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Asymptotic results for a class of Markovian self-exciting processes

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Abstract

Hawkes process is a class of self-exciting point processes with clustering effect whose jump rate relies on their entire past history. This process is usually defined as a continuous-time setting and has been widely applied in several fields, including insurance, finance, queueing theory, and statistics. The Hawkes model is generally non-Markovian because the future development of a self-exciting point process is determined by the timing of past events. However, it can be Markovian in special cases such as when the exciting function is an exponential function or a sum of exponential functions. Difficulty arises when the exciting function is not an exponential function or a sum of exponentials, in which case the process can be non-Markovian. The inverse Markovian case for Hawkes processes was introduced by Seol (Stat. Probab. Lett. 155:108580, 2019) who studied some asymptotic behaviors. An extended version of the inverse Markovian Hawkes process was also studied by Seol (J. Korean Math. Soc. 58(4):819–833, 2021). In the current work, we propose a class of Markovian self-exciting processes that interpolates between the Hawkes process and the inverse Hawkes process. We derived limit theorems for the newly considered class of Markovian self-exciting processes. In particular, we established both the law of large numbers (LLN) and central limit theorems (CLT) with some key results.

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Keywords: Hawkes process; Inverse Markovian; Self-exciting point processes; Central limit theorems; Law of large numbers

1 Introduction

Hawkes processes [15] are the most popular and useful models of simple point processes and are self-exciting with clustering effect. The intensity process for a point process is composed of the summation of the baseline intensity plus other terms that depend upon the history of whole past of the point process in comparison with a standard Poisson process. In applications, the Hawkes process is typically used as an expressive model for temporal phenomena of a stochastic process which evolve in continuous time, such as in modeling high-frequency trading. The Hawkes process is a natural generalization of the Poisson process and captures both the self-exciting property and the clustering effect. This process is a very variable model that is amenable to statistical analysis. Therefore, it has wide applications in insurance, social networks, neuroscience, criminology, seismology, DNA modeling, and finance. Typically, the self-exciting and clustering properties make the Hawkes process highly desirable for computations in financial applications [6], such as in modeling the associated defaults and evaluating the derivatives of credit in finance [5, 7]. There are many situations that require time-dependent frameworks when it comes to model adjustment. The Hawkes process generally can be categorized by linear and non-linear cases of Hawkes processes based on the intensity. Hawkes [15] introduced the linear process that can be studied via immigration-birth representation [16]. The stability [20], the law of large numbers (LLN) [4], the Bartlett spectrum [22], the central limit theorem (CLT) [1], and large deviation principles (LDP) [2] have all been studied and understood very well. Most applications for the Hawkes process consider exclusively the linear case. The nonlinear Hawkes process is much less studied mainly due to the deficiency of immigration-birth representation and computational tractability, although some efforts in this direction have been made. The first nonlinear case was studied by Brémaud and Massoulié [3]. Recently, Zhu [34–37, 39] investigated several results for both linear and nonlinear models. The central limit theorem for the nonlinear model was investigated by Zhu [34], and the large deviation principles were obtained by Zhu [37]. Jaisson and Rosenbaum [18, 19] studied some limit theorems and rough fractional diffusions as scaling limits of nearly unstable Hawkes processes. Some variations and extensions of the Hawkes process were studied by Dassios and Zhao [5], Ferro, Leiva, and Møller [8], Karabash and Zhu [21], Mehrdad and Zhu [24], and Zhu [38]. Seol [26] considered the arrival time τ_n, inverse of the Hawkes process, and studied the limit theorems for τ_n. Recently, data-driven models have gained attention due to the development of storage technology. In contrast to the continuous-time scheme, in real world, events are often recorded in a discrete-time scheme. It is more important that the data are collected in a fixed phase or that the data only show the aggregate results. For example, continuous-time Hawkes models can be spaced unevenly in time, whereas a discrete-time Hawkes model can be spaced evenly in time, and so a discrete-time Hawkes process has wide applications in many fields. Usually, the Hawkes process is considered as a continuous-time scheme. However, data are often recorded in a discrete-time scheme. Seol [25] proposed a 0-1 discrete Hawkes process starting from empty history and proved some limit behaviors such as the law of large numbers (LLN), the invariance principles, and the central limit theorem (CLT). Recently, Wang [31, 32] studied limit behaviors of a discrete-time Hawkes process with random marks and proved large and moderate deviations for a discrete-time Hawkes process with marks. Seol [27, 28] studied the moderate deviation principle of marked Hawkes processes and also studied asymptotic behaviors for the compensator processes of Hawkes models. Furthermore, Gao and Zhu [10, 12–14] made some progress in the direction of limit behaviors other than the large time scale limits. Studies have also been reported on modifying and extending the classical Hawkes process. First of all, the intensity of the baseline was given by time-inhomogeneous (see [11]). As the second case, the immigrants can arrive by a Cox process with shot noise intensity, which was known as the dynamic contagion model (see [5]). As the third case, the immigrants can arrive by a conditioned on renewal process instead of the Poisson process, which generalizes the classical Hawkes process. That is known as the renewal Hawkes process (see [33]). Recently, Seol [29] introduced the inverse case of Markovian Hawkes processes represented as several existing models of self-exciting point processes and proved some asymptotic behaviors of the inverse Markovian Hawkes processes. Seol [30] further studied an extended version of the inverse case of Markovian Hawkes model.

