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Approximation of G-variational inequality problems and fixed-point problems of G-κ-strictly pseudocontractive mappings by an intermixed method endowed with a graph

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Abstract

In this paper, we first study G-κ-strictly pseudocontractive mappings and we establish a strong convergence theorem for finding the fixed points of two G-κ-strictly pseudocontractive mappings, two G-nonexpansive mappings, and two G-variational inequality problems in a Hilbert space endowed with a directed graph without the Property G. Moreover, we prove an interesting result involving the set of fixed points of a G-κ-strictly pseudocontractive and G-variational inequality problem and if Λ is a G-κ-strictly pseudocontractive mapping, then $I - \Lambda$ is a $G - \frac{(1-\kappa)}{2}$-inverse strongly monotone mapping, shown in Lemma 3.3. In support of our main result, some examples are also presented.

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Keywords: G-κ-strictly pseudocontractive; G-variational inequality; G-inverse strongly monotone; G-nonexpansive mappings

1 Introduction

Let H be a real Hilbert space. Let ζ be a nonempty, closed, and convex subset of H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let $\mathfrak{T}$ be a self-mapping of ζ. We use $F(\mathfrak{T})$ to denote the set of fixed points of $\mathfrak{T}$ (i.e., $F(\mathfrak{T}) = \{x \in \zeta : \mathfrak{T}x = x\}$). The iterative schemes to approximate the fixed points of nonlinear mappings have a long history and have been studied intensively by many researchers [10, 11, 17, 23]. A mapping $\mathfrak{T}$ is called κ-strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|\mathfrak{T}x - \mathfrak{T}y\|^2 \leq \|x - y\|^2 + \kappa \left\|(I - \mathfrak{T})x - (I - \mathfrak{T})y\right\|^2, \quad \forall x, y \in \zeta. \tag{1}$$

If $\kappa = 1$, a mapping $\mathfrak{T}$ is called a pseudocontractive mapping.

Note that the class of κ-strictly pseudocontractive strictly includes the class of nonexpansive mappings that are self-mappings $\mathfrak{T}$ on ζ such that

$$\|\mathfrak{T}x - \mathfrak{T}y\| \leq \|x - y\|, \quad \forall x, y \in \zeta. \tag{2}$$

Strictly pseudocontractive was first introduced by Browder and Petryshyn [8] in 1967. It is well known that strictly pseudocontractive is more general than nonexpansive mappings and they have a wider range of applications. Therefore, it is important to develop the theory of iterative methods for strictly pseudocontractive. Many authors have proposed iterative algorithms and proved the strong convergence theorems for a nonexpansive mapping and a κ-strictly pseudocontractive mapping in Hilbert space to find their fixed points, see, for example, [1, 4, 10, 13, 15, 18, 29].

To prove the strong convergence of iterations determined by nonexpansive mapping, Moudafi [19] established a theorem for finding the fixed points of nonexpansive mappings. More precisely, he established the following result, known as the *viscosity approximation method*. To construct an iterative algorithm such that it converges strongly to the fixed points of a finite family of strictly pseudocontractive by using the concept of the viscosity approximation method, Yao et al. [31] proposed the intermixed algorithm for two strictly pseudocontractive mappings as follows:

Algorithm 1 For arbitrarily given $x_0 \in \zeta$, $y_0 \in \zeta$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_\zeta[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n], & n \geq 0, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_\zeta[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n], & n \geq 0, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of real numbers in (0,1), $T,S:\zeta \to \zeta$ are strictly λ-pseudocontractive mappings, $f:\zeta \to \mathcal{H}$ is a ρ_1-contraction, $g:\zeta \to \mathcal{H}$ is a ρ_2-contraction, and $k \in (0, 1-\lambda)$ is a constant.

We now move onto some basics and definitions in graph theory. Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of a graph and E(G) is a set of its edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of the edges. That is,

$$E(G^{-1}) = \{(x,y) : (x,y) \in E(G)\}.$$

The following basic definitions of domination in graphs are needed to prove the main theorem. Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if every $z \in V(G) \setminus X$ there exists $x \in X$ such that $(x,z) \in E(G)$ and we say that x dominates z or z is dominated by x. Let $z \in V$, a set $X \subseteq V$ is dominated by z if $(z,x) \in E(G)$ for any $x \in X$ and we say that X dominates z if $(x,z) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops.

In 2008, by combining the notions in fixed-point theory and graph theory, Jachymski [12] generalized the Banach contraction principle in a complete metric space endowed with a directed graph. He also introduced a contractive-type mapping with a directed graph as follows:

Definition 1.1 Let (X,d) be a metric space and $G = (V(G), E(G))$ be a directed graph, where $V(G) = X$ and $E(G)$ contains all loops, that is $\Delta \subseteq E(G)$. We say that a mapping

$\mathbf{f}: X \to X$ is a Banach *G-contraction* if $\mathbf{f}$ preserves the edges of G, i.e.,

$$\text{for any } x, y \in X \text{ such that } (x, y) \in E(G) \text{ implies } (\mathbf{f}x, \mathbf{f}y) \in E(G)$$

and there exists $k \in (0, 1)$ such that

$$d(\mathbf{f}x, \mathbf{f}y) \leq kd(x, y) \quad \text{for all } x, y \in X \text{with } (x, y) \in E(G).$$

Definition 1.2 Let ζ be a nonempty convex subset of a Banach space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = \zeta$ and $\mathcal{T}: \zeta \to \zeta$, then $\mathcal{T}$ is said to be a *G-nonexpansive* mapping if the following conditions hold:

(i) $\mathcal{T}$ is edge preserving, i.e., for any $x, y \in \zeta$ such that $(x, y) \in E(G) \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in E(G)$;
(ii) $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$, where $(x, y) \in E(G)$ for any $x, y \in \zeta$.

This mapping was introduced by Tiammee et al. [25].

The variational inequality problem (VIP) is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0,$$

for all $v \in C$. The set of all solutions of the variational inequality is denoted by $VI(C, A)$. Historically, the variational inequality was introduced by Stampachhia [22]. Since then, variational inequalities have been used in various topics such as physic, optimization, and applied sciences, see, for example, [3, 20, 30].

Recently, in 2019, Kangtunyakarn [14] introduced *G-variational inequality* problems and *G-α-inverse strongly monotone mappings* as follows:

Definition 1.3 Let ζ be a nonempty, convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$. The *G-variational inequality* problem is to find a point $x^* \in \zeta$ such that

$$\left\langle y - x^*, Ax^* \right\rangle \geq 0, \tag{4}$$

for all $y \in \zeta$ with $(x^*, y) \in E(G)$ and $A: \zeta \to H$ is a mapping. The set of all solutions of (4) is denoted by $G - VI(\zeta, A)$.

Definition 1.4 Let ζ be a nonempty, convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$. The mapping $A: \zeta \to H$ is said to be *G-α-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in \zeta$ with $(x, y) \in E(G)$.

Let ζ be a nonempty, convex subset of a Banach space X and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = \zeta$. Then, ζ is said to have *Property G* [25] if every sequence $\{a_n\}$ in ζ converges weakly to $x \in \zeta$, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $(a_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. During the course of this research, when investigating the literature on research methods, it was found that many researchers were using the Property G to prove the strong convergence theorems, see for example [14, 16, 26, 28].

In this paper, we will use some suitable conditions instead of the Property G.

Question 1 Can we prove a strong convergence theorem for finding the fixed points of two $G-\kappa$-strictly pseudocontractive mappings, two G-nonexpansive mappings, and two G-variational inequality problems in a Hilbert space endowed with a directed graph without the Property G?

By using the concept of (3), we introduce a new iterative method as follows:

Algorithm 2 Starting with $x_1, y_1 \in \zeta$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_\zeta(I-\lambda_1\mathcal{S}_1)x_n + \mu_n P_\zeta(\alpha_n f(y_n) + (1-\alpha_n)\mathcal{B}_1 x_n), \\ y_{n+1} = \delta_n y_n + \eta_n P_\zeta(I-\lambda_2\mathcal{S}_2)y_n + \mu_n P_\zeta(\alpha_n g(x_n) + (1-\alpha_n)\mathcal{B}_2 y_n). \end{cases}$$

By putting $\mathcal{S}_1 = \mathcal{S}_2 = 0$, we obtain

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n x_n + \mu_n P_\zeta(\alpha_n f(y_n) + (1-\alpha_n)\mathcal{B}_1 x_n), \\ y_{n+1} = \delta_n y_n + \eta_n y_n + \mu_n P_\zeta(\alpha_n g(x_n) + (1-\alpha_n)\mathcal{B}_2 y_n), \end{cases}$$

which is a modified version of [31].

To answer Question 1, we prove a strong convergence theorem for solving fixed-point problems in a Hilbert space endowed with a directed graph by using Algorithm 2, where $\mathcal{S}_1, \mathcal{S}_2 : \zeta \to H$ are the $G-\alpha$-inverse strongly monotone mappings, $f, g : H \to H$ are G-contraction mappings, and $\mathcal{B}_1, \mathcal{B}_2 : \zeta \to \zeta$ are G-nonexpansive mappings with some extra conditions in Theorem 3.2.

Inspired by the concept above, we introduce the definition of a $G-\kappa$-strictly pseudocontractive mapping that is different from κ-strictly pseudocontractive, see Example 2.5, and prove a strong convergence theorem for finding the fixed points of two G-κ-strictly pseudocontractive mappings, two G-nonexpansive mappings, and two G-variational inequality problems in a Hilbert space endowed with a directed graph without the Property G. Moreover, we prove an interesting Lemma involving the set of fixed points of a G-κ-strictly pseudocontractive and G-variational inequality problem and if Λ is a G-κ-strictly pseudocontractive mappings, then $I-\Lambda$ is a $G-\frac{(1-\kappa)}{2}$-inverse strongly monotone mapping, shown in Lemma 3.3. Finally, we give some examples for the main theorem.

2 Preliminaries

In this paper, we denote the weak convergence and the strong convergence by "$\rightharpoonup$" and "$\to$", respectively. For every $x \in H$, there exists a unique nearest point $P_\zeta x$ in ζ such that $\|x - P_\zeta x\| \leq \|x - y\|$ for all $y \in \zeta$. P_ζ is called the metric projection of H onto ζ. Furthermore, P_ζ is a firmly nonexpansive mapping of H onto ζ, i.e.,

$$\|P_\zeta x - P_\zeta y\|^2 \leq \langle P_\zeta x - P_\zeta y, x - y\rangle, \quad \forall x, y \in H.$$

In a real Hilbert space H, it is well known that H satisfies *Opial's condition* [21], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

The following Definitions and Lemmas are needed to prove the main theorem.

Lemma 2.1 ([27]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2 ([24]) *For a given $z \in H$ and $u \in \zeta$,*

$$u = P_\zeta z \quad \Leftrightarrow \quad \langle u - z, v - u\rangle \geq 0, \quad \forall v \in \zeta.$$

Definition 2.3 ([25]) Let $G = (V(G), E(G))$ be a directed graph. A graph G is called *transitive* if for any $x, y, z \in V(G)$ with (x,y) and (y,z) are in $E(G)$, then $(x,z) \in E(G)$.

Next, we introduce the definition of a $G - \kappa$*-strictly pseudocontractive* mapping.

Definition 2.4 Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$ and $\Lambda : \zeta \to \zeta$. Then, Λ is said to be $G - \kappa$*-strictly pseudocontractive* if there exists a constant $\kappa \in [0,1)$ and the following conditions hold;

(i) Λ is edge preserving, i.e., for any $x, y \in \zeta$ such that $(x,y) \in E(G) \Rightarrow (\Lambda x, \Lambda y) \in E(G)$;

(ii) $\|\Lambda x - \Lambda y\|^2 \leq \|x-y\|^2 + \kappa \|(I-\Lambda)x - (I-\Lambda)y\|^2$, where $(x,y) \in E(G)$ for any $x, y \in \zeta$.

