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Inequalities for integral operators in Hölder–Morrey spaces on differential forms

Xuexin Li[1,2], Jianwei Wang[2*] and Ning Pan[2]

*Correspondence:
wjwanghit@163.com
[2]Department of Mathematics, Northeast Forestry University, Harbin, 150040, P.R. China
Full list of author information is available at the end of the article

Abstract
The Hölder–Morrey spaces $\Lambda_\kappa^{p,\tau}(\Omega, \wedge^l)$ are proposed in this paper. The imbedding inequalities for homotopy operator are derived in Hölder–Morrey spaces on differential forms. The Hölder continuity for Riesz potential with envelope function is deduced. As application, some composite theorems, which are associated with conjugate *A*-harmonic equations on differential forms, are given.

Keywords: Hölder–Morrey spaces; Homotopy operator; Riesz potential; Differential forms

1 Introduction

The purpose of this paper is to study the imbedding theory for homotopy operator in Hölder–Morrey spaces on differential forms. We first give a new definition for the Hölder–Morrey spaces $\Lambda_\kappa^{p,\tau}(\Omega, \wedge^l)$ that is based on the classical Hölder theory and Sobolev spaces $W^{p,\kappa}(\Omega, \wedge^l)$ on differential forms. The theory of Hölder continuity of order $\kappa \in (0,1]$ has been widely used in various fields such as partial differential equations, harmonic analysis, and basic geometrical theory, see [1–3].

The concept of Morrey space was proposed in 1938, and many scholars have conducted in-depth research on it. Morrey space theory was extended to generalized Morrey spaces, weighted Morrey spaces, variable exponent Morrey spaces, weak Morrey spaces, and so on, see [4–7]; with the deepening of research, to composite spaces of Morrey spaces such as Morrey–Herz spaces and Orlicz–Morrey spaces, see [8, 9]. The concept of Hölder–Morrey spaces we present is in Sect. 2. Furthermore, the theory of envelope functions is introduced in the same section, and we give the definition for a class of admissible envelope functions. Envelope functions have been widely studied in super-lattice band structure, medical science, and nonlinear periodic structure, see [10–12]. In the present paper, the real envelope function $\lambda : \mathbb{R}^n \to \mathbb{R}$ satisfies the condition of locally Hölder-continuous in $L^{p,\tau}$-norm.

Homotopy operator T is a key tool used in the decomposition theorem and Poincaré-type inequality on differential forms, see [13, 14] for more details. Homotopy operator T on differential forms is a linear mapping from $\wedge^l(\mathbb{R}^n)$ to $\wedge^{l-1}(\mathbb{R}^n)$. In fact, by using the

Poincaré lemma in [15], we obtain

$$u = dTu + T\,du = dTu$$

for a closed form u on Ω. The primary properties for homotopy operator T on differential forms can be found in [15].

In this paper, we introduce the definition of Hölder–Morrey spaces and envelope functions on differential forms in Sect. 2. The imbedding inequality for homotopy operator T and Poincaré-type inequality are given in Sect. 3. In Sect. 4, the Hölder continuity for Riesz potential operator with envelope functions is derived. Finally, some estimates that are closely related to homotopy operator applied to the solutions of conjugate A-harmonic equations on differential forms are obtained.

2 Preliminary

Before stating our main results, we introduce some notations and basic theory for differential forms.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain, $n \geq 2$ and an l-form u be a locally integrable function on Ω with values in $\wedge^l(\mathbb{R}^n)$. If $u(x) \in \wedge^l(\mathbb{R}^n)$, then the value of $u(x)$ at the vectors $\xi_1, \dots, \xi_l \in \mathbb{R}^n$ is denoted by $u(x)(\xi_1, \dots, \xi_l) = u(x; \xi_1, \dots, \xi_l)$. For more details on differential forms, see [16].

Arbitrary l-form $u : \Omega \to \wedge^l(\mathbb{R}^n)$ can be denoted as $u(x) = \sum_I u_I(x)\,dx_I$, and each u_I is the coefficient function. Moreover, u is called a differential form, the coefficient functions u_I are differentiable, see [17, 18]. The operator $\star : \wedge^l(\mathbb{R}^n) \to \wedge^{n-l}(\mathbb{R}^n)$ is the Hodge-star operator, and the linear operator $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$, $0 \leq l \leq n-1$, is called the exterior differential. $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ denotes the Hodge codifferential operator, which is the formal adjoint of d.

Denote by $L^p(\Omega, \wedge^l)$ the space of differential l-forms satisfying $\int_\Omega |u_I|^p < \infty$ with the norm

$$\|u\|_{L^p(\Omega,\wedge^l)} = \left(\int_\Omega \left(\sum_I \left|u_I(x)\right|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}.$$

Definition 2.1 (Hölder space) If $1 < p < \infty$ and $\kappa \in (0,1]$. Let $\Lambda^{p,\kappa}(\Omega, \wedge^l)$ denote the Hölder space of all differential l-forms u whose norm

$$\|u\|_{\Lambda^{p,\kappa}(\Omega,\wedge^l)} := \|u\|_{L^p(\Omega,\wedge^l)} + \sup_{x,y\in\Omega} \frac{\|u(x) - u(y)\|_{L^p(\Omega,\wedge^l)}}{|x-y|^\kappa} < \infty.$$

The Hölder space on differential forms will reduce to the classical type when $l = 0$.

Clearly, Sobolev spaces $W^{p,\kappa}(\Omega, \wedge^l)$ in [19] could replace Hölder spaces in what follows, but we selected Hölder spaces because they are slightly more elementary.

The following definition is contributed in Sect. 4.

Definition 2.2 (Envelope function) A real envelope function is a function $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that for every $1 < p < \infty$ there exists $\kappa \in (0,1)$ such that $\lambda \in \Lambda^{p,\kappa}(\mathbb{R}^n)$. Thus, the space of all envelope functions is

$$\Lambda =: \bigcap_{1<p<\infty} \bigcup_{0<\kappa<1} \Lambda^{p,\kappa}(\mathbb{R}^n).$$

In this paper, we consider that each envelope function $\lambda \in \Lambda$ satisfies the condition: there exists a constant $C_1 \geq 1$ such that

$$C_1^{-1} \leq \lambda(\cdot) \leq C_1. \tag{2.1}$$

Remark 1 Any function in the Schwartz class is an envelope function, and the product of two envelope functions $\lambda_1, \lambda_2 \in \Lambda$ is again an envelope function.

