac{\mu(t)M(t)G_M(t,y)}{2}+\frac{\mu(t)M(t)G_M(\sigma(t),y)}{2}; \\ &= G_M\big(\sigma(t),y\big)-\frac{\mu(t)M(t)G_M(\sigma(t),y)}{2}-\left(I+\frac{\mu}{2}M\right)G_M(t,y). \end{aligned}$$

Therefore, we have

$$\left(I-\frac{\mu}{2}M\right)G_M\big(\sigma(t),y\big)=\left(I+\frac{\mu}{2}M\right)G_M(t,y).$$

Since $M\in\mathcal{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, it implies that

$$G_M\big(\sigma(t),y\big)=\left(I-\frac{\mu}{2}M\right)^{-1}\left(I+\frac{\mu}{2}M\right)G_M(t,y).$$

5. Now let $Y(t)=G_M(t,y)G_M(y,r^+)$, $0\le r\le y\le t$. It follows that:

$$\begin{aligned} Y^{\Delta}(t) &= G_M^{\Delta}(t,y)G_M\big(y,r^+\big) \\ &= M(t)\big\langle G_M(t,y)\big\rangle G_M\big(y,r^+\big) \\ &= M(t)\big\langle G_M(t,y)G_M\big(y,r^+\big)\big\rangle \\ &= M(t)\big\langle Y(t)\big\rangle \quad \text{for } t\neq x_k. \end{aligned}$$

Also

$$Y\big(r^+\big)=G_M\big(r^+,y\big)G_M\big(y,r^+\big)=G_M\big(r^+,y\big)G_M^{-1}\big(y,r^+\big)=I.$$

and

$$Y\big(x_k^+\big)=G_M\big(x_k^+,y\big)G_M\big(y,r^+\big)=(I+B_k)Y(x_k)$$

$(\forall)$ $x_k\ge y$. Uniqueness result implies that $G_M(t,r)=G_M(t,y)G_M(y,r)$, $0\le r\le y\le t$. □

Theorem 4.8 *If* (H) *holds, then*

$$\frac{\partial}{\Delta s}G_M(x,s)=-G_M\big(x,\sigma(s)\big)M(s)\big\langle G_M(s,x)\big\rangle G_M^{-1}(s,x), \qquad s\neq x_k.$$

Proof By [4, Theorem A.2], we have

$$\begin{aligned}\frac{\partial}{\Delta s}G_M(x,s) &= \frac{\partial}{\Delta s}G_M^{-1}(s,x)\\ &= -G_M^{-1}\big(s,\sigma(x)\big)\frac{\partial}{\Delta s}G_M(s,x)G_M^{-1}(s,x)\\ &= -G_M^{-1}\big(s,\sigma(x)\big)M(x)\big\langle G_M(s,x)\big\rangle G_M^{-1}(s,x).\end{aligned}$$

Therefore, $\frac{\partial}{\Delta t}G_M(x,s) = -G_M(x,\sigma(s))M(s)\langle G_M(s,x)\rangle G_M^{-1}(s,x)$ for all $s \in \mathbb{T}_+$, $s \neq x_k$, $k = 1,2\cdots$. □

Let $l^\infty(\mathbb{R}^n) := \{c := \{c_k\}_{k=1}^\infty, c_k \in \mathbb{R}^n\ k = 1,2,\dots,$ such that $\sup_{k\geq 1}\|c_k\| < \infty\}$. Then $l^\infty(\mathbb{R}^n)$ is a complete normed space with the norm $\|c\| := \sup_{k\geq 1}\|c_k\|$.

Consider the following nonhomogeneous IVP

$$\begin{cases} a^\Delta(x) = M(x)\langle a(x)\rangle + (I - \frac{\mu(x)}{2}M(x))f(x),\ x \in \mathbb{T}_{(\tau)}, \quad x \neq x_k;\\ a(x_k^+) = a(x_k) + B_k a(x_k) + c_k, \quad k = 1,2,3,\dots;\\ a(\tau^+) = \eta, \quad \tau \geq 0,\end{cases} \tag{17}$$

with a given vector-valued function f.