Example 2.5 Let $\mathbb{R}$ be the set of real numbers, $\zeta = [-10, 10]$ with $G = (\zeta, E(G))$ and $E(G) = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x, y \in [0,10]\}$. Define the mapping $\Lambda : \zeta \to \zeta$ by

$$\Lambda x = \begin{cases} \max(-x, x); & x \in (-10, 10], \\ 0; & x = -10, \end{cases}$$

for all $x \in \zeta$ with

$$\max(a,b) = \frac{1}{2}\big(a + b + |a-b|\big).$$

Then, Λ is a G-$(\frac{1}{2})$-strictly pseudocontractive mapping, but Λ is not a $\frac{1}{2}$-strictly pseudocontractive mapping.

Solution. Let $x, y \in \zeta$ and $(x,y) \in E(G)$. Then, we have $x, y \in [0, 10]$.

It follows that $\Lambda x, \Lambda y \in [0, 10]$. Thus,

$$(\Lambda x, \Lambda y) \in E(G).$$

From the definition of Λ, we have

$$\begin{aligned}
\big\langle (I-\Lambda)x-(I-\Lambda)y, x-y\big\rangle &= \big\langle x-\max(-x,x)-y+\max(-y,y), x-y\big\rangle \\
&= \bigg\langle x-y-\frac{1}{2}\big(-x+x+|-x-x|\big) \\
&\quad + \frac{1}{2}\big(-y+y+|-y-y|\big), x-y\bigg\rangle \\
&= \big\langle x-y+\big(|y|-|x|\big), x-y\big\rangle \\
&= \big[x-y+\big(|y|-|x|\big)\big](x-y)
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\big|(I-\Lambda)x-(I-\Lambda)y\big|^2 &= \big|x-\max(-x,x)-y+\max(-y,y)\big|^2 \\
&= \bigg|x-\frac{1}{2}\big(-x+x+|-x-x|\big)-y+\frac{1}{2}\big(-y+y+|-y-y|\big)\bigg|^2 \\
&= \big|(x-y)+\big(|y|-|x|\big)\big|^2 \\
&\le \big|(x-y)+\big(|y|-|x|\big)\big|\cdot\big\{|x-y|+\big|\big(|x|-|y|\big)\big|\big\} \\
&\le \big|(x-y)+\big(|y|-|x|\big)\big|\cdot\big\{|x-y|+|x-y|\big\} \\
&= 2\big|(x-y)+\big(|y|-|x|\big)\big|\cdot|x-y|.
\end{aligned} \tag{6}$$

From (5) and (6), we have

$$\begin{aligned}
\big\langle (I-\Lambda)x-(I-\Lambda)y, x-y\big\rangle &\ge \frac{1}{2}\big|(I-\Lambda)x-(I-\Lambda)y\big|^2 \\
&\ge \frac{1-\frac{1}{2}}{2}\big|(I-\Lambda)x-(I-\Lambda)y\big|^2,
\end{aligned} \tag{7}$$

for all $x, y \in \zeta$ with $(x, y) \in E(G)$.

From the definition of Λ and (7), we have

$$\begin{aligned}
|\Lambda x-\Lambda y|^2 &= \big|(I-I+\Lambda)x-(I-I+\Lambda)y\big|^2 \\
&= \big|x-(I-\Lambda)x-y+(I-\Lambda)y\big|^2 \\
&= \big|(x-y)-\big[(I-\Lambda)x-(I-\Lambda)y\big]\big|^2 \\
&= \big|(x-y)\big|^2-2\big\langle x-y,(I-\Lambda)x-(I-\Lambda)y\big\rangle+\big[(I-\Lambda)x-(I-\Lambda)y\big]^2 \\
&\le \big|(x-y)\big|^2-2\bigg[\frac{1-\frac{1}{2}}{2}\big|(I-\Lambda)x-(I-\Lambda)y\big|^2\bigg]+\big[(I-\Lambda)x-(I-\Lambda)y\big]^2 \\
&= \big|(x-y)\big|^2+\frac{1}{2}\big[(I-\Lambda)x-(I-\Lambda)y\big]^2.
\end{aligned}$$

Hence, we obtain that Λ is a G-$(\frac{1}{2})$-strictly pseudocontractive mapping.

Next, we will show that Λ is not a $\frac{1}{2}$-strictly pseudocontractive mapping. Choose $x = -9$ and $y = -10$. Then,

$$\left|\Lambda(-9) - \Lambda(-10)\right|^2 = |9 - 0|^2 = 81 > 33 = |-9 + 10|^2 + \frac{1}{2}\left|-9 - \Lambda(-9) + 10 + \Lambda(-10)\right|^2.$$

Hence, Λ is not a $\frac{1}{2}$-strictly pseudocontractive mapping.

Example 2.6 Let $H = \mathbb{R}$, $\zeta = [-10, 10]$ with $G = (\zeta, E(G))$ and $E(G) = \{(x, y)|x \cdot y > 0\}$. Define the mapping $\bar{\Lambda} : \zeta \to \zeta$ by $\bar{\Lambda}_1 : [-10, 10] \to [-10, 10]$ defined by

$$\bar{\Lambda}_1 x = \operatorname{sgn}\bigl(\operatorname{sgn}(x)\bigr) = \begin{cases} 1; & x \in (0, 10], \\ 0; & x = 0, \\ -1; & x \in [-10, 0), \end{cases}$$

for all $x \in [-10, 10]$ with

$$\operatorname{sgn}(x) = \begin{cases} -1; & x < 0, \\ 0; & x = 0, \\ 1; & x > 0. \end{cases}$$

Then, Λ is a G-($\frac{1}{2}$)-strictly pseudocontractive mapping, but Λ is not a $\frac{1}{2}$-strictly pseudocontractive mapping.

Solution. Let $x, y \in \zeta$ and $(x, y) \in E(G)$. Then, we have $x, y > 0$.

Case I; $x, y > 0$, we have $\bar{\Lambda}x = \bar{\Lambda}y = 1$. Then, $\bar{\Lambda}x \cdot \bar{\Lambda}y > 0$.

Case II; $x, y < 0$, we have $\bar{\Lambda}x = \bar{\Lambda}y = -1$. Then, $\bar{\Lambda}x \cdot \bar{\Lambda}y > 0$.

Thus, $(\bar{\Lambda}x, \bar{\Lambda}y) \in E(G)$.

From case I; $x, y > 0$, then

$$\begin{aligned} |x - y|^2 + \frac{1}{2}\left|(I - \bar{\Lambda})x - (I - \bar{\Lambda})y\right|^2 &= |x - y|^2 + \frac{1}{2}|x - y - \bar{\Lambda}x + \bar{\Lambda}y|^2 \\ &= \frac{3}{2}|x - y|^2 \\ &\geq 0 \\ &= |\bar{\Lambda}x - \bar{\Lambda}y|^2. \end{aligned}$$

From case II; $x, y < 0$, by using the same technique as in Case I, we obtain that

$$|x - y|^2 + \frac{1}{2}\left|(I - \bar{\Lambda})x - (I - \bar{\Lambda})y\right|^2 \geq |\bar{\Lambda}x - \bar{\Lambda}y|^2.$$

Thus, $\bar{\Lambda}$ is a G-($\frac{1}{2}$)-strictly pseudocontractive mapping.

Next, we will show that $\bar{\Lambda}$ is not a $\frac{1}{2}$-strictly pseudocontractive mapping. Choose $x = 1$ and $y = \frac{-1}{5}$. Then,

$$|\bar{\Lambda}x - \bar{\Lambda}y|^2 = 4 \geq 1.76 = \left|1 + \frac{1}{5}\right|^2 + \frac{1}{2}\left|1 - \bar{\Lambda}(1) + \frac{1}{5} + \bar{\Lambda}\left(-\frac{1}{5}\right)\right|^2.$$

Hence, $\bar{\Lambda}$ is not a $\frac{1}{2}$-strictly pseudocontractive mapping.

Lemma 2.7 ([14]) *Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$. Let $E(G)$ be convex and G be transitive with $E(G) = E(G)^{-1}$ and let $A : \zeta \to H$ be a $G - \alpha$-inverse strongly monotone operator with $A^{-1}(0) \neq 0$. Then, $G - VI(\zeta, A) = A^{-1}(0) = F(P_\zeta(I - \lambda A))$, for all $\lambda > 0$.*

Lemma 2.8 ([14]) *Let H be a Hilbert space and ζ be a nonempty, closed, and convex subset of H with ζ having a property G. Let $G = (V(G), E(G))$ be a directed graph where $V(G) = \zeta$ and $E(G)$ is a convex set. Let $A : \zeta \to H$ be a $G - \alpha$-inverse strongly monotone mapping with $F(P_\zeta(I - \lambda A)) \times F(P_\zeta(I - \lambda A)) \subseteq E(G)$, for all $\lambda \in (0, 2\alpha)$. Then, $F(P_\zeta(I - \lambda A))$ is closed and convex.*

Lemma 2.9 ([25]) *Let $\mathcal{X}$ be a normed space and ζ be a subset of $\mathcal{X}$ having a property G. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = \zeta$ and $E(G)$ is convex. Suppose $\mathcal{T} : \zeta \to \zeta$ is a G-nonexpansive mapping and $F(\mathcal{T}) \times F(\mathcal{T}) \subseteq E(G)$. Then, $F(\mathcal{T})$ is closed and convex.*

Lemma 2.10 *Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$. Let $E(G)$ be convex and G be transitive with $E(G) = E(G)^{-1}$. Let $\mathcal{T} : \zeta \to \zeta$ be a G- nonexpansive mapping and $\Lambda : \zeta \to \zeta$ be a $G - \kappa$-strictly pseudocontractive mapping. Define a mapping $\mathcal{B} : \zeta \to \zeta$ by $\mathcal{B}x = \mathcal{T}((1-b)I + b\Lambda)x$ for all $x \in \zeta$ and $b \in (0, 1-k)$. Then, $\mathcal{B}$ is a G-nonexpansive mapping, for all $x, y \in \zeta$ with $(x, y) \in E(G)$.*

Proof Let $x, y \in \zeta$ and $(x, y) \in E(G)$.

Since Λ is edge preserving and $(x, y) \in E(G)$, then $(\Lambda x, \Lambda y) \in E(G)$.

As $(x, y), (\Lambda x, \Lambda y) \in E(G)$ and $E(G)$ is convex, we obtain

$$\bigl((1-b)x + b\Lambda x, (1-b)y + b\Lambda y\bigr) \in E(G).$$

Since $\mathcal{T}$ is edge preserving and $((1-b)x + b\Lambda x, (1-b)y + b\Lambda y) \in E(G)$, then

$$\bigl(\mathcal{T}\bigl((1-b)x + b\Lambda x\bigr), \mathcal{T}\bigl((1-b)y + b\Lambda y\bigr)\bigr) \in E(G).$$

Thus, $\mathcal{B}$ is edge preserving.

We have

$$\begin{aligned}
\|\mathcal{B}x - \mathcal{B}y\|^2 &= \bigl\|\mathcal{T}\bigl((1-b)x + b\Lambda x\bigr) - \mathcal{T}\bigl((1-b)y + b\Lambda y\bigr)\bigr\|^2 \\
&\leq \bigl\|(1-b)x + b\Lambda x - (1-b)y - b\Lambda y\bigr\|^2 \\
&= \bigl\|(1-b)(x-y) + b(\Lambda x - \Lambda y)\bigr\|^2 \\
&= (1-b)\|x-y\|^2 + b\|\Lambda x - \Lambda y\|^2 - b(1-b)\bigl\|(x-y) - (\Lambda x - \Lambda y)\bigr\|^2 \\
&\leq (1-b)\|x-y\|^2 + b\bigl\{\|x-y\|^2 + \kappa\bigl\|(I-\Lambda)x - (I-\Lambda)y\bigr\|^2\bigr\} \\
&\quad - b(1-b)\bigl\|(x-y) - (\Lambda x - \Lambda y)\bigr\|^2 \\
&\leq (1-b)\|x-y\|^2 + b\|x-y\|^2 + kb\bigl\|(x-y) - (\Lambda x - \Lambda y)\bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - b(1-b)\big\|(x-y)-(\Lambda x-\Lambda y)\big\|^2\\
&= \|x-y\|^2 + \big(kb-b(1-b)\big)\big\|(x-y)-(\Lambda x-\Lambda y)\big\|^2\\
&= \|x-y\|^2 + b\big(k-(1-b)\big)\big\|(x-y)-(\Lambda x-\Lambda y)\big\|^2\\
&\le \|x-y\|^2,
\end{aligned}
$$

for all $x,y\in\zeta$ with $(x,y)\in E(G)$.

