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Correction to: On statistical convergence and strong Cesàro convergence by moduli for double sequences

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Abstract

We correct a logic mistake in our paper "On statistical convergence and strong Cesàro convergence by moduli for double sequences" (León-Saavedra et al. in J. Inequal. Appl. 2022:62, 2022).

Keywords: Strong Cesàro convergence; f-density; f-statistical convergence; f-strong Cesàro convergence

1 Introduction

It has come to our attention that there is a logic mistake with the converse of some results in our paper [1]. These converse of these results are not central in the papers, but they could be interested in its own right.

Let us denote $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$. The following result correct Theorem 3.5 in [1].

Proposition 1.1

a) *If all statistical convergent double sequences* $(x_{i,j})$ *are* f *statistical convergent then* f *must be compatible.*
b) *If all strong Cesàro convergent double sequences* $(x_{i,j})$ *are* f*-strong Cesàro convergent then* f *is compatible.*

Proof If f is not compatible, then there exists $c > 0$ such that $\limsup_n \frac{f(n\varepsilon)}{f(n)} > c$. Assume that ε_k is a decreasing sequence converging to 0, for each k we can construct inductively an increasing sequence m_k satisfying $f(m_k\varepsilon_k) > cf(m_k)$ and

$$\varepsilon_{k+1} - \frac{m_k\varepsilon_k - 1}{m_{k+1}} < \left(1 - \sqrt{\frac{m_k}{m_{k+1}}} - \frac{2}{\sqrt{m_{k+1}}}\right)^2. \tag{1.1}$$

Let us define by $n_k = \lfloor m_k\varepsilon_k \rfloor + 1$, and we set $\ell_k = \lfloor \sqrt{m_k} \rfloor$. An easy check using (1.1) yields $(\ell_{k+1} - \ell_k)^2 > n_{k+1} - n_k$.

Let us fix $A_{k+1} \subset [\ell_{k+1} - \lfloor \sqrt{n_{k+1} - n_k} \rfloor - 1] \times [\ell_{k+1} - \lfloor \sqrt{n_{k+1} - n_k} \rfloor - 1]$ a subset of $\mathbb{N} \times \mathbb{N}$ such that $\#(A_{k+1}) = n_{k+1} - n_k$. Let us denote $A = \bigcup_k A_k$ and set $x_{i,j} = \chi_A(i,j)$. Let us see

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that $x_{i,j}$ is statistically convergent to 0, but not f-statistically convergent. Indeed, for any $m, n \in \mathbb{N}$, there exist p and q, such that $\ell_p < m \leq \ell_{p+1}$ and $\ell_q < n \leq \ell_{q+1}$. Set $k = \min\{p, q\}$. We can suppose without loss that $k \geq \ell_{k+1} - \lfloor\sqrt{n_{k+1} - n_k}\rfloor - 1$. Hence, since $(\sqrt{m_{k+1}} - 1)^2 \leq \ell_{k+1}^2 \leq m_k$, for any $\varepsilon > 0$:

$$\begin{aligned}\frac{\#\{(i,j) : i \leq m, j \leq n \text{ and } |x_{i,j}| > \varepsilon\}}{mn} &\leq \frac{n_k}{\ell_k^2} + \frac{n_{k+1} - n_k}{[\ell_{k+1} - \lfloor\sqrt{n_{k+1} - n_k}\rfloor - 1]^2} \\ &\leq \frac{n_k}{m_k} + \frac{n_{k+1} - n_k}{(\sqrt{m_{k+1}} - \sqrt{n_{k+1} - n_k} - 2)^2} \\ &\leq \frac{n_k}{m_k} + \frac{\frac{n_{k+1}-n_k}{m_{k+1}}}{(1 - \sqrt{\frac{n_{k+1}-n_k}{m_{k+1}}} - \frac{2}{\sqrt{m_{k+1}}})^2}\end{aligned}$$

which goes to zero as $r \to \infty$ as desired. On the other hand, if we set

$$A_\varepsilon(p, q, m, n) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : p \leq i \leq m, q \leq j \leq n, |x_{ij}| > \varepsilon\},$$

we shall show that there exists ε such that the limit

$$\lim_{p,q} \lim_{m,n} \frac{f(\#A_\varepsilon(p, q, m, n))}{f(mn)}$$

is not zero. Indeed,

$$\frac{f(\#\{(i,j) : i \leq \ell_k, j \leq \ell_k, |x_{i,j}| > 1/2\})}{f(\ell_k^2)} \geq \frac{f(n_k)}{f(m_k)} \geq \frac{f(m_k \varepsilon_k)}{f(m_k)} \geq c,$$

