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On a more accurate half-discrete multidimensional Hilbert-type inequality involving one derivative function of m-order

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Abstract

By means of the weight functions, the idea of introduced parameters, using the transfer formula and Hermite–Hadamard's inequality, a more accurate half-discrete multidimensional Hilbert-type inequality with the homogeneous kernel as $\frac{1}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}$ $(x, \lambda > 0)$ involving one derivative function of m-order is given. The equivalent conditions of the best possible constant factor related to several parameters are considered. The equivalent forms. the operator expressions and some particular inequalities are obtained.

MSC: 26D15

Keywords: Weight function; Half-discrete multidimensional Hilbert-type inequality; Derivative function of m-order; Parameter; Beta function

1 Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. We have the following discrete Hardy–Hilbert's inequality with the best possible constant factor $\pi / \sin(\frac{\pi}{p})$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{1}$$

The integral analogues of (1) named in Hardy–Hilbert's integral inequality was provided as follows (cf. [1], Theorem 316):

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} \, dx, dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x) \, dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y) \, dy \right)^{\frac{1}{q}}, \tag{2}$$

with the same best possible factor. The more accurate form of (1) was given as follows (cf. [1], Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{3}$$

Inequalities (1)–(3) with their extensions played an important role in analysis and its applications (cf. [2–13]).

The following half-discrete Hilbert-type inequality was provided in 1934 (cf. [1], Theorem 351): If $K(x)$ $(x>0)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^\infty K(x)x^{s-1}\,dx<\infty$, $f(x)\geq 0, 0<\int_0^\infty f^p(x)\,dx<\infty$, then

$$\sum_{n=1}^{\infty}n^{p-2}\left(\int_0^\infty K(nx)f(x)\,dx\right)^p<\phi^p\left(\frac{1}{q}\right)\int_0^\infty f^p(x)\,dx. \tag{4}$$

Some new extensions of (3) were given by [14–19].

In 2006, by using Euler–Maclaurin summation formula, Krnic et al. [20] gave an extension of (1) with the kernel as $\frac{1}{(m+n)^\lambda}$ $(0<\lambda\leq 4)$. In 2019–2020, following the results of [20], Adiyasuren et al. [21] provided an extension of (1) involving partial sums, and Mo et al. [22] gave an extension of (2) involving the upper limit functions, which a new application of the way in [21]. In 2016–2017, Hong et al. [23, 24] considered some equivalent statements of the extensions of (1) and (2) with a few parameters. Some further results were provided by [25–28]. In 2023, Hong et. al. [29] published a more accurate half-discrete multidimensional Hilbert-type inequality involving one multiple upper limit function.

In this paper, following the way of [22], by means of the weight functions, the idea of introduced parameters, using the transfer formula and Hermite–Hadamard's inequality, a more accurate half-discrete multidimensional Hilbert-type inequality with the homogeneous kernel as $\frac{1}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}$ $(x,\lambda>0)$ involving one derivative function of m-order and the beta function is given. The equivalent conditions of the best possible constant factor related to several parameters are considered. The equivalent forms, the operator expressions and some particular inequalities are obtained. Our new work is different to [29], which is involving one higher-order derivative function but not involving one multiple upper limit function.

2 Some formulas and lemmas

Hereinafter in this paper, we suppose that $p>1, \frac{1}{p}+\frac{1}{q}=1, \lambda>0, \lambda_1, \lambda_2\in(0,\lambda), m,n\in \mathrm{N}=\{1,2,\ldots\}, \alpha\in(0,1], \xi\in[0,\frac{1}{2}], \hat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \hat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}, \|y\|_\alpha:=(\sum_{k=1}^n|y_i|^\alpha)^{\frac{1}{\alpha}}$ $(y=(y_1,\ldots,y_n)\in\mathrm{R}^n)$. We also assume that $f(x)$ (≥ 0) is a differentiable function of m-order unless at finite points in $\mathrm{R}_+=(0,\infty)$,

$$f^{(k-1)}(x)=o\left(e^{tx}\right)\quad(t>0;x\to\infty),\qquad f^{(k-1)}\left(0^+\right)=0\quad(k=1,\ldots,m),$$

$$f^{(m)}(x),\ a_k=(a_{k_1},\ldots,a_{k_n})\geq 0\ \left(x\in\mathrm{R}_+=(0,\infty),\ k=(k_1,\ldots,k_n)\in\mathrm{N}^n\right),\ \text{such that}$$

$$0<\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\,dx<\infty\quad\text{and}\quad 0<\sum_k\|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n}a_k^q<\infty.$$

For $M > 0, \psi(u)$ $(u > 0)$ is a nonnegative measurable function, we have the following transfer formula (cf. [2], (9.3.3)):

$$\begin{aligned}
&\int \cdots \int_{\{y \in R_+^n; 0 < \sum_{i=1}^n (\frac{y_i}{M})^\alpha \leq 1\}} \psi\left(\sum_{i=1}^n \left(\frac{y_i}{M}\right)^\alpha\right) dy_1 \cdots dy_n \\
&\quad = \frac{M^n \Gamma^n(\frac{1}{\alpha})}{\alpha^n \Gamma(\frac{n}{\alpha})} \int_0^1 \psi(u) u^{\frac{n}{\alpha}-1}\, du. \qquad (5)
\end{aligned}$$