This implies that $\|\mathcal{B}x-\mathcal{B}y\|\le\|x-y\|$.

Hence, $\mathcal{B}$ is a G-nonexpansive mapping. □

Lemma 2.11 *Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G=(V(G),E(G))$ be a directed graph with $\zeta=V(G)$. Let $E(G)$ be convex and G be transitive with $E(G)=E(G)^{-1}$. Let $\mathcal{T}:\zeta\to\zeta$ be a G- nonexpansive mapping and $\Lambda:\zeta\to\zeta$ be a $G-\kappa$-strictly pseudocontractivea mapping. Define a mapping $\mathcal{B}:\zeta\to\zeta$ by $\mathcal{B}x=\mathcal{T}((1-b)I+b\Lambda)x$ for all $x\in\zeta$, $b\in(0,1-k)$ and $F(\mathcal{B})\times(F(\mathcal{T})\cap F(\Lambda))\subseteq E(G)$. Then,*

$$F(\mathcal{B})=F(\mathcal{T})\cap F(\Lambda).$$

Proof It is obvious that $F(\mathcal{T})\cap F(\Lambda)\subseteq F(\mathcal{B})$. Next, we claim that $F(\mathcal{B})\subseteq F(\mathcal{T})\cap F(\Lambda)$. To show this, let $x_0\in F(\mathcal{B})$, $x^*\in F(\mathcal{T})\cap F(\Lambda)$, Then, we have $(x_0,x^*)\in E(G)$. Since Λ is edge preserving and $(x_0,x^*)\in E(G)$, then $(\Lambda x_0,x^*)\in E(G)$. Since $E(G)=E(G)^{-1}$ and $(\Lambda x_0,x^*)\in E(G)$, then we have $(x^*,\Lambda x_0)\in E(G)$. By the transitivity of G and $(x_0,x^*)\in E(G)$ and $(x^*,\Lambda x_0)\in E(G)$, then we have $(x_0,\Lambda x_0)\in E(G)$. As (x_0,x^*) and $(\Lambda x_0,x^*)$ are in $E(G)$ and $E(G)$ is convex, we obtain

$$\big((1-b)x_0+b\Lambda x_0,(1-b)x^*+bx^*\big)=\big((1-b)x_0+b\Lambda x_0,x^*\big)\in E(G).$$

Then, we have

$$
\begin{aligned}
\|x_0=x^*\|^2 &= \big\|\mathcal{T}\big((1-b)x_0+b\Lambda x_0\big)-\mathcal{T}x^*\big\|^2\\
&\le \big\|(1-b)x_0+b\Lambda x_0-x^*\big\|^2\\
&= \big\|(1-b)\big(x_0-x^*\big)+b\big(\Lambda x_0-x^*\big)\big\|^2\\
&= (1-b)\big\|x_0-x^*\big\|^2+b\big\|\Lambda x_0-x^*\big\|^2\\
&\quad - b(1-b)\big\|\big(x_0-x^*\big)-\big(\Lambda x_0-x^*\big)\big\|^2\\
&\le (1-b)\big\|x_0-x^*\big\|^2+b\big\{\big\|x_0-x^*\big\|^2+k\big\|(I-\Lambda)x_0-(I-\Lambda)x^*\big\|^2\big\}\\
&\quad - b(1-b)\|x_0-\Lambda x_0\|^2\\
&= \big\|x_0-x^*\big\|^2+bk\big\|(I-\Lambda)x_0-(I-\Lambda)x^*\big\|^2-b(1-b)\|x_0-\Lambda x_0\|^2\\
&= \big\|x_0-x^*\big\|^2+bk\|x_0-\Lambda x_0\|^2-b(1-b)\|x_0-\Lambda x_0\|^2\\
&= \big\|x_0-x^*\big\|^2-b\big((1-k)-b\big)\|x_0-\Lambda x_0\|^2,
\end{aligned}
\tag{8}
$$

for all $x_0,x^*\in\zeta$ with $(x_0,x^*)\in E(G)$.

From (8), this implies that

$$b\big((1-k)-b\big)\|x_0-\Lambda x_0\|^2 \le \|x_0=x^*\|^2-\|x_0=x^*\|^2=0,$$

for all $x_0, x^* \in \zeta$ with $(x_0, x^*) \in E(G)$.

Then, we have $\Lambda x_0 = x_0$.

That is, $x_0 \in F(\Lambda)$.

Since $x_0 \in F(\mathcal{B})$ from the definition of $\mathcal{B}$, we have

$$x_0 = \mathcal{B}x_0 = \mathcal{T}\big((1-b)x_0 + b\Lambda x_0\big) = \mathcal{T}x_0.$$

Then, we have $x_0 \in F(\mathcal{T})$, therefore $x_0 \in F(\mathcal{T}) \cap F(\Lambda)$.

It follows that $F(\mathcal{B}) \subseteq F(\mathcal{T}) \cap F(\Lambda)$.

Hence, $F(\mathcal{B}) = F(\mathcal{T}) \cap F(\Lambda)$. □

3 Main results

In this section, we prove a strong convergence theorem for solving the fixed-point problem of two G-nonexpansive mappings, two $G-\kappa_i$-strictly pseudocontractive mappings, and two G-variational inequality problems in Hilbert space endowed with a directed graph.

The following Proposition is needed to prove the main theorem.

Proposition 3.1 *Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$. Let $E(G)$ be convex and G be transitive with $E(G) = E(G)^{-1}$. For every $i = 1, 2$, let $\mathcal{S}_i : \zeta \to H$ be a $G-\alpha_i$-inverse strongly monotone mapping with*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_\zeta(I-\lambda_1\mathcal{S}_1)x_n + \mu_n P_\zeta(\alpha_n f(y_n) + (1-\alpha_n)\mathcal{B}_1 x_n), \\ y_{n+1} = \delta_n y_n + \eta_n P_\zeta(I-\lambda_2\mathcal{S}_2)y_n + \mu_n P_\zeta(\alpha_n g(x_n) + (1-\alpha_n)\mathcal{B}_2 y_n), \end{cases} \tag{9}$$

where (x_0, x_0) and (y_0, y_0) are in $E(G)$. If $\zeta = V(G)$ dominates x_0 and y_0, then (x_n, x_{n+1}), (y_n, y_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$.

Proof Since $\zeta = V(G)$ dominates x_0 and $P_\zeta(I-\lambda_1\mathcal{S}_1)x_0 \in \zeta$, then we have $(P_\zeta(I-\lambda_1\mathcal{S}_1)x_0, x_0) \in E(G)$. From $E(G) = E(G)^{-1}$ and $(P_\zeta(I-\lambda_1\mathcal{S}_1)x_0, x_0) \in E(G)$, we have $(x_0, P_\zeta(I-\lambda_1\mathcal{S}_1)x_0) \in E(G)$. Since $\zeta = V(G)$ dominates x_0 and $U_0 = P_\zeta(\alpha_0 f(y_0) + (1-\alpha_0)\mathcal{B}_1 x_0) \in \zeta$, then we have $(U_0, x_0) \in E(G)$. From $E(G) = E(G)^{-1}$ and $(U_0, x_0) \in E(G)$, we have $(x_0, U_0) \in E(G)$. Since (x_0, x_0), $(x_0, P_\zeta(I-\lambda_1\mathcal{S}_1)x_0)$ and (x_0, U_0) are in $E(G)$ and $E(G)$ is convex, we obtain

$$\big(\delta_0 x_0 + \eta_0 x_0 + \mu_0 x_0, \delta_0 x_0 + \eta_0 P_\zeta(I-\lambda_1\mathcal{S}_1)x_0 + \mu_0 U_0\big) = (x_0, x_1) \in E(G).$$

Since $\zeta = V(G)$ dominates x_0 and $P_\zeta(I-\lambda_1\mathcal{S}_1)x_1 \in \zeta$, then we have $(P_\zeta(I-\lambda_1\mathcal{S}_1)x_1, x_0) \in E(G)$. From $E(G) = E(G)^{-1}$ and $(P_\zeta(I-\lambda_1\mathcal{S}_1)x_1, x_0) \in E(G)$, we have $(x_0, P_\zeta(I-\lambda_1\mathcal{S}_1)x_1) \in E(G)$. Since $\zeta = V(G)$ dominates x_0 and $U_1 = P_\zeta(\alpha_1 f(y_1) + (1-\alpha_1)\mathcal{B}_1 x_1) \in \zeta$, then we have $(U_1, x_0) \in E(G)$. From $E(G) = E(G)^{-1}$ and $(U_1, x_0) \in E(G)$, we have $(x_0, U_1) \in E(G)$. Since (x_0, x_1), $(x_0, P_\zeta(I-\lambda_1\mathcal{S}_1)x_1)$ and (x_0, U_1) are in $E(G)$ and $E(G)$ is convex, we obtain

$$\big(\delta_0 x_0 + \eta_0 x_0 + \mu_0 x_0, \delta_0 x_1 + \eta_0 P_\zeta(I-\lambda_1\mathcal{S}_1)x_1 + \mu_0 U_1\big) = (x_0, x_2) \in E(G).$$

Since $\zeta = V(G)$ dominates x_0 and $P_\zeta(I-\lambda_1\mathcal{S}_1)x_k \in \zeta$, then we have $(P_\zeta(I-\lambda_1\mathcal{S}_1)x_k, x_0) \in E(G)$, for $k \in \mathbb{N}$. From $E(G) = E(G)^{-1}$ and $(P_\zeta(I-\lambda_1\mathcal{S}_1)x_k, x_0) \in E(G)$, we have $(x_0, P_\zeta(I-\lambda_1\mathcal{S}_1)x_k) \in E(G)$, for $k \in \mathbb{N}$. Since $\zeta = V(G)$ dominates x_0 and $U_k = P_\zeta(\alpha_k f(y_k) + (1-\alpha_k)\mathcal{B}_k x_k) \in \zeta$, then we have $(U_k, x_0) \in E(G)$, for $k \in \mathbb{N}$. From $E(G) = E(G)^{-1}$ and $(U_k, x_0) \in E(G)$, we have $(x_0, U_k) \in E(G)$, for $k \in \mathbb{N}$.

Next, Assume that $(x_0, x_k) \in E(G)$, for $k \in \mathbb{N}$. Since (x_0, x_k), $(x_0, P_\zeta(I-\lambda_1\mathcal{S}_1)x_k)$, and (x_0, U_k) are in $E(G)$ and $E(G)$ is convex, then we obtain $(\delta_0 x_0 + \eta_0 x_0 + \mu_0 x_0, \delta_0 x_k + \eta_0 P_\zeta(I-\lambda_1\mathcal{S}_1)x_k + \mu_0 U_k) = (x_0, x_{k+1}) \in E(G)$.