In the current paper, we consider a class of Markovian self-exciting processes, which combines a general Markovian Hawkes process and an inverse Markovian Hawkes process and has remarkable properties to be more active and useful models. We also study the limit theorems of a class of Markovian self-exciting processes. This paper has been organized into mainly two parts. The general review of the Hawkes process and the statement of main theorems are reported in Sect. 1. The proofs of the main theorems with some auxiliary results are provided in Sect. 2.

1.1 The general Hawkes process

In this section, we formally introduce the general Hawkes process that was introduced by Brémaud and Massoulié [3].

Let $\Upsilon_t^{-\infty} := \sigma(\mathbb{N}(C), C \subset (-\infty, t], C \in \mathcal{B}(\mathbb{R}))$ be an increasing function of the family of σ-algebras with $\mathbb{N}$ being a simple point process on $\mathbb{R}$. Any nonnegative $\Upsilon_t^{-\infty}$-progressively measurable process λ_t with

$$E\big[\mathbb{N}(a,b]|\Upsilon_a^{-\infty}\big] = E\biggl[\int_a^b \lambda_s\,ds \Big| \Upsilon_a^{-\infty}\biggr]$$

a.s. for all interval $(a,b]$ is called an $\Upsilon_t^{-\infty}$-intensity of $\mathbb{N}$. We use the notation $\mathbb{N}_t := \mathbb{N}(0,t]$ to present the number of points in the interval $(0,t]$. The general definition of Hawkes process is a simple point process $\mathbb{N}$ admitting an $\Upsilon_t^{-\infty}$-intensity

$$\lambda_t := \lambda\left(\int_{-\infty}^t h(t-s)\mathbb{N}(ds)\right),$$

where $\lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is left continuous and locally integrable, $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with the condition $\|h\|_{L^1} = \int_0^\infty h(t)\,dt < \infty$. In the literature, $\lambda(\cdot)$ and $h(\cdot)$ are usually referred to as a rate function and an exciting function, respectively. Assumption for local integrability of $\lambda(\cdot)$ makes sure that the process is nonexplosive, while the left continuity assumption makes sure that λ_t is Υ_t-predictable. The Hawkes process is generally non-Markovian because the future development of a self-exciting point process is determined by timing of the past events, whereas it is Markovian as a special case. If the exciting function h is an exponential function or a sum of exponential functions, then the process is Markovian with a generator of the process. However, the difficulty arises when h is neither an exponential function nor a sum of exponentials, in which case the process becomes non-Markovian. When $h(t) = pe^{-qt}$, the structure of the Hawkes process is Markovian in the manner that $Z_t = \int_{-\infty}^{t-} pe^{-q(t-s)}\,d\mathbb{N}_s$ is Markovian satisfying the dynamics

$$dZ_t = -qZ_t\,dt + p\,d\mathbb{N}_t, \tag{1}$$

where $\mathbb{N}_t$ has the intensity $\nu + Z_{t-}$ at time t and Z_t has the infinitesimal generator

$$\Gamma f(z) = -qzf'(z) + (\nu + z)\big[f(z+p) - f(z)\big]. \tag{2}$$

It is well known (see [14]) that

$$\frac{1}{t}\int_0^t Z_s\,ds \to \frac{\nu}{q-p}, \tag{3}$$

and

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s\,ds - \frac{\nu}{q-p}\cdot t\right] \to N\left(0, \frac{p^2\nu q}{(q-p)^3}\right) \tag{4}$$

in distribution as $t \to \infty$.