In particular, (i) in view of $\|y\|_\alpha = M[\sum_{i=1}^n (\frac{y_i}{M})^\alpha]^{\frac{1}{\alpha}}$, by (5), we have

$$\begin{aligned}
&\int_{R_+^n} \phi(\|y\|_\alpha)\, dy \\
&\quad = \lim_{M\to\infty} \int \cdots \int_{\{y \in R_+^n; 0 < \sum_{i=1}^n (\frac{y_i}{M})^\alpha \leq 1\}} \phi\left(M\left[\sum_{i=1}^n \left(\frac{y_i}{M}\right)^\alpha\right]^{\frac{1}{\alpha}}\right) dy_1 \cdots dy_n \\
&\quad = \lim_{M\to\infty} \frac{M^n \Gamma^n(\frac{1}{\alpha})}{\alpha^n \Gamma(\frac{n}{\alpha})} \int_0^1 \phi(M u^{\frac{1}{\alpha}}) u^{\frac{n}{\alpha}-1}\, du \overset{v = M u^{\frac{1}{\alpha}}}{=} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \int_0^\infty \phi(v) v^{n-1}\, dv; \qquad (6)
\end{aligned}$$

(ii) for $\psi(u) = \phi(M u^{\frac{1}{\alpha}}) = 0.u < \frac{b^\alpha}{M^\alpha} (b > 0)$, by (5), we have

$$\begin{aligned}
\int_{\{y \in R_+^n, \|y\|_\alpha \geq b\}} \phi(\|y\|_\alpha)\, dy &= \lim_{M\to\infty} \frac{M^n \Gamma^n(\frac{1}{\alpha})}{\alpha^n \Gamma(\frac{n}{\alpha})} \int_{\frac{b^\alpha}{M^\alpha}}^1 \phi(M u^{\frac{1}{\alpha}}) u^{\frac{n}{\alpha}-1}\, du \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \int_b^\infty \phi(v) v^{n-1}\, dv. \qquad (7)
\end{aligned}$$

Lemma 1 *For* $s > 0, \alpha \in (0,1], \xi \in [0, \frac{1}{2}], A_\xi := \{y = \{y_1, \ldots, y_n\}; y_i > \xi\ (i = 1, \ldots, n)\}$, *define the following function*:

$$g_x(y) := \frac{1}{(x + \|y - \xi\|_\alpha)^s} = \frac{1}{\{x + [\sum_{i=1}^n (y_i - \xi)^\alpha]^{1/\alpha}\}^s} \quad \bigl(x > 0, y = (y_1, \ldots, y_n) \in A_\xi\bigr).$$

Then we have $\frac{\partial}{\partial y_j} g_x(y) < 0, \frac{\partial^2}{\partial y_j^2} g_x(y) > 0$ $(y \in A_\xi; j = 1, \ldots, n)$.

Proof We obtain that for $s > 0, \alpha \in (0,1], \xi \in [0, \frac{1}{2}], y \in A_\xi$,

$$\begin{aligned}
\frac{\partial}{\partial y_j} g_x(y) &= \frac{-s[\sum_{i=1}^n (y_i - \xi)^\alpha]^{\frac{1}{\alpha}-1}(y_j - \xi)^{\alpha-1}}{\{x + [\sum_{i=1}^n (y_i - \xi)^\alpha]^{1/\alpha}\}^{s+1}} < 0, \\
\frac{\partial^2}{\partial y_j^2} g_x(y) &= \frac{s(s+1)[\sum_{i=1}^n (y_i - \xi)^\alpha]^{\frac{2}{\alpha}-2}(y_j - \xi)^{2\alpha-2}}{\{x + [\sum_{i=1}^n (y_i - \xi)^\alpha]^{1/\alpha}\}^{s+2}} \\
&\quad + \frac{s(1-\alpha)[\sum_{i=1}^n (y_i - \xi)^\alpha]^{\frac{1}{\alpha}-2}(y_j - \xi)^{\alpha-2}}{\{x + [\sum_{i=1}^n (y_i - \xi)^\alpha]^{1/\alpha}\}^{s+1}} \left[\sum_{i=1}^n (y_i - \xi)^\alpha - (y_j - \xi)^\alpha\right] > 0.
\end{aligned}$$

The lemma is proved. □

Note. In the same way, for $s_2 \leq n, \alpha \in (0,1], \xi \in [0,\frac{1}{2}], y \in A_\xi$, we can prove that

$$\frac{\partial}{\partial y_j}\|y-\xi\|_\alpha^{s_2-n} \leq 0, \frac{\partial^2}{\partial y_j^2}\|y-\xi\|_\alpha^{s_2-n} \geq 0 \quad (j=1,\ldots,n), \tag{8}$$

and then for $s_2 \leq n, \alpha \in (0,1], \xi \in [0,\frac{1}{2}], h_x(y) := g_x(y)\|y-\xi\|_\alpha^{s_2-n}$ $(x>0, y \in A_\xi)$, by Lemma 1, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial y_j}h_x(y) = \|y-\xi\|_\alpha^{s_2-n}\frac{\partial}{\partial y_j}g_x(y) + g_x(y)\frac{\partial}{\partial y_j}\|y-\xi\|_\alpha^{s_2-n} < 0, \\
&\frac{\partial^2}{\partial y_j^2}h_x(y) = \frac{\partial}{\partial y_j}\|y-\xi\|_\alpha^{s_2-n}\frac{\partial}{\partial y_j}g_x(y) + \|y-\xi\|_\alpha^{s_2-n}\frac{\partial^2}{\partial y_j^2}g_x(y) \\
&\qquad + \frac{\partial}{\partial y_j}g_x(y)\frac{\partial}{\partial y_j}\|y-\xi\|_\alpha^{s_2-n} + g_x(y)\frac{\partial^2}{\partial y^2}\|y-\xi\|_\alpha^{s_2-n} > 0, \quad (j=1,\ldots,n).
\end{aligned} \tag{9}$$

Lemma 2 *For $c>0$, we have the following inequalities*:

$$\frac{\Gamma^n(\frac{1}{\alpha})}{c\alpha^{n-1}\Gamma(\frac{n}{\alpha})} < \sum_k \|k\|_\alpha^{-c-n} < \frac{2^c\Gamma^n(\frac{1}{\alpha})}{c\alpha^{n-1}\Gamma(\frac{n}{\alpha})}, \tag{10}$$

where, $\sum_k G(k) = \sum_{k_n=1}^\infty \cdots \sum_{k_1=1}^\infty G(k_1,\ldots,k_n)$.

