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Convergence of linear processes generated by negatively dependent random variables under sub-linear expectations

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Abstract

In this paper, we study the complete convergence and complete moment convergence of linear processes generated by negatively dependent random variables under sub-linear expectations. The obtained results complement the ones of Meng, Wang, and Wu (Commun. Stat., Theory Methods 52(9):2931–2945, 2023) in the case of negatively dependent random variables under sub-linear expectations.

MSC: 60F15; 60F05

Keywords: Negatively dependent random variables; Linear processes; Complete convergence; Complete moment convergence; Sub-linear expectation

1 Introduction

To investigate the uncertainty in probability, Peng [11–13] introduced the important concepts of the sub-linear expectations space. Peng's works [11–13] motivated many scientists to extend the results of classic probability space to that of the sub-linear expectations space. Zhang [27–29] obtained the exponential inequalities, Rosenthal's inequalities, and Donsker's invariance principle under sub-linear expectations. Wu [16] studied precise asymptotics for complete integral convergence under sub-linear expectations. Under sub-linear expectations, Xu and Cheng [24] investigated how small the increments of G-Brownian motion are. Xu and Zhang [18, 19] studied a three-series theorem of independent random variables and a law of logarithm for arrays of row-wise extended negatively dependent random variables under the sub-linear expectations. Zhong and Wu [33] obtained the complete convergence and complete moment convergence for weighted sums of extended negatively dependent random variables under sub-linear expectations. For more limit theorems under sub-linear expectations, interested readers could refer to Wu and Jiang [17], Huang and Wu [8], Zhang and Lin [31], Zhong and Wu [33], Hu and Yang [7], Chen [2], Zhang [30], Hu, Chen, and Zhang [6], Gao and Xu [3], Kuczmaszewska [9], Chen and Wu [1], Xu and Cheng [20–24, 26], and the references therein.

Meng, Wang, and Wu [10] studied convergence of asymptotically almost negatively associated random variables with random coefficients. Hosseini and Nezakati [4] obtained complete moment convergence of the dependent linear processes with random coefficients. For more results about complete moment convergence of moving average processes, interested readers could refer to Zhang and Ding [32], Hosseini and Nezakati [5], and the refercences therein. The main conclusions of Meng, Wang, and Wu [10] are that under proper conditions, the complete convergence and complete moment convergence of almost negatively associated random variables with random coefficients hold. It is natural to ask whether or not the relevant results hold under sub-linear expectations. Here, we try to establish convergence of linear processes generated by negatively dependent random variables under sub-linear expectations, complementing the corresponding results obtained in Meng, Wang, and Wu [10]. In other words, we here investigate the complete convergence and complete moment convergence related to the infinitely weighted sums of negatively dependent random variables under sub-linear expectations. Our results differ from that of Xu et al. [25], Xu and Kong [26], who studied the complete convergence and complete moment convergence relevant to finitely weighted sums of negatively dependent random variables under sub-linear expectations.

We organize the rest of this paper as follows. We give some necessary basic notions, concepts, and corresponding properties, and cite necessary lemma under sub-linear expectations in Sect. 2. In Sect. 3, we give our main results, Theorems 3.1-3.3, their proofs will be presented in Sect. 4. Some of the lemmas and their proofs are also given in Sect. 4.

2 Preliminaries

As in Xu and Cheng [21], we use similar notations as in the work by Peng [12, 13], Chen [2], and Zhang [29]. Suppose that $(\Omega,\mathcal{F})$ is a given measurable space. Assume that $\mathcal{H}$ is a subset of all random variables on $(\Omega,\mathcal{F})$ such that $X_1,\ldots,X_n\in\mathcal{H}$ implies $\varphi(X_1,\ldots,X_n)\in\mathcal{H}$ for each $\varphi\in\mathcal{C}_{l,Lip}(\mathbb{R}^n)$, where $\mathcal{C}_{l,Lip}(\mathbb{R}^n)$ represents the linear space of (local Lipschitz) function φ fulfilling

$$\left|\varphi(\mathbf{x})-\varphi(\mathbf{y})\right|\le C\left(1+|\mathbf{x}|^m+|\mathbf{y}|^m\right)\left(|\mathbf{x}-\mathbf{y}|\right),\quad \forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n$$

for some $C>0$, $m\in\mathbb{N}$ depending on φ.

Definition 2.1 A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E}:\mathcal{H}\mapsto\bar{\mathbb{R}}:=[-\infty,\infty]$ fulfilling the following properties: for all $X,Y\in\mathcal{H}$, we have

(a) Monotonicity: If $X\ge Y$, then $\mathbb{E}[X]\ge\mathbb{E}[Y]$;
(b) Constant preserving: $\mathbb{E}[c]=c$, $\forall c\in\mathbb{R}$;
(c) Positive homogeneity: $\mathbb{E}[\lambda X]=\lambda\mathbb{E}[X]$, $\forall\lambda\ge 0$;
(d) Sub-additivity: $\mathbb{E}[X+Y]\le\mathbb{E}[X]+\mathbb{E}[Y]$ whenever $\mathbb{E}[X]+\mathbb{E}[Y]$ is not of the form $+\infty-\infty$ or $-\infty+\infty$.

Remark 2.1 By Theorem 1.2.1 of Peng [13], the sub-linear expectation $\mathbb{E}$ could be represented as a supremum of linear expectations. By Theorem 1.2.2 of Peng [13], if $X_i(\omega)\downarrow 0$ as $i\to\infty$ implies $\lim_{i\to\infty}\mathbb{E}(X_i)=0$, then the sub-linear expectation $\mathbb{E}$ is the upper expectation, i.e., there exists a set of probability measures $\mathscr{P}$ in $(\Omega,\mathcal{F})$ such that $\mathbb{E}(\xi)=\sup_{P\in\mathscr{P}}\mathbf{E}_P(\xi)$, $\xi\in\mathcal{H}$, where $\mathbf{E}_P$ denotes the linear expectation under probability P. Hence, the difference between the sub-linear expectation and the upper expectation is that the former extends the later in some sense.

A set function $V : \mathcal{F} \mapsto [0,1]$ is named to be a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \leq V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

A capacity V is called sub-additive if $V(A \cup B) \leq V(A) + V(B)$, $A, B \in \mathcal{F}$.

In what follows, given a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\} = \mathbb{E}[I_A]$, $\forall A \in \mathcal{F}$ (see (2.3) and the definitions of $\mathbb{V}$ above (2.3) in Zhang [28]). $\mathbb{V}$ is a sub-additive capacity. Set

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\,\mathrm{d}x + \int_{-\infty}^0 \bigl(\mathbb{V}(X > x) - 1\bigr)\,\mathrm{d}x.$$

As in 4.3 of Zhang [28], throughout this paper, define an extension of $\mathbb{E}$ on the space of all random variables by

$$\mathbb{E}^*(X) = \inf\bigl\{\mathbb{E}[Y] : X \leq Y, Y \in \mathcal{H}\bigr\}.$$

Then $\mathbb{E}^*$ is a sub-linear expectation on the space of all random variables, $\mathbb{E}[X] = \mathbb{E}^*[X]$, $\forall X \in \mathcal{H}$, and $\mathbb{V}(A) = \mathbb{E}^*(I_A)$, $\forall A \in \mathcal{F}$.

Suppose that $\mathbf{X} = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $\mathbf{Y}$ is said to be negatively dependent on $\mathbf{X}$, if for each function $\psi_1 \in \mathcal{C}_{l,Lip}(\mathbb{R}^m)$, $\psi_2 \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)$, we have $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] \leq \mathbb{E}[\psi_1(\mathbf{X})]\mathbb{E}[\psi_2(\mathbf{Y})]$ whenever $\psi_1(\mathbf{X}) \geq 0$, $\mathbb{E}[\psi_2(\mathbf{Y})] \geq 0$, $\mathbb{E}[|\psi_1(\mathbf{X})\psi_2(\mathbf{Y})|] < \infty$, $\mathbb{E}[|\psi_1(\mathbf{X})|] < \infty$, $\mathbb{E}[|\psi_2(\mathbf{Y})|] < \infty$, and either ψ_1 and ψ_2 are coordinatewise nondecreasing, or ψ_1 and ψ_2 are coordinatewise nonincreasing (see Definition 2.3 of Zhang [28], Definition 1.5 of Zhang [29], Definition 2.5 in Chen [2]). $\{X_n; -\infty < n < \infty\}$ is called to be negatively dependent, if X_{l+n} is negatively dependent on $(X_l, \ldots, X_{l+n-1})$ for each $n \geq 1$, $-\infty < l < +\infty$. The existence of negatively dependent random variables $\{X_n; -\infty < n < \infty\}$ under sub-linear expectations could be yielded by Example 1.6 of Zhang [29] and Kolmogorov's existence theorem in classic probability space. As in discussed in Zhang [28], if $\{X_n\}_{n=-\infty}^{+\infty}$ is sequence of negatively dependent random variables, and $f_n(x) \in \mathcal{C}_{l,Lip}(\mathbb{R})$, $-\infty < n < +\infty$ are nondecreasing (resp. nonincreasing) functions, then $\{f_n(X_n); -\infty < n < \infty\}$ is also a sequence of negatively dependent random variables.

