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On Orlicz sequence algebras

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Dedicated to the memory of Professors Paweł Domański and Henryk Hudzik

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Abstract

We study Orlicz sequence algebras and their properties. In particular, we fully characterize biflat and biprojective Orlicz sequence algebras as well as weakly amenable and approximately (semi-)amenable Orlicz sequence algebras. As a consequence, we show the existence of a wide class of sequence algebras that behave differently—in terms of the amenability properties—from any of the algebras ℓ_p, $1 \le p \le \infty$.

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1 Introduction

The aim of this note is to study Orlicz sequence spaces ℓ_φ and h_φ as Banach algebras. Recall that an Orlicz function space $L^\varphi(X)$ is a Banach algebra with respect to pointwise multiplication if and only if $b_\varphi < \infty$ (the definition of the number b_φ is provided at the beginning of Sect. 2) or X is at most a countable union of atoms—see [7, Theorem B]. Consequently, Orlicz sequence spaces are always Banach algebras with respect to pointwise multiplication. On the other hand, if G is a locally compact group then $L^\varphi(G)$ is a Banach algebra if and only if $L^\varphi(G)$ embeds continuously into $L^1(G)$—see [8, Theorem 2]. The situation changes substantially—see, e.g., [22, 25]—if one considers weighted Orlicz spaces $L^\varphi_w(G)$, where by a *weight* we mean a measurable function $w\colon G \to (0,\infty)$ satisfying

$$w(st) \le w(s)w(t) \quad (s,t \in G).$$

One can also consider the so-called *twisted Orlicz algebras*, which are Orlicz spaces $L^\varphi(G)$ with multiplication arising from a specific 2-cocycle—see [23, 24] for details. Their weighted variants are also considered. In this note, we focus entirely on Orlicz sequence algebras ℓ_φ and h_φ.

The paper is organized as follows. Section 2 gives some preliminaries and notation. Section 3 contains a list of auxiliary but very useful results. The final section presents the main results of the paper. It emphasizes in particular the power of the $\Delta_2(0)$ condition.

For unexplained details from Banach algebra theory we refer the reader to [3, 27] and from Orlicz space theory to [12–16, 18, 20, 26].

2 Notation and preliminaries

In the whole paper φ denotes an *Orlicz function* (see [12, 16, 18, 20, 26]), that is, $\varphi\colon [0,\infty) \to [0,\infty]$ and φ is convex, vanishing, and right continuous at zero, not identically equal to zero and left continuous on the interval $(0,\infty)$. By the convexity of φ, it is nondecreasing on $[0,\infty)$. Let

$$a_\varphi := \sup\{u \geq 0 : \varphi(u) = 0\},$$
$$b_\varphi := \sup\{u \geq 0 : \varphi(u) < \infty\}.$$

Note that the left continuity of φ on $(0,\infty)$ is equivalent to the fact that

$$\lim_{u\to(b_\varphi)^-} \varphi(u) = \varphi(b_\varphi).$$

Recall that an Orlicz function φ is an *N-function at* 0 if $a_\varphi = 0$ and $\lim_{u\to 0} \frac{\varphi(u)}{u} = 0$.

The *generalized inverse* φ^{-1} of the Orlicz function φ is defined as follows:

$$\varphi^{-1}(v) := \inf\{u \geq 0\colon \varphi(u) > v\}$$

if $v \in [0,\infty)$ and

$$\varphi^{-1}(\infty) = \lim_{v\to\infty} \varphi^{-1}(v)$$

(see [21] and [11]).

We say that an Orlicz function φ satisfies the condition $\Delta_2(0)$ (in short, $\varphi \in \Delta_2(0)$) if there exist $u_0 > 0$ and a constant $K > 0$ such that $\varphi(u_0) > 0$ and $\varphi(2u) \leq K\varphi(u)$ for every $u \leq u_0$ (then we also have $a_\varphi = 0$).

Given any Orlicz function φ, we define its *complementary function* in the sense of Young by the formula

$$\psi(u) := \sup_{v>0}\{uv - \varphi(v)\} \tag{1}$$

for all $u \geq 0$. It is easy to show that ψ is also an Orlicz function and φ is complementary to ψ (see [16, p. 147]).

For any Orlicz function φ we define Orlicz sequence spaces ℓ_φ by

$$\ell_\varphi := \left\{x = \big(x(i)\big)_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}\colon \sum_{i=1}^{\infty} \varphi\big(\lambda\big|x(i)\big|\big) < \infty \text{ for some } \lambda > 0\right\},$$

where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. It is well known that the space ℓ_φ equipped with the Luxemburg norm $\|\cdot\|_\varphi$, defined by

$$\|x\|_\varphi := \inf\left\{\lambda > 0\colon \sum_{i=1}^{\infty} \varphi\big(\lambda^{-1}\big|x(i)\big|\big) \leq 1\right\},$$

is a Banach space. Moreover, as has been shown in [7], it is also a Banach algebra.

Recall that the subspace h_φ of the space ℓ_φ is defined by the formula

$$h_\varphi := \left\{ x \in \ell_\varphi \mid \forall \lambda > 0\ \exists i_0 \in \mathbb{N}\colon \sum_{i=i_0}^{\infty} \varphi\big(\lambda \big|x(i)\big|\big) < \infty \right\}.$$

Let $E \subset \mathbb{K}^{\mathbb{N}}$ be a Banach lattice. An element $x \in E$ is said to be order continuous if for any sequence (x_n) in E with $0 \leq x_n \leq |x|$ and $x_n \to 0$ coordinatewise one obtains $\|x_n\|_E \to 0$. The subspace E_a of all order-continuous elements in E is an order ideal in E. The space E is called order continuous if $E_a = E$, see [17]. It is well known that $h_\varphi = (\ell_\varphi)_a$. Hence, the standard unit vectors $e_n := (\delta_{jn})_{j \in \mathbb{N}}, n \in \mathbb{N}$ constitute a Schauder basis in h_φ (see also [16, Proposition 4.a.2]). The sequence $(e_n^*)_{n \in \mathbb{N}}$ of evaluation functionals is defined as

$$\langle x, e_n^* \rangle := x(n) \quad (x \in \ell_\varphi, n \in \mathbb{N}).$$