Proof By (8) (for $\xi = 0$), in view of $-c-n<0$, we find that

$$\frac{\partial}{\partial y_j}\|y\|_\alpha^{-c-n} < 0, \frac{\partial^2}{\partial y_j^2}\|y\|_\alpha^{-c-n} > 0 \quad (j=1,\ldots,n),$$

and then by Hermite–Hadamard's inequality (cf. [30]) and (7), we have

$$\sum_k \|k\|_\alpha^{-c-n} < \int_{\{y\in R_+^n, \|y\|_\alpha \geq \frac{1}{2}\}} \|y\|_\alpha^{-c-n}\,dy = \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_{\frac{1}{2}}^\infty v^{-c-n}v^{n-1}\,dv = \frac{2^c\Gamma^n(\frac{1}{\alpha})}{c\alpha^{n-1}\Gamma(\frac{n}{\alpha})}.$$

By the decreasingness property of series and (7), it follows that

$$\sum_k \|k\|_\alpha^{-c-n} > \int_{\{y\in R_+^n, \|y\|_\alpha \geq 1\}} \|y\|_\alpha^{-c-n}\,dy = \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_1^\infty v^{-c-1}\,dv = \frac{\Gamma^n(\frac{1}{\alpha})}{c\alpha^{n-1}\Gamma(\frac{n}{\alpha})},$$

namely, inequalities (10) follow.

The lemma is proved. □

Lemma 3 *For $s>0$, we fine the following weight functions*:

$$\varpi_s(s_2,x) := x^{s-s_2}\sum_k \frac{\|k-\xi\|_\alpha^{s_2-n}}{(x+\|k-\xi\|_\alpha)^s} \quad (x \in \mathrm{R}_+), \tag{11}$$

$$\omega_s(s_1,k) := \|k-\xi\|_\alpha^{s-s_1}\int_0^\infty \frac{x^{s_1-1}}{(x+\|k-\xi\|_\alpha)^s}\,dx \quad \left(k \in \mathrm{N}^n\right), \tag{12}$$

(i) *For* $0 < s_2 < s, s_2 \leq n$, *we have the following inequality*:

$$\varpi_s(s_2,x) < \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(s_2, s-s_2) \quad (x \in \mathrm{R}_+); \tag{13}$$

(ii) *for* $0 < s_1 < s$, *the following expression follows*:

$$\omega_s(s_1,k) = B(s_1, s-s_1) \quad \big(y \in \mathrm{R}_+^{\mathrm{n}}\big), \tag{14}$$

where, $B(u,v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt$ $(u,v>0)$ *is the beta function* (*cf.* [31]).

Proof (i) For $0 < s_2 < s, s_2 \leq n$, by (9) and Hermite–Hadamard's inequality, we have

$$\begin{aligned}\varpi_s(s_2,x) &< x^{s-s_2} \int_{A_{1/2}} \frac{\|y-\xi\|_\alpha^{s_2-n}}{(x+\|y-\xi\|_\alpha)^s}\,dy \leq x^{s-s_2}\int_{A_\xi} \frac{\|y-\xi\|_\alpha^{s_2-n}}{(x+\|y-\xi\|_\alpha)^s}\,dy \\ &= x^{s-s_2}\int_{R_+^n} \frac{\|y\|_\alpha^{s_2-n}}{(x+\|y\|_\alpha)^s}\,dy.\end{aligned}$$

Setting $\phi(v) := \frac{v^{s_2-n}}{(x+v)^s}$, by (6), it follows that

$$\begin{aligned}\varpi_s(s_2,x) &< x^{s-s_2}\int_{R_+^n} \phi\big(\|y\|_\alpha\big)\,dy = x^{s-s_2}\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \phi(v)v^{n-1}\,dv \\ &= x^{s-s_2}\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \frac{v^{s_2-1}}{(x+v)^s}\,dv \overset{u=v/x}{=} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \frac{u^{s_2-1}}{(1+u)^s}\,du \\ &= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(s_2, s-s_2),\end{aligned}$$

namely, (13) follows.

(ii) Setting $u = \frac{x}{\|k-\xi\|_\alpha}$ in (12), we find

$$\begin{aligned}\omega_s(s_1,k) &= \|k-\xi\|_\alpha^{s-s_1}\int_0^\infty \frac{(u\|k-\xi\|_\alpha)^{s_1-1}\|k-\xi\|_\alpha}{(u\|k-\xi\|_\alpha+\|k-\xi\|_\alpha)^s}\,du \\ &= \int_0^\infty \frac{u^{s_1-1}}{(u+1)^s}\,du = B(s_1, s-s_1),\end{aligned}$$

and then we have (17).