which gives that

$$\lim_{m,n} \frac{f(\#A_{1/2}(0, 0, m, n))}{f(mn)} \geq c.$$

On the other hand, for each p, q there exists p', q' such that $\ell'_p \leq p \leq \ell_{p'+1}$ and $\ell'_q \leq q \leq \ell_{q'+1}$. Set $s = \max\{p', q'\}$. Since $A_{1/2}(0, 0, m, n) \subset A_{1/2}(p, q, m, n) \cup A_{1/2}(s, s, m, n)$, we get that for any $\delta > 0$

$$\lim_{m,n} \frac{A(p, q, m, n)}{f(mn)} \geq \lim_{m,n} \frac{f(A_{1/2}(0, 0, m, n))}{f(mn)} - \frac{f(n_s)}{f(m, n)} \geq c - \delta$$

which yields part a). Again the part b) is the same proof. □

The next result fixed the converse of Theorem 3.7 in [1].

Proposition 1.2 *If all f-strong Cesàro convergent double sequences are f-statistically and bounded then f must be compatible.*

Proof If f is not compatible then there exist two sequences (ε_k), (m_k) satisfying $f(m_k \varepsilon_k) \geq cf(m_k)$ for some $c > 0$. We set $\ell_k = \lfloor\sqrt{m_k}\rfloor$, we can select m_k inductively, such that the sequence

$$r_{k+1} = \frac{m_{k+1}\varepsilon_{k+1} - m_k\varepsilon_k}{(\ell_{k+1} - \ell_k)^2}$$

is decreasing and converges to zero. Again it is direct to show that $x_{i,j} = \sum_{i,j} r_{k+1}\chi_{(\ell_k,\ell_{k+1}]}(i,j)$ is f-statistically convergent to zero, but not f-strong Cesàro convergent. □

Let us recall that f is a compatible modulus function provided $\lim_{\varepsilon\to 0}\limsup_n \frac{f(n\varepsilon)}{f(n)} = 0$. We will say that a modulus function f *is compatible of second order or* 2*-compatible*, provided $\lim_{\varepsilon\to 0}\limsup_n \frac{f(n\varepsilon)}{f(n^2)} = 0$. Clearly, if f is compatible, then f is 2-compatible. The next result correct Theorem 2.6 in [1].

Proposition 1.3 *Assume that for any f-statistical convergent double sequence $(x_{i,j})$ we have that for any $\varepsilon > 0$*

$$\lim_{m,n}\frac{f(\#A_\varepsilon(0,0,m,n))}{f(mn)} = 0$$

then f must be 2*-compatible.*

Proof Indeed, assume that f is not compatible. Let ε_n be a decreasing sequence converging to 0. Since f is not compatible, there exists $c > 0$ such that, for each k, there exists m_k such that $f(m_k\varepsilon_k) > cf(m_k)$. Moreover, we can select m_k inductively satisfying

$$1 - \varepsilon_{k+1} - \frac{1}{m_{k+1}} > \frac{(1-\varepsilon_k)m_k}{m_{k+1}}. \tag{1.2}$$

Now we use an standard argument used to construct subsets with prescribed densities. Set $n_k = \lfloor m_k\varepsilon_k\rfloor + 1$. And extracting a subsequence if it is necessary, we can assume that $n_1 < n_2 < \cdots$, $m_1 < m_2 < \cdots$. Thus, set $A_k = [m_{k+1} - (n_{k+1} - n_k)] \cap \mathbb{N}$. Condition (1.2) guarantee that $A_k \subset [m_k, m_{k+1}]$.

Let us denote $A = \bigcup_k A_k$, and $x_n = \chi_A(n)$.

An easy check show that the sequence $x_{1,n} = x_n$ is f-statistical convergent to zero, but $\frac{f(\#A_\varepsilon(0,0,m_k,m_k))}{f(m_k^2)} \geq c$ which yields the desired result. □

It is worthy to find a 2-compatible function that is not compatible, and to improve Proposition 1.3 replacing 2-compatibility by compatibility.

The corrections have been indicated in this article and the original article [1] has been corrected.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

All authors read and approved the final manuscript.

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