It is easy to show that if $a_\varphi > 0$, then $\ell_\varphi = \ell_\infty$ ($h_\varphi = c_0$) as sets and the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ are equivalent. On the other hand, if $a_\varphi = 0$, then $\ell_\varphi \hookrightarrow c_0$. Moreover, in this case $\ell_\varphi = h_\varphi$ if and only if $\varphi \in \Delta_2(0)$ (see [16, Proposition 4.a.4]). Recall that if φ vanishes only at zero but is not an N-function at 0, then $\ell_\varphi = \ell_1$ as sets and the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_1$ are equivalent (see [16, p. 147]). On the other hand, there exist N-functions at 0 such that ℓ_φ is not isomorphic to any ℓ_p, $1 \leq p \leq \infty$. Obviously, we have this situation whenever $\varphi \notin \Delta_2(0)$, because then h_φ is a proper subspace of ℓ_φ. A more sophisticated construction is provided in [14, Theorem 3].

We end this section by recalling a number of well-known ideas. If X is a Banach space then by $\iota\colon X \hookrightarrow X''$ we denote the canonical embedding into the second dual. If A is a Banach algebra then the *product map* $\pi_A\colon A \widehat{\otimes} A \to A$ is the unique linear mapping arising from the bilinear map $A \times A \ni (a,b) \mapsto ab \in A$. We will simply write π when there is no risk of confusion. If, furthermore, X is a Banach A-bimodule then X' becomes the so-called *dual A-bimodule* with the bimodule operations defined as

$$\langle x, a \cdot \lambda \rangle := \langle x \cdot a, \lambda \rangle \quad \text{and} \quad \langle x, \lambda \cdot a \rangle := \langle a \cdot x, \lambda \rangle \quad \big(a \in A, x \in X, \lambda \in X'\big). \tag{2}$$

A projective tensor product $X \widehat{\otimes} X$ becomes canonically an A-bimodule with the bimodule operations defined as

$$a \cdot (x \otimes y) := a \cdot x \otimes y, \qquad (x \otimes y) \cdot a := x \otimes y \cdot a \quad (a \in A, x, y \in X).$$

A bounded linear map $T\colon X \to Y$ between A-bimodules X and Y is a *bimodule map* if

$$T(a \cdot x \cdot b) = a \cdot Tx \cdot b \quad (a, b \in A, x \in X).$$

If A is a Banach algebra and X is a Banach A-bimodule then a linear (not necessarily bounded) mapping $\delta\colon A \to X$ is called a *derivation* if it satisfies the so-called "derivation rule", i.e.,

$$\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b \quad (a, b \in A).$$

A derivation $\delta\colon A \to X$ is *inner* if there is some $x \in X$ such that

$$\delta(a) = \mathrm{ad}_x(a) := a \cdot x - x \cdot a \quad (a \in A).$$

A Banach algebra A is said to be *amenable* if for any Banach A-bimodule X every continuous derivation $\delta\colon A \to X'$ into the dual A-bimodule X' (see (2)) is inner.

3 Auxiliary results

Recall that if A is a Banach algebra then we denote $A^2 := \mathrm{span}\{ab\colon a, b \in A\}$ and $\overline{A^2}$ is the norm closure of A^2.

Proposition 3.1 *If φ is an Orlicz function and $a_\varphi = 0$, then $\overline{(\ell_\varphi)^2} = h_\varphi$.*

Proof Clearly, $e_n = e_n \cdot e_n \in (\ell_\varphi)^2$, thus $\sum_{i=1}^{j} x(i)e_i \in (\ell_\varphi)^2$ for any $x \in h_\varphi$ and any $j \in \mathbb{N}$ and, in consequence, $h_\varphi \subset \overline{(\ell_\varphi)^2}$.

Conversely, let $x, y \in \ell_\varphi$ and let $\mu > 0$ satisfy $\|y\|_\varphi < \mu$. Since $\ell_\varphi \subset c_0$, for an arbitrary $\lambda > 0$ there exists $i_0 \in \mathbb{N}$ such that

$$\lambda |x(i)| \le \mu^{-1} \quad \text{for any } i \ge i_0.$$

Hence,

$$\sum_{i=i_0}^{\infty} \varphi\big(\lambda |x(i)y(i)|\big) \le \sum_{i=i_0}^{\infty} \varphi\big(\mu^{-1} |y(i)|\big) \le 1 < \infty.$$

Consequently, $\overline{(\ell_\varphi)^2} \subset h_\varphi$. □

Theorem 3.2 *If φ is an Orlicz function and $a_\varphi = 0$ then the Orlicz sequence algebras ℓ_φ and h_φ are not closed with respect to taking square roots.*

Proof Let $u_1 \le 10^{-2}$ satisfy $\varphi(u_1) \le \frac{1}{2}$ and let $k_1 \in \mathbb{N}$ be the largest number such that

$$\frac{1}{4} < k_1 \varphi(u_1) \le \frac{1}{2}.$$

Let $u_2 \le \min\{u_1, 10^{-4}\}$ satisfy $\varphi(2u_2) \le \frac{1}{4}$ and let $k_2 \in \mathbb{N}$ be the largest number such that

$$\frac{1}{8} < k_2 \varphi(2u_2) \le \frac{1}{4}.$$

Proceeding recursively, we obtain two sequences $(u_n)_{n\in\mathbb{N}} \subset (0, \infty)$ and $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that

$$u_n \le \min\{u_{n-1}, 10^{-2n}\}, \qquad \frac{1}{2^{n+1}} < k_n \varphi(n u_n) \le \frac{1}{2^n} \quad (n \in \mathbb{N}).$$