The lemma is proved. □

We indicate the following gamma function (cf. [31]): $\Gamma(\alpha) := \int_0^\infty e^{-t}t^{\alpha-1}\,dt$ $(\alpha > 0)$, satisfying $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $(\alpha > 0)$ and $B(u,v) = \frac{1}{\Gamma(u+v)}\Gamma(u)\Gamma(v)$ $(u,v>0)$. By the definition of the gamma function, for $\lambda, x > 0$, the following expression holds:

$$\frac{1}{(x+\|k-\xi\|_\alpha)^{\lambda+m}} = \frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}e^{-(x+\|k-\xi\|_\alpha)t}\,dt. \tag{15}$$

Lemma 4 *For* $t > 0$, *we have the following expression*:

$$\int_0^\infty e^{-tx}f(x)\,dx = t^{-m}\int_0^\infty e^{-tx}f^{(m)}(x)\,dx. \tag{16}$$

Proof Since $f(0)=0, f(x)=o(e^{tx})$ $(t>0; x\to\infty)$, we find

$$\int_0^\infty e^{-tx}f'(x)\,dx=\int_0^\infty e^{-tx}\,df(x)=e^{-tx}f(x)|_0^\infty-\int_0^\infty f(x)\,de^{-tx}$$
$$=\lim_{x\to\infty}\frac{f(x)}{e^{tx}}+t\int_0^\infty e^{-tx}f(x)\,dx=t\int_0^\infty e^{-tx}f(x)\,dx.$$

Inductively, for $f^{(i)}(0^+)=0, f^{(i)}(x)=o(e^{tx})$ $(t>0, i=1,\ldots,m; x\to\infty)$, we still have

$$\int_0^\infty e^{-tx}f(x)\,dx=t^{-1}\int_0^\infty e^{-tx}f'(x)\,dx=\cdots=t^{-i}\int_0^\infty e^{-tx}f^{(i)}(x)\,dx,$$

namely, expression (17) follows.

The lemma is proved. □

Lemma 5 *We have the following inequality*:

$$\begin{aligned}I_\lambda&:=\sum_k\int_0^\infty\frac{f^{(m)}(x)a_k}{(x+\|k-\xi\|_\alpha)^\lambda}\,dx\\&<\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\\&\quad\times\left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\big(f^{(m)}(x)\big)^p\,dx\right]^{\frac{1}{p}}\left[\sum_k\|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n}a_k^q\right]^{\frac{1}{q}}.\end{aligned}\tag{17}$$

Proof By Hölder's inequality (cf. [30]), and Lebesgue term by term integration theorem (cf. [32]), we obtain

$$\begin{aligned}I_\lambda&=\sum_k\int_0^\infty\frac{1}{(x+\|k-\xi\|_\alpha|)^\lambda}\left[\frac{\|k-\xi\|_\alpha^{(\lambda_2-n)/p}}{x^{(\lambda_1-1)/q}}f^{(m)}(x)\right]\left[\frac{x^{(\lambda_1-1)/q}}{\|k-\xi\|_\alpha^{(\lambda_2-n)/p}}a_k\right]dx\\&\le\left\{\int_0^\infty\left[\sum_k\frac{1}{(x+\|k-\xi\|_\alpha|)^\lambda}\frac{\|k-\xi\|_\alpha^{\lambda_2-n}}{x^{(\lambda_1-1)(p-1)}}\right]\big(f^{(m)}(x)\big)^p\,dx\right\}^{\frac{1}{p}}\\&\quad\times\left\{\sum_k\left[\int_0^\infty\frac{1}{(x+\|k-\xi\|_\alpha|)^\lambda}\frac{x^{\lambda_1-1}}{\|k-\xi\|_\alpha^{(\lambda_2-n)(q-1)}}\,dx\right]a_k^q\right\}^{\frac{1}{q}}\\&=\left[\int_0^\infty\varpi_\lambda(\lambda_2,x)x^{p(1-\hat{\lambda}_1)-1}\big(f^{(m)}(x)\big)^p\,dx\right]^{\frac{1}{p}}\\&\quad\times\left[\sum_k\omega_\lambda(\lambda_1,k)\|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n}a_k^q\right]^{\frac{1}{q}}.\end{aligned}$$

Therefore, by (13) and (15) (for $s=\lambda, s_1=\lambda_1, s_2=\lambda_2$), we have (18).

The lemma is proved. □

3 Main results

Theorem 1 *We have the following more accurate half-discrete multidimensional Hilbert-type inequality involving one derivative function of m-order*:

$$\begin{aligned} I &:= \sum_k \int_0^\infty \frac{f(x)a_k}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}\,dx \\ &< \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right))^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\ &\quad\times\left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}\left[\sum_k \|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n}a_k^q\right]^{\frac{1}{q}}. \end{aligned} \tag{18}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, we reduce (19) *to the following*:

$$\begin{aligned} I &= \sum_k \int_0^\infty \frac{f(x)a_k}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}\,dx \\ &< \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}B(\lambda_1,\lambda_2) \\ &\quad\times\left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}\left[\sum_k \|k-\xi\|_\alpha^{q(n-\lambda_2)-n}a_k^q\right]^{\frac{1}{q}}. \end{aligned} \tag{19}$$

where, the constant factor $(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})})^{\frac{1}{p}}[\prod_{i=0}^{m-1}(\lambda+i)]^{-1}B(\lambda_1,\lambda_2)$ *is the best possible.*

Proof Using (16) and (17), in view of Lebesgue term by term integration theorem (cf. [32]), we find

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+m)}\sum_k \int_0^\infty f(x)a_k\left[\int_0^\infty t^{\lambda+m-1}e^{-(x+\|k-\xi\|_\alpha)t}\,dt\right]dx \\ &= \frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}\left(\int_0^\infty e^{-xt}f(x)\,dx\right)\left(\sum_k e^{-\|k-\xi\|_\alpha t}a_k\right)dt \\ &= \frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}\left(t^{-m}\int_0^\infty e^{-xt}f^{(m)}(x)\,dx\right)\left(\sum_k e^{-\|k-\xi\|_\alpha t}a_k\right)dt \\ &= \frac{1}{\Gamma(\lambda+m)}\sum_k \int_0^\infty f^{(m)}(x)a_k\left[\int_0^\infty t^{\lambda-1}e^{-(x+\|k-\xi\|_\alpha)t}\,dt\right]dx \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\sum_k \int_0^\infty \frac{f^{(m)}(x)a_k}{(x+\|k-\xi\|_\alpha)^\lambda}\,dx = \left[\prod_{i=0}^{m-1}(\lambda+i)\right]]^{-1}I_\lambda. \end{aligned}$$

Then by (18), we have (19).