Let $u \in L^1_{\mathrm{loc}}(\Omega, \wedge^l)$ be a differential l-form on Ω. For $1 \leq p < \infty$, $0 < \tau \leq n$, and $r > 0$, we denote the Morrey space on differential forms by $L^{p,\tau}(\Omega, \wedge^l)$ with the norm

$$\|u(x)\|_{L^{p,\tau}(\Omega,\wedge^l)} = \left(\sup_{x\in\Omega, r>0} \frac{r^\tau}{|B(x,r)|} \int_{B(x,r)} \left(\sum_I |u_I(y)|^2\right)^{p/2} dy\right)^{1/p} < \infty,$$

where $B(x,r) \subset \Omega$ is an open ball centered at x of radius r.

Easily we see that $L^{p,\tau}(\Omega, \wedge^l)$ is an expansion of $L^p(\Omega, \wedge^l)$ in the sense that $L^{p,n}(\Omega, \wedge^l) = L^p(\Omega, \wedge^l)$. If $l = 0$, then the Morrey space on differential forms reduces to the classical Morrey space, which was first introduced by C. Morrey in [20]. In recent years, Morrey spaces and generalized Morrey spaces have received much investigation. The boundedness for Hardy–Littlewood maximal function and singular integrals in Morrey spaces were given in [21–23].

Definition 2.3 (Hölder–Morrey space) Assume $1 \leq p < \infty$, $0 < \tau \leq n$, and $\kappa \in (0,1]$. Let $\Lambda^{p,\tau}_\kappa(\Omega, \wedge^l)$ denote the Hölder–Morrey space of all differential l-forms u whose norm

$$\|u\|_{\Lambda^{p,\tau}_\kappa(\Omega,\wedge^l)} := \|u\|_{L^{p,\tau}(\Omega,\wedge^l)} + \sup_{x,y\in\Omega} \frac{\|u(x) - u(y)\|_{L^{p,\tau}(\Omega,\wedge^l)}}{|x-y|^\kappa} < \infty.$$

We choose $l = 0$ and $\tau = n$, then $\Lambda^{p,\tau}_\kappa(\mathbb{R}^n, \wedge^l)$ coincides with $\Lambda^{p,\kappa}(\mathbb{R}^n)$. Furthermore, the space

$$\Lambda^\tau =: \bigcap_{1<p<\infty} \bigcup_{0<\kappa<1} \Lambda^{p,\tau}_\kappa(\mathbb{R}^n)$$

denotes all envelope functions with Morrey norms. Recall some properties of BMO spaces and integral average on differential forms. For $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y) - b_{B(x,r)}\right| dy,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)\, dy.$$

Define $\mathrm{BMO}(\mathbb{R}^n) = \{b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|b\|_* < \infty\}$. Analogically, we show integral average on differential forms. The following lemma appeared in [24].

Lemma 2.4 *Let Ω be a bounded, convex domain in $\mathbb{R}^n$. To each $z \in \Omega$, there corresponds a linear operator $K_z : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ defined by*

$$(K_z u)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + z - tz; x - z, \xi_1, \ldots, \xi_{l-1})\, dt,$$

and the decomposition

$$u = d(K_z u) + K_z(du)$$

holds at any point z in Ω.

Assume that $\varphi \in C_0^\infty(\Omega)$ is a test function and satisfies $\int_\Omega \varphi(z)\, dz = 1$. We define $T : L^p(\Omega, \wedge^l) \to L^p(\Omega, \wedge^{l-1})$ is a homotopy operator by averaging K_z in Ω:

$$Tu = \int_\Omega \varphi(z) K_z u\, dz = \int_0^1 t^{l-1} \int_\Omega \varphi(z) u(tx + z - tz; x - z, \xi_1, \ldots, \xi_l)\, dz\, dt.$$

It is obvious that

$$u = d(Tu) + T(du). \tag{2.2}$$

Then we denote the integral average of u over Ω by $u_\Omega = d(Tu)$ for $l = 1, 2, \ldots, n$. If $l = 0$, then the differential form u reduces to the function type and integral average $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(y)\, dy$. Clearly, we see that

$$u = Td(u) + u_\Omega. \tag{2.3}$$

From Poincaré lemma in [15], u_Ω is a closed form. Suppose that u is a differential l-form in $L^1_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 0, 1, 2, \ldots, n$, and for $1 < p < \infty$, $0 < \tau \le n$. Let $\mathrm{BMO}^{p,\tau}(\Omega, \wedge^l)$ be a space on differential form with the finite norm

$$\|u\|_{\mathrm{BMO}^{p,\tau}(\Omega, \wedge^l)} := \sup_{x \in \Omega, r > 0} \left(\frac{r^\tau}{|B(x,r)|} \int_{B(x,r)} \left| u(y) - u_{B(x,r)} \right|^p dy \right)^{1/p},$$

where u_B coincides with (2.3). If $l = 0$ and $\tau = 0$, then $\mathrm{BMO}^{p,\tau}(\Omega, \wedge^l)$ spaces reduce to the classical BMO spaces.

3 Imbedding inequalities for homotopy operator

In this section, we are concerned with the imbedding inequality for homotopy operator. The following lemma appeared in [13, 14].

Lemma 3.1 *Let u be a differential l-form on Ω, $l = 0, 1, \ldots, n$. Let $T : L^p(\Omega, \wedge^l) \to L^p(\Omega, \wedge^{l-1})$ be a homotopy operator and $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain. Then there exists a constant C, independent of u, such that*

$$\left(\int_B \left| T(u) \right|^p dx \right)^{1/p} \le C|B|^{1/n} \left(\int_B |u|^p\, dx \right)^{1/p},$$

where $B := B(x, r) \subset \Omega$ is an open ball centered at x of radius r.

Now, we give the imbedding inequality for homotopy operator on differential forms.