Hence, by induction, we have $(x_0, x_n) \in E(G)$. From $E(G) = E(G)^{-1}$ and $(x_0, x_n) \in E(G)$, we have $(x_n, x_0) \in E(G)$, for all $n \in \mathbb{N}$. By the transitivity of G and $(x_n, x_0), (x_0, x_{n+1}) \in E(G)$, we have

$$(x_n, x_{n+1}) \in E(G), \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

Applying the same arguments as for deriving (10), we also obtain

$$(y_n, y_{n+1}) \in E(G), \quad \text{for all } n \in \mathbb{N}.$$ □

Theorem 3.2 *Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$. Let $E(G)$ be convex and G be transitive with $E(G) = E(G)^{-1}$. For every $i = 1,2$, let $\mathcal{S}_i : \zeta \to H$ be a $G-\alpha_i$-inverse strongly monotone mapping with $\alpha = \min\{\alpha_1, \alpha_2\}$ and let $g, f : H \to H$ be an a_g and $a_f - G$-contraction mapping with $a = \max\{a_g, a_f\}$. For every $i = 1,2$, let $\mathcal{T}_i : \zeta \to \zeta$ be a G-nonexpansive mapping and $\Lambda_i : \zeta \to \zeta$ be a $G-\kappa_i$-strictly pseudocontractive mapping with $k = \max\{\kappa_1, \kappa_2\}$. Define a mapping $\mathcal{B} : \zeta \to \zeta$ by $\mathcal{B}_i x = \mathcal{T}_i((1-b)I + b\Lambda_i)x$ for all $x \in \zeta$ and $i = 1,2$, $b \in (0, 1-k)$. Assume that $\mathfrak{F}_i = F(\mathcal{B}_i) \cap G - VI(\zeta, \mathcal{S}_i) \neq \emptyset$ for all $i = 1,2$ with $F(\mathcal{B}_i) \times F(\mathcal{B}_i) \subseteq E(G)$, $G - VI(\zeta, \mathcal{S}_i) \times G - VI(\zeta, \mathcal{S}_i) \subseteq E(G)$ and $F(\mathcal{B}) \times (F(\mathcal{T}) \cap F(\Lambda)) \subseteq E(G)$ for all $i = 1,2$ and there exists $x_0, y_0 \in \zeta$ such that (x_0, x_0) and (y_0, y_0) are in $E(G)$ and $\zeta = V(G)$ dominates x_0 and y_0. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_0, y_0 \in \zeta$ and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_\zeta(I-\lambda_1\mathcal{S}_1)x_n + \mu_n P_\zeta(\alpha_n f(y_n) + (1-\alpha_n)\mathcal{B}_1 x_n), \\ y_{n+1} = \delta_n y_n + \eta_n P_\zeta(I-\lambda_2\mathcal{S}_2)y_n + \mu_n P_\zeta(\alpha_n g(x_n) + (1-\alpha_n)\mathcal{B}_2 y_n), \end{cases} \tag{11}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \eta_n + \mu_n = 1$ and $\lambda \in (0, 2\alpha)$ with $\lambda = \min\{\lambda_1, \lambda_2\}$. Assume the following conditions hold;

(i) $0 < \xi \le \delta_n, \eta_n, \mu_n \le \bar{\xi}$ *for all $n \in N$ and for some $\xi, \bar{\xi} > 0$;*
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

If $\{x_n\}$ dominates $P_{\mathfrak{F}_1} f(y_0)$ and $\{y_n\}$ dominates $P_{\mathfrak{F}_2} g(x_0)$, then $\{x_n\}$ converges strongly to $x^* = P_{\mathfrak{F}_1} f(y_0)$ and $\{y_n\}$ converges strongly to $y^* = P_{\mathfrak{F}_2} g(x_0)$, where $P_{\mathfrak{F}_i}$ is a metric projection on $\mathfrak{F}_i$, for all $i = 1,2$.

Proof The proof of this theorem will be divided into five steps.

Step 1: We will show that $\{x_n\}$ is bounded.

First, we will prove that

$$\left\|P_\zeta(I-\lambda\mathcal{S})x-P_\zeta(I-\lambda\mathcal{S})y\right\|\leq\|x-y\|, \tag{12}$$

for all $x,y\in\zeta$ and $(x,y)\in E(G)$.

Letting $x,y\in\zeta$ and $(x,y)\in E(G)$, we have

$$\begin{aligned}\left\|P_\zeta(I-\lambda\mathcal{S})x-P_\zeta(I-\lambda\mathcal{S})y\right\|^2&\leq\left\|x-y-\lambda(\mathcal{S}x-\mathcal{S}y)\right\|^2\\&=\|x-y\|^2-2\lambda\langle x-y,\mathcal{S}y-\mathcal{S}y\rangle\\&\quad+\lambda^2\|\mathcal{S}x-\mathcal{S}y\|^2\\&\leq\|x-y\|^2-2\alpha\lambda\|\mathcal{S}x-\mathcal{S}y\|^2\\&\quad+\lambda^2\|\mathcal{S}x-\mathcal{S}y\|^2\\&=\|x-y\|^2-\lambda(2\alpha-\lambda)\|\mathcal{S}x-\mathcal{S}y\|^2\\&\leq\|x-y\|^2,\end{aligned}$$

for all $x,y\in\zeta$ with $(x,y)\in E(G)$.

From Lemmas 2.7 and 2.8, we have $G-VI(\zeta,\mathcal{S}_i)$ is closed and convex, for all $i=1,2$.

From Lemma 2.9, we have $F(\mathcal{B}_i)$ is closed and convex. Then, $\mathfrak{F}_i$ is closed and convex, for all $i=1,2$.

Let $x^*=P_{\mathfrak{F}_1}f(y_0)$ be dominated by $\{x_n\}$ and $y^*=P_{\mathfrak{F}_2}g(x_0)$ be dominated by $\{y_n\}$, we have (x_n,x^*) and (y_n,y^*) are in $E(G)$ for all $n\in\mathbb{N}$. From Lemma 2.7, we have $G-VI(\zeta,\mathcal{S}_1)=\mathcal{S}_1^{-1}(0)$.

Then, $x^*\in\mathcal{S}_1^{-1}(0)$. Since $\mathcal{S}_1x^*=0$, we have

$$\begin{aligned}\left\|P_\zeta(I-\lambda_1\mathcal{S}_1)x_n-x^*\right\|^2&\leq\left\|x_n-x^*-\lambda_1\mathcal{S}_1x_n\right\|^2\\&=\left\|x_n-x^*-\left(\lambda_1\mathcal{S}_1x_n-\lambda_1\mathcal{S}_1x^*\right)\right\|^2\\&=\left\|x_n-x^*\right\|^2-2\lambda_1\left\langle x_n-x^*,\mathcal{S}_1x_n-\mathcal{S}_1x^*\right\rangle\\&\quad+\lambda_1^2\left\|\mathcal{S}_1x_n-\mathcal{S}_1x^*\right\|^2\\&=\left\|x_n-x^*\right\|^2-2\lambda_1\alpha\left\|\mathcal{S}_1x_n-\mathcal{S}_1x^*\right\|^2+\lambda_1^2\|\mathcal{S}_1x_n\|^2\\&=\left\|x_n-x^*\right\|^2-\lambda_1(2\alpha-\lambda_1)\|\mathcal{S}_1x_n\|^2\\&\leq\left\|x_n-x^*\right\|^2.\end{aligned}$$

From the definition of $\{x_n\}$, we have

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|&=\big\|\delta_nx_n+\eta_nP_\zeta(I-\lambda_1\mathcal{S}_1)x_n+\mu_nP_\zeta\left(\alpha_nf(y_n)+(1-\alpha_n)\mathcal{B}_1x_n\right)\\&\quad-(\delta_n+\eta_n+\mu_n)x^*\big\|\\&\leq\delta_n\left\|x_n-x^*\right\|+\eta_n\left\|P_\zeta(I-\lambda_1\mathcal{S}_1)x_n-x^*\right\|+\mu_n\big\|P_\zeta\big(\alpha_nf(y_n)\\&\quad+(1-\alpha_n)\mathcal{B}_1x_n\big)-x^*\big\|\\&\leq(1-\mu_n)\left\|x_n-x^*\right\|+\mu_n\left\|\alpha_n\left(f(y_n)-x^*\right)+(1-\alpha_n)\left(\mathcal{B}_1x_n-x^*\right)\right\|\end{aligned}$$

$$\begin{aligned}
&\le (1-\mu_n)\|x_n - x^*\| + \mu_n\alpha_n\|f(y_n) - x^*\| + \mu_n(1-\alpha_n)\|x_n - x^*\| \\
&\le (1-\mu_n)\|x_n - x^*\| + \mu_n\alpha_n a\|y_n - y^*\| + \mu_n\alpha_n\|f(y^*) - x^*\| \\
&\quad + \mu_n(1-\alpha_n)\|x_n - x^*\| \\
&= (1-\mu_n\alpha_n)\|x_n - x^*\| + \mu_n\alpha_n a\|y_n - y^*\| + \mu_n\alpha_n\|f(y^*) - x^*\|. \qquad (13)
\end{aligned}$$

Similarly, we obtain

$$\|y_{n+1} - y^*\| \le (1-\mu_n\alpha_n)\|y_n - y^*\| + \mu_n\alpha_n a\|x_n - x^*\| + \mu_n\alpha_n\|g(x^*) - y^*\|. \qquad (14)$$

Combining (13) and (14), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| &\le (1-\mu_n\alpha_n)\{\|x_n - x^*\| + \|y_n - y^*\|\} \\
&\quad + \mu_n\alpha_n a\{\|x_n - x^*\| + \|y_n - y^*\|\} \\
&\quad + \mu_n\alpha_n\{\|g(x^*) - y^*\| + \|f(y^*) - x^*\|\} \\
&= (1-\mu_n\alpha_n(1-a))\{\|x_n - x^*\| + \|y_n - y^*\|\} \\
&\quad + \frac{\mu_n\alpha_n\{\|g(x^*) - y^*\| + \|f(y^*) - x^*\|\}}{(1-a)}.
\end{aligned}$$

By induction, we can derive that

$$\|x_n - x^*\| + \|y_n - y^*\| \le \max\left\{\|x_1 - x^*\| + \|y_1 - y^*\|, \frac{\|g(x^*) - y^*\| + \|f(y^*) - x^*\|}{1-a}\right\},$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded.

Step 2: We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.

From Proposition 3.1, we have (x_n, x_{n+1}) and (y_n, y_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$. First, we let $U_n = P_\zeta(\alpha_n f(y_n) + (1-\alpha_n)\mathcal{B}_1 x_n)$ and $V_n = P_\zeta(\alpha_n g(x_n) + (1-\alpha_n)\mathcal{B}_2 y_n)$. Then, we observe that

$$\begin{aligned}
\|U_n - U_{u-1}\| &= \|P_\zeta(\alpha_n f(y_n) + (1-\alpha_n)\mathcal{B}_1 x_n) - P_\zeta(\alpha_{n-1} f(y_{n-1}) \\
&\quad + (1-\alpha_{n-1})\mathcal{B}_1 x_{n-1})\| \\
&\le \alpha_n\|f(y_n) - f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| \\
&\quad + (1-\alpha_n)\|\mathcal{B}_1 x_n - \mathcal{B}_1 x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|\mathcal{B}_1 x_{n-1}\| \\
&\le \alpha_n a\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\{\|f(y_{n-1})\| + \|\mathcal{B}_1 x_{n-1}\|\} \\
&\quad + (1-\alpha_n)\|x_n - x_{n-1}\|. \qquad (15)
\end{aligned}$$