For $\lambda_1+\lambda_2=\lambda$ in (19), we have (20). For any $0<\varepsilon<p\lambda_1$, we set $\tilde{a}_k := \|k\|_\alpha^{\lambda_2-\frac{\varepsilon}{q}-n}(k\in N^n), \tilde{f}(x) := 0, 0<x<1$,

$$\tilde{f}(x) := \prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\int_1^x\left(\int_1^{t_m}\cdots\int_1^{t_2} t_1^{\lambda_1-\frac{\varepsilon}{p}-1}\,dt_1\cdots dt_{m-1}\right)dt_m$$

$$= x^{\lambda_1-\frac{\varepsilon}{p}+m-1} - O(x^{m-1}), \quad x \geq 1,$$

where, for $m \in \mathrm{N}, O(x^{m-1})$ is indicated a nonnegative polynomial of $(m-1)$-order.

If there exists a positive constant $M(\leq (\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})})^{\frac{1}{p}}[\prod_{i=0}^{m-1}(\lambda+i)]^{-1}B(\lambda_1,\lambda_2))$, such that (20) is valid when we replace $(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})})^{\frac{1}{p}}[\prod_{i=0}^{m-1}(\lambda+i)]^{-1}B(\lambda_1,\lambda_2)$ by M, then in particular, for $\xi = 0$, we still have

$$\tilde{I} := \sum_k \int_0^\infty \frac{\tilde{f}(x)\tilde{a}_k}{(x+\|k\|_\alpha)^{\lambda+m}}\,dx < M\left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(\tilde{f}^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \times \left[\sum_k \|k\|_\alpha^{q(n-\lambda_2)-n}\tilde{a}_k^q\right]^{\frac{1}{q}}. \tag{20}$$

By (10), we obtain

$$\begin{aligned}\tilde{J} &:= \left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(\tilde{f}^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}}\left[\sum_k \|k\|_\alpha^{q(n-\lambda_2)-n}\tilde{a}_k^q\right]^{\frac{1}{q}} \\ &< \prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\left(\int_1^\infty x^{-\varepsilon-1}\,dx\right)^{\frac{1}{p}}\left(\sum_k \|k\|_\alpha^{-\varepsilon-n}\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\left(\frac{2^\varepsilon\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}}.\end{aligned} \tag{21}$$

We obtain

$$\tilde{I} = \sum_k \int_1^\infty \frac{(x^{\lambda_1-\frac{\varepsilon}{p}+m-1}-O(x^{m-1}))}{(x+\|k\|_\alpha)^{\lambda+m}}\|k\|_{\mathrm{A}}^{\lambda_2-\frac{\varepsilon}{q}-n}\,dx = I_0 - I_1,$$

where, $I_0 := \sum_k \int_1^\infty \frac{x^{\lambda_1-\frac{\varepsilon}{p}+m-1}}{(x+\|k\|_\alpha)^{\lambda+m}}\|k\|_{\mathrm{A}}^{\lambda_2-\frac{\varepsilon}{q}-n}\,dx, I_1 := \sum_k \int_1^\infty \frac{O(x^{m-1})}{(x+\|k\|_\alpha)^{\lambda+m}}\|k\|_{\mathrm{A}}^{\lambda_2-\frac{\varepsilon}{q}-n}\,dx$.

By (10), we also find that $\frac{1}{c-\varepsilon}\sum_k \|k\|_\alpha^{-c-n} = O(1)$ $(c = \lambda_1+m+\frac{\varepsilon}{q})$. For $s = \lambda+m > 0, s_1 = \lambda_1+m-\frac{\varepsilon}{p} \in (0,s)$ in (12) and (15), by (10), we find

$$\begin{aligned}I_0 &= \sum_k \|k\|_\alpha^{-\varepsilon-n}\left[\|k\|_\alpha^{(\lambda_2+\frac{\varepsilon}{p})}\int_1^\infty \frac{x^{(\lambda_1+m-\frac{\varepsilon}{p})-1}}{(x+\|k\|_\alpha)^{\lambda+m}}\,dx\right] \\ &= \sum_k \|k\|_\alpha^{-\varepsilon-n}\left[\|k\|_\alpha^{(\lambda_2+\frac{\varepsilon}{p})}\int_0^\infty \frac{x^{(\lambda_1+m-\frac{\varepsilon}{p})-1}}{(x+\|k\|_\alpha)^{\lambda+m}}\,dx - \|k\|_\alpha^{(\lambda_2+\frac{\varepsilon}{p})}\int_0^1 \frac{x^{(\lambda_1+m-\frac{\varepsilon}{p})-1}}{(x+\|k\|_\alpha)^{\lambda+m}}\,dx\right] \\ &\geq \sum_k \|k\|_\alpha^{-\varepsilon-n}\left[\omega_{\lambda+m}\left(\lambda_1+m-\frac{\varepsilon}{p},k\right) - \|k\|_\alpha^{(\lambda_2+\frac{\varepsilon}{p})}\int_0^1 \frac{x^{(\lambda_1+m-\frac{\varepsilon}{p})-1}}{\|k\|_\alpha^{\lambda+m}}\,dx\right] \\ &= \sum_k \|k\|_\alpha^{-\varepsilon-n}\omega_{\lambda+m}\left(\lambda_1+m-\frac{\varepsilon}{p},k\right) - \frac{1}{\lambda_1+m-\frac{\varepsilon}{p}}\sum_k \|k\|_\alpha^{-(\lambda_1+m+\frac{\varepsilon}{q})-n} \\ &= B\left(\lambda_1+m-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right)\sum_k \|k\|_\alpha^{-\varepsilon-n} - O(1) \\ &> \frac{1}{\varepsilon}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B\left(\lambda_1+m-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right)-\varepsilon O(1)\right).\end{aligned}$$