Assume that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors defined in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed if for every function $\psi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)$ such that $\psi(\mathbf{X}_1) \in \mathcal{H}_1$, $\psi(\mathbf{X}_2) \in \mathcal{H}_2$,

$$\mathbb{E}_1\bigl[\psi(\mathbf{X}_1)\bigr] = \mathbb{E}_2\bigl[\psi(\mathbf{X}_2)\bigr],$$

whenever the sub-linear expectations are finite. $\{X_n\}_{n=1}^\infty$ is named to be identically distributed if for each $i \geq 1$, X_i and X_1 are identically distributed.

In the paper we assume that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^\infty \mathbb{E}(X_n)$, whenever $X \leq \sum_{n=1}^\infty X_n$, $X, X_n \in \mathcal{H}$, and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Hence $\mathbb{E}^*$ is also countably sub-additive. Let C stand for a positive constant, which may change from place to place. $I(A)$ or I_A represent the indicator function of A. Write $\log(x) = \ln\max\{\mathrm{e}, x\}$, $x > 0$.

We cite the following lemma.

Lemma 2.1 (See Lemma 2.4 of Xu et al. [25]) *Let $\{X_n; n \geq 1\}$ be a sequence of negatively dependent random variables in sub-linear expectations space $(\Omega, \mathcal{H}, \mathbb{E})$. Then there exists a positive constant $C = C_p$ depending on p such that for $n \geq 1$, and $1 \leq p \leq 2$,*

$$\mathbb{E}\left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right] \leq C_p(\log n)^p\left[\sum_{k=1}^{n}\mathbb{E}\left[|X_k|^p\right] + \left(\sum_{k=1}^{n}\left[\left|\mathbb{E}(X_k)\right| + \left|\mathbb{E}(-X_k)\right|\right]\right)^p\right]. \quad (2.1)$$

We also present a useful lemma.

Lemma 2.2 *If for a random variable X on $(\Omega, \mathcal{F})$, $C_{\mathbb{V}}\{|X|\} < \infty$, then*

$$\mathbb{E}^*\left[|X|\right] \leq C_{\mathbb{V}}\left\{|X|\right\}.$$

Proof The proof is similar to that of Lemma 4.5 of Zhang [28]. By $C_{\mathbb{V}}\{|X|\} < \infty$, we see that $\lim_{n\to\infty} n\mathbb{V}\{|X| > n\} = 0$. Since

$$\begin{aligned}
|X| &= \lim_{n\to\infty}\sum_{k=1}^{n}|X|I\left\{k-1 < |X| \leq k\right\} \\
&\leq \lim_{n\to\infty}\sum_{k=1}^{n}kI\left\{k-1 < |X| \leq k\right\} = \lim_{n\to\infty}\sum_{k=1}^{n}k\left(I\left\{|X| > k-1\right\} - I\left\{|X| > k\right\}\right) \\
&\leq \lim_{n\to\infty} 1 + \sum_{k=1}^{n-1}I\left\{|X| > k\right\} - nI\left\{|X| > n\right\} \leq 1 + \sum_{k=1}^{\infty}I\left\{|X| > k\right\},
\end{aligned}$$

by the countable sub-additivity of $\mathbb{E}^*$, we obtain

$$\mathbb{E}^*\left[|X|\right] \leq 1 + \sum_{k=1}^{\infty}\mathbb{E}^*\left[I\left\{|X| > k\right\}\right] = 1 + \sum_{k=1}^{\infty}\mathbb{V}\left\{|X| > k\right\} \leq 1 + C_{\mathbb{V}}\left\{|X|\right\}.$$

By considering $|X|/\epsilon$ instead of $|X|$, we have

$$\mathbb{E}^*\left[|X|/\epsilon\right] \leq 1 + C_{\mathbb{V}}\left\{|X|/\epsilon\right\}, \qquad \mathbb{E}^*\left[|X|\right] \leq \epsilon + C_{\mathbb{V}}\left\{|X|\right\},$$

which by taking $\epsilon \to 0$ implies

$$\mathbb{E}^*\left[|X|\right] \leq C_{\mathbb{V}}\left\{|X|\right\}.$$

The proof is finished. □

3 Main results

Our main results, considered as an extension of Chen and Wu [1] in some way, are as follows.

Theorem 3.1 *Suppose that $\gamma > 1$, $1 \leq p < 2$, $1 < \gamma p < q \leq 2$, and $X_t = \sum_{i=-\infty}^{\infty} a_i\epsilon_{t-i}$ is a linear process in sub-linear expectations space $(\Omega, \mathcal{H}, \mathbb{E})$. Let $\{\epsilon_n, n \in \mathbb{Z}\}$ be a sequence of negatively dependent random variables, identically distributed as ϵ with $\mathbb{E}(\epsilon) = \mathbb{E}(-\epsilon) = 0$*

in sub-linear expectations space $(\Omega, \mathcal{H}, \mathbb{E})$. *Suppose that* $\{a_n; n \in \mathbb{Z}\}$ *is a sequence of constants satisfying* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. *If* $C_{\mathbb{V}}(|\epsilon|^{\gamma p}(\log|\epsilon|)^q) < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\gamma-2} \mathbb{V}\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| > \varepsilon n^{1/p}\right) < \infty. \tag{3.1}$$

Theorem 3.2 *Assume that* $\alpha > 0$, $1 < p < q \leq 2$, *and* $X_t = \sum_{i=-\infty}^{\infty} a_i \epsilon_{t-i}$ *is a linear process in sub-linear expectations space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Let* $\{\epsilon_n, n \in \mathbb{Z}\}$ *be a sequence of negatively dependent random variables, identically distributed as* ϵ *with* $\mathbb{E}(\epsilon) = \mathbb{E}(-\epsilon) = 0$ *in sub-linear expectations space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Suppose that* $\{a_n; n \in \mathbb{Z}\}$ *is a sequence of constants satisfying* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. *If* $C_{\mathbb{V}}(|\epsilon|^p \log^q(|\epsilon|)) < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \mathbb{E}^*\left(\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| - \varepsilon n^{\alpha}\right)^+\right) < \infty. \tag{3.2}$$

Remark 3.1 Under the assumptions of Theorem 3.2, we see that

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \mathbb{E}^*\left(\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| - \varepsilon n^{\alpha}\right)^+\right)\\ &\geq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \mathbb{E}^*\left(\varepsilon n^{\alpha} I\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| > 2\varepsilon n^{\alpha}\right)\right)\\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{V}\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| > 2\varepsilon n^{\alpha}\right), \quad \text{for all } \varepsilon > 0.\end{aligned} \tag{3.3}$$

Therefore, from (3.3), we conclude that the complete moment convergence implies the complete convergence. In particular, set $\alpha p = 1$, $1 < p < q \leq 2$, we conclude that

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| > \varepsilon n^{1/p}\right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{3.4}$$

Theorem 3.3 *Suppose that* $X_t = \sum_{i=-\infty}^{\infty} a_i \epsilon_{t-i}$ *is a linear process in sub-linear expectations space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Assume that* $\{\epsilon_n, n \in \mathbb{Z}\}$ *is a sequence of negatively dependent random variables, identically distributed as* ϵ *with* $\mathbb{E}(\epsilon) = \mathbb{E}(-\epsilon) = 0$ *in sub-linear expectations space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Suppose that* $\{a_n; n \in \mathbb{Z}\}$ *is a sequence of constants satisfying* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. *If* $C_{\mathbb{V}}(|\epsilon| \log^2(1 + |\epsilon|)) < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-2} \mathbb{E}^*\left(\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| - \varepsilon n\right)^+\right) < \infty. \tag{3.5}$$

Remark 3.2 Under the assumptions of Theorem 3.3, we can conclude that

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} X_t\right| > \varepsilon n\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{3.6}$$

In fact, if the assumptions of Theorem 3.2 hold, then (3.4) yields (3.6). However, the assumptions of Theorem 3.3 are weaker than that of Theorem 3.2.

By (3.6), we obtain the following corollary.

Corollary 3.1 *If the assumptions of Theorem 3.3 hold, then*

$$\mathbb{V}\left(\lim_{n\to\infty} n^{-1}\sum_{t=1}^{n} X_t \neq 0\right) = 0.$$

By similar proofs of Theorems 3.1, 3.2, 3.3, Remarks 3.1, 3.2, and Corollary 3.1, we could obtain the following.