The Hawkes process generally can be classified as linear and nonlinear case models based on the intensity $\lambda(\cdot)$. When $\lambda(\cdot)$ is linear, we call the process linear Hawkes process; furthermore, for $\lambda(l) = \nu + l$, for some $\nu > 0$ and $\|h\|_{L^1} < 1$, we can use a useful method immigration-birth representation, also known as Galton–Watson theory. The limit results are well understood and more explicitly represented. The limit behaviors of the linear Hawkes processes with marks were reported by Karabash and Zhu [21]. Daley and Vere-Jones [4] investigated the law of large numbers (LLN) of the linear case model as shown in equation (5).

$$\frac{\mathbb{N}_t}{t} \to \frac{\nu}{1-\|h\|_{L^1}} \quad \text{as } t \to \infty. \tag{5}$$

The functional central limit theorem (FCLT) of linear multivariate Hawkes model under certain assumptions was investigated by Bacry et al. [1], and the results are given by

$$\frac{\mathbb{N}_{\cdot t} - \cdot\mu t}{\sqrt{t}} \to \sigma B(\cdot) \quad \text{as } t \to \infty,$$

where $B(\cdot)$ is the standard Brownian motion and

$$\mu = \frac{\nu}{1-\|h\|_{L^1}} \quad \text{and} \quad \sigma^2 = \frac{\nu}{(1-\|h\|_{L^1})^3}.$$

Throughout the paper, we use a weak convergence on $D[0,1]$, and the space of càdlàg function on $[0,1]$ is equipped with Skorokhod topology. Bordenave and Torrisi [2] showed that under the conditions $0 < \|h\|_{L^1} < 1$ and $\int_0^\infty th(t)\,dt < \infty$, $\mathbb{P}(\frac{\mathbb{N}_t}{t} \in \cdot)$ satisfies the large deviation principle with the good rate function $I(\cdot)$, which means that for any closed set $C \subset \mathbb{R}$,

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}(\mathbb{N}_t/t \in C) \le -\inf_{x\in C} I(x),$$

and for any open set $G \subset \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}(\mathbb{N}_t/t \in G) \ge -\inf_{x\in G} I(x),$$

where

$$I(x) = \begin{cases} x\theta_x + \nu - \frac{\nu x}{\nu + \|h\|_{L^1} x} & \text{if } x \in (0,\infty), \\ \nu & \text{if } x = 0, \\ +\infty & \text{if } x \in (-\infty, 0), \end{cases}$$

where $\theta = \theta_x$ is the unique solution in $(-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1})$ of

$$\mathbb{E}\bigl(e^{\theta S}\bigr) = \frac{x}{\nu + x\|h\|_{L^1}}, \quad x > 0, \tag{6}$$

where S in the above equation is $S(\infty)$, the total number of descendants with $\|h\|_{L^1}$. Zhu [35] showed that under the conditions $\|h\|_{L^1} < 1$ and $\sup_{t>0} t^{3/2} h(t) \le C < \infty$, for any Borel set $\mathcal{B}$ and time sequence $\sqrt{n} \ll \kappa(n) \ll n$, there exists a moderate deviation principle

$$\begin{aligned} -\inf_{x \in \mathcal{B}^\circ} L(x) &\le \liminf_{t\to\infty} \frac{t}{\kappa(t)^2} \log \mathbb{P}\biggl(\frac{1}{\kappa(t)}(\mathbb{N}_t - \mu t) \in \mathcal{B}\biggr) \\ &\le \limsup_{t\to\infty} \frac{t}{\kappa(t)^2} \log \mathbb{P}\biggl(\frac{1}{\kappa(t)}(\mathbb{N}_t - \mu t) \in \mathcal{B}\biggr) \le -\inf_{x\in\bar{\mathcal{B}}} L(x), \end{aligned} \tag{7}$$

where $L(x) = \frac{x^2(1-\|h\|_{L^1})^3}{2\nu}$.