Let $l_0 = 0$, $l_n = \sum_{m=1}^{n} k_m$ and

$$x := \sum_{n=1}^{\infty} \sum_{i=l_{n-1}+1}^{l_n} u_n e_i.$$

We will show that $x \in h_\varphi$ and $\sqrt{x} \notin \ell_\varphi$. To this end, let $\lambda > 0$ be fixed and let n_0 be the smallest natural number such that $\lambda \leq n_0$. Then,

$$\sum_{i=l_{n_0-1}+1}^{\infty} \varphi\big(\lambda x(i)\big) \leq \sum_{n=n_0}^{\infty} \sum_{i=l_{n-1}+1}^{l_n} \varphi(nu_n) \leq \sum_{n=n_0}^{\infty} k_n \varphi(nu_n) \leq \sum_{n=n_0}^{\infty} \frac{1}{2^n} \leq 1 < \infty.$$

Consequently, $x \in h_\varphi$. To obtain the other claim we again fix $\lambda > 0$ and choose $n_0 \in \mathbb{N}$ such that $\lambda > \frac{n}{5^n}$ for every $n \geq n_0$. Then,

$$\varphi(\lambda\sqrt{u_n}) \geq \varphi\left(\frac{n}{5^n}\frac{u_n}{\sqrt{u_n}}\right) \geq \varphi\big(2^n nu_n\big) \geq 2^n \varphi(nu_n),$$

where the last inequality follows from convexity of φ. Thus,

$$\sum_{i=1}^{\infty} \varphi\big(\lambda\sqrt{x(i)}\big) = \sum_{n=1}^{\infty} \sum_{i=l_{n-1}+1}^{l_n} \varphi(\lambda\sqrt{u_n}) \geq \sum_{n=n_0}^{\infty} 2^n k_n \varphi(nu_n) \geq \sum_{n=n_0}^{\infty} \frac{1}{2} = \infty.$$

Consequently, $\sqrt{x} \notin \ell_\varphi$. $\square$

Remark 3.3 From [27, Proposition 2.2.1] and [3, Corollary 2.9.25] it follows that if $a_\varphi = 0$ then neither ℓ_φ nor h_φ is amenable.

Recall that a Banach algebra A *factors weakly* if $A^2 = A$ as sets.

Corollary 3.4 *If φ is an Orlicz function and $a_\varphi = 0$, then the Orlicz sequence algebras ℓ_φ and h_φ do not factor weakly.*

Proof Let $x \in h_\varphi$ be such that $\sqrt{x} \notin \ell_\varphi$. Assume towards a contradiction that $x \in (\ell_\varphi)^2$, i.e.,

$$x = \sum_{k=1}^{n} a_k b_k \quad (a_k, b_k \in \ell_\varphi).$$

Then,

$$\forall i \in \mathbb{N}\ \exists k = 1, \ldots, n\colon \quad \big|a_k(i)b_k(i)\big| \geq \frac{1}{n}\big|x(i)\big|.$$

We can therefore decompose $\mathbb{N}$ into a finite disjoint union $\mathbb{N} = \bigcup_{k=1}^{n} N_k$ and

$$\forall k = 1, \ldots, n\ \forall i \in N_k\colon \quad \big|a_k(i)b_k(i)\big| \geq \frac{1}{n}\big|x(i)\big|.$$

This implies

$$\forall k = 1, \ldots, n\ \forall i \in N_k\colon \quad \big|x(i)\big|^{\frac{1}{2}} \leq \sqrt{n}\max\big\{\big|a_k(i)\big|, \big|b_k(i)\big|\big\} \leq \sqrt{n}\big(\big|a_k(i)\big| + \big|b_k(i)\big|\big).$$

Let

$$\lambda := 2\sqrt{n}\max\bigl\{\|a_k\|_\varphi, \|b_k\|_\varphi : k = 1,\dots,n\bigr\}.$$

Using the convexity of φ we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\varphi\bigl(\lambda^{-1}\bigl|x(i)\bigr|^{1/2}\bigr) &= \sum_{k=1}^{n}\sum_{i\in N_k}\varphi\bigl(\lambda^{-1}\bigl|x(i)\bigr|^{1/2}\bigr)\\ &\le \sum_{k=1}^{n}\sum_{i\in N_k}\varphi\left(\frac{|a_k(i)|}{2\|a_k\|_\varphi}+\frac{|b_k(i)|}{2\|b_k\|_\varphi}\right)\\ &\le \frac{1}{2}\sum_{k=1}^{n}\sum_{i\in N_k}\bigl(\varphi\bigl(\|a_k\|_\varphi^{-1}\bigl|a_k(i)\bigr|\bigr)+\varphi\bigl(\|b_k\|_\varphi^{-1}\bigl|b_k(i)\bigr|\bigr)\bigr)\\ &\le \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{\infty}\bigl(\varphi\bigl(\|a_k\|_\varphi^{-1}\bigl|a_k(i)\bigr|\bigr)+\varphi\bigl(\|b_k\|_\varphi^{-1}\bigl|b_k(i)\bigr|\bigr)\bigr) \le n < \infty.\end{aligned}$$

Consequently, $\sqrt{x} \in \ell_\varphi$, which contradicts the choice of x. □

We are now going to focus on Arens regularity. Recall that if A is a Banach algebra then A'' can be made into a Banach algebra in two canonical ways. These are the so-called *Arens products* and they are defined as follows. Let $\Phi, \Psi \in A''$ be given. Then, the *first Arens product* is defined as

$$\langle \lambda, \Phi\Box\Psi\rangle := \langle \Psi\cdot\lambda, \Phi\rangle \quad \bigl(\lambda \in A'\bigr),$$

where

$$\langle a, \Psi\cdot\lambda\rangle := \langle \lambda\cdot a, \Psi\rangle \quad (a \in A)$$

and

$$\langle b, \lambda\cdot a\rangle := \langle ab, \lambda\rangle \quad (b \in A).$$

The *second Arens product* is defined as

$$\langle \lambda, \Phi\diamond\Psi\rangle := \langle \lambda\cdot\Phi, \Psi\rangle \quad \bigl(\lambda \in A'\bigr),$$

where

$$\langle a, \lambda\cdot\Phi\rangle := \langle a\cdot\lambda, \Phi\rangle \quad (a \in A)$$

and

$$\langle b, a\cdot\lambda\rangle := \langle ba, \lambda\rangle \quad (b \in A).$$

A Banach algebra A is called *Arens regular* if $\Phi\Box\Psi = \Phi\diamond\Psi$ for all $\Phi, \Psi \in A''$.