Theorem 3.2 *Let $u \in \Lambda_\kappa^{p,\tau}(\Omega, \wedge^l)$ be a differential l-form on Ω, $l = 1, 2, \dots, n$, $0 < \tau \le n - p$, and $\kappa \in (0, 1]$. Let also $T : L^p(\Omega, \wedge^l) \to L^p(\Omega, \wedge^{l-1})$ be a homotopy operator. Then there exists a constant C, independent of u, such that*

$$\|Tu\|_{\Lambda_\kappa^{p,\tau}(\Omega,\wedge^{l-1})} \lesssim \begin{cases} \|u\|_{\Lambda_\kappa^{p,\tau+p}(\Omega,\wedge^l)} + \|u\|_{L^{p,\tau}(\Omega,\wedge^l)}, & \textit{for } l = 1, \\ \|u\|_{\Lambda_\kappa^{p,\tau+p}(\Omega,\wedge^l)}, & \textit{for } l = 2, \dots, n. \end{cases}$$

Proof For arbitrary $y \in \Omega$, we have

$$\|Tu\|_{\Lambda_\kappa^{p,\tau}(\Omega,\wedge^{l-1})} = \|Tu\|_{L^{p,\tau}(\Omega,\wedge^{l-1})} + \sup_{x,y\in\Omega} \frac{\|Tu(y) - Tu(x)\|_{L^{p,\tau}(\Omega,\wedge^{l-1})}}{|y - x|^\kappa}. \tag{3.1}$$

By using Lemma 3.1, we get the first part of the right-hand side for (3.1)

$$\begin{aligned} \|Tu\|_{L^{p,\tau}(\Omega,\wedge^{l-1})} &= \left(\sup_{x_0\in\Omega, r>0} \frac{r^\tau}{|B(x_0, r)|} \int_{B(x_0,r)} \left|Tu(x)\right|^p dx \right)^{1/p} \\ &\lesssim \sup_{x_0\in\Omega, r>0} r^{(\tau-n)/p} r \left(\int_{B(x_0,r)} \left|Tu(x)\right|^p dx \right)^{1/p} \\ &\lesssim \|u\|_{L^{p,\tau+p}(\Omega,\wedge^l)}. \end{aligned}$$

For any $\xi_1, \dots, \xi_{l-1} \in \mathbb{R}^n$, we denote $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{l-1})$ for notational simplicity. Then we deduce that

$$\begin{aligned} &\left|Tu(y) - Tu(x)\right| \\ &\quad = \bigg| \int_0^1 t^{l-1} \int_\Omega \varphi(z) u(ty + z - tz; y - z, \boldsymbol{\xi})\, dz\, dt \\ &\qquad - \int_0^1 t^{l-1} \int_\Omega \varphi(z) u(tx + z - tz; x - z, \boldsymbol{\xi})\, dz\, dt \bigg| \\ &\quad \le \left| \int_0^1 t^{l-1} \int_\Omega \varphi(z) \big[u(ty + z - tz; y - z, \boldsymbol{\xi}) - u(ty + z - tz; x - z, \boldsymbol{\xi}) \big]\, dz\, dt \right| \\ &\qquad + \left| \int_0^1 t^{l-1} \int_\Omega \varphi(z) \big[u(ty + z - tz; x - z, \boldsymbol{\xi}) - u(tx + z - tz; x - z, \boldsymbol{\xi}) \big]\, dz\, dt \right| \\ &\quad = \mathcal{T}_1 + \mathcal{T}_2. \end{aligned}$$

Note

$$\mathcal{T}_1 = \left| \int_0^1 t^{l-1} \int_\Omega \varphi(z) u(ty + z - tz; x - y, \boldsymbol{\theta})\, dz\, dt \right|, \tag{3.2}$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{l-1})$ and each θ_i for $i = 1, 2, \dots, l - 1$ is a 0-vector in $\mathbb{R}^n$. For (3.2), we only consider the condition for $l = 1$. In fact, if $l > 1$, then the determinant of $(x - y, \boldsymbol{\theta}) = 0$. For $l = 1$ and each u_i is a locally integrable function, we obtain

$$\mathcal{T}_1 = \left| \int_0^1 t^{l-1} \int_\Omega \varphi(z) u(ty + z - tz; x - y)\, dz\, dt \right|$$

$$
\begin{aligned}
&\leq \int_{\Omega} \varphi(z)\biggl(\sum_{1<i<n} |u_i|^2\biggr)^{1/2} |x-y|\,dz \\
&\lesssim \biggl(\sum_{1<i<n} |u_i|^2\biggr)^{1/2}.
\end{aligned}
$$

For $\mathcal{T}_2$, we have

$$
\begin{aligned}
\mathcal{T}_2 &= \biggl|\int_0^1 t^{l-1}\int_{\Omega}\varphi(z)\bigl[u(ty+z-tz)-u(tx+z-tz)\bigr](x-z,\boldsymbol{\xi})\,dz\,dt\biggr| \\
&= \bigl|T\bigl(u(y)-u(x)\bigr)(x;\boldsymbol{\xi})\bigr|.
\end{aligned}
\tag{3.3}
$$

Combining (3.2) and (3.3), we obtain

$$
\begin{aligned}
&\bigl\|Tu(y)-Tu(x)\bigr\|_{L^{p,\tau}(\Omega,\wedge^{l-1})} \\
&\quad\leq \|\mathcal{T}_1\|_{L^{p,\tau}(\Omega,\wedge^{l-1})} + \|\mathcal{T}_2\|_{L^{p,\tau}(\Omega,\wedge^{l-1})} \\
&\quad\lesssim |y-x|\biggl(\sup_{x_0\in\Omega,r>0}\frac{r^{\tau}}{|B(x_0,r)|}\int_{B(x_0,r)}\biggl(\sum_{1<i<n}|u_i|^2\biggr)^{p/2}dx\biggr)^{1/p} \\
&\qquad + \biggl(\sup_{x_0\in\Omega,r>0}\frac{r^{\tau}}{|B(x_0,r)|}\int_{B(x_0,r)}\bigl|T\bigl(u(y)-u(x)\bigr)(x;\boldsymbol{\xi})\bigr|^p\,dx\biggr)^{1/p} \\
&\quad\lesssim \|u\|_{L^{p,\tau}(\Omega,\wedge^l)} + \bigl\|u(y)-u(x)\bigr\|_{L^{p,\tau+p}(\Omega,\wedge^l)}.
\end{aligned}
\tag{3.4}
$$

Collecting the above facts, we deduce that

$$
\|Tu\|_{\Lambda_{\kappa}^{p,\tau}(\Omega,\wedge^{l-1})} \lesssim
\begin{cases}
\|u\|_{\Lambda_{\kappa}^{p,\tau+p}(\Omega,\wedge^l)} + \|u\|_{L^{p,\tau}(\Omega,\wedge^l)}, & \text{for } l=1,\\
\|u\|_{\Lambda_{\kappa}^{p,\tau+p}(\Omega,\wedge^l)}, & \text{for } l=2,\dots,n.
\end{cases}
$$

The proof of Theorem 3.2 has been completed. □

Next, we show the Poincaré-type inequality in Hölder–Morrey spaces on differential forms.