When $\lambda(\cdot)$ is nonlinear, we call the process nonlinear Hawkes process, and the general Galton–Watson theory cannot be used to work. The nonlinear model is much harder to study because of the lack of immigration-birth representation with computational tractability. Brémaud and Massoulié [3] provided the unique stationary of nonlinear Hawkes processes under certain conditions with convergence to equilibrium of a nonstationary version. Massoulié [23] extended the stability results of the nonlinear case of Hawkes processes with random marks and also considered the Markovian case. The author also proved stability without the Lipschitz condition for $\lambda(\cdot)$. Furthermore, Brémaud [3] considered the rate of extinction for the nonlinear case of Hawkes process. A functional central limit theorem(FCLT) of the nonlinear case of Hawkes process was reported by Zhu [34]. Zhu [39] also proved large deviation principles for a special case of nonlinear Hawkes process when $h(\cdot)$ was an exponential function or a sum of exponential functions. Zhu [37] provided a large deviation principle level-3 of nonlinear Hawkes processes for the general $h(\cdot)$.

1.2 Inverse Markovian Hawkes process

In the recent paper of Seol [29], an inverse version of Markovian Hawkes process was developed and studied. This new model has some particular remarks compared with the general Hawkes process. For the general Hawkes process, the more jumps can be expected in the future, the more jumps one has in the past. However, for the inverse version of Hawkes process, the larger jumps can be expected in the future, the more jumps one has in the past. It is worth mentioning that, for the general Hawkes process, the self-excitation depends upon the intensity for the general Hawkes process, while for the inverse version of Hawkes process, the self-excitation depends upon the jump size. That is, for the general Hawkes process, self-excitation represents frequency, whereas for the inverse version of Markovian Hawkes process, self-excitation represents severity. The inverse Markovian

Hawkes process can be represented as several existing models of the self-exciting process, which means that if $p = 0$, then Z_t can be expressed as a shot-noise process, such as $Z_t = Z_0 e^{-qt} + \int_0^t \nu e^{-q(t-s)}\, d\mathbb{N}_s$, and if $\nu = 0$, then it can be represented as a jump-diffusion process with no diffusions, such as the following model $Z_t = Z_0 \exp(-qt + \log(1+p)\mathbb{N}_t)$.

Seol [29] first proposed an inverse version of the Markovian Hawkes process, which was defined as

$$dZ_t = -qZ_t\, dt + (\nu + pZ_{t-})\, d\mathbb{N}_t, \tag{8}$$

where N_t is Poisson with intensity 1 and $p > 0$, $q > 0$, and $\nu > 0$, and it follows that

$$d\left(e^{qt} Z_t\right) = (pZ_{t-} + \nu)e^{qt}\, d\mathbb{N}_t, \tag{9}$$

and since we assumed $Z_0 = 0$, we get

$$Z_t = \int_0^t (pZ_{s-} + \nu)e^{-q(t-s)}\, d\mathbb{N}_s. \tag{10}$$

The Z_t process has the infinitesimal generator

$$\Gamma f(z) = -qzf'(z) + f(z + pz + \nu) - f(z). \tag{11}$$

Under certain assumptions, Seol [29] obtained the law of large numbers

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Z_s\, ds = \frac{\nu}{q-p} \tag{12}$$

in probability and the central limit theorem

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s\, ds - \frac{\nu}{q-p} \cdot t\right] \to N\left(0, \frac{\nu^2 + 2\nu p \frac{\nu}{q-p} + p^2 \frac{\nu^2(p+q+2)}{(q-p)(2q-2p-p^2)}}{(q-p)^2}\right) \tag{13}$$

in distribution as $t \to \infty$. Furthermore, Seol [30] introduced a model combining the Hawkes process and the inverse Hawkes process, which is an extended version of the inverse Markovian Hawkes process. The extended model can be defined as

$$dZ_t = -qZ_t\, dt + p_1\, d\mathbb{N}_t^{(1)} + (\nu_2 + p_2 Z_{t-})\, d\mathbb{N}_t^{(2)}, \tag{14}$$

where $\mathbb{N}_t^{(1)}$ is a simple point process with intensity $\nu_1 + Z_{t-}$ at time t and $\mathbb{N}_t^{(2)}$ is a Poisson process with intensity 1, where p_1, p_2, q, ν_1, and ν_2 are all positive constants. The infinitesimal generator of Z_t process is given by

$$\Gamma f(z) = -qzf'(z) + (\nu_1 + z)\left[f(z + p_1) - f(z)\right] + f(z + \nu_2 + p_2 z) - f(z). \tag{15}$$