By the definition of x_n and (15) we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\delta_n x_n + \eta_n P_\zeta(I - \lambda_1\mathcal{S}_1)x_n + \mu_n U_n - \delta_{n-1}x_{n-1} \\
&\quad - \eta_{n-1}P_\zeta(I - \lambda_1\mathcal{S}_1)x_{n-1} - \mu_{n-1}U_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + \eta_n\big\|P_\zeta(I - \lambda_1\mathcal{S}_1)x_n \\
&\quad - P_\zeta(I - \lambda_1\mathcal{S}_1)x_{n-1}\big\| + |\eta_n - \eta_{n-1}|\big\|P_\zeta(I - \lambda_1\mathcal{S}_1)x_{n-1}\big\| \\
&\quad + \mu_n\|\mathcal{U}_n - \mathcal{U}_{n-1}\| + |\mu_n - \mu_{n-1}|\|\mathcal{U}_{n-1}\| \\
&\leq (1 - \mu_n)\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
&\quad + |\eta_n - \eta_{n-1}|\big\|P_\zeta(I - \lambda_1\mathcal{S}_1)x_{n-1}\big\| \\
&\quad + \mu_n|\alpha_n - \alpha_{n-1}|\big\{\big\|f(y_{n-1})\big\| + \|G_1x_{n-1}\|\big\} \\
&\quad + \mu_n(1 - \alpha_n)\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|\mathcal{U}_{n-1}\| \\
&\quad + \mu_n\alpha_n a\|y_n - y_{n-1}\|.
\end{aligned} \tag{16}
$$

Using the same method as derived in (16), we have

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\leq (1 - \mu_n)\|y_n - y_{n-1}\| + |\delta_n - \delta_{n-1}|\|y_{n-1}\| \\
&\quad + |\eta_n - \eta_{n-1}|\big\|P_\zeta(I - \lambda_2\mathcal{S}_2)y_{n-1}\big\| \\
&\quad + \mu_n|\alpha_n - \alpha_{n-1}|\big\{\big\|g(x_{n-1})\big\| + \|\mathcal{B}_2y_{n-1}\|\big\} \\
&\quad + \mu_n(1 - \alpha_n)\|y_n - y_{n-1}\| + |\mu_n - \mu_{n-1}|\|V_{n-1}\| \\
&\quad + \mu_n\alpha_n a\|x_n - x_{n-1}\|.
\end{aligned} \tag{17}
$$

From (16) and (17), we obtain

$$
\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq (1 - \mu_n)\big[\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\big] \\
&\quad + |\delta_n - \delta_{n-1}|\big[\|x_{n-1}\| + \|y_{n-1}\|\big] \\
&\quad + |\eta_n - \eta_{n-1}|\big[\big\|P_\zeta(I - \lambda_1\mathcal{S}_1)x_{n-1}\big\| \\
&\quad + \big\|P_\zeta(I - \lambda_2\mathcal{S}_2)y_{n-1}\big\|\big] \\
&\quad + |\mu_n - \mu_{n-1}|\big[\|\mathcal{U}_{n-1}\| + \|V_{n-1}\|\big] \\
&\quad + \mu_n\alpha_n a\big[\|y_n - y_{n-1}\| + \|x_n - x_{n-1}\|\big] \\
&\quad + \mu_n|\alpha_n - \alpha_{n-1}|\big[\big\|f(y_{n-1})\big\| + \|\mathcal{B}_1x_{n-1}\| \\
&\quad + \big\|g(x_{n-1})\big\| + \|\mathcal{B}_2y_{n-1}\|\big] \\
&\quad + \mu_n(1 - \alpha_n)\big[\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\big] \\
&\leq \big(1 - \alpha_n\bar{\xi}(1 - a)\big)\big[\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\big] \\
&\quad + |\delta_n - \delta_{n-1}|\big[\|x_{n-1}\| + \|y_{n-1}\|\big] \\
&\quad + |\eta_n - \eta_{n-1}|\big[\big\|P_\zeta(I - \lambda_1\mathcal{S}_1)x_{n-1}\big\| \\
&\quad + \big\|P_\zeta(I - \lambda_2\mathcal{S}_2)y_{n-1}\big\|\big] \\
&\quad + |\mu_n - \mu_{n-1}|\big[\|\mathcal{U}_{n-1}\| + \|V_{n-1}\|\big] \\
&\quad + \xi|\alpha_n - \alpha_{n-1}|\big[\big\|f(y_{n-1})\big\| + \|\mathcal{B}_1x_{n-1}\| \\
&\quad + \big\|g(x_{n-1})\big\| + \|\mathcal{B}_2y_{n-1}\|\big].
\end{aligned}
$$

Applying Lemma 2.1 and the conditions (ii) and (iii) we can conclude that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{and} \quad \|y_{n+1} - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{18}$$

Step 3: We prove that $\lim_{n\to\infty} \|\mathcal{U}_n - P_\zeta(I - \lambda_1\mathcal{S}_1)\mathcal{U}_n\| = \lim_{n\to\infty} \|\mathcal{U}_n - \mathcal{B}_1\mathcal{U}_n\| = 0$.

To show this, take $\tilde{u}_n = \alpha_n f(y_n) + (1 - \alpha_n)\mathcal{B}_1 x_n$, $\forall n \in \mathbb{N}$.

Since x^* is dominated by $\{x_n\}$, y^* is dominated by $\{y_n\}$ and (x_n, x_{n+1}), and (y_n, y_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$, then we derive that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\delta_n(x_n - x^*) + \eta_n(P_\zeta(I - \lambda_1\mathcal{S}_1)x_n - x^*) + \mu_n(\mathcal{U}_n - x^*)\|^2 \\
&\le \delta_n\|x_n - x^*\|^2 + \eta_n\|P_\zeta(I - \lambda_1\mathcal{S}_1)x_n - x^*\|^2 \\
&\quad - \delta_n\eta_n\|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\|^2 + \mu_n\|\tilde{u}_n - x^*\|^2 \\
&\le (1 - \mu_n)\|x_n - x^*\|^2 - \delta_n\eta_n\|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\|^2 \\
&\quad + \mu_n\|\alpha_n(f(y_n) - \mathcal{B}_1x_n) + (\mathcal{B}_1x_n - x^*)\|^2 \\
&\le (1 - \mu_n)\|x_n - x^*\|^2 - \delta_n\eta_n\|x_n - P_\zeta(I - \lambda_1 A_1)x_n\|^2 \\
&\quad + \mu_n\{\|\mathcal{B}_1x_n - x^*\|^2 + 2\alpha_n\langle f(y_n) - \mathcal{B}_1x_n, \tilde{u}_n - x^*\rangle\} \\
&\le \|x_n - x^*\|^2 - \delta_n\eta_n\|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\|^2 \\
&\quad + 2\mu_n\alpha_n\|f(y_n) - \mathcal{B}_1x_n\|\|\tilde{u}_n - x^*\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_n\eta_n\|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\mu_n\alpha_n\|f(y_n) - \mathcal{B}_1x_n\|\|\tilde{u}_n - x^*\| \\
&\le \|x_n - x_{n+1}\|\{\|x_n - x^*\| + \|x_{n+1} - x^*\|\} \\
&\quad + 2\mu_n\alpha_n\|f(y_n) - \mathcal{B}_1x_n\|\|\tilde{u}_n - x^*\|.
\end{aligned}$$

Then, we have

$$\|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\| \to 0 \quad \text{as } n \to \infty. \tag{19}$$

Observe that

$$x_{n+1} - x_n = \eta_n(P_\zeta(I - \lambda_1\mathcal{S}_1)x_n - x_n) + \mu_n(\mathcal{U}_n - x_n).$$

It follows that

$$\mu_n\|\mathcal{U}_n - x_n\| \le \eta_n\|P_\zeta(I - \lambda_1\mathcal{S}_1)x_n - x_n\| + \|x_{n+1} - x_n\|.$$

From (18) and (19), we obtain

$$\|\mathcal{U}_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{20}$$

Since $\zeta = V(G)$ dominates x_0 and $U_n, x_n \in \zeta$, then we have (U_n, x_0) and (x_n, x_0) are in $E(G)$, for all $n \in \mathbb{N}$. Since $E(G) = E(G)^{-1}$ and $(x_n, x_0) \in E(G)$, then $(x_0, x_n) \in E(G)$, for all $n \in \mathbb{N}$. By the transitivity of G and $(U_n, x_0), (x_0, x_n) \in E(G)$, for all $n \in \mathbb{N}$, we obtain $(U_n, x_n) \in E(G)$, for all $n \in \mathbb{N}$. Since $E(G) = E(G)^{-1}$ and $(U_n, x_n) \in E(G)$, then $(x_n, U_n) \in E(G)$, for all $n \in \mathbb{N}$.

Observe that

$$\begin{aligned}\|U_n - P_\zeta(I - \lambda_1\mathcal{S}_1)U_n\| &\leq \|U_n - x_n\| + \|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\| \\ &\quad + \|P_\zeta(I - \lambda_1\mathcal{S}_1)x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)U_n\| \\ &\leq \|U_n - x_n\| + \|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\| + \|x_n - U_n\| \\ &= 2\|U_n - x_n\| + \|x_n - P_\zeta(I - \lambda_1\mathcal{S}_1)x_n\|,\end{aligned}$$

by (12), (19), and (20), we obtain

$$\|U_n - P_\zeta(I - \lambda_1\mathcal{S}_1)U_n\| \to 0 \quad \text{as } n \to \infty. \tag{21}$$

Applying the same arguments as for deriving (21), we also obtain

$$\|V_n - P_\zeta(I - \lambda_2\mathcal{S}_2)V_n\| \to 0 \quad \text{as } n \to \infty.$$

Consider

$$\|x_{n+1} - U_n\| \leq \|x_{n+1} - x_n\| + \|x_n - U_n\|.$$

From (18) and (20), we have

$$\|x_{n+1} - U_n\| \to 0 \quad \text{as } n \to \infty. \tag{22}$$

Since

$$\begin{aligned}\|x_n - \mathcal{B}_1x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|U_n - \mathcal{B}_1x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|\tilde{u}_n - \mathcal{B}_1x_n\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| \\ &\quad + \|\alpha_nf(y_n) + (1 - \alpha_n)\mathcal{B}_1x_n - \mathcal{B}_1x_n\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \alpha_n\|f(y_n) - \mathcal{B}_1x_n\|,\end{aligned}$$

from (18), (22), and condition (ii), we obtain

$$\|x_n - \mathcal{B}_1x_n\| \to 0 \quad \text{as } n \to \infty. \tag{23}$$

Consider

$$\begin{aligned}\|U_n - \mathcal{B}_1U_n\| &\leq \|U_n - x_n\| + \|x_n - \mathcal{B}_1x_n\| + \|\mathcal{B}_1x_n - \mathcal{B}_1U_n\| \\ &\leq \|U_n - x_n\| + \|x_n - \mathcal{B}_1x_n\| + \|x_n - U_n\|\end{aligned}$$

$$\leq 2\|\mathcal{U}_n - x_n\| + \|x_n - \mathcal{B}_1 x_n\|.$$

From (20) and (23), we have

$$\|\mathcal{U}_n - \mathcal{B}_1\mathcal{U}_n\| \to 0 \quad \text{as } n \to \infty. \tag{24}$$

Applying the same arguments as for deriving (24), we also obtain

$$\|V_n - \mathcal{B}_2 V_n\| \to 0 \quad \text{as } n \to \infty.$$

Step 4: We claim that $\limsup_{n\to\infty}\langle f(y^*) - x^*, \mathcal{U}_n - x^*\rangle \leq 0$, where $x^* = P_{\mathfrak{F}_1} f(y^*)$.

First, take a subsequence $\{\mathcal{U}_{n_k}\}$ of $\{\mathcal{U}_n\}$ such that

$$\limsup_{n\to\infty}\langle f(y^*) - x^*, \mathcal{U}_n - x^*\rangle = \lim_{k\to\infty}\langle f(y^*) - x^*, \mathcal{U}_{n_k} - x^*\rangle. \tag{25}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in \zeta$ as $k \to \infty$. From (20), we obtain $\mathcal{U}_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$. Since ζ dominates x_0, $\hat{x}$ and $\mathcal{U}_{n_k}$ are in ζ, then $(\hat{x}, x_o)$ and $(\mathcal{U}_{n_k}, x_o)$ are in $E(G)$. From $E(G) = E(G)^{-1}$ and $(\mathcal{U}_{n_k}, x_o) \in E(G)$, then $(x_o, \mathcal{U}_{n_k}) \in E(G)$. By the transitivity of G, $(\hat{x}, x_o)$ and $(x_o, \mathcal{U}_{n_k})$ are in $E(G)$, we have $(\hat{x}, \mathcal{U}_{n_k}) \in E(G)$. From $E(G) = E(G)^{-1}$ and $(\hat{x}, \mathcal{U}_{n_k}) \in E(G)$, then $(\mathcal{U}_{n_k}, \hat{x}) \in E(G)$.