Theorem 3.3 *Let $u\in\Lambda_{\kappa}^{p,\tau}(\Omega,\wedge^l)$ be a differential l-form and $du\in\Lambda_{\kappa}^{p,\tau}(\Omega,\wedge^{l+1})$, $l=1,2,\dots,n-1$, $0<\tau\leq n-p$. Then there exists a constant C, independent of u, such that*

$$
\|u-u_B\|_{\Lambda_{\kappa}^{p,\tau}(B,\wedge^l)} \leq C\|du\|_{\Lambda_{\kappa}^{p,\tau+p}(B,\wedge^{l+1})},
$$

where B is any ball in Ω.

Proof Using Lemma 3.1, we get

$$
\|u-u_B\|_{L^{p,\tau}(B,\wedge^l)} = \|T\,du\|_{L^{p,\tau}(B,\wedge^l)} \lesssim \|du\|_{L^{p,\tau+p}(B,\wedge^{l+1})}.
$$

Note that

$$
\bigl|\bigl(u(y)-u(y)_B\bigr)-\bigl(u(x)-u(x)_B\bigr)\bigr| = \biggl|\int_B\varphi(z)\bigl[K_z\bigl(du(y)\bigr)-K_z\bigl(du(x)\bigr)\bigr]\,dz\,dt\biggr|.
$$

From Lemma 2.4, we have

$$\begin{aligned}K_z\big(du(y;\xi_1\cdots,\xi_l)\big) &= \int_0^1 t^l\big[u'(ty+z-tz)(y-z)\big](\xi_1,\dots,\xi_l)\,dt\\&\quad+\sum_{i=1}^{l}(-1)^i\int_0^1 t^l\big[u'(ty+z-tz)\xi_i\big](y-z,\xi_1,\dots,\hat{\xi}_i,\dots,\xi_l)\,dt\\&=\mathcal{K}_1+\mathcal{K}_2\end{aligned}$$

and

$$\begin{aligned}K_z\big(du(x;\xi_1\cdots,\xi_l)\big) &= \int_0^1 t^l\big[u'(tx+z-tz)(x-z)\big](\xi_1,\dots,\xi_l)\,dt\\&\quad+\sum_{i=1}^{l}(-1)^i\int_0^1 t^l\big[u'(tx+z-tz)\xi_i\big](x-z,\xi_1,\dots,\hat{\xi}_i,\dots,\xi_l)\,dt\\&=\mathcal{K}_3+\mathcal{K}_4,\end{aligned}$$

where $u'(x):\mathbb{R}^n\to\wedge^l(\mathbb{R}^n)$ is a derivative mapping. Moreover, $u'(x)\xi_i$ is an l-linear anti-symmetric function on $\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ for each $\xi_i\in\mathbb{R}^n$. We first estimate $\mathcal{K}_1-\mathcal{K}_3$:

$$\begin{aligned}\mathcal{K}_1-\mathcal{K}_3\le{}&\int_0^1 t^l\big\{\big[u'(ty+z-tz)(y-z)\big](\xi_1,\dots,\xi_l)\\&-\big[u'(ty+z-tz)(x-z)\big](\xi_1,\dots,\xi_l)\big\}\,dt\\&+\int_0^1 t^l\big\{\big[u'(ty+z-tz)(x-z)\big](\xi_1,\dots,\xi_l)\\&-\big[u'(tx+z-tz)(x-z)\big](\xi_1,\dots,\xi_l)\big\}\,dt.\end{aligned}\tag{3.5}$$

Similarly, for $\mathcal{K}_2-\mathcal{K}_4$, we obtain

$$\begin{aligned}&\mathcal{K}_2-\mathcal{K}_4\\&\quad\le\sum_{i=1}^{l}(-1)^i\int_0^1 t^l\big[u'(ty+z-tz)\theta_i\big](y-x,\theta_1,\dots,\hat{\theta}_i,\dots,\theta_l)\,dt\\&\qquad+\sum_{i=1}^{l}(-1)^i\int_0^1 t^l\big[\big[u'(ty+z-tz)-u'(tx+z-tz)\big]\xi_i\big]\\&\qquad\times(x-y,\xi_1,\dots,\hat{\xi}_i,\dots,\xi_l)\,dt.\end{aligned}\tag{3.6}$$

Combining (3.5) and (3.6), we deduce that

$$\begin{aligned}&K_z\,du(y)-K_z\,du(x)=(\mathcal{K}_1-\mathcal{K}_3)+(\mathcal{K}_2-\mathcal{K}_4)\\&\quad=\int_0^1 t^l\big[\big[u'(ty+z-tz)-u'(tx+z-tz)\big](x-z)\big](\xi_1,\dots,\xi_l)\,dt\\&\qquad+\sum_{i=1}^{l}(-1)^i\int_0^1 t^l\big[\big[u'(ty+z-tz)-u'(tx+z-tz)\big]\xi_i\big]\end{aligned}$$

$$\begin{aligned}&\times (x-z,\xi_1,\dots,\hat{\hat{\xi}}_i,\dots,\xi_l)\,dt\\&=K_z\big(du(y)-du(x)\big).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\|u-u_B\|_{\Lambda_\kappa^{p,\tau}(B,\wedge^l)}&\lesssim \|du\|_{L^{p,\tau+p}(B,\wedge^{l+1})}+\sup_{x,y\in\Omega}\frac{\|T(du(y))-T(du(x))\|_{L^{p,\tau}(\Omega,\wedge^{l+1})}}{|y-x|^\kappa}\\&=\|du\|_{\Lambda_\kappa^{p,\tau+p}(B,\wedge^{l+1})}.\end{aligned}$$