Proposition 3.5 *Orlicz sequence algebras ℓ_φ and h_φ are Arens regular.*

Proof If $a_\varphi > 0$ then $\ell_\varphi = \ell_\infty$ (equivalent norms) and ℓ_∞, being a unital C^*-algebra, is Arens regular by [2, Theorem 7.1] (cf. [3, Theorem 3.2.36]). From now on we restrict ourselves to the case where $a_\varphi = 0$. Assume for a moment that $a \in \ell_\varphi$ and $\theta \in \ell'_\varphi$ are given so that $a \cdot \theta \in \ell'_\varphi$. From [26, Proposition 2] it follows that

$$\ell'_\varphi = \ell_\psi \oplus h_\varphi^\perp,$$

where ψ denotes the complementary function to φ (see formula (1)). Therefore, there exists a unique decomposition

$$\theta = (y,f), \quad y \in \ell_\psi, f \in h_\varphi^\perp.$$

Thus,

$$\langle x, a \cdot \theta\rangle = \langle xa, y\rangle + \langle xa, f\rangle \quad (x \in \ell_\varphi).$$

From Proposition 3.1 it now follows that $xa \in h_\varphi$, whence

$$\langle xa, f\rangle = 0$$

and, consequently,

$$\langle x, a \cdot \theta\rangle = \langle x, ay\rangle \quad (x \in \ell_\varphi).$$

Since $ay \in h_\psi$ (again the argument from the proof of Proposition 3.1 applies), we may assume without loss of generality that

$$\forall a \in \ell_\varphi, \theta \in \ell'_\varphi : \quad a \cdot \theta \in \iota(h_\psi).$$

We now proceed with the proof of Arens regularity. Since $h'_\psi = \ell_\varphi$ (see [16, Proposition 4.b.1], recall that φ is the complementary function of ψ) we may apply the Dixmier projection to obtain the decomposition

$$\ell''_\varphi = \ell_\varphi \oplus \iota(h_\psi)^\perp.$$

Let $(a,F),(b,G) \in \ell''_\varphi$ with $a, b \in \ell_\varphi, F, G \in \iota(h_\psi)^\perp$ be given. For any $\theta \in \ell'_\varphi$ we obtain

$$\big\langle \theta, (a,F)\Box(b,G)\big\rangle = \langle a, b\cdot\theta\rangle + \langle a, G\cdot\theta\rangle + \langle b\cdot\theta, F\rangle + \langle G\cdot\theta, F\rangle.$$

By the choice of the elements involved we obtain

$$\langle a, G\cdot\theta\rangle = \langle \theta\cdot a, G\rangle = 0, \qquad \langle b\cdot\theta, F\rangle = 0, \qquad \langle G\cdot\theta, F\rangle = 0.$$

Therefore,

$$\big\langle \theta, (a,F)\Box(b,G)\big\rangle = \langle ab, \theta\rangle,$$

whence

$$(a,F)\square(b,G) = (ab,0).$$

Similarly, we show that

$$(a,F) \diamond (b,G) = (ab,0). \qquad \square$$

Recall that a *character space* Φ_A of a Banach algebra A is a set of nonzero multiplicative functionals on A.

Proposition 3.6 *Let φ be an Orlicz function and $a_\varphi = 0$. Then,*

$$\Phi_{\ell_\varphi} = \Phi_{h_\varphi} = \{e_n^*\colon n \in \mathbb{N}\}.$$

Proof Let $f \in \ell'_\varphi$ be a multiplicative functional. We have

$$f(e_n) = f(e_n^2) = f(e_n)^2 \quad (n \in \mathbb{N})$$

therefore,

$$f(e_n) = 0 \quad \text{or} \quad f(e_n) = 1 \quad (n \in \mathbb{N}).$$

If $f(e_n) = 0$ for all $n \in \mathbb{N}$ then $f \in h_\varphi^\perp$. This implies

$$f(x)^2 = f(x^2) = 0 \quad (x \in \ell_\varphi)$$

since $x^2 \in h_\varphi$, see Proposition 3.1. Thus, $f = 0$. If $f(e_n) = f(e_m) = 1$ for some $n \neq m$, then

$$0 = f(e_n e_m) = f(e_n)f(e_m) = 1,$$

a contradiction. Consequently, the only nonzero multiplicative functionals are the evaluation ones. □

Let A be a Banach algebra and let $\lambda \in \Phi_A \cup \{0\}$. A Banach A-bimodule X is called *left λ-linked*, resp. *right λ-linked*, if

$$a \cdot x = \langle a, \lambda\rangle x \quad (a \in A, x \in X),$$

resp.

$$x \cdot a = \langle a, \lambda\rangle x \quad (a \in A, x \in X).$$

A is said to be *left λ-amenable*, resp. *right λ-amenable*, if for any left λ-linked, resp. right λ-linked, Banach A-bimodule X every continuous derivation $\delta\colon A \to X'$ is inner. A is said to be *λ-amenable* if it is both left and right λ-amenable. A is said to be *left character amenable*, resp. *right character amenable,* if it is left λ-amenable, resp. right λ-amenable, for every $\lambda \in \Phi_A \cup \{0\}$. A more detailed account on character amenability can be found in [27, Sect. 4.3].