Under certain assumptions, Seol [30] obtained the law of large numbers

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Z_s\, ds = \frac{\nu_1 p_1 + \nu_2}{q - p_1 + p_2} \tag{16}$$

in probability a.s. as $t \to \infty$, and the central limit theorem

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s\,ds - \frac{\nu_1 p_1 + \nu_2}{q - p_1 + p_2}\cdot t\right] \to N\left(0,\sigma^2\right) \tag{17}$$

in distribution as $t \to \infty$, where

$$\begin{aligned}\sigma^2 &:= \frac{1}{(q-p_1-p_2)^2}\left[p_1^2\left(\nu_1 + \mathbb{E}[Z_\infty]\right) + \mathbb{E}\left[(\nu_1 + p_2 Z_\infty)^2\right]\right] \\ &= \frac{(p_1^2\nu_1 + \nu_1^2)K_1K_2 + (p_1^2\nu_1 + 2\nu_1 p_2)K_2K_3 + p_2^2(K_3K_5 + K_1K_4)}{K_1^3K_2},\end{aligned}$$

and K_i ($i \in 1,2,3,4,5$) are constants and

$$\begin{aligned}K_1 &= q - p_1 + p_2, \\ K_2 &= 2q - 2p_1 - p_2 - p_2^2, \\ K_3 &= \nu_1 p_1 + \nu_2, \\ K_4 &= \nu_1 p_1^2 + \nu_2^2, \\ K_5 &= p_1^2 + 2\nu_1 p_1 + 2\nu_2 p_2 + 2\nu_2.\end{aligned}$$

1.3 Main results of this paper

We now give the statement of the main part for this paper. We investigate asymptotic results for a more general Markovian self-exciting process that interpolates between the Hawkes process and the inverse Hawkes process. Our results mainly consist of both the central limit theorems (CLT) and the law of large numbers (LLN). We developed a more general and newly considered model, which is a class of Markovian self-exciting processes that interpolates between the general Hawkes process and the inverse Hawkes process.

We first define Z_t as a Markov process satisfying the dynamics

$$dZ_t = -qZ_t\,dt + p(\nu + Z_{t-})^{1-\gamma}\,d\mathbb{N}_t, \tag{18}$$

where we assume that $q > p$, and N_t is a simple point process with intensity

$$\lambda_t = (\nu + Z_{t-})^\gamma \tag{19}$$

at time t, where $0 \le \gamma \le 1$ is the interpolation coefficient. The infinitesimal generator of Z_t process is given by

$$\mathcal{A}f(z) = -qzf'(z) + (\nu + z)^\gamma\left[f\left(z + p(\nu + z)^{1-\gamma}\right) - f(z)\right]. \tag{20}$$

Note that when $\gamma = 1$, it reduces to the Markovian Hawkes process, and when $\gamma = 0$, it reduces to the inverse Markovian Hawkes process.

The assumptions that we use throughout the paper are stated below.

Assumption 1.1

1. $N(-\infty,0] = 0$, which means that Hawkes model has empty history;

2. $q > p > 0$;
3. $0 \leq \gamma \leq 1$, where γ is the interpolation coefficient;
4. $\nu > 0$.

The first asymptotic result is a law of large numbers for our considered model.

Theorem 1.2 *Let Z_t be defined in (18). Under Assumption 1.1, we have*

$$\frac{1}{t}\int_0^t Z_s\,ds \to \frac{p\nu}{q-p} \tag{21}$$

a.s. as $t \to \infty$.

The second asymptotic result is the central limit theorem.