The proof of Theorem 3.3 has been completed. □

Remark 2 Replacing $u\in L^p_{\mathrm{loc}}(\Omega,\wedge^l)$ with the solution for A-harmonic equation on differential forms in [25, 26]

$$d^\star A(x,du)=B(x,du),$$

we can get the high order Poincaré-type estimate with Hölder norm by using the reverse Hölder inequality

$$\big\|(u-u_B)\big\|_{\Lambda^{np/(n-p),\kappa}(B,\wedge^{l-1})}\le C\|du\|_{\Lambda^{p,\kappa}(\sigma B,\wedge^{l+1})}$$

for $\sigma B\subset\Omega$ with $\sigma>1$.

Next, we show the vectorial differential forms

$$U=\big(u^1,u^2,\dots,u^k\big):\Omega\to\wedge^{l_1}\times\cdots\times\wedge^{l_k},$$

where each u_i is a differential l_i form for $i=1,2,\dots,k$ and all l_i satisfy the condition $1\le l_1\le\cdots\le l_k\le n$, $\sum_1^k l_i\le n$. We denote the spaces of all vectorial differential forms by $\mathcal{L}^P(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})$ with the norm

$$\|U\|_{\mathcal{L}^P(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})}=\sum_1^k\|u_i\|_{L^{p_i}(\Omega,\wedge^{l_i})}<\infty,$$

where $P=(p_1,p_2,\dots,p_k)$ and each p_i belongs to $(1,\infty)$. Analogically, we denote the vectorial Hölder–Morrey space by $\Lambda_\kappa^{P,\tau}(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})$ with the finite norm

$$\|U\|_{\Lambda_\kappa^{P,\tau}(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})}:=\|U\|_{\mathcal{L}^{P,\tau}(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})}+\sup_{x,y\in\Omega}\frac{\|U(y)-U(x)\|_{\mathcal{L}^{P,\tau}(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})}}{|y-x|^\kappa}.$$

Then we can obtain the following corollary.

Corollary 3.4 *Suppose that $U=(u^1,u^2,\dots,u^k)$ is a vectorial differential form on Ω. Each u^i belongs to $\Lambda_\kappa^{p_i,\tau}(\Omega,\wedge^{l_i})$ with $P=(p_1,\dots,p_k)$ and each du^i belongs to $\Lambda_\kappa^{q_i,\tau}(\Omega,\wedge^{l_i+1})$ with $Q=(q_1,\dots,q_k)$, $l=l_1+\cdots+l_k$. $P=(p_1,\dots,p_k)$ and $Q=(q_1,\dots,q_k)$ satisfying $\max\{1,np_i/(n+p_i)\}\le q_i\le p_i$ for $i=1,2,\dots,k$. Let $T^k=T\times\cdots\times T$ be a k-order vectorial homotopy operator and $T^kU=(Tu^1,\dots,Tu^k)$. Then*

(i) *There exists a constant $C > 0$ independent of u such that*

$$\|T^k U\|_{\Lambda^{P,\tau}(\Omega,\wedge^{l_1-1}\times\cdots\times\wedge^{l_k-1})} \le C\|U\|_{\Lambda^{P,\tau+P}(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})},$$

where $\tau + P =: (p_1 + \tau, \dots, p_k + \tau)$.

(ii) *There exists a test form* $\eta \in C_0^\infty(\Omega, \wedge^{n-l})$ *that satisfies* $\|\eta\|_\infty \le 1$ *and* $\|d\eta\|_\infty \le 1$ *with* $l = \sum_1^k l_i$ *such that*

$$\begin{aligned}&\left|\int_\Omega \eta \wedge \bigl(u^1(y) \wedge \cdots \wedge u^k(y) - u^1(x) \wedge \cdots \wedge u^k(x)\bigr)\right| \\ &\quad\lesssim \|U\|^{k-2}_{L^P(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})}\bigl\{\|U\|_{L^P(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})} + \|dU\|_{L^Q(\Omega,\wedge^{l_1}\times\cdots\times\wedge^{l_k})}\bigr\}.\end{aligned}$$

4 Hölder continuity with envelope function

For $0 < \alpha < n$ and a differential l-form u on Ω, we define Riesz potential operator $I_\alpha u$ of order α by

$$I_\alpha u(x) = \sum_I \left(\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} u_I(y)\,dy\right) dx_I.$$

The operator $I_\alpha u$ will reduce to function type if $l = 0$. The boundedness and Sobolev-type imbedding inequality in classic Morrey spaces for Riesz potential operator were given by Adams and Xiao in [27, 28]. The Hölder continuity for I_α with envelope functions is as follows.

Theorem 4.1 *Suppose that* $1 < p < q < \infty$, $\tau \in (\alpha + \kappa - 1, \alpha + \kappa)$ *and* $\tau \in (\alpha p - np/q, \alpha q - nq/p)$, $h = |x - z|$. *Let* $u \in L^1_{\mathrm{loc}}(\Omega, \wedge^l)$ *be a differential l-form and* $\lambda(\cdot) \in \Lambda^\tau$ *be an envelope function. Then there exists a constant* $C > 0$ *such that*

$$\bigl\|\bigl(\lambda(x)I_\alpha u(x)\bigr) - \lambda(z)I_\alpha u(z)\bigr\|_{L^{p,\tau}(\Omega,\wedge^l)} \le C\bigl\{h^{\alpha+\kappa-\tau} + h^{\alpha-\tau/q-n/p} + h^{\alpha+\kappa-\tau/p-n/q}\bigr\}$$

for all differential l-forms $u \in L^p_{\mathrm{loc}}(\Omega, \wedge^l)$ *with* $\|u\|_{L^{p,\tau}(\Omega,\wedge^l)} \le 1$.