Next, we need to show that $\hat{x} \in \mathfrak{F}_1 = F(\mathcal{B}_1) \cap VI(\zeta, \mathcal{S}_1)$. Assume $\hat{x} \notin F(\mathcal{B}_1)$. Then, we have $\hat{x} \neq \mathcal{B}_1\hat{x}$. By Opial's condition, we obtain

$$\begin{aligned}
\liminf_{k\to\infty}\|\mathcal{U}_{n_k} - \hat{x}\| &< \liminf_{k\to\infty}\|\mathcal{U}_{n_k} - \mathcal{B}_1\hat{x}\| \\
&\leq \liminf_{k\to\infty}\|\mathcal{U}_{n_k} - \mathcal{B}_1\mathcal{U}_{n_k}\| + \liminf_{k\to\infty}\|\mathcal{B}_1\mathcal{U}_{n_k} - \mathcal{B}_1\hat{x}\| \\
&\leq \liminf_{k\to\infty}\|\mathcal{U}_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction.

Therefore,

$$\hat{x} \in F(\mathcal{B}_1). \tag{26}$$

Assume $\hat{x} \notin VI(\zeta, \mathcal{S}_1)$, then we obtain $\hat{x} \neq P_\zeta(I - \lambda_1\mathcal{S}_1)\hat{x}$.

From Opial's condition, $(\mathcal{U}_{n_k}, \hat{x}) \in E(G)$ and (21), we have

$$\begin{aligned}
\liminf_{k\to\infty}\|\mathcal{U}_{n_k} - \hat{x}\| &< \liminf_{k\to\infty}\left\|\mathcal{U}_{n_k} - P_\zeta(I - \gamma_1\mathcal{S}_1)\hat{x}\right\| \\
&\leq \liminf_{k\to\infty}\left\|\mathcal{U}_{n_k} - P_\zeta(I - \lambda_1\mathcal{S}_1)\mathcal{U}_{n_k}\right\| \\
&\quad + \liminf_{k\to\infty}\left\|P_\zeta(I - \lambda_1\mathcal{S}_1)\mathcal{U}_{n_k} - P_C(I - \lambda_1\mathcal{S}_1)\hat{x}\right\| \\
&\leq \liminf_{k\to\infty}\|\mathcal{U}_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction.

Therefore,

$$\hat{x} \in VI(\zeta, \mathcal{S}_1). \tag{27}$$

By (26) and (27), this yields that

$$\hat{x} \in \mathfrak{F}_1 = F(\mathcal{B}_1) \cap VI(\zeta, \mathcal{S}_1). \tag{28}$$

Since $U_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$, (28) and Lemma 2.2, we can derive that

$$\begin{aligned} \limsup_{n\to\infty}\langle f(y^*) - x^*, U_n - x^*\rangle &= \lim_{k\to\infty}\langle f(y^*) - x^*, U_{n_k} - x^*\rangle \\ &= \langle f(y^*) - x^*, \hat{x} - x^*\rangle \\ &\leq 0. \end{aligned} \tag{29}$$

Following the same method as for (29), we obtain that

$$\limsup_{n\to\infty}\langle g(x^*) - y^*, V_n - y^*\rangle \leq 0. \tag{30}$$

Step 5: Finally, we prove that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{\mathfrak{F}_1} f(y^*)$ and $y^* = P_{\mathfrak{F}_2} g(x^*)$, respectively.

By the firm nonexpansiveness of P_ζ, (x_n, x_{n+1}) and (y_n, y_{n+1}) being in $E(G)$ we derive that

$$\begin{aligned} \|U_n - x^*\|^2 &= \|P_\zeta \tilde{u}_n - x^*\|^2 \\ &\leq \langle \tilde{u}_n - x^*, U_n - x^*\rangle \\ &= \langle \alpha_n(f(y_n) - x^*) + (1-\alpha_n)(\mathcal{B}_1 x_n - x^*), U_n - x^*\rangle \\ &= \alpha_n\langle f(y_n) - x^*, U_n - x^*\rangle + (1-\alpha_n)\langle \mathcal{B}_1 x_n - x^*, U_n - x^*\rangle \\ &= \alpha_n\langle f(y_n) - f(y^*), U_n - x^*\rangle + \alpha_n\langle f(y^*) - x^*, U_n - x^*\rangle \\ &\quad + (1-\alpha_n)\|\mathcal{B}_1 x_n - x^*\|\|U_n - x^*\| \\ &\leq \alpha_n a\|y_n - y^*\|\|U_n - x^*\| + \alpha_n\langle f(y^*) - x^*, U_n - x^*\rangle \\ &\quad + (1-\alpha_n)\|x_n - x^*\|\|U_n - x^*\| \\ &\leq \frac{\alpha_n a}{2}\{\|y_n - y^*\|^2 + \|U_n - x^*\|^2\} + \alpha_n\langle f(y^*) - x^*, U_n - x^*\rangle \\ &\quad + \frac{(1-\alpha_n)}{2}\{\|x_n - x^*\|^2 + \|U_n - x^*\|^2\} \\ &= \frac{\alpha_n a}{2}\|y_n - y^*\|^2 + \frac{(1-\alpha_n)}{2}\|x_n - x^*\|^2 \\ &\quad + \left(\frac{\alpha_n a}{2} + \frac{(1-\alpha_n)}{2}\right)\|U_n - x^*\|^2 \\ &\quad + \alpha_n\langle f(y^*) - x^*, U_n - x^*\rangle \\ &= \frac{\alpha_n a}{2}\|y_n - y^*\|^2 + \frac{(1-\alpha_n)}{2}\|x_n - x^*\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1-\alpha_n(1-a)}{2}\right)\left\|U_n-x^*\right\|^2 \\
&+\alpha_n\left\langle f\left(y^*\right)-x^*, U_n-x^*\right\rangle,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\left\|U_n-x^*\right\|^2 \leq{}& \frac{\alpha_n a}{1+\alpha_n(1-a)}\left\|y_n-y^*\right\|^2+\frac{(1-\alpha_n)}{1+\alpha_n(1-a)}\left\|x_n-x^*\right\|^2 \\
&+\frac{\alpha_n}{1+\alpha_n(1-a)}\left\langle f\left(y^*\right)-x^*, U_n-x^*\right\rangle.
\end{aligned}
\tag{31}
$$

From the definition of x_n and (31), we obtain

$$
\begin{aligned}
\left\|x_{n+1}-x^*\right\|^2 \leq{}& \delta_n\left\|x_n-x^*\right\|^2+\eta_n\left\|P_\zeta\left(I-\lambda_1 \mathcal{S}_1\right)x_n-x^*\right\|^2+\mu_n\left\|U_n-x^*\right\|^2 \\
\leq{}& (1-\mu_n)\left\|x_n-x^*\right\|^2+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\left\|y_n-y^*\right\|^2 \\
&+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left\langle f\left(y^*\right)-x^*, U_n-x^*\right\rangle \\
&+\frac{\mu_n(1-\alpha_n)}{1+\alpha_n(1-a)}\left\|x_n-x^*\right\|^2 \\
={}& \left(1-\frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\left\|x_n-x^*\right\|^2+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\left\|y_n-y^*\right\|^2 \\
&+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left\langle f\left(y^*\right)-x^*, U_n-x^*\right\rangle.
\end{aligned}
\tag{32}
$$

Similarly, as derived above, we also have

$$
\begin{aligned}
\left\|y_{n+1}-y^*\right\|^2 \leq{}& \left(1-\frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\left\|y_n-y^*\right\|^2+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\left\|x_n-x^*\right\|^2 \\
&+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left\langle g\left(x^*\right)-y^*, V_n-y^*\right\rangle.
\end{aligned}
\tag{33}
$$

From (32) and (33), we deduce that

$$
\begin{aligned}
&\left\|x_{n+1}-x^*\right\|^2+\left\|y_{n+1}-y^*\right\|^2 \\
&\quad\leq\left(1-\frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\left\{\left\|x_n-x^*\right\|^2+\left\|y_n-y^*\right\|^2\right\} \\
&\qquad+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\left\{\left\|x_n-x^*\right\|^2+\left\|y_n-y^*\right\|^2\right\} \\
&\qquad+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\left\langle f\left(y^*\right)-x^*, U_n-x^*\right\rangle+\left\langle g\left(x^*\right)-y^*, V_n-y^*\right\rangle\right) \\
&\quad=\left(1-\frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\right)\left\{\left\|x_n-x^*\right\|^2+\left\|y_n-y^*\right\|^2\right\} \\
&\qquad+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\left\langle f\left(y^*\right)-x^*, U_n-x^*\right\rangle+\left\langle g\left(x^*\right)-y^*, V_n-y^*\right\rangle\right) \\
&\quad=\left(1-\frac{2\mu_n\alpha_n(1-a)}{1+\alpha_n(1-a)}\right)\left\{\left\|x_n-x^*\right\|^2+\left\|y_n-y^*\right\|^2\right\}
\end{aligned}
$$

$$+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\left\langle f\left(y^*\right)-x^*,U_n-x^*\right\rangle+\left\langle g\left(x^*\right)-y^*,V_n-y^*\right\rangle\right).$$

Applying the condition (ii), (29), (30), and Lemma 2.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^*=P_{\mathfrak{F}_1}f(y^*)$ and $y^*=P_{\mathfrak{F}_2}g(x^*)$, respectively. This completes the proof. □

Lemma 3.3 *Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G=(V(G),E(G))$ be a directed graph with $\zeta=V(G)$. Let $\Lambda:\zeta\to\zeta$ be edge preserving and Λ be a $G-\kappa$-strictly pseudocontractive. Then, the following conditions hold;*

(i) *Λ is $G-\kappa$-strictly pseudocontractive $\Leftrightarrow$ $(I-\Lambda)$ is $G-(\frac{1-\kappa}{2})$-inverse strongly monotone;*
(ii) *$G-VI(\zeta,I-\Lambda)=F(\Lambda)$.*

Proof *i)* ($\Leftarrow$) Let $(x,y)\in E(G)$.

Since Λ is edge preserving and $(x,y)\in E(G)$, then $(\Lambda x,\Lambda y)\in E(G)$.

Suppose that $M=I-\Lambda$ is $G-(\frac{1-\kappa}{2})-$ inverse strongly monotone, i.e.,

$$\langle x-y,Mx-My\rangle\geq\left(\frac{1-\kappa}{2}\right)\|Mx-My\|^2,$$

for all $x,y\in\zeta$ with $(x,y)\in E(G)$. Then,

$$\begin{aligned}\|\Lambda x-\Lambda y\|^2&=\left\|(I-M)x-(I-M)y\right\|^2\\&=\|Mx-My\|^2+\|x-y\|^2-2\langle x-y,Mx-My\rangle\\&\leq\|Mx-My\|^2+\|x-y\|^2-2\left(\frac{1-\kappa}{2}\right)\|Mx-My\|^2\\&=\|x-y\|^2+\kappa\|Mx-My\|^2,\end{aligned}$$

for all $x,y\in\zeta$ with $(x,y)\in E(G)$.

Thus, Λ is a $G-\kappa$-strictly pseudocontractive mapping.

($\Rightarrow$) Conversely, suppose that $M=I-\Lambda$ and Λ is a $G-\kappa$-strictly pseudocontractive mapping.