Theorem 1.3 *Let Z_t be defined in (18). Under Assumption 1.1, we have*

$$\frac{1}{\sqrt{t}}\left(\int_0^t Z_s\,ds - \frac{p\nu}{q-p}t\right) \to N\left(0, \frac{p^2}{(q-p)^2}\mathbb{E}\left[(\nu + Z_\infty)^{2-\gamma}\right]\right) \tag{22}$$

in distribution as $t \to \infty$.

2 Proofs of the main results

In the current section, we give the proofs of our main theorems and related auxiliary results. The following are the key results to prove the main results. The key result is devoted to the distributional properties of non-Markovian inverse Hawkes processes. Both the first and the second moments of Z_t have been computed in Sect. 2.1. The main theorems of the paper are validated in Sects. 2.2 and 2.3.

2.1 Some auxiliary results

In this section, we obtain closed formulae for the moments of Z_t. In particular, the first moments can be discussed.

Proposition 2.1 *Let Z_t be defined in (18). Under Assumption 1.1, we have: Given $Z_0 > 0$,*

$$\mathbb{E}[Z_t] = Z_0 e^{-(q-p)t} + \frac{p\nu}{q-p}\left(1 - e^{-(q-p)t}\right). \tag{23}$$

In particular,

$$\mathbb{E}[Z_\infty] = \frac{p\nu}{q-p}. \tag{24}$$

Proof To show this, we will use the following:

$$\mathbb{E}f(Z_t) = f(Z_0) + \int_0^t \mathbb{E}\mathcal{A}f(Z_s)\,ds. \tag{25}$$

Taking $f(z) = z$ gives us two explicit forms

$$\mathbb{E}[Z_t] = Z_0 + \int_0^t \mathbb{E}\mathcal{A}Z_s\,ds. \tag{26}$$

We can compute that

$$\begin{aligned}\mathcal{A}z &= -qz + (\nu + z)^{\gamma} p(\nu + z)^{1-\gamma} \\ &= -(q-p)z + p\nu.\end{aligned}$$

This implies that

$$\mathbb{E}[Z_t] = Z_0 + \int_0^t \mathbb{E}\mathcal{A}Z_s\,ds = Z_0 + \int_0^t \bigl(p\nu + (p-q)\mathbb{E}[Z_s]\bigr)\,ds. \tag{27}$$

Using the derivative with respect to t to both sides, we have

$$\frac{d}{dt}\mathbb{E}[Z_t] = -(q-p)\mathbb{E}[Z_t] + p\nu. \tag{28}$$

Solving differential equation yields

$$\mathbb{E}[Z_t] = Z_0 e^{-(q-p)t} + \frac{p\nu}{q-p}\bigl(1 - e^{-(q-p)t}\bigr). \tag{29}$$

In particular, we have, as $t \to \infty$,

$$\mathbb{E}[Z_\infty] = \frac{p\nu}{q-p}, \tag{30}$$

since Z_t is uniformly integrable. □

Remark 2.2 We notice that in the above result, $\mathbb{E}[Z_t]$ is independent of the interpolation coefficient $\gamma \in [0,1]$. This means that for the Markovian Hawkes process, the inverse Markovian Hawkes process and any interpolation in between share the same first moment.

2.2 Proof of the law of large numbers

The following are the proofs of the first main theorems.

Note that

$$\mathcal{A}z = -(q-p)z + p\nu, \tag{31}$$

where $q - p > 0$. Using the definition of Z_t process, the Foster–Lyapunov criterion (see [9] for details), and (i) of Assumption 1.1, we conclude that Z_t is ergodic. Therefore, by ergodic theorem and equation (30), we have

$$\frac{1}{t}\int_0^t Z_s\,ds \to \mathbb{E}[Z_\infty] = \frac{p\nu}{q-p} \tag{32}$$

a.s. as $t \to \infty$. This completes the proof of Theorem 1.2.