Theorem 4.9 *If* (H) *holds,* $c := \{c_k\}_{k=1}^\infty \in l^\infty(\mathbb{R}^n)$. *For regressive vector-valued function* f*, the IVP* (17) *has only one solution*:

$$\begin{aligned} a(x) = {}& G_M(x,\tau)\eta + \int_\tau^x G_M\big(x,\sigma(s)\big)f(s)\Delta s\\ &+ \sum_{\tau<x_j<x} G_M\big(x,x_j^+\big)c_j, \quad x \geq \tau.\end{aligned} \tag{18}$$

Proof Let $(\tau,\eta) \in \mathbb{T}_+ \times R^n$. Then there exists $i \in \{1,2,\dots\}$ such that $\tau \in [x_{i-1},x_i)$. Theorem 3.5 implies the unique solution of (17) on $[\tau,x_i)$:

$$\begin{aligned} a(x) &= E_M(x,\tau)\eta + \int_\tau^x E_M\big(x,\sigma(s)\big)f(s)\Delta s\\ &= G_M(x,\tau)\eta + \int_\tau^x G_M\big(x,\sigma(s)\big)f(s)\Delta s, \quad x \in [\tau,x_i).\end{aligned}$$

For $x \in [x_i,x_{i+1})$ the Cauchy problem

$$\begin{cases} a^\Delta = M(x)\langle a(x)\rangle + (I - \frac{\mu}{2}M)f(x), \quad x \in (x_i,x_{i+1}),\\ a(x_i^+) = a(x_i) + B_i a(x_i) + c_i,\end{cases}$$

has unique solution

$$a(x) = E_M\big(x,x_i^+\big)a\big(x_i^+\big) + \int_{x_i}^x E_M\big(x,\sigma(s)\big)f(s)\Delta s, \quad x \in [x_i,x_{i+1}).$$

It follows that

$$\begin{aligned}
a(x) &= E_M\bigl(x,x_i^+\bigr)\bigl[(I+B_i)a(x_i)+c_i\bigr] \\
&\quad + \int_{x_i}^{x} E_M\bigl(x,\sigma(s)\bigr)f(s)\Delta s \\
&= E_M\bigl(x,x_i^+\bigr)(I+B_i)E_M(x_i,\tau)\eta \\
&\quad + \int_{\tau}^{x_i} G_M\bigl(x_i,\sigma(s)\bigr)f(s)\Delta s] + E_M\bigl(x,x_i^+\bigr)c_i \\
&\quad + \int_{x_i}^{x} E_M\bigl(x,\sigma(s)\bigr)f(s)\Delta s \\
&= E_M\bigl(x,x_i^+\bigr)(I+B_i)E_M(x_i,\tau)\eta \\
&\quad + \int_{\tau}^{x_i} E_M\bigl(x,x_i^+\bigr)(I+B_i)E_M\bigl(x_i,\sigma(s)\bigr)f(s)\Delta s + E_M\bigl(x,x_i^+\bigr)c_i \\
&\quad + \int_{x_i}^{x} E_M\bigl(x,\sigma(s)\bigr)f(s)\Delta s.
\end{aligned}$$

Using (16), we get that

$$\begin{aligned}
a(x) &= G_M(x,\tau)\eta + \int_{\tau}^{x_k} G_M\bigl(x,\sigma(s)\bigr)f(s)\Delta s \\
&\quad + \int_{x_k}^{x} G_M\bigl(x,\sigma(s)\bigr)f(s)\Delta s + G_M\bigl(x,x_i^+\bigr)c_i,
\end{aligned}$$

and so

$$\begin{aligned}
a(x) &= G_M(x,\tau)\eta + \int_{\tau}^{x} G_M\bigl(x,\sigma(s)\bigr)f(s)\Delta s \\
&\quad + G_M\bigl(x,x_i^+\bigr)c_i, \quad x\in(x_i,x_{i+1}].
\end{aligned}$$

Assume for $k>i+2$, (17) has a solution on an interval $[x_{k-1},x_k)$:

$$\begin{aligned}
a(x) &= G_M(x,\tau)\eta + \int_{\tau}^{x} G_M\bigl(x,\sigma(s)\bigr)f(s)\Delta s \\
&\quad + \sum_{\tau<x_j<x_k} G_M\bigl(x,x_j^+\bigr)c_j, \quad x\in[x_k,x_{k+1}).
\end{aligned}$$