Proof Since $\lambda(\cdot) \in \Lambda^\tau$ and each $\lambda(\cdot)$ on $\mathbb{R}^n$ satisfies condition (2.1), we get

$$\bigl\|\lambda(x) - \lambda(z)\bigr\|_{L^{p,\tau}(\mathbb{R}^n)} \le C|h|^\kappa,$$

where C is a constant independent of λ. Assume that each u_I for $u = \sum_I u_I\,dx_I$ is nonnegative on Ω and $\|u\|_{L^{p,\tau}(\Omega,\wedge^l)} \le 1$. Let $B(x,h) \subset \Omega$ be an open ball centered at x of radius h. Then we have

$$\begin{aligned}&\bigl|\lambda(x)I_\alpha u(x) - \lambda(z)I_\alpha u(z)\bigr| \\ &\quad\le \Bigl|\lambda(x)\sum_I \int_{B(x,2h)} \frac{1}{|x-y|^{n-\alpha}} u_I(y)\,dy\,dx_I \\ &\qquad + \lambda(x)\sum_I \int_{\mathbb{R}^n\setminus B(x,2h)} \frac{1}{|x-y|^{n-\alpha}} u_I(y)\,dy\,dx_I\end{aligned}$$

$$
\begin{aligned}
&- \lambda(z)\sum_I \int_{B(x,2h)} \frac{1}{|z-y|^{n-\alpha}} u_I(y)\,dy\,dx_I \\
&- \lambda(z)\sum_I \int_{\mathbb{R}^n\setminus B(x,2h)} \frac{1}{|z-y|^{n-\alpha}} u_I(y)\,dy\,dx_I \bigg| \\
\lesssim{}& \bigg|\lambda(x)\sum_I \int_{B(x,3h)} \frac{1}{|x-y|^{n-\alpha}} u_I(y)\,dy\,dx_I\bigg| \\
&+ \bigg|\lambda(z)\sum_I \int_{B(z,3h)} \frac{1}{|z-y|^{n-\alpha}} u_I(y)\,dy\,dx_I\bigg| \\
&+ \bigg|\sum_I \int_{\mathbb{R}^n\setminus B(x,3h)} \big(\lambda(x)-\lambda(z)\big)|z-y|^{\alpha-n} u_I(y)\,dy\,dx_I\bigg| \\
&+ \bigg|\lambda(x)\sum_I \int_{\mathbb{R}^n\setminus B(x,2h)} \Big||x-y|^{\alpha-n} - |z-y|^{\alpha-n}\Big| u_I(y)\,dy\,dx_I\bigg| \\
={}& \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.
\end{aligned}
$$

We first estimate $\mathcal{S}_1$

$$
\begin{aligned}
\mathcal{S}_1 &= \bigg|\lambda(x)\sum_I \int_{B(x,3h)} \frac{1}{|x-y|^{n-\alpha}} u_I(y)\,dy\,dx_I\bigg| \\
&\le \int_{B(x,3h)} \frac{|\lambda(x)-\lambda(y)|}{|x-y|^{n-\alpha}} \Big(\sum_I \big|u_I(y)\big|^2\Big)^{1/2} dy \\
&\quad + \int_{B(x,3h)} \frac{\lambda(y)}{|x-y|^{n-\alpha}} \Big(\sum_I \big|u_I(y)\big|^2\Big)^{1/2} dy \\
&= \mathcal{S}_{11} + \mathcal{S}_{12}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{S}_{11} &\lesssim \int_0^{3h} t^{\alpha-n} \int_{B(x,t)} \big|\lambda(x)-\lambda(y)\big| \Big(\sum_I \big|u_I(y)\big|^2\Big)^{1/2} dy \frac{dt}{t} \\
&\lesssim \int_0^{3h} t^{\alpha-\tau} \big\|\lambda(x)-\lambda(y)\big\|_{L^{p,\tau}(B(x,t))} \frac{dt}{t} \\
&\lesssim h^{\alpha+\kappa-\tau},
\end{aligned}
$$

where $\alpha+\kappa-\tau>0$. For $\mathcal{S}_{12}$,

$$
\begin{aligned}
\mathcal{S}_{12} &\lesssim \int_0^{3h} t^{\alpha} \bigg(\frac{1}{|B(x,t)|} \int_{B(x,t)} \lambda(y) \Big(\sum_I \big|u_I(y)\big|^2\Big)^{1/2} dy\bigg) \frac{dt}{t} \\
&\lesssim h^{\alpha-\tau/q-n/p},
\end{aligned}
$$

where $\alpha-\tau/q-n/p>0$. Similar to $\mathcal{S}_1$, we omit the calculation of $\mathcal{S}_2$. For $\mathcal{S}_3$, we have

$$
\mathcal{S}_3 \lesssim \big|\lambda(x)-\lambda(z)\big| \int_{\mathbb{R}^n\setminus B(x,3h)} |z-y|^{\alpha-n} \Big(\sum_I \big|u_I(y)\big|^2\Big)^{1/2} dy
$$

$$\begin{aligned}
&\lesssim \left|\lambda(x)-\lambda(z)\right| \int_{\mathbb{R}^n\setminus B(z,2h)} |z-y|^{\alpha-n}\left(\sum_I \left|u_I(y)\right|^2\right)^{1/2} dy \\
&\lesssim \left|\lambda(x)-\lambda(z)\right| \int_{2h}^{\infty} t^{\alpha-n}\left(\sum_I \left|u_I(y)\right|^2\right)^{1/2} dy\frac{dt}{t} \\
&\lesssim h^{\alpha-\tau/p-n/q}\left|\lambda(x)-\lambda(z)\right|
\end{aligned}$$

since $\alpha-\tau/p-n/q<0$. Next, we estimate $\mathcal{S}_4$:

$$\begin{aligned}
\mathcal{S}_4 &\le h\left|\lambda(x)\sum_I \int_{\mathbb{R}^n\setminus B(x,3h)} |x-y|^{\alpha-n-1}u_I(y)\,dy\,dx_I\right| \\
&\lesssim h\int_{\mathbb{R}^n\setminus B(x,3h)} \frac{|\lambda(x)-\lambda(y)|}{|x-y|^{n-\alpha+1}}\left(\sum_I \left|u_I(y)\right|^2\right)^{1/2} dy \\
&\quad + h\int_{\mathbb{R}^n\setminus B(x,3h)} \frac{\lambda(y)}{|x-y|^{n-\alpha+1}}\left(\sum_I \left|u_I(y)\right|^2\right)^{1/2} dy \\
&= \mathcal{S}_{41}+\mathcal{S}_{42}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{S}_{41} &\lesssim h\int_{3h}^{\infty} t^{\alpha-n-1}\int_{B(x,t)} \left|\lambda(x)-\lambda(y)\right|\left(\sum_I \left|u_I(y)\right|^2\right)^{1/2} dy\frac{dt}{t} \\
&\lesssim h\int_{3h}^{\infty} t^{\alpha-\tau-1}\left\|\lambda(x)-\lambda(y)\right\|_{L^{p,\tau}(B(x,t))}\frac{dt}{t} \\
&\lesssim h^{\alpha+\kappa-\tau}
\end{aligned}$$

for $\alpha+\kappa-\tau-1<0$ and

$$\begin{aligned}
\mathcal{S}_{42} &\lesssim h\int_{3h}^{\infty} t^{\alpha-n-1}\int_{B(x,t)} \lambda(y)\left(\sum_I \left|u_I(y)\right|^2\right)^{1/2} dy\frac{dt}{t} \\
&\lesssim h^{\alpha-\tau/q-n/p}
\end{aligned}$$

for $\alpha-\tau/p-n/q-1<0$.

Let $B(x_0,r_0)\supset\Omega$ centered at x_0 and radius of r_0. Then r_0 is a finite constant independent of x. Combining all the above facts, we deduce that

$$\begin{aligned}
&\left\|\lambda(x)I_\alpha u(x)-\lambda(z)I_\alpha u(z)\right\|_{L^{p,\tau}(\Omega,\wedge^l)} \\
&\quad\le \sum_{i=1}^{4}\|\mathcal{S}_i\|_{L^{p,\tau}(\Omega,\wedge^l)} \\
&\quad= \sum_{i=1}^{4}\sup_{x_0\in\Omega,r>0}\|\mathcal{S}_i\|_{L^{p}(B(x_0,r),\wedge^l)} \le C\left\{h^{\alpha+\kappa-\tau}+h^{\alpha-\tau/q-n/p}+h^{\alpha+\kappa-\tau/p-n/q}\right\},
\end{aligned}$$

where C is a constant independent of x and z. The proof of Theorem 4.1 has been completed. □

The inequality for composite operator with Morrey norms on differential forms is as follows.

Theorem 4.2 *Let $u \in \Lambda_\kappa^{p,\tau}(\Omega, \wedge^l)$ be a differential l-form and $du \in \Lambda_\kappa^{p,\tau}(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$, $h = |x-z|$, let $T : L^p(\Omega, \wedge^l) \to L^p(\Omega, \wedge^{l-1})$ be a homotopy operator and $\lambda(\cdot) \in \Lambda^\tau$. Then there exists a constant C, independent of u, such that*

$$\begin{aligned} &\big\| \lambda(x) I_\alpha\big(u(x) - u(x)_B\big) - \lambda(z) I_\alpha\big(u(z) - u(z)_B\big)\big\|_{L^{p,\tau}(\Omega,\wedge^l)} \\ &\quad \le C\big\{h^{\alpha+\kappa-\tau} + h^{\alpha-\tau/q-n/p} + h^{\alpha+\kappa-\tau/p-n/q}\big\}, \end{aligned} \tag{4.1}$$

where B is any ball in Ω.

Proof Assume that $l \ge 1$, arbitrary $x_0 \in \Omega$, and $B(x_0, r) \subset \Omega$ is any ball centered at x_0 and radius of r. Then we have

$$\begin{aligned} &\big\| \lambda(x) I_\alpha\big(u(x) - u(x)_B\big) - \lambda(z) I_\alpha\big(u(z) - u(z)_B\big)\big\|_{L^{p,\tau}(\Omega,\wedge^l)} \\ &\quad \lesssim \sup_{r>0} r^{(\tau-n)/p} \big\| \lambda(x) I_\alpha\big(u(x) - u(x)_B\big) - \lambda(z) I_\alpha\big(u(z) - u(z)_B\big)\big\|_{L^p(B(x_0,r),\wedge^l)}. \end{aligned}$$

$|\lambda(x) I_\alpha(du(x)) - \lambda(z) I_\alpha(du(z))|$ is expressed by

$$\begin{aligned} &\big|\lambda(x) I_\alpha\big(du(x)\big) - \lambda(z) I_\alpha\big(du(z)\big)\big| \\ &\quad = \Bigg| \lambda(x) \sum_k^n \sum_{1\le i_1<\cdots<i_l\le n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \left(\frac{\partial u_I(y)}{\partial x_k}\right) dy\, dx_k \wedge dx_I \\ &\qquad - \lambda(z) \sum_k^n \sum_{1\le i_1<\cdots<i_l\le n} \int_{\mathbb{R}^n} \frac{1}{|z-y|^{n-\alpha}} \left(\frac{\partial u_I(y)}{\partial x_k}\right) dy\, dx_k \wedge dx_I \Bigg|. \end{aligned}$$

Assume that each $\frac{\partial u_I(y)}{\partial x_k} = v_I(x)$ is a coefficient function of differential $(l+1)$ form du. By using Theorem 4.1, we obtain (4.1) holds. □

5 Homotopy operator and *A*-harmonic equations

As applications for homotopy operator on differential forms, we study the solutions for nonhomogeneous A-harmonic equation on differential forms

$$A(x, g + du) = h + d^\star v, \tag{5.1}$$

where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and g, h are differential l-forms on Ω. To study the property of (5.1), we need the following conditions for operator A:

$$\big|A(x, \zeta)\big| \le a|\zeta|^{p-1}$$

and

$$\big\langle A(x, \zeta), \zeta\big\rangle \le |\zeta|^p,$$

where arbitrary $x \in \Omega$, $\zeta \in \wedge^l(\mathbb{R}^n)$ and a is a positive constant. In the present paper, we choose $g = h = 0$ in (5.1), then we get the nonhomogeneous conjugate A-harmonic equation

$$A(x, du) = d^\star v. \tag{5.2}$$

If u, v is a pair of solutions for equation (5.2), then we say u and v are a pair of conjugate A-harmonic tensors. Furthermore, u was called homogeneous A-harmonic tensor if u is a solution for homogeneous A-harmonic equation

$$d^\star A(x, du) = 0. \tag{5.3}$$

The following lemma appeared in [24].