Let $x,y\in\zeta$ with $(x,y)\in E(G)$, i.e.,

$$\|\Lambda x-\Lambda y\|^2\leq\|x-y\|^2+\kappa\|Mx-My\|^2,$$

for all $x,y\in\zeta$ with $(x,y)\in E(G)$. Then,

$$\begin{aligned}\|\Lambda x-\Lambda y\|^2&=\left\|(I-M)x-(I-M)y\right\|^2\\&=\|Mx-My\|^2+\|x-y\|^2-2\langle x-y,Mx-My\rangle\\&\leq\|x-y\|^2+\kappa\|Mx-My\|^2.\end{aligned}$$

Hence,

$$\langle x-y,Mx-My\rangle\geq\left(\frac{1-\kappa}{2}\right)\|Mx-My\|^2,$$

for all $x, y \in \zeta$ with $(x, y) \in E(G)$. This implies that M is a $G - \kappa$-strictly pseudocontractive mapping.

ii) Let $\hat{x} \in G - VI(\zeta, I - \Lambda)$.

From *i*) and Lemma 2.7, we have

$$\hat{x} \in (I - \Lambda)^{-1}(0).$$

This implies that $(I - \Lambda)\hat{x} = 0$.

Thus, $\hat{x} \in F(\Lambda)$. Therefore,

$$G - VI(\zeta, I - \Lambda) \subseteq F(\Lambda). \tag{34}$$

Let $\breve{x} \in F(\Lambda)$ and $y \in \zeta$ with $(\breve{x}, y) \in E(G)$.

Since $\Lambda\breve{x} = \breve{x}$, we have $(I - \Lambda)\breve{x} = 0$.

This implies that $\breve{x} \in (I - \Lambda)^{-1}(0)$.

From Lemma 2.7 and $\breve{x} \in (I - \Lambda)^{-1}(0)$, we have

$$\langle y - \breve{x}, (I - \Lambda)\breve{x} \rangle = 0,$$

for all $y \in \zeta$ with $(\breve{x}, y) \in E(G)$. Then, $\breve{x} \in G - VI(\zeta, I - \Lambda)$.

Therefore,

$$F(\Lambda) \subseteq G - VI(\zeta, I - \Lambda). \tag{35}$$

From (34) and (35), we have

$$G - VI(\zeta, I - \Lambda) = F(\Lambda). \qquad \square$$

Corollary 3.4 *Let ζ be a nonempty, closed, and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph with $\zeta = V(G)$. Let $E(G)$ be convex and G be transitive with $E(G) = E(G)^{-1}$. For every $i = 1, 2$, let $\mathcal{A}_i : \zeta \to \zeta$ be a $G - \kappa$-strictly pseudocontractive mapping with $F(\mathcal{A}_i) \neq \emptyset$ and let $g, f : H \to H$ be an a_g and a_f – G-contraction mapping with $a = \max\{a_g, a_f\}$. For every $i = 1, 2$, let $\mathcal{T}_i : \zeta \to \zeta$ be a G-nonexpansive mapping and $\Lambda_i : \zeta \to \zeta$ be a $G - \kappa_i$-strictly pseudocontractive mapping with $k = \max\{\kappa_1, \kappa_2\}$. Define a mapping $\mathcal{B}_i : \zeta \to \zeta$ by $\mathcal{B}_i x = \mathcal{T}_i((1 - b)I + b\Lambda_i)x$ for all $x \in \zeta$ and $i = 1, 2$, $b \in (0, 1 - k)$. Assume that $\mathfrak{F}_i = F(\mathcal{B}_i) \cap F(\mathcal{A}_i) \neq \emptyset$ for all $i = 1, 2$ with $F(\mathcal{B}_i) \times F(\mathcal{B}_i) \subseteq E(G)$ and $G - VI(\zeta, \mathcal{S}_i) \times G - VI(\zeta, \mathcal{S}_i) \subseteq E(G)$ for all $i = 1, 2$ and there exists $x_0, y_0 \in \zeta$ such that (x_0, x_0) and (y_0, y_0) are in $E(G)$ and $\zeta = V(G)$ dominates x_0 and y_0. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_0, y_0 \in \zeta$ and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_\zeta(I - \lambda_1(I - \mathcal{A}_1))x_n + \mu_n P_\zeta(\alpha_n f(y_n) + (1 - \alpha_n)\mathcal{B}_1 x_n), \\ y_{n+1} = \delta_n y_n + \eta_n P_\zeta(I - \lambda_2(I - \mathcal{A}_2))y_n + \mu_n P_\zeta(\alpha_n g(x_n) + (1 - \alpha_n)\mathcal{B}_2 y_n), \end{cases} \tag{36}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0, 1]$ with $\delta_n + \eta_n + \mu_n = 1$ and $\lambda \in (0, 2\alpha)$ with $\alpha = \min\{\frac{1-\kappa_1}{2}, \frac{1-\kappa_2}{2}\}$. Assume the following conditions hold;

(i) $0 < \xi \leq \delta_n, \eta_n, \mu_n \leq \bar{\bar{\xi}}$ *for all $n \in N$ and for some $\xi, \bar{\bar{\xi}} > 0$;*

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| < \infty$, $\sum_{n=1}^{\infty}|\eta_{n+1}-\eta_n| < \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$.

If $\{x_n\}$ *dominates* $P_{\mathfrak{F}_1}f(y_0)$ *and* $\{y_n\}$ *dominates* $P_{\mathfrak{F}_2}g(x_0)$, *then* $\{x_n\}$ *converges strongly to* $x^* = P_{\mathfrak{F}_1}f(y_0)$ *and* $\{y_n\}$ *converges strongly to* $y^* = P_{\mathfrak{F}_2}g(x_0)$, *where* $P_{\mathfrak{F}_i}$ *is a metric projection on* $\mathfrak{F}_i$, *for all* $i = 1, 2$.

Proof From Theorem 3.2 and Lemma 3.3, we have the desired conclusion. □

4 Example

In this section, we give some examples to support our main theorem.

Example 4.1 Let $H = \mathbb{R}$ and $\zeta = [-10, 10]$ and let $G = (V(G), E(G))$ be such that $V(G) = \zeta$, $E(G) = \{(x, y) : xy > 0\}$. Let $\mathcal{T}_1 : [-10, 10] \to [-10, 10]$ be defined by

$$\mathcal{T}_1 x = \frac{|x|}{x},$$

for all $x \in [-10, 10]$ and let $\mathcal{T}_2 : [-10, 10] \to [-10, 10]$ be defined by

$$\mathcal{T}_2 x = \begin{cases} \frac{1-x}{2}; & x < 0, \\ 0.5; & x = 0, \\ \frac{x+1}{2}; & x > 0, \end{cases}$$

for all $x \in [-10, 10]$.

Let $\Lambda_1 : [-10, 10] \to [-10, 10]$ be defined by

$$\Lambda_1 x = \operatorname{sgn}\big(\operatorname{sgn}(x)\big) = \begin{cases} 1; & x \in (0, 10], \\ 0; & x = 0, \\ -1; & x \in [-10, 0), \end{cases}$$

for all $x \in [-10, 10]$ and let $\Lambda_2 : [-10, 10] \to [-10, 10]$ be defined by

$$\Lambda_2 x = \begin{cases} \frac{2x+3}{5}; & x \in (0, 10], \\ 0; & x = 0, \\ \frac{2x-11}{5}; & x \in [-10, 0), \end{cases}$$

for all $x \in [-10, 10]$.

Let $\mathcal{S}_1 : [-10, 10] \to \mathbb{R}$ be defined by

$$\mathcal{S}_1 x = \begin{cases} \frac{x+5}{12} - \frac{1}{2}; & x \in (0, 10], \\ 0; & x = 0, \\ \frac{x+5}{12} + \frac{1}{2}; & x \in [-10, 0), \end{cases}$$

for all $x \in [-10, 10]$, and let $\mathcal{S}_2 : [-10, 10] \to \mathbb{R}$ be defined by

$$\mathcal{S}_2 x = \begin{cases} \frac{2(x-1)}{\sqrt{19}}; & x \in (0, 10], \\ 0; & x = 0, \\ \frac{2(x+1)}{\sqrt{19}}; & x \in [-10, 0). \end{cases}$$

For every $i = 1, 2$, let $\mathcal{B}_i : \zeta \to \zeta$ be defined by $\mathcal{B}_i x = \mathcal{T}_i(0.5I + 0.5\Lambda_i)x$ for all $x \in [-10, 10]$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{\operatorname{sgn}(x)}{2} = \begin{cases} \frac{1}{2}; & x > 0, \\ 0; & x = 0, \\ -\frac{1}{2}; & x < 0 \end{cases}$$

and

$$g(x) = \frac{\operatorname{sgn}(x)}{3} + \frac{1}{2} = \begin{cases} \frac{5}{6}; & x > 0, \\ 0; & x = 0, \\ \frac{1}{6}; & x < 0. \end{cases}$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_0, y_0 \in \zeta$ and

$$\begin{cases} x_{n+1} = (\frac{n}{3n+2})x_n + (\frac{n+\frac{1}{2}}{3n+2})P_\zeta(I - 0.7\mathcal{S}_1)x_n + (\frac{n+\frac{3}{2}}{3n+2})P_\zeta(\frac{1}{8n}f(y_n) + (1 - \frac{1}{8n})\mathcal{B}_1 x_n), \\ y_{n+1} = (\frac{n}{3n+2})y_n + (\frac{n+\frac{1}{2}}{3n+2})P_\zeta(I - 0.5\mathcal{S}_2)y_n + (\frac{n+\frac{3}{2}}{3n+2})P_\zeta(\frac{1}{8n}g(x_n) + (1 - \frac{1}{8n})\mathcal{B}_2 y_n). \end{cases}$$

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

Solution. For every $i = 1, 2$, it is clear that $\mathcal{S}_i^{-1}(0) \neq \emptyset$, since $1 \in \mathcal{S}_i^{-1}(0)$, $E(G) = E(G)^{-1}$ and $E(G)$ is convex. By the definitions of $\mathcal{T}_1$, $\mathcal{T}_2$, f, g, $\mathcal{S}_1$, $\mathcal{S}_2$, Λ_1, and Λ_2, we have $\mathcal{T}_1$ and $\mathcal{T}_2$ are G-nonexpansive mappings, f and g are G-contraction mappings, $\mathcal{S}_1$ and $\mathcal{S}_2$ are G-12, $\frac{\sqrt{19}}{2}$-inverse strongly monotone mappings, respectively, Λ_1 and Λ_2 are G-$\frac{1}{2}$, $\frac{1}{3}$-strictly pseudocontractive mappings, respectively. However, $\mathcal{T}_1$ and $\mathcal{T}_2$ are not nonexpansive mappings, as if we choose $x_1 = 1$, $y_1 = -\frac{1}{5}$, $x_2 = 0$ and $y_2 = 1$, then we see that $|\mathcal{T}_1 x_1 - \mathcal{T}_1 y_1| = 2 > \frac{6}{5} = |x_1 - y_1|$ and $|\mathcal{T}_2 x_2 - \mathcal{T}_2 y_2| = 1.5 > 1 = |x_2 - y_2|$.

f and g are not contraction mappings, as if we choose $x = \frac{1}{4}$ and $y = -\frac{1}{5}$, then we see that $|f(x) - f(y)| = 1 > \frac{7}{10} = |x - y|$ and $|g(x) - g(y)| = |\frac{5}{6} - \frac{1}{6}| = \frac{4}{6} > \frac{9}{20} = |\frac{1}{4} + \frac{1}{5}| = |x - y|$.