Remark 3.3 Suppose that in Theorems 3.1, 3.2, 3.3, Remarks 3.1, 3.2, and Corollary 3.1, with the conditions that there exists a constant C satisfying $\mathbb{E}(h(\epsilon_n)) \le C\mathbb{E}(h(\epsilon))$, for all $n \in \mathbb{Z}$, $0 \le h \in \mathcal{C}_{l,Lip}(\mathbb{R})$, and $\mathbb{E}(-\epsilon_n) = \mathbb{E}(\epsilon_n) = 0$ in place of the assumption that for each $n \in \mathbb{Z}$, ϵ_n is identically distributed as ϵ, the other conditions remain the same. Then the conclusions also hold in Theorems 3.1, 3.2, 3.3, Remarks 3.1, 3.2, Corollary 3.1. Moreover, in what follows, the assumption that $\mathbb{E}$ is countably sub-additive could be replaced by that $\mathbb{E}^*$ is countably sub-additive, all conclusions also hold in Theorems 3.1, 3.2, 3.3, Remarks 3.1, 3.2, and Corollary 3.1.

4 Proofs

To prove our main results, we need the lemmas below.

Lemma 4.1 *Suppose that* $1 \le p \le 2$ *and* $\{\epsilon_n, n \in \mathbb{Z}\}$ *is a sequence of negatively dependent random variables in sub-linear expectations space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Assume that* $\{a_n; n \in \mathbb{Z}\}$ *is a sequence of constants satisfying* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. *Then*

$$\begin{aligned}
&\mathbb{E}^*\left(\max_{1<m\le n}\left|\sum_{j=-\infty}^{\infty}\left(\sum_{i=1-j}^{m-j} a_i\right)\epsilon_j\right|^p\right) \\
&\quad\le C(\log n)^p \sup_{i\in\mathbb{Z}}\left\{\sum_{j=1-i}^{n-i}\mathbb{E}|\epsilon_j|^p + \left(\sum_{j=1-i}^{n-i}\left[|\mathbb{E}\epsilon_j| + \left|\mathbb{E}(-\epsilon_j)\right|\right]\right)^p\right\}. \qquad (4.1)
\end{aligned}$$

Proof By Hölder's inequality, countable sub-additivity of $\mathbb{E}^*$, and Lemma 2.1, we obtain for $p > 1$,

$$\begin{aligned}
&\mathbb{E}^*\left(\max_{1<m\le n}\left|\sum_{j=-\infty}^{\infty}\left(\sum_{i=1-j}^{m-j} a_i\right)\epsilon_j\right|^p\right) \\
&\quad= \mathbb{E}^*\left(\max_{1<m\le n}\left|\sum_{i=-\infty}^{\infty}\left(\sum_{j=1-i}^{m-i} \epsilon_j\right)a_i\right|^p\right) \le \mathbb{E}^*\left(\left|\sum_{i=-\infty}^{\infty}|a_i|\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|\right|^p\right) \\
&\quad= \mathbb{E}^*\left(\left|\sum_{i=-\infty}^{\infty}|a_i|^{1-1/p}|a_i|^{1/p}\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|\right|^p\right) \\
&\quad\le \mathbb{E}^*\left(\left|\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{1-1/p}\left(\sum_{i=-\infty}^{\infty}|a_i|\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|^p\right)^{1/p}\right|^p\right)
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}^*\left(\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|^p\right)=C\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}\left(\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|^p\right)\\
&\le C\sum_{i=-\infty}^{\infty}|a_i|(\log n)^p\left[\sum_{j=1-i}^{n-i}\mathbb{E}|\epsilon_j|^p+\left(\sum_{j=1-i}^{n-i}\bigl[|\mathbb{E}\epsilon_j|+\bigl|\mathbb{E}(-\epsilon_j)\bigr|\bigr]\right)^p\right]\\
&\le C(\log n)^p\sum_{i=-\infty}^{\infty}|a_i|\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}|\epsilon_j|^p+\left(\sum_{j=1-i}^{n-i}\bigl[|\mathbb{E}\epsilon_j|+\bigl|\mathbb{E}(-\epsilon_j)\bigr|\bigr]\right)^p\right]\\
&\le C(\log n)^p\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}|\epsilon_j|^p+\left(\sum_{j=1-i}^{n-i}\bigl[|\mathbb{E}\epsilon_j|+\bigl|\mathbb{E}(-\epsilon_j)\bigr|\bigr]\right)^p\right],
\end{aligned}
$$

while $p=1$, by the countable sub-additivity of $\mathbb{E}^*$ and Lemma 2.1, we obtain

$$
\begin{aligned}
&\mathbb{E}^*\left(\max_{1<m\le n}\left|\sum_{j=-\infty}^{\infty}\left(\sum_{i=1-j}^{m-j}a_i\right)\epsilon_j\right|\right)\\
&=\mathbb{E}^*\left(\max_{1<m\le n}\left|\sum_{i=-\infty}^{\infty}\left(\sum_{j=1-i}^{m-i}\epsilon_j\right)a_i\right|\right)\\
&\le\mathbb{E}^*\left(\left|\sum_{i=-\infty}^{\infty}|a_i|\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|\right|\right)\le\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}^*\left(\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|\right)\\
&=\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}\left(\max_{1<m\le n}\left|\sum_{j=1-i}^{m-i}\epsilon_j\right|\right)\\
&\le C(\log n)\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}|\epsilon_j|+\left(\sum_{j=1-i}^{n-i}\bigl[|\mathbb{E}\epsilon_j|+\bigl|\mathbb{E}(-\epsilon_j)\bigr|\bigr]\right)\right].
\end{aligned}
$$

Hence the proof of this Lemma is finished. □

Lemma 4.2 *Suppose that* $\{X_n;n\ge 1\}$ *and* $\{Y_n;n\ge 1\}$ *are two sequences of random variables on* $(\Omega,\mathcal{F})$. *Then for any* $n\ge 1$, $M>1$, $\varepsilon>0$, *and* $a>0$,

$$
\begin{aligned}
\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j}(X_t+Y_t)\right|-\varepsilon a\right)^+\le\left(\frac{1}{\varepsilon^M}+\frac{1}{M-1}\right)\frac{1}{a^{M-1}}\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j}X_t\right|^M\right)\\
+\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j}Y_t\right|\right).
\end{aligned}
\tag{4.2}
$$

Proof By Lemma 2.2 of Sung [15], we obtain

$$
\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j}(X_t+Y_t)\right|-\varepsilon a\right)^+\le\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j}X_t\right|-\varepsilon a\right)^++\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j}Y_t\right|\right).\tag{4.3}
$$

On the other hand, by Markov's inequality under sub-linear expectations and Lemma 2.2, we see that

$$\begin{aligned}
&\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right| - \varepsilon a\right)^+ \\
&\le C_{\mathbb{V}}\left(\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right| - \varepsilon a\right)^+\right) = \int_0^\infty \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right| - \varepsilon a > t\right) \mathrm{d}t \\
&= \int_0^a \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right| > \varepsilon a + t\right) \mathrm{d}t + \int_a^\infty \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right| > \varepsilon a + t\right) \mathrm{d}t \\
&\le a\mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right| > \varepsilon a\right) + \int_a^\infty \mathbb{V}\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right| > t\right) \mathrm{d}t \qquad (4.4)\\
&\le \frac{\mathbb{E}^*(\max_{1\le j\le n}|\sum_{t=1}^{j} X_t|^M)}{\varepsilon^M a^{M-1}} + \mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right|^M\right)\int_a^\infty \frac{1}{t^M}\,\mathrm{d}t \\
&= \left(\frac{1}{\varepsilon^M} + \frac{1}{M-1}\right)\frac{1}{a^{M-1}}\mathbb{E}^*\left(\max_{1\le j\le n}\left|\sum_{t=1}^{j} X_t\right|^M\right).
\end{aligned}$$

Combining (4.3) and (4.4) results in (4.2) immediately. □

Proof of Theorem 3.1 Observe that $\gamma p < q \le 2$. Notice that $\mathbb{E}(\epsilon_j) = \mathbb{E}(-\epsilon_j) = 0$, for all $j \in \mathbb{Z}$ and for each $n \ge 1$, write

$$\begin{aligned}
\epsilon'_{nj} &= -n^{1/p} I(\epsilon_j < -n^{1/p}) + \epsilon_j I(|\epsilon_j| \le n^{1/p}) + n^{1/p} I(\epsilon_j > n^{1/p}); \\
\epsilon''_{nj} &= \epsilon_j - \epsilon'_{nj} = (\epsilon_j - n^{1/p}) I(\epsilon_j > n^{1/p}) + (\epsilon_j + n^{1/p}) I(\epsilon_j < -n^{1/p}); \\
\epsilon_j &= \epsilon'_{nj} - \mathbb{E}\epsilon'_{nj} + \epsilon''_{nj} - \mathbb{E}\epsilon''_{nj} + \mathbb{E}\epsilon'_{nj} + \mathbb{E}\epsilon''_{nj}.
\end{aligned}$$