Lemma 5.1 *Let u and v be a pair of solutions for equation (5.1) on Ω. If $g \in L^p(B, \wedge^l)$ and $h \in L^q(B, \wedge^l)$, then $du \in L^p(B, \wedge^l)$ if and only if $d^\star v \in L^q(B, \wedge^l)$. Moreover, we have the following inequalities*:

$$\begin{aligned} \left\| d^\star v \right\|^q_{L^q(B, \wedge^{l+1})} &\le C_1 \left\{ \|h\|^q_{L^q(B, \wedge^l)} + \|g\|^p_{L^p(B, \wedge^l)} + \|du\|^p_{L^p(B, \wedge^{l+1})} \right\}, \\ \|du\|^p_{L^p(B, \wedge^{l+1})} &\le C_2 \left\{ \|h\|^q_{L^q(B, \wedge^l)} + \|g\|^p_{L^p(B, \wedge^l)} + \left\| d^\star v \right\|^q_{L^q(B, \wedge^{l+1})} \right\} \end{aligned}$$

for C_1 and C_2 are two constants independent of u and v.

Now, we show the Caccioppoli-type estimate for homotopy operator with $L^{p,\tau}(\Omega, \wedge^l)$-norms.

Theorem 5.2 *Assume that $1 < \tau, \upsilon < n$ and $1 < p < \infty$. Let u and v be a pair of conjugate A-harmonic tensors on Ω and $T : L^p(\Omega, \wedge^l) \to L^p(\Omega, \wedge^{l-1})$ be a homotopy operator. Then there exist a constant $C > 0$, independent of u and v, such that*

$$\left\| T d^\star v \right\|^q_{L^{q,\tau}(\Omega, \wedge^l)} \le C \|u - c\|^p_{L^{p,\upsilon}(\Omega, \wedge^l)},$$

where c is any closed form on Ω and $\upsilon = \tau - p + q$.

Proof From Theorem 2.9 in [29], we have the following Caccioppoli inequality in L^p-spaces:

$$\|du\|_{L^p(B, \wedge^{l+1})} \le C|B|^{-1/n} \|u - c\|_{L^p(\sigma B, \wedge^l)}$$

for a homogeneous A-harmonic tensor u, B with σB in Ω and all closed forms c. Since u and v are a pair of conjugate A-harmonic tensors on Ω with $g = h = 0$, by using Lemma 5.1, we have

$$\left\| d^\star v \right\|^q_{L^q(B, \wedge^{l+1})} \le C \|du\|^p_{L^p(B, \wedge^{l+1})}.$$

Hence

$$\left\| T d^\star v \right\|^q_{L^q(B, \wedge^l)} \le C|B|^{q/n} \left\| d^\star v \right\|^p_{L^p(B, \wedge^l)} \le C|B|^{-p/n} \|u - c\|^p_{L^p(\sigma B, \wedge^l)}.$$

Clearly, $d^\star = (-1)^{nl+1} \star d \star u$ and $\star : \wedge^l \to \wedge^{n-l}$ is an isometric isomorphism mapping. Combining all the facts, we obtain

$$\begin{aligned}\|Td^\star v\|^q_{L^{q,\tau}(\Omega,\wedge^l)} &= \sup_{r>0} r^{\tau-n} \|Td^\star v\|^q_{L^q(B,\wedge^l)} \\ &\lesssim \sup_{r>0} r^{\tau-n} r^{q-p} \|u-c\|^p_{L^p(\sigma B,\wedge^l)} \\ &\lesssim \|u-c\|^p_{L^{p,\upsilon}(\Omega,\wedge^l)}\end{aligned}$$

for all B with $\sigma B \subset \Omega$ and $\upsilon = \tau - p + q$. The proof of Theorem 5.2 has been completed. □

Next, we give the Poincaré-type estimate for homotopy operator with $L^{p,\tau}(\Omega,\wedge^l)$-norms.

Theorem 5.3 *Assume that* $1 < \tau, \upsilon < n$ *and* $1 < p < \infty$. *Let* u *and* v *be a pair of conjugate* A*-harmonic tensors on* Ω *and* $T : L^p(\Omega,\wedge^l) \to L^p(\Omega,\wedge^{l-1})$ *be a homotopy operator. Then there exists a constant* $C > 0$, *independent of* u *and* v, *such that*

$$\|T(u-u_B)\|^p_{L^{p,\upsilon}(\Omega,\wedge^{l-1})} \le C\|d^\star v\|^q_{L^{q,\tau}(\Omega,\wedge^{l+1})},$$

where c *is any closed form on* Ω *and* $\upsilon = \tau + p$.

Proof Let u and v be a pair of conjugate A-harmonic tensors on Ω and $g = h = 0$. Using the second relation of Lemma 5.1, we have

$$\|du\|^p_{L^p(B,\wedge^{l+1})} \le C\|d^\star v\|^q_{L^q(B,\wedge^{l+1})}.$$

By using Lemma 3.1, we have

$$\|T(u-u_B)\|_{L^{p,\upsilon}(\Omega,\wedge^{l-1})} \lesssim \sup_{r>0} r^{(\upsilon-n)/p}|B|^{1/n}\|du\|_{L^p(B,\wedge^{l+1})} = \|du\|_{L^{p,\tau}(\Omega,\wedge^{l+1})}$$

for $\tau = \upsilon + p$ and any ball B in Ω. Collecting these facts, we obtain

$$\|T(u-u_B)\|^p_{L^{p,\upsilon}(\Omega,\wedge^{l-1})} \le C\|d^\star v\|^q_{L^{q,\tau}(\Omega,\wedge^{l+1})}.$$

The proof of Theorem 5.3 has been completed. □

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All authors worked on the manuscript.

Author details
[1]Institute of Cold Regions Science and Engineering, Northeast Forestry University, Harbin, 150040, P.R. China.
[2]Department of Mathematics, Northeast Forestry University, Harbin, 150040, P.R. China.

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