Remark 3.7 Let φ be an Orlicz function such that $a_\varphi = 0$ and let $A \in \{\ell_\varphi, h_\varphi\}$. From Proposition 3.6 and [10, Theorem 1.1] (cf. [27, Theorem 4.3.5]) it follows that A is λ-amenable for every $\lambda \in \Phi_A$. From [27, Theorem 4.3.4] it follows that A is never 0-amenable. Consequently, A is never character amenable.

4 Main results

Before proceeding with the main results we recall the following well-known facts from the theory of Orlicz functions. Let φ be an Orlicz function such that $a_\varphi = 0$ and let ℓ_φ be the Orlicz sequence space. Obviously, φ is continuous and increasing on the interval $[0, b_\varphi)$. Consequently, φ^{-1} is continuous and increasing on the interval $[0, \varphi(b_\varphi))$. Thus, $\varphi^{-1}(\varphi(u)) = u$ for all $u \in [0, b_\varphi)$ and $\varphi(\varphi^{-1}(v)) = v$ for all $v \in [0, \varphi(b_\varphi))$. Then, a straightforward computation shows that

$$\exists k \in \mathbb{N}\ \forall n \geq k: \quad \left\| \sum_{j=1}^{n} e_j \right\|_\varphi = \frac{1}{\varphi^{-1}(\frac{1}{n})}. \tag{3}$$

In such a case, we will say that the equality (3) is satisfied for large $n \in \mathbb{N}$.

We will also be using the following notation. If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences of nonnegative numbers, then $(a_n)_{n\in\mathbb{N}} \approx (b_n)_{n\in\mathbb{N}}$ means that there is a constant $C > 0$ such that

$$C^{-1} a_n \leq b_n \leq C a_n \quad \text{for all } n \in \mathbb{N}.$$

Recall that a Banach algebra A:

(1) is called *biprojective* if the product map π_A has a right inverse bimodule map;
(2) is called *biflat* if the dual map π_A' has a left inverse bimodule map;
(3) has the *π-property* if $\pi_A(A \widehat{\otimes} A) = \overline{A^2}$.

Theorem 4.1 *Let ℓ_φ be an Orlicz sequence algebra. TFAE*:

(i) *ℓ_φ is biflat*;
(ii) *ℓ_φ has the π-property*;
(iii) *φ is not an N-function at* 0.

Remark 4.2 Recall that if φ is not an N-function at 0 then either $a_\varphi > 0$ (i.e., $\ell_\varphi = \ell_\infty$ with equivalent norms) or $\lim_{u\to 0} \frac{\varphi(u)}{u} > 0$ (i.e., $\ell_\varphi = \ell_1$ with equivalent norms). Examples of Orlicz functions with the above properties will be provided in Example 4.8.

Proof (*i*) $\Rightarrow$ (*ii*): Let $\pi : \ell_\varphi \widehat{\otimes} \ell_\varphi \to \ell_\varphi$ be the product map and let $\sigma : (\ell_\varphi \widehat{\otimes} \ell_\varphi)' \to \ell_\varphi'$ be the left module inverse to π'. We will show that $\operatorname{im}\pi$ is closed. To this end, let

$$\lim_{n\to\infty} \pi' f_n = F$$

for some sequence $(f_n)_{n\in\mathbb{N}} \subset \ell_\varphi'$. Then,

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \sigma\pi' f_n = \sigma F.$$

Therefore,

$$\pi'\sigma F = \lim_{n\to\infty} \pi' f_n = F.$$

Consequently, $F \in \operatorname{im}\pi'$ and the latter space is closed. From [19, Theorem 9.4] it follows that $\operatorname{im}\pi$ is closed as well. However, then

$$\operatorname{im}\pi = \overline{\operatorname{im}\pi} = \overline{\ell^2_\varphi}.$$

From [3, p. 166] it now follows that ℓ_φ has the π-property.

$(ii) \Rightarrow (iii)$: Assume that ℓ_φ has the π-property. If

$$a_\psi = \lim_{u\to 0^+} \frac{\varphi(u)}{u} > 0,$$

where ψ denotes the complementary function of φ (see formula (1)), then $\ell_\varphi = \ell_1$ (equivalent norms). Let us therefore assume that

$$a_\psi = \lim_{u\to 0^+} \frac{\varphi(u)}{u} = 0. \tag{4}$$

We will now show that $a_\varphi > 0$ that is $\ell_\varphi = \ell_\infty$ as sets and the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ are equivalent. By assumption

$$\hat{\pi} : \ell_\varphi \widehat{\otimes} \ell_\varphi / \ker\pi \to h_\varphi$$

is an isomorphism. Since $h'_\varphi = \ell_\psi$ (see [16, Proposition 4.b.1]) and $(\ell_\varphi \widehat{\otimes} \ell_\varphi / \ker\pi)' = (\ker\pi)^\perp$ we obtain that

$$\hat{\pi}' : \ell_\psi \to (\ker\pi)^\perp \tag{5}$$

is an isomorphism as well. Let

$$B_j : \ell_\varphi \times \ell_\varphi \to \mathbb{K}, \qquad B_j(x,y) := x(j)y(j) \quad (j \in \mathbb{N})$$

be a continuous bilinear form. Clearly, $B_j \in \ker\pi^\perp$. Moreover,

$$\langle u + \ker\pi, \hat{\pi}' e_j \rangle = \langle \pi(u), e_j \rangle = \langle u + \ker\pi, B_j \rangle \quad (u \in \ell_\varphi \widehat{\otimes} \ell_\varphi).$$

Therefore,

$$\hat{\pi}' e_j = B_j \quad \text{or} \quad (\hat{\pi}')^{-1}(B_j) = e_j \quad (j \in \mathbb{N}).$$