Define ϵ'_n, ϵ''_n as ϵ'_{nj}, ϵ''_{nj} only with ϵ in place of ϵ_j. We easily see that

$$\begin{aligned}
\left|\epsilon'_{nj}\right| &= |\epsilon_j| I(|\epsilon_j| \le n^{1/p}) + n^{1/p} I(|\epsilon_j| > n^{1/p}); \\
\left|\epsilon''_{nj}\right| &= (\epsilon_j - n^{1/p}) I(\epsilon_j > n^{1/p}) - (\epsilon_j + n^{1/p}) I(\epsilon_j < -n^{1/p}) \qquad (4.5)\\
&\le |\epsilon_j| I(|\epsilon_j| > n^{1/p}).
\end{aligned}$$

Thus, for each $n \ge 1$, we see that

$$\begin{aligned}
\sum_{t=1}^{m} X_t &= \sum_{t=1}^{m}\sum_{i=-\infty}^{\infty} a_i \epsilon_{t-i} = \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i \epsilon_j \\
&= \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i(\epsilon'_{nj} - \mathbb{E}\epsilon'_{nj}) + \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i(\epsilon''_{nj} - \mathbb{E}\epsilon''_{nj}) \qquad (4.6)\\
&\quad + \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i(\mathbb{E}\epsilon'_{nj} + \mathbb{E}\epsilon''_{nj}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma-2}\mathbb{V}\left(\max_{1\le m\le n}\left|\sum_{t=1}^{m} X_t\right| > \varepsilon n^{1/p}\right) \\
&\le \sum_{n=1}^{\infty} n^{\gamma-2}\mathbb{V}\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\left(\epsilon'_{nj} - \mathbb{E}\epsilon'_{nj}\right)\right| > \varepsilon n^{1/p}/3\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\gamma-2}\mathbb{V}\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\left(\mathbb{E}\epsilon'_{nj} + \mathbb{E}\epsilon''_{nj}\right)\right| > \varepsilon n^{1/p}/3\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\gamma-2}\mathbb{V}\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\left(\epsilon''_{nj} - \mathbb{E}\epsilon''_{nj}\right)\right| > \varepsilon n^{1/p}/3\right) \qquad (4.7) \\
&=: L_1 + L_2 + L_3.
\end{aligned}$$

First, we establish $L_1 < \infty$. By Markov's inequality under sub-linear expectations, Lemma 2.1, Lemma 4.1, $\mathbb{E}(\epsilon_1) = \mathbb{E}(-\epsilon_1) = 0$, and (4.5), we see that

$$\begin{aligned}
L_1 &= \sum_{n=1}^{\infty} n^{\gamma-2}\mathbb{V}\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\left(\epsilon'_{nj} - \mathbb{E}\epsilon'_{nj}\right)\right| > \varepsilon n^{1/p}/3\right) \\
&\le C\sum_{n=1}^{\infty} n^{\gamma-2-q/p}\mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\left(\epsilon'_{nj} - \mathbb{E}\epsilon'_{nj}\right)\right|^q\right) \\
&\le C\sum_{n=1}^{\infty} n^{\gamma-2-q/p}(\log n)^q \sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}\left|\epsilon'_{nj}\right|^q + \left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\left(\epsilon'_{nj}\right)\right| + \left|\mathbb{E}\left(-\epsilon'_{nj}\right)\right|\right]\right)^q\right] \\
&= C\sum_{n=1}^{\infty} n^{\gamma-2-q/p}(\log n)^q \sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}\left|\epsilon'_{n}\right|^q + \left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\left(\epsilon'_{n}\right)\right| + \left|\mathbb{E}\left(-\epsilon'_{n}\right)\right|\right]\right)^q\right] \\
&= C\sum_{n=1}^{\infty} n^{\gamma-2-q/p}(\log n)^q \sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}^*\left|\epsilon'_{n}\right|^q + \left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\left(\epsilon'_{n}\right)\right| + \left|\mathbb{E}\left(-\epsilon'_{n}\right)\right|\right]\right)^q\right] \\
&\le C\sum_{n=1}^{\infty} n^{\gamma-2-q/p}(\log n)^q \sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{E}^*\left(|\epsilon|^q I\left(|\epsilon| \le n^{1/p}\right)\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\gamma-2}(\log n)^q \sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{V}\left(|\epsilon| > n^{1/p}\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\gamma-2-q/p}(\log n)^q\left(n\mathbb{E}\left(\left|\epsilon'_n - \epsilon\right|\right)\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{\gamma-1-q/p}(\log n)^q\mathbb{E}^*\left(|\epsilon|^q I\left(|\epsilon| \le n^{1/p}\right)\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\gamma-2-1/p}(\log n)^q \sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{E}^*\left(|\epsilon| I\left(|\epsilon| > n^{1/p}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty} n^{\gamma-2-q/p+q}(\log n)^q \big(\mathbb{E}^*|\epsilon| I\big(|\epsilon| > n^{1/p}\big)\big)^q \\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-1-q/p}(\log n)^q \mathbb{E}^*\big(|\epsilon|^q I\big(|\epsilon| \leq n^{1/p}\big)\big) \\
&\quad + C\sum_{n=1}^{\infty} n^{\gamma-1-1/p}(\log n)^q \mathbb{E}^*\big(|\epsilon| I\big(|\epsilon| > n^{1/p}\big)\big) \\
&\quad + C\sum_{n=1}^{\infty} n^{\gamma-2-q/p+q}(\log n)^q \big(\mathbb{E}^*|\epsilon| I\big(|\epsilon| > n^{1/p}\big)\big)^q \\
&=: L_{11} + L_{12} + L_{13}.
\end{aligned} \tag{4.8}
$$

Note that $\gamma p < q$, by Lemma 2.2, we see that

$$
\begin{aligned}
L_{11} &= C\sum_{n=1}^{\infty} n^{\gamma-1-q/p}(\log n)^q \mathbb{E}^*\big(|\epsilon|^q I\big(|\epsilon| \leq n^{1/p}\big)\big) \\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-1-q/p}(\log n)^q C_{\mathbb{V}}\big(|\epsilon|^q I\big(|\epsilon| \leq n^{1/p}\big)\big) \\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-1-q/p}(\log n)^q \int_0^{n^{q/p}} \mathbb{V}\big(|\epsilon|^q > x\big)\,\mathrm{d}x \\
&\leq C\int_1^{\infty} y^{\gamma-1-q/p}(\log y)^q \,\mathrm{d}y \int_0^{y^{q/p}} \mathbb{V}\big(|\epsilon|^q > x\big)\,\mathrm{d}x \\
&\leq C\int_1^{\infty} y^{\gamma-1-q/p}(\log y)^q \,\mathrm{d}y \int_0^{y} \mathbb{V}\big(|\epsilon|^p > x\big)x^{q/p-1}\,\mathrm{d}x \\
&\leq C\int_0^{\infty} \mathbb{V}\big(|\epsilon|^p > x\big)x^{q/p-1}\,\mathrm{d}x \int_{1\vee x}^{\infty} y^{\gamma-1-q/p}(\log y)^q \,\mathrm{d}y \\
&\leq C\int_0^{\infty} \mathbb{V}\big(|\epsilon|^p > x\big)x^{q/p-1}(1\vee x)^{\gamma-q/p}\big(\log(1\vee x)\big)^q \,\mathrm{d}x \\
&\leq C\int_0^{\infty} \mathbb{V}\big(|\epsilon|^p > x\big)x^{\gamma-1}(\log x)^q \,\mathrm{d}x \\
&\leq CC_{\mathbb{V}}\big(|\epsilon|^{\gamma p}\log^q\big(|\epsilon|\big)\big) < \infty.
\end{aligned} \tag{4.9}
$$

Since $\gamma p > 1$, we also obtain

$$
\begin{aligned}
L_{12} &= C\sum_{n=1}^{\infty} n^{\gamma-1-1/p}(\log n)^q \mathbb{E}^*\big(|\epsilon| I\big(|\epsilon| > n^{1/p}\big)\big) \\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-1-1/p}(\log n)^q C_{\mathbb{V}}\big(|\epsilon| I\big(|\epsilon| > n^{1/p}\big)\big) \\
&\leq C\int_1^{\infty} x^{\gamma-1-1/p}(\log x)^q \,\mathrm{d}x \int_0^{\infty} \mathbb{V}\big(|\epsilon| I\big(|\epsilon| > x^{1/p}\big) > y\big)\,\mathrm{d}y \\
&= C\int_1^{\infty} x^{\gamma-1-1/p}(\log x)^q \,\mathrm{d}x \bigg[\int_0^{x^{1/p}} \mathbb{V}\big(|\epsilon| > x^{1/p}\big)\,\mathrm{d}y + \int_{x^{1/p}}^{\infty} \mathbb{V}\big(|\epsilon| > y\big)\,\mathrm{d}y\bigg]
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
&\leq C\int_1^\infty x^{\gamma-1}(\log x)^q \mathbb{V}\big(|\epsilon| > x^{1/p}\big)\,\mathrm{d}x \\
&\quad + C\int_1^\infty \mathbb{V}\big(|\epsilon|^p > y\big) y^{1/p-1}\,\mathrm{d}y \int_1^y x^{\gamma-1-1/p}(\log x)^q\,\mathrm{d}x \\
&\leq CC_{\mathbb{V}}\big(|\epsilon|^{\gamma p}\log^q\big(|\epsilon|\big)\big) + C\int_1^\infty \mathbb{V}\big(|\epsilon|^p > y\big) y^{\gamma-1}(\log y)^q\,\mathrm{d}y \\
&\leq CC_{\mathbb{V}}\big(|\epsilon|^{\gamma p}\log^q\big(|\epsilon|\big)\big) < \infty.
\end{aligned}
$$