Then

$$a(x) = E_M\bigl(x,x_k^+\bigr)a\bigl(x_k^+\bigr) + \int_{x_k}^{x} E_M\bigl(x,\sigma(s)M(s)f(s)\Delta s, \quad x\in[x_k,x_{k+1}\bigr).$$

is the solution of the following IVP:

$$\begin{cases}
a^{\Delta} = M(x)\langle a(x)\rangle + (I-\frac{\mu}{2}M)f(x), & x\in(x_k,x_{k+1}) \\
a(x_k^+) = a(x_k) + B_k a(x_k) + c_k.
\end{cases}$$

More explicitly, we have

$$\begin{aligned} a(x) &= E_M(x,x_k^+)\big[(I+B_k)a(x_k)+c_k\big] + \int_{x_k}^{x} E_M\big(x,\sigma(s)\big)f(s)\Delta s \\ &= E_M(x,x_k^+)(I+B_k)G_M(x_k,\tau)\eta + \int_{\tau}^{x_k} G_M\big(x_k,\sigma(s)\big)f(s)\Delta s \\ &\quad + \sum_{\tau<x_j<x_k} G_M(x_k,x_j^+)c_j + E_M(x,x_k^+)c_k \\ &\quad + \int_{x_k}^{x} E_M\big(x,\sigma(s)\big)f(s)\Delta s, \end{aligned}$$

Hence, using Remark 4.6, we get

$$\begin{aligned} a(x) &= G_M(x,\tau)\eta + \int_{\tau}^{x_k} G_M\big(x,\sigma(s)\big)f(s)\Delta s \\ &\quad + \int_{x_k}^{x} G_M\big(x,\sigma(s)\big)f(s)\Delta s + \sum_{\tau<x_j<x_k} G_M(x,x_j^+)c_j, \end{aligned}$$

and so

$$a(x) = G_M(x,\tau)\eta + \int_{\tau}^{x} G_M\big(x,\sigma(s)\big)f(s)\Delta s + \sum_{\tau<x_j<x_k} G_M(x,x_j^+)c_j. \qquad \square$$

For $c_k = 0$, for each k

$$a(x) = G_M(x,\tau)\eta + \int_{\tau}^{x} G_M\big(x,\sigma(s)\big)f(s)\Delta s, \quad x \in \mathbb{T}_{(\tau)}$$

is the only solution for the following IVP:

Corollary 4.10 *If* (H) *holds, then the IVP* (11) *has at most one solution, given by*

$$a(x) = G_M(x,\tau)\eta, \quad x \geq \tau, \textit{ for each } (\tau,\eta) \in \mathbb{T}_+ \times \mathbb{R}^n.$$

Corollary 4.11 *If* (H) *holds, then generalized transition matrix* $G_M(x,y)$, $0 \leq y \leq x$, *uniquely satisfied IVP*:

$$\begin{cases} Y^{\Delta}(x) = M(x)\langle Y(x)\rangle, & x \in \mathbb{T}_{(+),}\, x \neq x_k, \\ Y(x_k^+) = (I+B_k)Y(x_k), & k = 1,2,\dots, \\ Y(y^+) = I, & y \geq 0. \end{cases} \tag{19}$$

Moreover, the following properties hold:

1. $G_M(x_i^+,s) = (I+B_i)G_M(x_i,s)$, $x_i \geq s$, $i = 1,2,3,\dots$;
2. If $(I+B_i)^{-1}$ exists for each i, then $G_M(x,x_i^+) = G_M(x,x_i)(I+B_i)^{-1}$, $x_i \leq x$, $i = 1,2,3,\dots$;
3. $G_M(x,x_i^+)G_M(x_i^+,y) = G_M(x,y)$, $0 \leq y \leq x_i \leq x$, $i = 1,2,3,\dots$.

Corollary 4.12 *If (H) holds, then we have the following estimate*

$$\left\| G_M(x,\tau)\right\| \leq \prod_{\tau<x_k\leq x}\left(I+\|B_k\|\right)\exp\left(\int_\tau^x \frac{\|M(s)\|}{1-\frac{\mu(s)\|M(s)\|}{2}}\Delta s\right), \qquad \mu(s)>0, \tag{20}$$

for $\tau, x\in\mathbb{T}_+$ with $x\geq\tau$.

Remark 4.13 From inequality (20), it is easy to see that G_M is a bounded operator.