If we now denote

$$p_n := \sum_{j=1}^{n} e_j, \qquad \widehat{B}_n := \sum_{j=1}^{n} B_j \quad (n \in \mathbb{N}),$$

then

$$\left(\hat{\pi}'\right)^{-1}(\widehat{B_n}) = p_n \quad (n \in \mathbb{N}).$$

From (5) we obtain a constant $D > 0$ such that

$$\|p_n\|_\psi \le D\|\widehat{B_n}\| \quad (n \in \mathbb{N}), \tag{6}$$

where on the right-hand side we consider the norm of a bilinear form, i.e., if $B\colon X \times Y \to Z$ is a bilinear mapping and X, Y, Z are Banach spaces, then $\|B\| := \sup\{\|B(x,y)\|_Z\colon \|x\|_X = \|y\|_Y = 1\}$. Let us now compute these norms. From (3) and (4) it follows that

$$\|p_n\|_\psi = \frac{1}{\psi^{-1}(\frac{1}{n})} \quad (\text{large } n \in \mathbb{N}).$$

As for the other norms, let us first observe that

$$\|\widehat{B_n}\| = \|p_n\|_{\mathcal{M}(\ell_\varphi,\ell_\psi)} \quad (n \in \mathbb{N}),$$

where $\mathcal{M}(\ell_\varphi, \ell_\psi)$ is the multiplier space. From [4, Theorem 3] it now follows that

$$\mathcal{M}(\ell_\varphi, \ell_\psi) \simeq \ell_\tau \quad (\text{equivalent norms}),$$

where τ is the Orlicz function defined by

$$\tau(s) := \max\bigl\{0, \sup\bigl\{\psi(st) - \varphi(t)\colon t \in [0,1]\bigr\}\bigr\}.$$

Therefore, we obtain that $(\|\widehat{B_n}\|)_{n\in\mathbb{N}} \approx (\|p_n\|_\tau)_{n\in\mathbb{N}}$. Assume for the moment that $a_\tau > 0$, i.e., $\ell_\tau = \ell_\infty$ (equivalent norms). Then, there exist constants $D_1, D_2 > 0$ such that

$$D_1 \le \|p_n\|_\tau \le D_2 \quad (n \in \mathbb{N}).$$

In particular, condition (6) takes the form

$$\frac{1}{\psi^{-1}(\frac{1}{n})} \le D_3$$

for some constant $D_3 > 0$ and all large $n \in \mathbb{N}$. Equivalently (see the discussion at the beginning of this section),

$$\psi\left(\frac{1}{D_3}\right) \le \frac{1}{n} \xrightarrow[n\to\infty]{} 0,$$

which contradicts the fact that $a_\psi = 0$. Therefore, $a_\tau = 0$ and

$$\|p_n\|_\tau = \frac{1}{\tau^{-1}(\frac{1}{n})} \quad (\text{large } n \in \mathbb{N}).$$

Coming back to (6) we obtain another constant $C > 0$ such that

$$\tau^{-1}\left(\frac{1}{n}\right) \leq C\psi^{-1}\left(\frac{1}{n}\right) \quad (\text{large } n \in \mathbb{N}). \tag{7}$$

The convexity of ψ now implies that

$$\psi\bigl(t\psi^{-1}(s)\bigr) - \varphi(t) \leq ts - \varphi(t) \quad \bigl(t \in [0,1], s \geq 0\bigr).$$

Recall that from [16, Proposition 4.a.5] it follows that we may assume that $b_\psi = \infty$ that then implies that ψ^{-1} is well defined for every $s \geq 0$. Hence,

$$\sup\bigl\{\psi\bigl(t\psi^{-1}(s)\bigr) - \varphi(t) \colon t \in [0,1]\bigr\} \leq \sup\bigl\{ts - \varphi(t) \colon t \geq 0\bigr\} = \psi(s).$$

Consequently,

$$\tau \circ \psi^{-1} \leq \psi$$

or, equivalently,

$$\psi^{-1} \leq \tau^{-1} \circ \psi. \tag{8}$$

If we denote $u_n := \psi^{-1}(\frac{1}{n})$ (equivalently, $\psi(u_n) = \frac{1}{n}$, see the discussion at the beginning of this section) then (7) and (8) imply that

$$\psi^{-1}(u_n) \leq \tau^{-1} \circ \psi(u_n) \leq Cu_n \quad (\text{large } n \in \mathbb{N}),$$

whence

$$u_n \leq \psi(Cu_n) \quad (\text{large } n \in \mathbb{N}).$$

Equivalently,

$$\frac{\psi(Cu_n)}{Cu_n} \geq \frac{1}{C} \quad (\text{large } n \in \mathbb{N}).$$

We now recall that $u_n = \psi^{-1}(\frac{1}{n}) \xrightarrow[n\to\infty]{} 0$, which implies that

$$a_\varphi = \lim_{u\to 0^+} \frac{\psi(u)}{u} \geq \frac{1}{C} > 0.$$

Consequently, $\ell_\varphi = \ell_\infty$ as sets and the respective norms are equivalent.