Since $\gamma > 1$, $q > 1$, by monotonicity of $\mathbb{E}^*$ and Lemma 2.2, we have

$$
\begin{aligned}
L_{13} &\leq C\sum_{n=1}^\infty n^{\gamma-2-q/p+q}(\log n)^q\big(\mathbb{E}^*|\epsilon|^{\gamma p} I\big(|\epsilon| > n^{1/p}\big)/n^{(\gamma p-1)/p}\big)^q \\
&\leq C\sum_{n=1}^\infty n^{\gamma-2-\gamma q+q}(\log n)^q\big(\mathbb{E}^*|\epsilon|^{\gamma p}\big)^q \\
&\leq C\sum_{n=1}^\infty n^{(\gamma-1)(1-q)-1}(\log n)^q\big(C_{\mathbb{V}}\big\{|\epsilon|^{\gamma p}\big\}\big)^q < \infty.
\end{aligned}
\tag{4.11}
$$

By (4.8)-(4.11), we conclude that $L_1 < \infty$.

Next, we prove $L_2 < \infty$.

By similar proofs of Lemma 2.1 here and Theorem 2.1 (a) of Zhang [29], and $\mathbb{E}(\epsilon'_{nj}) = \mathbb{E}(-\epsilon''_{nj})$ (cf. Proposition 1.3.7 of Peng [13]), we could see that

$$
\begin{aligned}
&\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^\infty \sum_{i=1-j}^{m-j} a_i\big(\mathbb{E}\big(\epsilon'_{nj}\big) + \mathbb{E}\big(\epsilon''_{nj}\big)\big)\right|^q\right) \\
&\quad \leq C(\log n)^q \sup_{i\in\mathbb{Z}} \sum_{j=1-i}^{n-i}\big|\mathbb{E}\big(\epsilon'_{nj}\big) + \mathbb{E}\big(\epsilon''_{nj}\big)\big|^q.
\end{aligned}
\tag{4.12}
$$

Hence, by Markov's inequality under sub-linear expectations, (4.12), (4.5), and the proof of $L_1 < \infty$, we obtain

$$
\begin{aligned}
L_2 &= \sum_{n=1}^\infty n^{\gamma-2}\mathbb{V}\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^\infty \sum_{i=1-j}^{m-j} a_i\big(\mathbb{E}\big(\epsilon'_{nj}\big) + \mathbb{E}\big(\epsilon''_{nj}\big)\big)\right| > \varepsilon n^{1/p}/3\right) \\
&\leq C\sum_{n=1}^\infty n^{\gamma-2-q/p}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^\infty \sum_{i=1-j}^{m-j} a_i\big(\mathbb{E}\big(\epsilon'_{nj}\big) + \mathbb{E}\big(\epsilon''_{nj}\big)\big)\right|^q\right) \\
&\leq C\sum_{n=1}^\infty n^{\gamma-2-q/p}(\log n)^q \sup_{i\in\mathbb{Z}} \sum_{j=1-i}^{n-i}\mathbb{E}\big|\epsilon'_{nj}\big|^q < \infty.
\end{aligned}
$$

Now, we establish $L_3 < \infty$. By Markov's inequality under sub-linear expectations, Lemma 2.1, Lemma 4.1, $C_{\mathbb{V}}\{|\epsilon|^p\} < CC_{\mathbb{V}}\{|\epsilon|^{\gamma p}(\log|\epsilon|)^p\} < \infty$, and (4.5), we see that

$$
L_3 = \sum_{n=1}^\infty n^{\gamma-2}\mathbb{V}\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^\infty \sum_{i=1-j}^{m-j} a_i\big(\epsilon''_{nj} - \mathbb{E}\big(\epsilon''_{nj}\big)\big)\right| > \varepsilon n^{1/p}/3\right)
$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{\gamma-3}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\big(\epsilon''_{nj}-\mathbb{E}(\epsilon''_{nj})\big)\right|^p\right)\\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-3}(\log n)^p\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}\left|\epsilon''_{nj}-\mathbb{E}\epsilon''_{nj}\right|^p+\left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}(\epsilon''_{nj})\right|+\left|\mathbb{E}(-\epsilon''_{nj})\right|\right]\right)^p\right]\\
&= C\sum_{n=1}^{\infty} n^{\gamma-3}(\log n)^p\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}\left|\epsilon''_{n}-\mathbb{E}\epsilon''_{n}\right|^p+\left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}(\epsilon''_{n})\right|+\left|\mathbb{E}(-\epsilon''_{n})\right|\right]\right)^p\right]\\
&= C\sum_{n=1}^{\infty} n^{\gamma-3}(\log n)^p\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}^*\left|\epsilon''_{n}-\mathbb{E}\epsilon''_{n}\right|^p+\left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}^*(\epsilon''_{n})\right|+\left|\mathbb{E}^*(-\epsilon''_{n})\right|\right]\right)^p\right]\\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-3}(\log n)^p\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}^*\big(|\epsilon|^pI(|\epsilon|>n^{1/p})\big)+\left(\sum_{j=1-i}^{n-i}\mathbb{E}^*\big(|\epsilon|I(|\epsilon|>n^{1/p})\big)\right)^p\right]\\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-2}(\log n)^p\mathbb{E}^*\big(|\epsilon|^pI(|\epsilon|>n^{1/p})\big)\\
&\quad + C\sum_{n=1}^{\infty} n^{\gamma-3+p}(\log n)^p\left(\mathbb{E}^*\big(|\epsilon|^pI(|\epsilon|>n^{1/p}\big)/n^{(p-1)/p})\right)^p\\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-2}(\log n)^pC_{\mathbb{V}}\big(|\epsilon|^pI(|\epsilon|>n^{1/p})\big)\\
&\quad + C\sum_{n=1}^{\infty} n^{\gamma-2}(\log n)^pC_{\mathbb{V}}\big(|\epsilon|^pI(|\epsilon|>n^{1/p})\big)\big(C_{\mathbb{V}}(|\epsilon|^p)\big)^{p-1}\\
&\leq C\sum_{n=1}^{\infty} n^{\gamma-2}(\log n)^pC_{\mathbb{V}}\big(|\epsilon|^pI(|\epsilon|>n^{1/p})\big)\\
&\leq C\int_1^{\infty} x^{\gamma-2}(\log x)^p\,\mathrm{d}x\int_0^{\infty}\mathbb{V}\big(|\epsilon|^pI(|\epsilon|>x^{1/p})>y\big)\,\mathrm{d}y\\
&= C\int_1^{\infty} x^{\gamma-2}(\log x)^p\,\mathrm{d}x\left[\int_0^{x}\mathbb{V}\big(|\epsilon|>x^{1/p}\big)\,\mathrm{d}y+\int_x^{\infty}\mathbb{V}\big(|\epsilon|>y^{1/p}\big)\,\mathrm{d}y\right]\\
&= C\int_1^{\infty} x^{\gamma-1}(\log x)^p\mathbb{V}\big(|\epsilon|>x^{1/p}\big)\,\mathrm{d}x+C\int_1^{\infty}\mathbb{V}\big(|\epsilon|>y^{1/p}\big)\,\mathrm{d}y\int_1^{y} x^{\gamma-2}(\log x)^p\,\mathrm{d}x\\
&\leq CC_{\mathbb{V}}\big(|\epsilon|^{\gamma p}(\log|\epsilon|)^p\big)+C\int_1^{\infty}\mathbb{V}\big(|\epsilon|^p>y\big)y^{\gamma-1}(\log y)^p\,\mathrm{d}y\\
&\leq CC_{\mathbb{V}}\big(|\epsilon|^{\gamma p}(\log|\epsilon|)^p\big)<\infty.
\end{aligned}
$$

The proof of Theorem 3.1 is finished.