$\mathcal{S}_1$ and $\mathcal{S}_2$ are not 12, $\frac{\sqrt{19}}{2}$-inverse strongly monotone mappings, respectively. If $x = 2$ and $y = 10$ with $(2, 10) \in E(G)$, then

$$\langle \mathcal{S}_1(2) - \mathcal{S}_1(10), 2 - 10 \rangle = 5.33 < 5.38 = 12 |\mathcal{S}_1(2) - \mathcal{S}_1(10)|^2.$$

If $x = 10$ and $y = 8$ with $(10, 8) \in E(G)$, then

$$\langle \mathcal{S}_2(10) - \mathcal{S}_2(8), 10 - 8 \rangle = \left\langle \frac{18}{\sqrt{19}}, 2 \right\rangle = 8.26 < 37.16 = \frac{\sqrt{19}}{2} |\mathcal{S}_2(10) - \mathcal{S}_2(8)|^2.$$

Λ_1 and Λ_2 are not $\frac{1}{2}$, $\frac{1}{3}$-strictly pseudocontractive mappings, respectively, as if we choose $x = 1$ and $y = -\frac{1}{5}$, then we have

$$|\Lambda_1 x - \Lambda_1 y|^2 = 4 > 1.76 = \left|1 + \frac{1}{5}\right|^2 + \frac{1}{2}\left|1 - \Lambda_1(1) + \frac{1}{5} + \Lambda_1\left(-\frac{1}{5}\right)\right|^2$$

and

$$|\Lambda_2 x - \Lambda_2 y|^2 = 10.76 > 2.88 = \left|1 + \frac{1}{5}\right|^2 + \frac{1}{3}\left|(I - \Lambda_2)(1) - (I - \Lambda_2)\left(-\frac{1}{5}\right)\right|^2.$$

By the definitions of $\mathcal{S}_i$, $\mathcal{T}_i$, and Λ_i for every $i = 1, 2$, we have $1 = F(\mathcal{B}_i) \cap G - VI(\zeta, \mathcal{S}_i)$. We observed that the parameters satisfy all the conditions of Theorem 3.2. Hence, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

Newton's Method is a mathematical tool often used in numerical analysis, which serves to approximate numerical solutions (i.e., x-intercepts, zeros, or roots). Given a function $g(x)$ defined over the domain of real numbers x, and the derivative of said function $g'(x)$. If x_n is an approximation of a solution of $g(x) = 0$ and if $\acute{g}(x_n) \neq 0$, the next approximation defined for each $n = 0, 1, 2, \dots$ by:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \tag{37}$$

where x_0 is an initial point, is the most popular, studied, and used method for generating a sequence $\{x_n\}$ approximating the solution.

Newton's Method (37) is also an example of Picard iteration, for the equation

$$x_{n+1} = Tx_n,$$

where $T = I - \frac{g}{g'}$. Several authors have used the Picard iteration for approximation of fixed points (see [5, 6, 9]).

Π is an important mathematic constant. The search for the accurate value of Π led not only to more accuracy, but also to the development of new concepts and techniques. Many researches have been trying to approximate the value of Π (see [2, 7]). By using our main result, we introduce a new method to approximate the value of Π as shown in the following example.

Example 4.2 Let $H = \mathbb{R}$ and $\zeta = [3, 4]$ and let $G = (V(G), E(G))$ be a directed graph with $V(G) = \zeta$, $E(G) = \{(x, y) : x, y \in [3, \frac{11}{3}]\}$. For an approximate value of Π, for every $i = 1, 2$, define $F_i : C \to H$ by $F_i(x) = \frac{x}{5i} - \frac{\Pi}{5i}$ and F_i is a subdifferentiable. It is easy to show that $I - \frac{F_i}{F_i'}$ is a G-nonexpansive mapping.

For every $i = 1, 2$, define $\mathcal{T}_i : [3, 4] \to [3, 4]$ by

$$\mathcal{T}_i x = \begin{cases} x - \frac{F(x)}{\acute{F}(x)}; & \text{if } x \in [3, 4), \\ \frac{11}{3}; & \text{if } x = 4, \end{cases}$$

for all $x \in [3, 4]$.

For every $i = 1,2$, let $\Lambda_i : [3,4] \to [3,4]$ be defined by

$$\Lambda_i x = \begin{cases} \frac{2ix+\Pi}{2i+1}; & \text{if } x \in [3,4), \\ \frac{11}{3}; & \text{if } x = 4, \end{cases}$$

for all $x \in [3,4]$.

For every $i = 1,2$, let $\mathcal{S}_i : [3,4] \to \mathbb{R}$ be defined by

$$\mathcal{S}_i x = \begin{cases} 2i(\frac{x}{3\Pi} - \frac{1}{3}); & \text{if } x \in [3,4), \\ \frac{11}{3}; & \text{if } x = 4, \end{cases}$$

for all $x \in [3,4]$.

For every $i = 1,2$, let $\mathcal{B}_i : \zeta \to \zeta$ be defined by $\mathcal{B}_i x = \mathcal{T}_i(0.5I + 0.5\Lambda_i)x$ for all $x \in [3,4]$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{2x}{3} + 1; & \text{if } x \in [3,4], \\ \frac{11}{3}; & \text{if } x \notin [3,4] \end{cases}$$

and

$$g(x) = \begin{cases} \frac{x+7}{3}; & \text{if } x \in [3,4], \\ \frac{11}{3}; & \text{if } x \notin [3,4]. \end{cases}$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_0, y_0 \in \zeta$ and

$$\begin{cases} x_{n+1} = (\frac{n}{3n+2})x_n + (\frac{n+\frac{1}{2}}{3n+2})P_\zeta(I - 0.7\mathcal{S}_1)x_n + (\frac{n+\frac{3}{2}}{3n+2})P_\zeta(\frac{1}{8n}f(y_n) + (1 - \frac{1}{8n})\mathcal{B}_1 x_n), \\ y_{n+1} = (\frac{n}{3n+2})y_n + (\frac{n+\frac{1}{2}}{3n+2})P_\zeta(I - 0.5\mathcal{S}_2)y_n + (\frac{n+\frac{3}{2}}{3n+2})P_\zeta(\frac{1}{8n}g(x_n) + (1 - \frac{1}{8n})\mathcal{B}_2 y_n). \end{cases} \tag{38}$$

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to Π.

Solution. For every $i = 1,2$, it is clear that $\mathcal{S}_i^{-1}(0) \neq \emptyset$, since $\Pi \in \mathcal{S}_i^{-1}(0)$, $E(G) = E(G)^{-1}$ and $E(G)$ is convex. By the definitions of $\mathcal{T}_1$, $\mathcal{T}_2$, f, g, $\mathcal{S}_1$, $\mathcal{S}_2$, Λ_1, and Λ_2, we have $\mathcal{T}_1$ and $\mathcal{T}_2$ are G-nonexpansive mappings, f and g are G-contraction mappings, $\mathcal{S}_1$ and $\mathcal{S}_2$ are G-$\frac{3\Pi}{2}$, $\frac{3\Pi}{4}$-inverse strongly monotone mappings, respectively, Λ_1 and Λ_2 are G-$\frac{1}{15}$-strictly pseudocontractive mappings.

By the definitions of $\mathcal{S}_i$, $\mathcal{T}_i$, and Λ_i for every $i = 1,2$, we have $\Pi = F(\mathcal{B}_i) \cap G - VI(\zeta, \mathcal{S}_i)$. We observed that the parameters satisfy all the conditions of Theorem 3.2. Hence, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to Π.

Using the algorithm (38), we have the numerical result to approximate the value of Π as shown in Table 1 and Fig. 1, where $x_1 = y_1 = 4.00000$ and $n = N = 150$.

Remark 1 We show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are not nonexpansive mappings, as if we choose $x_1 = x_2 = 4$, $y_1 = \frac{11}{3}$ and $y_2 = 3.5$, then we see that $|\mathcal{T}_1 x_1 - \mathcal{T}_1 y_1| = 0.526 > 0.3 = |x_1 - y_1|$ and $|\mathcal{T}_2 x_2 - \mathcal{T}_2 y_2| = 0.526 > 0.5 = |x_2 - y_2|$.

f and g are not contraction mappings, as if we choose $x_1 = \frac{25}{10}$, $x_2 = \frac{29}{10}$ and $y_1 = y_2 = 3$, then we see that $|f(x_1) - f(y_1)| = 0.67 > 0.5 = |x_1 - y_1|$ and $|g(x_2) - g(y_2)| = 0.3 > 0.1 = |x_2 - y_2|$.

Table 1 The values of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = y_1 = 4.00000$ and $n = N = 150$

n	x_n	y_n
1	4.00000	4.00000
2	3.53333	3.58051
3	3.44013	3.44482
⋮	⋮	⋮
74	3.14162	3.14221
75	3.14162	3.14220
76	3.14162	3.14219
⋮	⋮	⋮
148	3.14158	3.14187
149	3.14158	3.14187
150	3.14158	3.14186

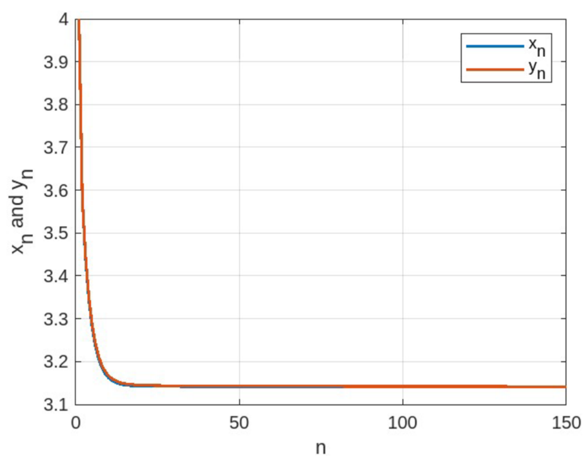

Figure 1 The convergence of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = y_1 = 4.00000$ and $n = N = 150$

$\mathcal{S}_1$ and $\mathcal{S}_2$ are not $\frac{3\Pi}{2}$, $\frac{3\Pi}{4}$-inverse strongly monotone mappings, respectively. If $x_1 = x_2 = 3.5$ and $y_1 = y_2 = 3$ with $(3.5, 3) \in E(G)$, then

$$\langle \mathcal{S}_1(x_1) - \mathcal{S}_1(y_1), x_1 - y_1 \rangle = 0.025 < 0.047 = \frac{3\Pi}{2} \left| \mathcal{S}_1(x_1) - \mathcal{S}_1(y_1) \right|^2$$

and

$$\langle \mathcal{S}_2(x_2) - \mathcal{S}_2(y_2), x_2 - y_2 \rangle = 0.025 < 9.42 = \frac{3\Pi}{4} \left| \mathcal{S}_2(x_2) - \mathcal{S}_2(y_2) \right|^2.$$

Λ_1 and Λ_2 are not $\frac{1}{15}$-strictly pseudocontractive mappings, as if we choose $x_1 = x_2 = 4$ and $y_1 = y_2 = \frac{11}{3}$, then we have $|\Lambda_1 x_1 - \Lambda_1 y_1|^2 = 0.24 > 0.22 = |x_1 - y_1|^2 + \frac{1}{15}|(I - \Lambda_1)x_1 - (I - \Lambda_1)y_1|^2$, and $|\Lambda_2 x_2 - \Lambda_2 y_2|^2 = 0.32 > 0.22 = |x_2 - y_2|^2 + \frac{1}{15}|(I - \Lambda_1)x_2 - (I - \Lambda_1)y_2|^2$.

5 Conclusion

In this work, we introduce the definition of a $G - \kappa$-strictly pseudocontractive mapping that is different from a κ-strictly pseudocontractive mapping and prove a strong convergence theorem for finding the fixed points of two G-κ-strictly pseudocontractive mappings, two G-nonexpansive mappings, and two G-variational inequality problems in a

Hilbert space endowed with a directed graph. However, we should like to note the following:

(1) Our result is proved without the Property G.

(2) We give some examples to support our main theorem and we have the numerical result to approximate the value of Π as shown in Table 1 and Fig. 1.

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Author contributions
AK dealt with the conceptualization, formal analysis, supervision, writing-review and editing. AS writing-original draft, formal analysis, writing-review and editing. Both authors have read and approved the manuscript.

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