5 Boundedness and exponential stability

Lemma 5.1 *For any constant $p>0$, such that $p\in\mathcal{R}_+^C(\mathbb{T}_+,\mathbb{R})$. Then*

$$1+p(x-s)\leq E_p(x,s) \quad \textit{for all } x\geq s.$$

Proof Let $a(x)=p(x-s)$, then it implies that $\langle a(x)\rangle = \frac{p}{2}(x^\sigma + x-2s)$. Since $x\geq s$, it follows that $x^\sigma + x - 2s\geq 0$. Furthermore

$$p\bigl\langle a(x)\bigr\rangle + p = \frac{p^2}{2}\bigl(x^\sigma + x-2s\bigr)+p\geq p = a^\Delta(x)$$

or

$$a^\Delta(x)\leq p\bigl\langle a(x)\bigr\rangle + p.$$

By applying [26, Lemma 3.1], we have

$$a(x)\leq a(s)E_p(x,s)+\int_s^x p\bigl\langle E_{-p}(r,x)\bigr\rangle\Delta r.$$

As $a(s)=0$ and $\int_s^x p\langle E_{-p}(r,x)\rangle\Delta r = E_{-p}(r,x)-1$. After simplification, we obtain

$$1+p(x-s)\leq E_p(x,s).$$

□

Lemma 5.2 *For any constant $p>0$, such that $-p\in\mathcal{R}_+^C(\mathbb{T}_+,\mathbb{R})$. Then we have*

$$E_{-p}(x,s)\leq e^{\frac{-p}{2}(x-s)} \quad \textit{for all } x\geq s.$$

Theorem 5.3 *Let (H) holds and $\exists\,\theta>0$ such that $x_{i+1}-x_i<\theta$, $i=1,2,3,\dots$. If the solution of IVP*

$$\begin{cases} a^\Delta(x)=M(x)\langle a(x)\rangle,\ x\in\mathbb{T}_{(\tau),} \quad x\neq x_k,\\ a(x_k^+)=(I+B_k)a(x_k)+c_k, \quad k=1,2,\dots,\\ a(\tau^+)=0, \quad \tau\geq 0,\end{cases} \tag{21}$$

is bounded for every $c\in l^\infty(\mathbb{R})^n$, then $\exists\, N=N(\tau)\geq 1$ with $-\lambda\in\mathcal{R}_+^C(\mathbb{T}_+,\mathbb{R})$ such that

$$\left\| G_M(x,\tau)\right\| \leq NE_{-\lambda}(x,\tau) \quad \textit{for all } x\in\mathbb{T}_{(\tau)}.$$

Proof Theorem 4.9, yields

$$a(x) = \sum_{\tau < x_j < x} G_M\bigl(x, x_j^+\bigr) c_j, \quad x \in \mathbb{T}_{(\tau)}.$$

By using similar estimation as given in [22, Theorem 5.1], we can obtain:

$$\bigl\| G_M(x,\tau) \bigr\| \leq \frac{K^4}{(K-1)2} \beta_i \left(1 - \frac{1}{K}\right)^{\frac{1}{\theta}(x-\tau)} \quad \text{for } x \in [x_{i+j}, x_{i+j+1}).$$

Let us consider the positive function $\lambda(x)$, with $-\lambda(x) \in \mathcal{R}^+$, as the solution of inequality

$$E_{-\lambda}(x,\tau) \geq \left(1 - \frac{1}{k}\right)^{\frac{1}{\theta}(x-\tau)}, \quad \text{for } x \in [x_{i+j}, x_{i+j+1}),$$

where $E_{-\lambda}(x,\tau)$ is a Cayley exponential function. Set

$$N = \max\left\{ \frac{K^4}{(K-1)2} \beta_i, \sup_{\tau < x < x_i} \frac{\|G_M(x,\tau)\|}{E_{-\lambda}(x,\tau)} \right\}.$$

Then for all $x, \tau \in \mathbb{T}_+$, we have

$$\bigl\| G_M(x,\tau) \bigr\| \leq N E_{-\lambda}(x,\tau),$$

and so the theorem is proved. □

Corollary 5.4 *Let* (H) *holds and* $\exists\, \theta > 0$ *such that* $x_{i+1} - x_i < \theta$, $i = 1, 2, 3, \ldots$. *Then the boundedness of* (21), *for every* $c := \{c_k\}_{k=1}^{\infty}, \in l^{\infty}(\mathbb{R})^n$, *implies the exponentially stability of* (11).