Proof of Theorem 3.2 Let $p<q\leq 2$. Notice that $\mathbb{E}\epsilon_j=\mathbb{E}(-\epsilon_j)=0$, for all $j\in\mathbb{Z}$ and for each $n\geq 1$, write

$$
\begin{aligned}
&\tilde{\epsilon}'_{nj}=-n^{\alpha}I(\epsilon_j<-n^{\alpha})+\epsilon_jI(|\epsilon_j|\leq n^{\alpha})+n^{\alpha}I(\epsilon_j>n^{\alpha});\\
&\tilde{\epsilon}''_{nj}=\epsilon_j-\tilde{\epsilon}'_{nj}=(\epsilon_j-n^{\alpha})I(\epsilon_j>n^{\alpha})+(\epsilon_j+n^{\alpha})I(\epsilon_j<-n^{\alpha});\\
&\epsilon_j=\tilde{\epsilon}'_{nj}-\mathbb{E}\tilde{\epsilon}'_{nj}+\tilde{\epsilon}''_{nj}-\mathbb{E}\tilde{\epsilon}''_{nj}+\mathbb{E}\tilde{\epsilon}'_{nj}+\mathbb{E}\tilde{\epsilon}''_{nj}.
\end{aligned}
$$

And define $\tilde{\epsilon}'_n$, $\tilde{\epsilon}''_n$ as $\tilde{\epsilon}'_{nj}$, $\tilde{\epsilon}''_{nj}$ only with ϵ in place of ϵ_j. We can see that

$$\left|\tilde{\epsilon}'_{nj}\right| = |\epsilon_j| I\big(|\epsilon_j| \le n^\alpha\big) + n^\alpha I\big(|\epsilon_j| > n^\alpha\big); \tag{4.13}$$

$$\begin{aligned} \left|\tilde{\epsilon}''_{nj}\right| &= \big(\epsilon_j - n^\alpha\big) I\big(\epsilon_j > n^\alpha\big) - \big(\epsilon_j + n^\alpha\big) I\big(\epsilon_j < -n^\alpha\big); \\ &\le |\epsilon_j| I\big(|\epsilon_j| > n^\alpha\big). \end{aligned} \tag{4.14}$$

Therefore, for each $n \ge 1$, we get

$$\begin{aligned} \sum_{t=1}^{m} X_t = {} & \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\tilde{\epsilon}'_{nj} - \mathbb{E}\tilde{\epsilon}'_{nj}\big) + \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\tilde{\epsilon}''_{nj} - \mathbb{E}\tilde{\epsilon}''_{nj}\big) \\ & + \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\mathbb{E}\tilde{\epsilon}'_{nj} + \mathbb{E}\tilde{\epsilon}''_{nj}\big). \end{aligned} \tag{4.15}$$

Hence, by Lemma 4.2, we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{t=1}^{m} X_t\right| - \varepsilon n^\alpha\right)^+ \\ & \le \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right) \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \frac{1}{n^{\alpha(q-1)}} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\tilde{\epsilon}'_{nj} - \mathbb{E}\tilde{\epsilon}'_{nj}\big)\right|^q\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\tilde{\epsilon}''_{nj} - \mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\mathbb{E}\tilde{\epsilon}'_{nj} + \mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right) \\ & \le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\tilde{\epsilon}'_{nj} - \mathbb{E}\tilde{\epsilon}'_{nj}\big)\right|^q\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\tilde{\epsilon}''_{nj} - \mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right) \\ & \quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\mathbb{E}\tilde{\epsilon}'_{nj} + \mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right) \\ & =: K_1 + K_2 + K_3. \end{aligned} \tag{4.16}$$

First, we establish $K_1 < \infty$. By Markov's inequality under sub-linear expectations, Lemma 4.1, and (4.13), we see that

$$\begin{aligned} K_1 &= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \mathbb{E}^*\left(\max_{1\le m\le n}\left|\sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{m-j} a_i\big(\tilde{\epsilon}'_{nj} - \mathbb{E}\tilde{\epsilon}'_{nj}\big)\right|^q\right) \\ & \le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} (\log n)^q \sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i} \mathbb{E}\left|\tilde{\epsilon}'_{nj} - \mathbb{E}\tilde{\epsilon}'_{nj}\right|^q + \left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\big(\tilde{\epsilon}'_{nj}\big)\right| + \left|\mathbb{E}\big(-\tilde{\epsilon}'_{nj}\big)\right|\right]\right)^q\right] \end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}\left|\tilde{\epsilon}_n' - \mathbb{E}\tilde{\epsilon}_n'\right|^q + \left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\left(\tilde{\epsilon}_n'\right)\right| + \left|\mathbb{E}\left(-\tilde{\epsilon}_n'\right)\right|\right]\right)^q\right] \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}^*\left|\tilde{\epsilon}_n' - \mathbb{E}\tilde{\epsilon}_n'\right|^q + \left(\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\left(\tilde{\epsilon}_n'\right)\right| + \left|\mathbb{E}\left(-\tilde{\epsilon}_n'\right)\right|\right]\right)^q\right] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{E}^*\left|\tilde{\epsilon}_n'\right|^q \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{i\in\mathbb{Z}}\left(\sum_{j=1-i}^{n-i}\mathbb{E}\left|\tilde{\epsilon}_n' - \epsilon\right|\right)^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{E}^*\left(|\epsilon|^q I\left(|\epsilon| \leq n^\alpha\right)\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^q \sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{V}\left(|\epsilon| > n^\alpha\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{i\in\mathbb{Z}}\left(\sum_{j=1-i}^{n-i}\mathbb{E}^*|\epsilon| I\left(|\epsilon| > n^\alpha\right)\right)^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}(\log n)^q \mathbb{E}^*\left(|\epsilon|^q I\left(|\epsilon| \leq n^\alpha\right)\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}(\log n)^q \mathbb{E}^*\left(|\epsilon| I\left(|\epsilon| > n^\alpha\right)\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{i\in\mathbb{Z}}\left(\sum_{j=1-i}^{n-i}\mathbb{E}^*|\epsilon| I\left(|\epsilon| > n^\alpha\right)\right)^q \\
&=: K_{11} + K_{12} + K_{13}.
\end{aligned} \tag{4.17}
$$

Notice that $p < q$, by Lemma 2.2, we obtain

$$
\begin{aligned}
K_{11} &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}(\log n)^q C_{\mathbb{V}}\left(|\epsilon|^q I\left(|\epsilon| \leq n^\alpha\right)\right) \\
&\leq C\int_1^\infty x^{\alpha p-\alpha q-1}(\log x)^q \, \mathrm{d}x \int_0^{x^{\alpha q}} \mathbb{V}\left(|\epsilon|^q > y\right) \mathrm{d}y \\
&= C\int_0^\infty \mathbb{V}\left(|\epsilon| > y^\alpha\right) y^{\alpha q-1} \, \mathrm{d}y \int_{1\vee y}^\infty x^{\alpha p-\alpha q-1}(\log x)^q \, \mathrm{d}x \\
&\leq C\int_0^\infty \mathbb{V}\left(|\epsilon| > y^\alpha\right) y^{\alpha q-1+\alpha p-\alpha q}(\log y)^q \, \mathrm{d}y \\
&\leq CC_{\mathbb{V}}\left(|\epsilon|^p\left(\log|\epsilon|\right)^q\right) < \infty.
\end{aligned} \tag{4.18}
$$

Since $p > 1$, we also see that

$$
K_{12} = C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}(\log n)^q \mathbb{E}^*\left(|\epsilon| I\left(|\epsilon| > n^\alpha\right)\right)
$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}(\log n)^q C_{\mathbb{V}}\big(|\epsilon|I\big(|\epsilon|>n^{\alpha}\big)\big)\\
&\leq C\int_1^{\infty} x^{\alpha p-\alpha-1}(\log x)^q\,\mathrm{d}x\int_0^{\infty}\mathbb{V}\big(|\epsilon|I\big(|\epsilon|>x^{\alpha}\big)>y\big)\,\mathrm{d}y\\
&= C\int_1^{\infty} x^{\alpha p-\alpha-1}(\log x)^q\,\mathrm{d}x\left[\int_0^{x^{\alpha}}\mathbb{V}\big(|\epsilon|>x^{\alpha}\big)\,\mathrm{d}y+\int_{x^{\alpha}}^{\infty}\mathbb{V}\big(|\epsilon|>y\big)\,\mathrm{d}y\right] \qquad (4.19)\\
&\leq C\int_1^{\infty}\mathbb{V}\big(|\epsilon|>x^{\alpha}\big)x^{\alpha p-1}(\log x)^q\,\mathrm{d}x\\
&\quad + C\int_1^{\infty}\mathbb{V}\big(|\epsilon|>y^{\alpha}\big)y^{\alpha-1}\,\mathrm{d}y\int_1^{y} x^{\alpha p-\alpha-1}(\log x)^q\,\mathrm{d}x\\
&\leq CC_{\mathbb{V}}\big(|\epsilon|^p\big(\log|\epsilon|\big)^q\big)+C\int_1^{\infty}\mathbb{V}\big(|\epsilon|>y^{\alpha}\big)y^{\alpha p-1}(\log y)^q\,\mathrm{d}y\\
&\leq CC_{\mathbb{V}}\big(|\epsilon|^p\big(\log|\epsilon|\big)^q\big)<\infty.
\end{aligned}
$$