(iii) $\Rightarrow$ *(i)*: By assumption, the Orlicz algebra ℓ_φ is either ℓ_∞ (in the case where $a_\varphi > 0$) or ℓ_1 (in the case where $\lim_{u\to 0^+} \varphi(u)/u > 0$). That ℓ_∞ is biflat follows from [28] (cf. [3, Theorem 2.9.65]) since ℓ_∞ is a commutative C^*-algebra, therefore amenable by [9, Lemma 7.10]. As for the other case observe that ℓ_1 is even biprojective—see [3, Example 4.1.42]. □

Corollary 4.3 *Let ℓ_φ be an Orlicz sequence algebra. TFAE:*
(i) *ℓ_φ is biprojective;*
(ii) *$\ell_\varphi = \ell_1$ (equivalent norms).*

Proof That ℓ_1 is biprojective follows from [3, Example 4.1.42]. Assume now that ℓ_φ is biprojective. It is therefore biflat and Theorem 4.1 implies that it is either ℓ_∞ or ℓ_1. That ℓ_∞ is not biprojective follows from [3, Theorem 2.8.48] and [27, Corollary 4.1.5]. □

Recall that an *approximate identity* in a Banach algebra A is a net $(e_\alpha)_{\alpha\in\Lambda} \subset A$ such that

$$\lim_\alpha e_\alpha a = a = \lim_\alpha a e_\alpha \quad (a \in A).$$

It is called *sequential* if Λ is countable and *bounded* if the set $\{e_\alpha : \alpha \in \Lambda\}$ is bounded in A.

A Banach algebra A is called:
(1) *essential* if $\overline{A^2} = A$ as sets;
(2) *weakly amenable* if every continuous derivation $\delta : A \to A'$ is inner;
(3) *approximately semicontractible* if for any A-bimodule X and every continuous derivation $\delta : A \to X$ there are nets $(x_\alpha)_{\alpha\in\Lambda}, (y_\alpha)_{\alpha\in\Lambda} \subset X$ such that

$$\delta(a) = \lim_\alpha (a \cdot x_\alpha - y_\alpha \cdot a) \quad (a \in A).$$

If, in addition we can always choose $x_\alpha = y_\alpha, \alpha \in \Lambda$ then A is *approximately contractible*. If, moreover, the net $(\mathrm{ad}_{x_\alpha})_{\alpha\in\Lambda}$ is bounded in $\mathcal{B}(A, X)$ then A is *boundedly approximately contractible*. If the above properties hold only for dual A-bimodules then A is said to be *(boundedly) approximately (semi-)amenable*.

A more detailed account on (bounded) approximate (semi-)amenability/contractibility can be found in [27, Sect. 4.4].

Now, we are ready to prove the following characterization.

Theorem 4.4 *Let ℓ_φ be an Orlicz sequence algebra. TFAE:*
(i) *ℓ_φ admits a (sequential) approximate identity;*
(ii) *ℓ_φ is essential;*
(iii) *ℓ_φ is weakly amenable;*
(iv) *either $\varphi \in \Delta_2(0)$ or $a_\varphi > 0$;*
(v) *ℓ_φ is approximately semiamenable.*

Remark 4.5 Observe that in condition (iv) we have an "exclusive-or" relation, i.e., properties $\varphi \in \Delta_2(0)$ and $a_\varphi > 0$ cannot hold simultaneously for any Orlicz function φ.

Proof We will show two chains of implications, namely

$$(i) \quad \Rightarrow \quad (ii) \quad \Rightarrow \quad (iii) \quad \Rightarrow \quad (iv) \quad \Rightarrow \quad (i)$$

and

$$(iv) \quad \Leftrightarrow \quad (v).$$

(i) ⇒ *(ii)*: this implication is clear.

(ii) ⇒ *(iii)*: if $a_\varphi > 0$ then $\ell_\varphi = \ell_\infty$ is even amenable by [9, Lemma 7.10] therefore weakly amenable. If $a_\varphi = 0$ then from the assumption and Proposition 3.1 it follows that $\ell_\varphi = \overline{(\ell_\varphi)^2} = h_\varphi$, therefore $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis in ℓ_φ consisting of idempotents. From [3, Proposition 2.8.72] the conclusion follows.

(iii) ⇒ *(iv)*: Suppose that $\varphi \notin \Delta_2(0)$ and $a_\varphi = 0$. Since $\varphi \notin \Delta_2(0)$, we obtain that $h_\varphi \subsetneq \ell_\varphi$ (see Proposition [16, Proposition 4.a.4]). From the Hahn–Banach Theorem it now follows that there exists a continuous nonzero functional $g \in \ell'_\varphi$ that vanishes on h_φ. Then, the mapping

$$\delta\colon \ell_\varphi \to \ell'_\varphi, \quad \delta(a) := g(a)g$$

defines a nonzero continuous derivation. Indeed,

$$\|\delta(a)\| = |g(a)|\|g\| \le \|g\|^2\|a\|_\varphi \quad (a \in \ell_\varphi),$$

where for any $f \in \ell'_\varphi$ we have $\|f\| = \sup\{|f(a)|\colon \|a\|_\varphi = 1\}$. Moreover, since $a_\varphi = 0$, from Proposition 3.1 it follows that

$$\delta(ab) = g(ab)g = 0 \quad (a, b \in \ell_\varphi)$$

and

$$\langle c, a\cdot\delta(b)\rangle = g(b)g(ca) = 0, \qquad \langle c, \delta(a)\cdot b\rangle = g(a)g(bc) = 0 \quad (a, b, c \in \ell_\varphi).$$

Clearly, δ is nonzero since for $a \notin \ker g$ we have

$$\langle a, \delta(a)\rangle = g(a)^2 \neq 0.$$

Since ℓ_φ is commutative the only inner derivation is the trivial one. Thus, δ is not inner that contradicts the weak amenability of ℓ_φ. Consequently, $\varphi \in \Delta_2(0)$ or $a_\varphi > 0$.

(iv) ⇒ *(i)*: if $a_\varphi > 0$ then $\ell_\varphi = \ell_\infty$ (equivalent norms) is even unital and if $\varphi \in \Delta_2(0)$ then it follows that $\ell_\varphi = h_\varphi$ (see [16, Proposition 4.a.4]) and $p_n := \sum_{j=1}^n e_j$ is a sequential approximate identity.