We still find that

$$
\begin{aligned}
0 < I_1 &= \sum_k \frac{\|k\|_{\mathrm{A}}^{\lambda_2-\frac{\varepsilon}{q}-n}}{(x+\|k\|_\alpha)^{\lambda_2}} \int_1^\infty \frac{O(x^{m-1})}{(x+\|k\|_\alpha)^{\lambda_1+m}}\,dx \\
&\le \sum_k \frac{\|k\|_{\mathrm{A}}^{\lambda_2-\frac{\varepsilon}{q}-n}}{\|k\|_\alpha^{\lambda_2}} \int_1^\infty \frac{O(x^{m-1})}{x^{\lambda_1+m}}\,dx \le M_1 < \infty.
\end{aligned}
$$

Hence, by (21) and the above results, we have the following inequality

$$
\begin{aligned}
&\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B\left(\lambda_1+m-\frac{\varepsilon}{p}, \lambda_2+\frac{\varepsilon}{p}\right) - \varepsilon O(1) - \varepsilon I_1 \\
&\quad < \varepsilon \tilde{I} < \varepsilon M \tilde{J} \le M \prod_{i=0}^{m-1}\left(\lambda_1 + i - \frac{\varepsilon}{p}\right)\left(\frac{2^\varepsilon \Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}}.
\end{aligned} \tag{22}
$$

For $\varepsilon \to 0^+$ in (23), in view of the continuity of the beta function, we find

$$
\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_1+m, \lambda_2) \le M \prod_{i=0}^{m-1}(\lambda_1+i)\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}},
$$

namely, $(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})})^{\frac{1}{p}} [\prod_{i=0}^{m-1}(\lambda+i)]^{-1} B(\lambda_1, \lambda_2) \le M$. It follows that

$$
M = \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1} B(\lambda_1, \lambda_2)
$$

is the best possible constant factor of (20).

The theorem is proved. □

Remark 1 For $\hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \hat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda_2 + \frac{\lambda-\lambda_1-\lambda_2}{q}$, we find $\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda$,

$$
0 < \hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \qquad 0 < \hat{\lambda}_2 = \lambda - \hat{\lambda}_1 < \lambda.
$$

For $\lambda - \lambda_1 - \lambda_2 \le q(n-\lambda_2)$, we still can find $\hat{\lambda}_2 \le n$. In this case, we can rewrite (20) as follows:

$$
\begin{aligned}
&\sum_k \int_0^\infty \frac{f(x)a_k}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}\,dx \\
&\quad < \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1} B(\hat{\lambda}_1, \hat{\lambda}_2) \\
&\qquad \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}} \left[\sum_k \|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n} a_k^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{23}
$$

Theorem 2 *If* $\lambda - \lambda_1 - \lambda_2 \le q(n - \lambda_2)$, *the constant factor*

$$\left[\prod_{i=0}^{m-1}(\lambda + i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2, \lambda - \lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (19) *is the best possible, then we have* $\lambda - \lambda_1 - \lambda_2 = 0, \lambda_1 + \lambda_2 = \lambda$.

Proof By Hölder's inequality (cf. [29]), we obtain

$$\begin{aligned} B(\hat{\lambda}_1, \hat{\lambda}_2) &= \int_0^\infty \frac{u^{\hat{\lambda}_1 - 1}}{(1+u)^\lambda}\, du = \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} - 1}\, du \\ &= \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\lambda_2-1}{p}} u^{\frac{\lambda_1-1}{q}}\, du \le \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda}\, du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda}\, du\right]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1). \end{aligned} \tag{24}$$

In view of the assumption, compare with the constant factors in (19) and (24), we have the following inequality:

$$\begin{aligned} &\left[\prod_{i=0}^{m-1}(\lambda + i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ &\quad \le \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} \left[\prod_{i=0}^{m-1}(\lambda + i)\right]^{-1} B(\hat{\lambda}_1, \hat{\lambda}_2), \end{aligned}$$

namely, $B(\hat{\lambda}_1, \hat{\lambda}_2) \ge B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$, which follows that (25) retains the form of equality. We observe that (25) retains the form of equality if and only if there exist constants A and B, such that they are not both zero and $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ a.e. in R_+ (cf. [30]). Assuming that $A \ne 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e.in R_+, namely, $\lambda - \lambda_1 - \lambda_2 = 0$ and then $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. □

4 Equivalent forms and operator expressions

Theorem 3 *Inequality* (19) *is equivalent to the following inequality*:

$$\begin{aligned} J &:= \left\{\sum_k \|k - \xi\|_\alpha^{p\hat{\lambda}_2 - n}\left[\int_0^\infty \frac{f(x)}{(x + \|k - \xi\|_\alpha)^{\lambda+m}}\, dx\right]^p\right\}^{\frac{1}{p}} \\ &< \left[\prod_{i=0}^{m-1}(\lambda + i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2, \lambda - \lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ &\quad \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p\, dx\right]^{\frac{1}{p}}. \end{aligned} \tag{25}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, *we reduce* (26) *to the equivalent form of* (20) *as follows*:

$$\left\{\sum_k \|k - \xi\|_\alpha^{p\lambda_2 - n}\left[\int_0^\infty \frac{f(x)}{(x + \|k - \xi\|_\alpha)^{\lambda+m}}\, dx\right]^p\right\}^{\frac{1}{p}}$$

$$< \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1} B(\lambda_1,\lambda_2)\left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}}, \tag{26}$$

where, the constant factor $(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})})^{\frac{1}{p}}[\prod_{i=0}^{m-1}(\lambda+i)]^{-1}B(\lambda_1,\lambda_2)$ *is the best possible.*