Obviously, while $\alpha p>1$, by Lemma 2.2,we see that

$$
\begin{aligned}
K_{13}&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q\sup_{i\in\mathbb{Z}}\left(\sum_{j=1-i}^{n-i}\mathbb{E}^*|\epsilon|^p I\big(|\epsilon|>n^{\alpha}\big)/n^{\alpha(p-1)}\right)^q\\
&\leq C\sum_{n=1}^{\infty} n^{-(q-1)(\alpha p-1)-1}(\log n)^q\big(\mathbb{E}^*|\epsilon|^p\big)^q\\
&\leq C\sum_{n=1}^{\infty} n^{-(q-1)(\alpha p-1)-1}(\log n)^q\big(C_{\mathbb{V}}\big(|\epsilon|^p\big)\big)^q<\infty. \qquad (4.20)
\end{aligned}
$$

While $0<\alpha\leq 1/p$, by $C_{\mathbb{V}}(|\epsilon|^p(\log|\epsilon|)^q)<\infty$, we obtain $\lim_{x\to\infty}x^p(\log x)^q\mathbb{V}\{|\epsilon|>x\}=0$, which in combination with Lebesgue's dominated convergence theorem of Rudin [14] implies

$$
\begin{aligned}
&\lim_{n\to\infty} n^{1-\alpha}C_{\mathbb{V}}\big\{|\epsilon|I\big(|\epsilon|>n^{\alpha}\big)\big\}\\
&\quad=\lim_{x\to\infty} x^{1-\alpha}\left(x^{\alpha}\mathbb{V}\big\{|\epsilon|>x^{\alpha}\big\}+\int_{x^{\alpha}}^{\infty}\mathbb{V}\big\{|\epsilon|>y\big\}\,\mathrm{d}y\right)\\
&\quad\leq\lim_{x\to\infty} C\frac{\int_{x^{\alpha}}^{\infty}\mathbb{V}\{|\epsilon|>y\}\,\mathrm{d}y}{x^{\alpha-1}}=\lim_{x\to\infty}\frac{\sum_{k=0}^{\infty}\int_{(2^k x)^{\alpha}}^{(2^{k+1}x)^{\alpha}}\mathbb{V}\{|\epsilon|>y\}\,\mathrm{d}y}{x^{\alpha-1}}\\
&\quad\leq\lim_{x\to\infty} C\sum_{k=0}^{\infty}\mathbb{V}\big\{|\epsilon|>\big(2^k x\big)^{\alpha}\big\}\big(2^k x\big)^{p\alpha}x^{\alpha-p\alpha}\big(2^k\big)^{\alpha-p\alpha}\big(2^{\alpha}-1\big)\\
&\quad=C\sum_{k=0}^{\infty}\big(2^k\big)^{\alpha-p\alpha}\times 0=0,
\end{aligned}
$$

and $\sup_n n^{1-\alpha}C_{\mathbb{V}}\{|\epsilon|I(|\epsilon|>n^{\alpha})\}<\infty$. Hence, while $0<\alpha p\leq 1$, by $K_{12}<\infty$, we have

$$
\begin{aligned}
K_{13}&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q\big(n\mathbb{E}^*|\epsilon|I\big(|\epsilon|>n^{\alpha}\big)\big)^q\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}(\log n)^q\big(\mathbb{E}^*|\epsilon|I\big(|\epsilon|>n^{\alpha}\big)\big)\big(n^{1-\alpha}\mathbb{E}^*|\epsilon|I\big(|\epsilon|>n^{\alpha}\big)\big)^{q-1} \qquad (4.21)
\end{aligned}
$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}(\log n)^q\left(\mathbb{E}^*|\epsilon|I\left(|\epsilon|>n^\alpha\right)\right)\left(n^{1-\alpha}C_{\mathbb{V}}\left(|\epsilon|I\left(|\epsilon|>n^\alpha\right)\right)\right)^{q-1}<\infty.$$

By (4.17)-(4.21), we conclude that $K_1 < \infty$.

Next, we establish $K_2 < \infty$. By Lemma 4.1, (4.14), and the proof of (4.19), we see that

$$\begin{aligned}
K_2 &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\left(\tilde{\epsilon}''_{nj}-\mathbb{E}\tilde{\epsilon}''_{nj}\right)\right|\right)\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}\left|\tilde{\epsilon}''_{nj}-\mathbb{E}\tilde{\epsilon}''_{nj}\right|+\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\left(\tilde{\epsilon}''_{nj}\right)\right|+\left|\mathbb{E}\left(-\tilde{\epsilon}''_{nj}\right)\right|\right]\right]\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}\left|\tilde{\epsilon}''_{n}-\mathbb{E}\tilde{\epsilon}''_{n}\right|+\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}\left(\tilde{\epsilon}''_{n}\right)\right|+\left|\mathbb{E}\left(-\tilde{\epsilon}''_{n}\right)\right|\right]\right]\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)\sup_{i\in\mathbb{Z}}\left[\sum_{j=1-i}^{n-i}\mathbb{E}^*\left|\tilde{\epsilon}''_{n}-\mathbb{E}\tilde{\epsilon}''_{n}\right|+\sum_{j=1-i}^{n-i}\left[\left|\mathbb{E}^*\left(\tilde{\epsilon}''_{n}\right)\right|+\left|\mathbb{E}^*\left(-\tilde{\epsilon}''_{n}\right)\right|\right]\right]\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)\sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{E}^*\left|\tilde{\epsilon}''_{n}\right|\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}(\log n)\mathbb{E}^*\left(|\epsilon|I\left(|\epsilon|>n^\alpha\right)\right)\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}(\log n)C_{\mathbb{V}}\left(|\epsilon|I\left(|\epsilon|>n^\alpha\right)\right) \qquad (4.22)\\
&\leq CC_{\mathbb{V}}\left(|\epsilon|^p\left(\log|\epsilon|\right)\right)<\infty.
\end{aligned}$$

Now, we prove $K_3 < \infty$. By similar proofs of Lemma 4.1 here and Theorem 2.1 (a) of Zhang [29], and $\mathbb{E}(\tilde{\epsilon}'_{nj}) = \mathbb{E}(-\tilde{\epsilon}''_{nj})$ (cf. Proposition 1.3.7 of Peng [13]), we see that

$$\begin{aligned}
&\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j} a_i\left(\mathbb{E}\left(\tilde{\epsilon}'_{nj}\right)+\mathbb{E}\left(\tilde{\epsilon}''_{nj}\right)\right)\right|\right)\\
&\quad\leq C(\log n)\sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\left|\mathbb{E}\left(\tilde{\epsilon}'_{nj}\right)+\mathbb{E}\left(\tilde{\epsilon}''_{nj}\right)\right|. \qquad (4.23)
\end{aligned}$$

Hence, by Markov's inequality under sub-linear expectations, (4.23), (4.14), and the proof of $K_2 < \infty$, we obtain

$$\begin{aligned}
K_3 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)\sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\left|\mathbb{E}\left(\tilde{\epsilon}'_{nj}\right)+\mathbb{E}\left(\tilde{\epsilon}''_{nj}\right)\right|\\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)\sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\left|\mathbb{E}\left(\tilde{\epsilon}'_{n}\right)+\mathbb{E}\left(\tilde{\epsilon}''_{n}\right)\right|\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}(\log n)\sup_{i\in\mathbb{Z}}\sum_{j=1-i}^{n-i}\mathbb{E}\left|\tilde{\epsilon}''_{n}\right|
\end{aligned}$$

$$\leq CC_{\mathbb{V}}\big(|\epsilon|^{p}\big(\log|\epsilon|\big)\big)<\infty.$$

The proof of Theorem 3.2 is finished. □

Proof of Theorem 3.3 For for all $j\in\mathbb{Z}$ and for each $n\geq 1$, write

$$\tilde{\epsilon}'_{nj}=-nI(\epsilon_j<-n)+\epsilon_j I\big(|\epsilon_j|\leq n\big)+nI(\epsilon_j>n);$$

$$\tilde{\epsilon}''_{nj}=\epsilon_j-\tilde{\epsilon}'_{nj}=(\epsilon_j-n)I(\epsilon_j>n)+(\epsilon_j+n)I(\epsilon_j<-n);$$

$$\epsilon_j=\tilde{\epsilon}'_{nj}-\mathbb{E}\tilde{\epsilon}'_{nj}+\tilde{\epsilon}''_{nj}-\mathbb{E}\tilde{\epsilon}''_{nj}+\mathbb{E}\tilde{\epsilon}'_{nj}+\mathbb{E}\tilde{\epsilon}''_{nj}.$$