Proof Theorem 5.3 implies that $(\exists)$ $N = N(\tau) \geq 1$, λ with $-\lambda \in \mathbb{R}^+$ such that

$$\bigl\| G_M(x,\tau) a(\tau) \bigr\| \leq N E_{-\lambda}(x,\tau) \quad \text{for all } x \in \mathbb{T}_{(\tau)}.$$

For any $\tau \in \mathbb{T}_+$, the solution of (11) satisfies

$$\bigl\| a(x) \bigr\| = \bigl\| G_M(x,\tau) a(\tau) \bigr\| \leq \bigl\| G_M(x,\tau) \bigr\| . \bigl\| a(\tau) \bigr\| \leq N \bigl\| a(\tau) \bigr\| E_{-\lambda}(x,\tau)$$

for all $x \in \mathbb{T}_{(\tau)}$. □

Lemma 5.5 *If* $\exists\, \theta > 0$ *such that* $x_{i+1} - x_i < \theta$, $\forall\, i = 1, 2, 3, \ldots$, *then for every positive number* λ *with* $\theta < \frac{1}{\lambda}$, *we have* $-\lambda \in \mathcal{R}_+^C$.

Proof Since $x_i^{\sigma} = x_i$ $\forall i = 1, 2, 3, \ldots$, then for $x \in [x_i, x_{i+1}]$, we have that $x_i \leq x \leq \sigma(x) \leq x_{i+1}$.

It follows that $\mu(x) = \sigma(x) - x \leq x_{i+1} - x_i < \theta$ for $x \in [x_i, x_{i+1}]$. Therefore, $\mu(x) < \theta$ for $x \in \mathbb{T}_+$. If $\theta < \frac{1}{\lambda}$, then we have that

$2 - \lambda\mu(x) > 2 - \frac{1}{\theta}\mu(x) > 0$ and thus $-\lambda \in \mathcal{R}_+^C$. □

Theorem 5.6 *Let* (H) *holds and* $\exists\,\theta > 0$ *such that* $x_{i+1} - x_i < \theta$, $i = 1, 2, 3, \dots$, *and*

$$\sup_{i\geq 1} \|B_i\| \leq b, \quad \int_{x_i}^{x_{i+1}} \frac{\|M(s)\|}{1 - \frac{\mu(s)\|M(s)\|}{2}} \Delta s \leq M, \quad i = 1, 2, \dots. \tag{22}$$

Then the IVP (21) *is bounded for any* $c \in l^\infty(\mathbb{R})^n$, *implies that* $\exists\, N > 0$, $\lambda > 0$ *with* $-\lambda \in \mathcal{R}_+^C$:

$$\left\| G_M(x,\tau) \right\| \leq N E_{-\lambda}(x,\tau) \quad \textit{for all } x \in \mathbb{T}_{(\tau)}.$$

Proof By using Lemma 5.1, Lemma 5.5, and [22, Theorem 5.2], we can obtain

$$\left\| G_M(x,\tau) \right\| \leq N E_{-\lambda}(x,\tau) \quad \text{for all } x \in \mathbb{T}_{(\tau)}. \qquad \square$$

Example 5.7 Let us consider the following linear Cayley-type dynamic system on the time scale $P_{1,1} := \bigcup_{k=0}^{\infty} [2k, 2k+1]$

$$\begin{cases} a^\Delta(x) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} \langle a(x) \rangle + \begin{pmatrix} (1+\frac{\mu\alpha}{2}) E_{-\alpha}(x^\sigma,\tau) \\ (1+\frac{\mu\beta}{2}) E_{-\beta}(x^\sigma,\tau) \end{pmatrix}, & x \neq x_k; \\ a((2k)^+) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix} a(2k), \quad k = 1, 2, \dots; \\ a(\tau) = \eta, \quad \tau \geq 0 \end{cases} \tag{23}$$

with $\beta > \alpha > 0$, and $(a_{2k})_{k\geq 1}$ and $(b_{2k})_{k\geq 1}$ such that $a_{ij} := \prod_{l=1}^{j} a_{2(i+l)} < a$, $b_{ij} := \prod_{l=1}^{j} a_{2(i+l)} < b$ for each fixed $i \geq 1$, and $j = 1, 2, \dots$.