Proof Suppose that (26) is valid. By Hölder's inequality, we have

$$\begin{aligned} I &= \sum_k \left[\|k-\xi\|_\alpha^{\frac{-n}{p}+\hat{\lambda}_2} \int_0^\infty \frac{f(x)}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}\,dx\right]\left[\|k-\xi\|_\alpha^{\frac{n}{p}-\hat{\lambda}_2} a_k\right] \\ &\le J\left[\sum_k \|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n} a_k^q\right]^{\frac{1}{q}}. \end{aligned} \tag{27}$$

Then by (26), we have (19).

On the other hand, assuming that (19) is valid, we set

$$a_k := \|k-\xi\|_\alpha^{p\hat{\lambda}_2-n}\left[\int_0^\infty \frac{f(x)}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}\,dx\right]^{p-1}, \quad k\in N^n.$$

If $J=0$, then (26) is naturally valid; if $J=\infty$, then it is impossible to make (26) valid, namely $J<\infty$. Suppose that $0<J<\infty$. By (19), we have

$$\begin{aligned} &\sum_k \|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n} a_k^q \\ &\quad = J^p = I < \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\ &\qquad \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}}\left[\sum_k \|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n} a_k^q\right]^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} &\left[\sum_k \|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n} a_k^q\right]^{\frac{1}{p}} \\ &\quad = J < \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\ &\qquad \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}}, \end{aligned}$$

namely, (26) follows, which is equivalent to (19).

The constant factor $(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})})^{\frac{1}{p}}[\prod_{i=0}^{m-1}(\lambda+i)]^{-1}B(\lambda_1,\lambda_2)$ in (27) is the best possible. Otherwise, by (28) (for $\lambda_1+\lambda_2=\lambda$), we would reach a contradiction that the constant factor in (20) is not the best possible.

The theorem is proved. □

We set functions $\phi(x) := x^{p(1-\hat{\lambda}_1)-1}$, $\psi(k) := \|k-\xi\|_\alpha^{q(n-\hat{\lambda}_2)-n}$, then,

$$\psi^{1-p}(k) = (\|k-\xi\|_\alpha^{p\hat{\lambda}_2-n} \quad \left(x\in \mathrm{R}_+, k\in \mathrm{N}^{\mathrm{n}}\right).$$

Define the following real normed spaces:

$$L_{p,\phi}(\mathrm{R}_+) := \left\{ f = f(x); \|f\|_{p,\phi} := \left(\int_0^\infty \phi(x) \left| f(x) \right|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ a = \{a_{k_1,\ldots,k_n}\}; \|a\|_{q,\psi} := \left(\sum_k \psi(k) |a_k|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ b = \{b_{k_1,\ldots,k_n}\}; \|b\|_{q,\psi} := \left(\sum_k \psi^{1-p}(k) |b_k|^p \right)^{\frac{1}{p}} < \infty \right\},$$

and $\tilde{L}(R_+) := \{f \in L_{p,\phi}(R_+); f(x)$ is a nonnegative differentiable function of m-order, unless at finite points in $\mathrm{R}_+, f^{(k-1)}(x) = o(e^{tx})$ $(t > 0; x \to \infty), f^{(k-1)}(0^+) = 0$ $(k = 1, \ldots, m)\}$.

For any $f \in \tilde{L}(R_+)$, setting $b_k := \int_0^\infty \frac{f(x)}{(x+\|k-\xi\|_\alpha)^{\lambda+m}} dx, k \in \mathrm{N}^{\mathrm{n}}$, we can rewrite (26) as follows:

$$\|b\|_{p,\psi^{1-p}} \le \left[\prod_{i=0}^{m-1} (\lambda + i) \right]^{-1} \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda - \lambda_2) \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left\| f^{(m)} \right\|_{p,\phi} < \infty,$$

namely, $b \in l_{p,\psi^{1-p}}$.

Definition 1 Define a Hilbert-type operator $T : \tilde{L}(\mathrm{R}_+) \to l_{p,\psi^{1-p}}$ as follows: For any $f \in \tilde{L}(\mathrm{R}_+)$, there exists a unique representation $Tf = b \in l_{p,\psi^{1-p}}$, satisfying $Tf(k) = b_k$ $(k \in \mathrm{N}^{\mathrm{n}})$. Define the formal inner product of Tf and $a \in l_{q,\psi}$, and the norm of T as follows:

$$(Tf, a) := \sum_k a_k \left[\int_0^\infty \frac{f(x)}{(x + \|k - \xi\|_\alpha)^{\lambda+m}} dx \right] = I,$$

$$\|T\| := \sup_{f(\neq 0) \in L_{p,\phi}(R_+)} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|f^{(m)}\|_{p,\phi}}.$$

By Theorem 1-3, we have

Theorem 4 *If $f \in \tilde{L}(\mathrm{R}_+), a \in l_{q,\psi}, \|f^{(m)}\|_{p,\phi}, \|a\|_{q,\psi} > 0$, then we have the following equivalent inequalities*:

$$(Tf, a) < \left[\prod_{i=0}^{m-1} (\lambda + i) \right]^{-1} \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda - \lambda_2) \right)^{\frac{1}{p}} \times B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left\| f^{(m)} \right\|_{p,\phi} \|a\|_{q,\psi}, \quad (28)$$

$$\|Tf\|_{p,\psi^{1-p}} < \left[\prod_{i=0}^{m-1} (\lambda + i) \right]^{-1} \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda - \lambda_2) \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left\| f^{(m)} \right\|_{p,\phi}. \quad (29)$$