And define $\tilde{\epsilon}'_n$, $\tilde{\epsilon}''_n$ as $\tilde{\epsilon}'_{nj}$, $\tilde{\epsilon}''_{nj}$ only with ϵ in place of ϵ_j. First, by $C_{\mathbb{V}}\{|\epsilon|\log^2(1+|\epsilon|)\}<\infty$, we obtain $\lim_{x\to\infty}x(\log x)^2\mathbb{V}\{|\epsilon|>x\}=0$, which in combination with Lebesgue's dominated convergence theorem of Rudin [14] implies

$$\begin{aligned}
&\lim_{n\to\infty}(\log n)C_{\mathbb{V}}\big(|\epsilon|I\big(|\epsilon|>n\big)\big)\\
&\quad=\lim_{x\to\infty}(\log x)\left(x\mathbb{V}\big\{|\epsilon|>x\big\}+\int_x^{\infty}\mathbb{V}\big\{|\epsilon|>y\big\}\,\mathrm{d}y\right)\\
&\quad=\lim_{x\to\infty}\sum_{k=1}^{\infty}\frac{\int_{x^k}^{x^{k+1}}\mathbb{V}\{|\epsilon|>y\}\,\mathrm{d}y}{(\log x)^{-1}}\leq C\lim_{x\to\infty}\sum_{k=1}^{\infty}\mathbb{V}\big\{|\epsilon|>x^k\big\}x^k\big(\log x^k\big)^2\frac{1}{k^2(\log x)}=0,
\end{aligned}$$

and $\sup_{n\geq 1}(\log n)C_{\mathbb{V}}(|\epsilon|I(|\epsilon|>n))<\infty$. As in the proof of Theorem 3.2, by the proof of Lemma 2.2 of Zhong and Wu [33] and the discussion above, we could see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{-2}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{t=1}^{m}X_t\right|-\varepsilon n\right)^+\\
&\quad\leq\left(\frac{1}{\varepsilon^2}+1\right)\sum_{n=1}^{\infty}n^{-3}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}a_i\big(\tilde{\epsilon}'_{nj}-\mathbb{E}\epsilon'_{nj}\big)\right|^2\right)\\
&\qquad+\sum_{n=1}^{\infty}n^{-2}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}a_i\big(\tilde{\epsilon}''_{nj}-\mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right)\\
&\qquad+\sum_{n=1}^{\infty}n^{-2}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}a_i\big(\mathbb{E}\tilde{\epsilon}'_{nj}+\mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right)\\
&\quad\leq C\sum_{n=1}^{\infty}n^{-3}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}a_i\big(\tilde{\epsilon}'_{nj}-\mathbb{E}\tilde{\epsilon}'_{nj}\big)\right|^2\right)\\
&\qquad+\sum_{n=1}^{\infty}n^{-2}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}a_i\big(\tilde{\epsilon}''_{nj}-\mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right)\\
&\qquad+\sum_{n=1}^{\infty}n^{-2}\mathbb{E}^*\left(\max_{1\leq m\leq n}\left|\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}a_i\big(\mathbb{E}\tilde{\epsilon}'_{nj}+\mathbb{E}\tilde{\epsilon}''_{nj}\big)\right|\right)\\
&\quad\leq C\sum_{n=1}^{\infty}n^{-3}(\log n)^2\sup_{i\in\mathbb{Z}}\left\{\sum_{j=1-i}^{n-i}\mathbb{E}\big|\tilde{\epsilon}'_{nj}\big|^2+\left(\sum_{j=1-i}^{n-i}\big[\big|\mathbb{E}\tilde{\epsilon}'_{nj}\big|+\big|\mathbb{E}\big(-\tilde{\epsilon}'_{nj}\big)\big|\big]\right)^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty} n^{-1}\log n\mathbb{E}^*\big(|\epsilon|I\big(|\epsilon|>n\big)\big) + C\sum_{n=1}^{\infty} n^{-1}\log n\mathbb{E}^*\big(|\epsilon|I\big(|\epsilon|>n\big)\big) \\
&= C\sum_{n=1}^{\infty} n^{-3}(\log n)^2 \sup_{i\in\mathbb{Z}}\left\{\sum_{j=1-i}^{n-i}\mathbb{E}\big|\tilde{\epsilon}_n'\big|^2 + \left(\sum_{j=1-i}^{n-i}\big[\big|\mathbb{E}\tilde{\epsilon}_n'\big| + \big|\mathbb{E}\big(-\tilde{\epsilon}_n'\big)\big|\big]\right)^2\right\} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}\log n\mathbb{E}^*\big(|\epsilon|I\big(|\epsilon|>n\big)\big) \\
&= C\sum_{n=1}^{\infty} n^{-3}(\log n)^2 \sup_{i\in\mathbb{Z}}\left\{\sum_{j=1-i}^{n-i}\mathbb{E}^*\big|\tilde{\epsilon}_n'\big|^2 + \left(\sum_{j=1-i}^{n-i}\big[\mathbb{E}^*\big|\tilde{\epsilon}_n' - \epsilon\big|\big]\right)^2\right\} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}\log n\mathbb{E}^*\big(|\epsilon|I\big(|\epsilon|>n\big)\big) \\
&\le C\sum_{n=1}^{\infty} n^{-2}(\log n)^2\mathbb{E}^*\big(|\epsilon|^2 I\big(|\epsilon|\le n\big)\big) + C\sum_{n=1}^{\infty}(\log n)^2\mathbb{E}^*\big(I\big(|\epsilon|>n\big)\big) \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}(\log n)^2\big(\mathbb{E}^*\big(|\epsilon|I\big(|\epsilon|>n\big)\big)\big)^2 + C\sum_{n=1}^{\infty} n^{-1}\log n\mathbb{E}^*\big(|\epsilon|I\big(|\epsilon|>n\big)\big) \\
&\le C\sum_{n=1}^{\infty} n^{-2}(\log n)^2 C_{\mathbb{V}}\big(|\epsilon|^2 I\big(|\epsilon|\le n\big)\big) + C\sum_{n=1}^{\infty}(\log n)^2\mathbb{V}\big(|\epsilon|>n\big) \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}(\log n)^2\big(C_{\mathbb{V}}\big(|\epsilon|I\big(|\epsilon|>n\big)\big)\big)^2 + C\sum_{n=1}^{\infty} n^{-1}(\log n)C_{\mathbb{V}}\big(|\epsilon|I\big(|\epsilon|>n\big)\big) \\
&\le C\sum_{n=1}^{\infty} n^{-2}(\log n)^2 C_{\mathbb{V}}\big(|\epsilon|^2 I\big(|\epsilon|\le n\big)\big) + CC_{\mathbb{V}}\big(|\epsilon|\log^2\big(1+|\epsilon|\big)\big) \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}(\log n)C_{\mathbb{V}}\big(|\epsilon|I\big(|\epsilon|>n\big)\big) \\
&\le C\int_1^{\infty} x^{-2}(\log x)^2\,\mathrm{d}x\int_0^{x^2}\mathbb{V}\big(|\epsilon|^2>y\big)\,\mathrm{d}y \\
&\quad + C\int_1^{\infty} x^{-1}(\log x)\,\mathrm{d}x\int_0^{\infty}\mathbb{V}\big(|\epsilon|I\big(|\epsilon|>x\big)>y\big)\,\mathrm{d}y + C \\
&= C\int_0^{\infty}\mathbb{V}\big(|\epsilon|>y\big)y\,\mathrm{d}y\int_{1\vee y}^{\infty} x^{-2}(\log x)^2\,\mathrm{d}x \\
&\quad + C\int_1^{\infty} x^{-1}(\log x)\,\mathrm{d}x\left[\int_0^{x}\mathbb{V}\big(|\epsilon|>x\big)\,\mathrm{d}y + \int_x^{\infty}\mathbb{V}\big(|\epsilon|>y\big)\,\mathrm{d}y\right] + C \\
&\le C\int_0^{\infty}\mathbb{V}\big(|\epsilon|>y\big)(\log y)^2\,\mathrm{d}y + C\int_1^{\infty}\mathbb{V}\big(|\epsilon|>x\big)(\log x)\,\mathrm{d}x \\
&\quad + C\int_1^{\infty}\mathbb{V}\big(|\epsilon|>y\big)\,\mathrm{d}y\int_1^{y} x^{-1}(\log x)\,\mathrm{d}x + C \\
&\le CC_{\mathbb{V}}\big(|\epsilon|\log^2\big(1+|\epsilon|\big)\big) + C\int_0^{\infty}\mathbb{V}\big(|\epsilon|\log\big(1+|\epsilon|\big)>z\big)\,\mathrm{d}z \\
&\quad + C\int_1^{\infty}\mathbb{V}\big(|\epsilon|>y\big)(\log y)^2\,\mathrm{d}y + C
\end{aligned}
$$

$$\leq C C_{\mathbb{V}}\left(|\epsilon| \log^2\left(1+|\epsilon|\right)\right) + C < \infty.$$

The proof is complete. □

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The authors declare no competing interests.

Author contributions
MZX was a major contributor in writing the manuscript; KC and WKY provided some helpful discussions in writing the manuscript. All authors read and approved the manuscript.

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