Since (H) holds, then the generalized exponential matrix $G_M(x,\tau)$, $0 \leq \tau \leq x$, for impulsive effects $\{B_i, x_i\}_{i=1}^{\infty}$ is given by

$$G_M(x,\tau) = \begin{pmatrix} a_{ij} E_{-\alpha}(x,\tau) & 0 \\ 0 & b_{ij} E_{-\beta}(x,\tau) \end{pmatrix},$$

where $E_p(x,\tau) = e_{\frac{2p}{2+\mu p}}(x,\tau) = (1 + \frac{2p}{2+\mu p})^j e^{\frac{2p(x-\tau)}{2+\mu p}} e^{-\frac{2pj}{2+\mu p}}$, for $\tau \in [2i, 2i+2)$ and $x \in [2(i+j), 2(i+j+1)]$. It follows that

$$\left\| G_M(x,\tau) \right\| \leq K E_{-\alpha}(x,\tau),$$

where $K := \max\{a, b\}$.

The solution of (23) is given by

$$\begin{aligned} a(x) &= \begin{pmatrix} a_{ij} E_{-\alpha}(x,\tau) & 0 \\ 0 & b_{ij} E_{-\beta}(x,\tau) \end{pmatrix} \eta \\ &\quad + \int_\tau^x \begin{pmatrix} a_{ij} E_{-\alpha}(x,s^\sigma) & 0 \\ 0 & b_{ij} E_{-\beta}(x,s^\sigma) \end{pmatrix} \begin{pmatrix} E_{-\alpha}(s^\sigma,\tau) \\ E_{-\beta}(s^\sigma,\tau) \end{pmatrix} \Delta s \\ &= \begin{pmatrix} a_{ij} E_{-\alpha}(x,\tau) & 0 \\ 0 & b_{ij} E_{-\beta}(x,\tau) \end{pmatrix} \eta + \begin{pmatrix} E_{-\alpha}(x,\tau) \int_\tau^x a_{ij} \Delta s \\ E_{-\beta}(x,\tau) \int_\tau^x b_{ij} \Delta s \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} a_{ij}E_{-\alpha}(x,\tau)(\eta_1 + (x-\tau)) \\ b_{ij}E_{-\beta}(x,\tau)(\eta_2 + (x-\tau)) \end{pmatrix}.$$

In particular, if we consider the following system:

$$\begin{cases} a^{\Delta}(x) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} \langle a(x) \rangle, \quad x \neq x_k; \\ a((2k)^+) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix} a(2k) + c_{2k}, \quad k = 1,2,\ldots; \\ a(\tau) = 0, \quad \tau \geq 0, \end{cases} \tag{24}$$

then the solution is

$$a(x) = \sum_{l=1}^{j} G_M\big(x, \big(2(i+l)\big)^+\big) c_{2(i+l)}.$$

Moreover, it is easy to see that the solution of (24) is bounded for any $c = \{c_k\}_{k=1}^{\infty} \in l^{\infty}(\mathbb{R}^n)$. Consequently, the following impulsive dynamic system

$$\begin{cases} a^{\Delta}(x) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} \langle a(x) \rangle, \quad x \neq x_k; \\ a((2k)^+) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix} a(2k), \quad k = 1,2,\ldots; \\ a(\tau) = \eta, \quad \tau \geq 0, \end{cases}$$

is uniformly exponentially stable.

6 Conclusions and future directions

We proposed the solution of a linear time-varying Cayley impulsive dynamic system on time scales. We have introduced Cayley regressive matrices. We proved the basic properties of the transition matrix and the impulsive transition matrix. We have also given some estimates for the solution of the Cayley impulsive dynamic systems. We have established the necessary and sufficient conditions for exponential stability and boundedness.

Moreover, these results will be very useful for the analysis and synthesis of impulsive control systems on time scales. The discussion of stability and Hyers–Ulam stability for Cayley impulsive dynamic systems is possible. We can use the results for the study of the transfer function for linear Cayley dynamic systems.

Acknowledgements
The authors A. ALoqaily, N. Mlaiki and T. Abdeljawad would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Funding
This research did not receive any external funding.

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
A.Y., G.A., A.A., T.A. and N.M wrote the main manuscript text. All authors reviewed the manuscript.

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Received: 10 November 2022 Accepted: 26 May 2023 Published online: 05 June 2023

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