2.3 Proof of the central limit theorem

In the current section, we prove the second main result. First of all, let us prove that

$$\mathbb{E}\bigl[(\nu + Z_\infty)^{2-\gamma}\bigr] < \infty. \tag{33}$$

We can compute that

$$\mathcal{A}z^{2-\gamma} = -q(2-\gamma)z^{2-\gamma} + (\nu+z)^{\gamma}\big[\big(z+p(\nu+z)^{1-\gamma}\big)^{2-\gamma} - z^{2-\gamma}\big]. \tag{34}$$

If $\gamma = 1$, then

$$\mathcal{A}z^{2-\gamma} = \mathcal{A}z = -(q-p)z + p\nu = -(q-p)z^{2-\gamma} + p\nu. \tag{35}$$

If $\gamma < 1$, then

$$\big(z+p(\nu+z)^{1-\gamma}\big)^{2-\gamma} - z^{2-\gamma} = z^{2-\gamma}\left[\left(1+p\frac{(\nu+z)^{1-\gamma}}{z}\right)^{2-\gamma} - 1\right], \tag{36}$$

where $p\frac{(\nu+z)^{1-\gamma}}{z} \to 0$ as $z \to \infty$, and we know that

$$\lim_{x\to 0}\frac{(1+x)^{2-\gamma}-1}{x} = 2-\gamma. \tag{37}$$

Therefore

$$\begin{aligned}
&\lim_{z\to\infty}\frac{(\nu+z)^{\gamma}[(z+p(\nu+z)^{1-\gamma})^{2-\gamma}-z^{2-\gamma}]}{z^{2-\gamma}} \\
&\quad= \lim_{z\to\infty}\frac{(\nu+z)^{\gamma}}{z^{\gamma}}\frac{[(1+p\frac{(\nu+z)^{1-\gamma}}{z})^{2-\gamma}-1]}{p\frac{(\nu+z)^{1-\gamma}}{z}}\frac{p\frac{(\nu+z)^{1-\gamma}}{z}}{z^{-\gamma}} \\
&\quad= p(2-\gamma).
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exists some $C_\epsilon > 0$ so that

$$\mathcal{A}z^{2-\gamma} \le -(q-p-\epsilon)(2-\gamma)z + C_\epsilon. \tag{38}$$

Since $q > p$, we can choose $\epsilon > 0$ to be sufficiently small so that $q-p-\epsilon > 0$. Hence, $\mathbb{E}[Z_\infty^{2-\gamma}] < \infty$, which implies that $\mathbb{E}[(\nu+Z_\infty)^{2-\gamma}] < \infty$.

Note that

$$\mathcal{A}z = -(q-p)z + p\nu \tag{39}$$

and

$$M_t = f(Z_t) - f(Z_0) - \int_0^t \mathcal{A}f(Z_s)\,ds \tag{40}$$

is a martingale where we can take

$$f(z) = -\frac{z}{q-p}. \tag{41}$$

Therefore

$$\int_0^t\left(Z_s - \frac{p\nu}{q-p}\right)ds = -M_t - \frac{Z_t}{q-p} + \frac{Z_0}{q-p}. \tag{42}$$

M_t is a martingale with quadratic variation the same as the quadratic variation of $\frac{-Z_t}{q-p}$, which is given by

$$\frac{1}{(q-p)^2}\int_0^t p^2(\nu+Z_{s-})^{2-2\gamma}\,dN_s \tag{43}$$

and by ergodic theorem

$$\frac{1}{t}\frac{1}{(q-p)^2}\int_0^t p^2(\nu+Z_{s-})^{2-2\gamma}\,dN_s \to \frac{p^2}{(q-p)^2}\mathbb{E}\big[(\nu+Z_\infty)^{2-\gamma}\big] \tag{44}$$

a.s. as $t\to\infty$. Applying the central limit theorem for the martingales properties (see Theorem VIII-3.11 of [17] for details), we have

$$\frac{M_t}{\sqrt{t}} \to N\left(0, \frac{p^2}{(q-p)^2}\mathbb{E}\big[(\nu+Z_\infty)^{2-\gamma}\big]\right) \tag{45}$$

in distribution as $t\to\infty$. Therefore, by the Markov inequality and the fact that $\lim_{t\to\infty}\mathbb{E}[Z_t] = \frac{p\nu}{q-p} < \infty$, we have both $\frac{Z_t}{\sqrt{t}} \to 0$ in probability and $\frac{Z_0}{\sqrt{t}} \to 0$ in probability as $t\to\infty$. This completes the proof of Theorem 1.3.

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Declarations

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The authors declare no competing interests.

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