(iv) ⇒ *(v)*: if $a_\varphi > 0$ then $\ell_\varphi = \ell_\infty$ (equivalent norms) is even amenable by [9, Lemma 7.10] therefore approximately semiamenable. Let $\varphi \in \Delta_2(0)$. For any $n \in \mathbb{N}$ let $E_n := (p_n \cdot \ell_\varphi, \|\cdot\|_\varphi)$ and let X be an ℓ_φ-bimodule. If

$$\delta\colon \ell_\varphi \to X$$

is a continuous derivation, then X is (in a natural way) an E_n-bimodule and

$$\delta_n\colon E_n \to X, \qquad \delta_n(a) := \delta(a)$$

is also a continuous derivation. Since E_n is finite dimensional and semisimple from [3, Theorem 1.9.21] it follows that δ_n is inner, i.e., there is $\xi_n \in X$ such that

$$\delta_n(a) = a\cdot\xi_n - \xi_n\cdot a \quad (a \in E_n, n \in \mathbb{N}).$$

By assumption $\ell_\varphi = h_\varphi$, therefore

$$a = \lim_n p_n a \quad \text{and} \quad p_n a \in E_n$$

and

$$\delta(a) = \lim_n \delta_n(p_n a) = \lim_n (p_n a \cdot \xi_n - \xi_n \cdot p_n a) = \lim_n \bigl(a \cdot (p_n \cdot \xi_n) - (p_n \cdot \xi_n) \cdot a\bigr).$$

Consequently, δ is approximately semiinner.

$(v) \Rightarrow (iv)$: suppose that $\varphi \notin \Delta_2(0)$ and $a_\varphi = 0$ and define

$$\delta : \ell_\varphi \to \ell_\varphi / h_\varphi, \qquad \delta(a) := a + h_\varphi,$$

where ℓ_φ / h_φ is the quotient module. From Proposition 3.1 it follows that the module actions in this quotient module are trivial and that δ is a derivation. If it were to be approximately semiinner then it would have to be trivial. However, $\delta(a) \neq 0$ for any $a \notin h_\varphi$. □

Recall that a Banach algebra A is said to be *pseudoamenable* if there is a (possibly unbounded) net $(d_\alpha)_\alpha \subset A \widehat{\otimes} A$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a \underset{\alpha}{\to} 0 \quad \text{and} \quad a\pi(d_\alpha) \underset{\alpha}{\to} 0 \quad (a \in A).$$

A more detailed account on pseudoamenability can be found in [27, Sect. 4.4].

Remark 4.6 (1) From [1, Lemma 2.4 and Theorem 2.5] it follows that ℓ_φ as well as h_φ are never approximately amenable unless $a_\varphi > 0$. Indeed, $(p_n)_{n\in\mathbb{N}} \subset h_\varphi$ constitutes an unbounded but multiplier bounded sequence satisfying $p_n p_{n+1} = p_n = p_{n+1} p_n$, $n \in \mathbb{N}$.

(2) From [5, Theorem 13] it follows that if $\varphi \in \Delta_2(0)$ then ℓ_φ is even boundedly approximately contractible.

(3) From [6, Corollary 3.7] and [27, Proposition 4.4.2] it follows that h_φ is always pseudoamenable, whereas ℓ_φ is pseudoamenable if and only if $\varphi \in \Delta_2(0)$ or $a_\varphi > 0$.

Remark 4.7 Condition (iv) of the above theorem shows in particular that there is a wide class of Orlicz sequence algebras (i.e., $a_\varphi = 0$ and $\varphi \notin \Delta_2(0)$) that serve as nonexamples to a number of amenability properties.

We end this section with a list of examples illustrating properties of Orlicz functions considered in Theorems 4.1 and 4.4.

Example 4.8 (1) The Orlicz functions

$$\varphi_1(u) := \max\bigl\{0, u^p - 1\bigr\} \quad (p \geq 1)$$

and

$$\varphi_2(u) := \begin{cases} 0, & u \in [0,1], \\ \infty, & u \in (1,\infty), \end{cases}$$

satisfy $a_{\varphi_1} = a_{\varphi_2} = 1$.

(2) The Orlicz functions

$$\varphi_{a,b,c}(u) := au\ln(b+cu) \quad (a>0, b>1, c\geq 0)$$

are not N-functions at 0. Clearly, if φ is an Orlicz function such that $a_\varphi = 0$ and φ is not an N-function at 0 then it satisfies condition $\Delta_2(0)$.

(3) The Orlicz functions

$$\varphi_3(u) := u^p \quad (p>1)$$

and

$$\varphi_4(u) := u^p\ln(1+u) \quad (p\geq 1)$$

are N-functions at 0 and satisfy condition $\Delta_2(0)$.

(4) We finish with a construction of an Orlicz function for which $a_\varphi = 0$ but it does not satisfy condition $\Delta_2(0)$. Let

$$u_n := \frac{1}{2^n} \quad \bigl(n\in\mathbb{N}\cup\{0\}\bigr)$$

and let $p\colon [0,\infty)\to[0,\infty)$ be a function defined as

$$p(0) := 0, \qquad p(t) := \frac{1}{n!} \quad \text{for } t\in[u_n, u_{n-1}), n\in\mathbb{N}, \qquad p(t) := t \quad \text{for } t\geq 1.$$

The function p is nondecreasing and right-continuous. Now, we define the Orlicz function φ_5 as

$$\varphi_5(u) := \int_0^u p(t)\,dt.$$

Clearly, $a_{\varphi_5} = 0$. Moreover, for any $n\in\mathbb{N}$ we have

$$\begin{aligned}
\varphi_5(2u_n) &= \int_0^{2u_n} p(t)\,dt \\
&\geq \int_{u_n}^{2u_n} \frac{1}{n!}\,dt \\
&= (n+1)\int_0^{u_n} \frac{1}{(n+1)!}\,dt \\
&\geq (n+1)\int_0^{u_n} p(t)\,dt = (n+1)\varphi_5(u_n).
\end{aligned}$$

Therefore, the function φ_5 does not satisfy condition $\Delta_2(0)$.

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