Moreover, if $\lambda_1 + \lambda_2 = \lambda$ then the constant factor $[\prod_{i=0}^{m-1}(\lambda + i)]^{-1}(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda - \lambda_2))^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$ in (29) and (30) is the best possible, namely, $\|T\| = (\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})})^{\frac{1}{p}} [\prod_{i=0}^{m-1}(\lambda +$

$i)]^{-1}B(\lambda_1,\lambda_2)$. On the other hand, if $\lambda - \lambda_1 - \lambda_2 \le q(n-\lambda_2)$, the constant factor

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

in (29) or (30) is the best possible, then we have $\lambda-\lambda_1-\lambda_2=0$, namely, $\lambda_1+\lambda_2=\lambda$.

Remark 2 (i) For $\lambda=1,\lambda_1=\frac{1}{q},\lambda_2=\frac{1}{p}$ in (20) and (27), we have the following equivalent Hilbert-type inequalities:

$$\begin{aligned}&\sum_k\int_0^\infty\frac{f(x)a_k}{(x+\|k-\xi\|_\alpha)^{1+m}}\,dx\\&\quad<\frac{1}{m!}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}\frac{\pi}{\sin(\pi/p)}\left[\int_0^\infty\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}\left[\sum_k\|k-\xi\|_\alpha^{(q-1)(n-1)}a_k^q\right]^{\frac{1}{q}},\end{aligned}\tag{30}$$

$$\begin{aligned}&\left\{\sum_k|k-\xi\|_\alpha^{1-n}\left[\int_0^\infty\frac{f(x)}{(x+\|k-\xi\|_\alpha)^{1+m}}\,dx\right]^p\right\}^{\frac{1}{p}}\\&\quad<\frac{1}{m!}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}\frac{\pi}{\sin(\pi/p)}\left[\int_0^\infty\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}};\end{aligned}\tag{31}$$

(ii) for $\lambda=1,\lambda_1=\frac{1}{p},\lambda_2=\frac{1}{q}$ in (20) and (27), we have the following equivalent dual forms of (32) and (33):

$$\begin{aligned}&\sum_k\int_0^\infty\frac{f(x)a_k}{(x+\|k-\xi\|_\alpha)^{1+m}}\,dx\\&\quad<\frac{1}{m!}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}\frac{\pi}{\sin(\pi/p)}\left[\int_0^\infty x^{p-2}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}\\&\qquad\times\left[\sum_k\|k-\xi\|_\alpha^{(q-1)n-1}a_k^q\right]^{\frac{1}{q}},\end{aligned}\tag{32}$$

$$\begin{aligned}&\left\{\sum_k\|k-\xi\|_\alpha^{p-1-n}\left[\int_0^\infty\frac{f(x)}{(x+\|k-\xi\|_\alpha)^{1+m}}\,dx\right]^p\right\}^{\frac{1}{p}}\\&\quad<\frac{1}{m!}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}\frac{\pi}{\sin(\pi/p)}\left[\int_0^\infty x^{p-2}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}};\end{aligned}\tag{33}$$

(iii) for $p=q=2$, both (31) and (33) reduce to

$$\begin{aligned}&\sum_k\int_0^\infty\frac{f(x)a_k}{(x+\|k-\xi\|_\alpha)^{1+m}}\,dx\\&\quad<\frac{\pi}{m!}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{2}}\left[\int_0^\infty\left(f^{(m)}(x)\right)^2\,dx\sum_k\|k-\xi\|_\alpha^{n-1}a_k^2\right]^{\frac{1}{2}},\end{aligned}\tag{34}$$

and both (32) and (34) reduce to the equivalent form of (35) as follows:

$$\left\{\sum_{k}\|k-\xi\|_{\alpha}^{1-n}\left[\int_{0}^{\infty}\frac{f(x)}{(x+\|k-\xi\|_{\alpha})^{1+m}}\,dx\right]^{2}\right\}^{\frac{1}{2}} < \frac{\pi}{m!}\left(\frac{\Gamma^{n}(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{2}}\left[\int_{0}^{\infty}\left(f^{(m)}(x)\right)^{2}\,dx\right]^{\frac{1}{2}}, \qquad (35)$$

The constant factors in the above particular inequalities are all the best possible.

Remark 3 For $\alpha > 0$, we only obtain $\frac{\partial}{y_j}h_x(y) < 0$ $(j = 1,\dots,n)$ in (9). In this case, we can't use Hermite–Hadamard's inequality to obtain (11). But for $\xi = 0$, we still can obtain (11), and then the equivalent inequalities (19) and (26) for $\xi = 0$ with the best possible constant factor were proved.

5 Conclusions

In this paper, following the way of [22], by means of the weight functions, the idea of introduced parameters and the transfer formula, a more accurate half-discrete multidimensional Hilbert-type inequality with the homogeneous kernel as $\frac{1}{(x+\|k-\xi\|_{\alpha})^{\lambda+m}}$ $(x,\lambda > 0)$ involving one derivative function of m-order and the beta function is given in Theorem 1. The equivalent conditions of the best possible constant factor related to several parameters are considered in Theorem 2. The equivalent forms, the operator expressions and some particular Hilbert-type inequalities are obtained Theorem 3, Theorem 4 and Remark 2. The lemmas and theorems provide an extensive account of this type of inequalities.

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Availability of data and materials
We declare that the data and material in this paper can be used publicly.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. YH and YZ participated in the design of the study and performed the numerical analysis. All authors reviewed the manuscript.

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