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Meromorphic functions sharing three values with their derivatives in an angular domain

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Abstract

In this paper, we investigate the problem of uniqueness transcendental meromorphic functions sharing three values with their derivatives in an arbitrary small angular domain including a singular direction. The obtained results extend the corresponding results from Gundersen, Mues–Steinmetz, Zheng, Li–Liu–Yi, and Chen.

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1 Introduction and main result

Let $f : C \to \hat{C} = C \cup \{\infty\}$ be a meromorphic function, where C is the complex plane. We assume that the reader is familiar with the basic results and notations of Nevanlinna's value distribution theory (see [6, 14, 15]) such as $T(r;f)$, $N(r,f)$, and $m(r,f)$. Meanwhile, the lower order μ and the order λ of a meromorphic function f are defined as follows:

$$\mu := \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\lambda := \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Let f and g be nonconstant meromorphic functions in the domain $D \subseteq C$. If $f - c$ and $g - c$ have the same zeros with the same multiplicities in D, then $c \in C \cup \{\infty\}$ is called a *CM* shared value in D of f and g. If $f - c$ and $g - c$ have the same zeros in D, then $c \in C \cup \{\infty\}$ is called an *IM* shared value in D of f and g. The zeros of $f - c$ imply the poles of f when $c = +\infty$.

In 1979, Gundersen [5] and Mues and Steinmetz [10] considered the uniqueness of a meromorphic function f and its derivative f' and obtained the following result.

Theorem A *Let f be a nonconstant meromorphic function in C, and let a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers. If f and f' IM share a_j $(j = 1, 2, 3)$, then $f \equiv f'$.*

Later on, Frank and Schwick [3] generalized this result and proved the following result.

Theorem B *Let f be a nonconstant meromorphic function, and let k be a positive integer. If there exist three distinct finite complex numbers a, b, and c such that f and $f^{(k)}$ IM share a, b, c, then $f \equiv f^{(k)}$.*

In 2004, Zheng [16] first considered the uniqueness question of meromorphic functions with shared values in an angular domain and proved the following result (see [16, Theorem 3]).

Theorem C *Let f be a transcendental meromorphic function of finite lower order and such that $\delta = \delta(a, f^{(p)}) > 0$ for some $a \in C \cup \{\infty\}$ and an integer $p \geq 0$. Let the pairs of real numbers $\{\alpha_j, \beta_j\}$ $(j = 1, \dots, q)$ be such that*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_q < \beta_q \leq \pi,$$

with $\omega = \max\{\frac{\pi}{\beta_j - \alpha_j} : 1 \leq j \leq q\}$, and

$$\sum_{j=1}^{q} (\alpha_{j+1} - \beta_j) < \frac{4}{\delta} \arcsin \sqrt{\delta\bigl(a, f^{(p)}\bigr)/2},$$

where $\delta = \max\{\omega, \mu\}$. For a positive integer k, assume that f and $f^{(k)}$ IM share three distinct finite complex numbers a_j $(j = 1, 2, 3)$ in $X = \bigcup_{l=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f \equiv f^{(k)}$.

In 2015, Li, Liu, and Yi [8] observed that Theorem C is invalid for $q \geq 2$ and proved the following more general result, which extends Theorem C (see [8, p. 443]).

Theorem D ([8]) *Let f be a transcendental meromorphic function of finite lower order $\mu(f)$ in C such that $\delta(a, f) > 0$ for some $a \in C$. Assume that $q \geq 2$ pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfy the conditions*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_q < \beta_q \leq \pi,$$

with $\omega = \max\{\frac{\pi}{(\beta_j - \alpha_j)} : 1 \leq j \leq q\}$, and

$$\sum_{j=1}^{q} (\alpha_{j+1} - \beta_j) < \frac{4}{\delta} \arcsin \sqrt{\delta(a, f)/2}, \tag{1.1}$$

where $\delta = \max\{\omega, \mu\}$. For a kth-order linear differential polynomial $L[f]$ in f with constant coefficients given by

$$L[f] = b_k f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_1 f', \tag{1.2}$$

where k is a positive integer, and $b_k \neq 0, b_{k-1}, \cdots, b_1$ are constants, assume that f and $L[f]$ IM share a_j $(j = 1, 2, 3)$ in

$$X = \bigcup_{l=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\},$$

where a_j $(j = 1,2,3)$ *are three distinct finite complex numbers such that* $a \neq a_j$ $(j = 1,2,3)$. *If* $\lambda(f) \neq \omega$, *then* $f = L[f]$.

In 2019, Chen [1] proved the following result.

Theorem E *Let* f *be a nonconstant meromorphic function of lower order* $\mu(f) > 1/2$ *in* C, *let* a_j $(j = 1,2,3)$ *be three distinct finite complex numbers, and let* $L[f]$ *be given by Theorem* D. *Then there exists an angular domain* $D = \{z : \alpha \leq \arg z \leq \beta\}$, *where* $0 \leq \beta - \alpha \leq 2\pi$, *such that if* f *and* $L[f]$ *CM share* a_j $(j = 1,2,3)$ *in* D, *then* $f = L[f]$.

Question 1.1 From Theorems C–E a natural question arises: whether we can get the corresponding results if the restriction of f on deficiency and lower order is removed or if restriction (1.1) for the width of the angular domain is removed. What is the relationship between these angular regions and the value distribution properties of f?

In theory of meromorphic functions, a function is uniquely determined by its value on a set with a accumulation point. It is natural to ask if we can prove similar results under the conditions

$$\bar{E}_D(f, a_j) = \bar{E}_D(f', a_j), \quad j = 1,2,3,$$

for some typical set in C instead of a general angular domain in C, where $\bar{E}_D(a,f) = \{z : z \in D, f(z) = a\}$ (as a set in C).

In general, the answer of this question is negative. For $f(z) = e^{2z}$, it is clear that $f(z) \neq f'(z)$, but $|f(z)|$ is bounded by 1 on the left-half plane D. Thus

$$\bar{E}_D(f, n) = \bar{E}_D(f', n) = \emptyset \quad \text{for all } n > 1.$$

This example shows us that if such an angular domain D exists, then it must be a region whose image under f is dense in C.

Based on the theory on singular direction for a meromorphic function (see [14]) and the research results on shared values of a meromorphic function (see [9, 11]), combining with Theorems D and E, we may conjecture that the angular domain of the singular direction may be right. In this paper, we investigate the above question and prove the following result, which extends Theorems D and E.

Theorem 1.1 *Let* f *be a meromorphic function of finite order that satisfies* $\lim_{r\to\infty}\sup \frac{T(r,f)}{(\log r)^3} = +\infty$, *and let* ε *be an arbitrary small positive number. Then there exists a direction* $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ *such that if* f *and* f' *IM share three distinct finite complex numbers* a_j $(j = 1,2,3)$ *in* $A(\theta_0, \varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$, *then* $f \equiv f'$.

Theorem 1.2 *Let* f *be a meromorphic function of finite order that satisfies* $\lim_{r\to\infty}\sup \frac{T(r,f)}{(\log r)^3} = +\infty$, *let* ε *be an arbitrary small positive number, and let* k *be a positive integer, Then there exists a direction* $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ *such that if* f *and* $f^{(k)}$ *CM share three distinct finite complex numbers* a_j $(j = 1,2,3)$ *in* $A(\theta_0, \varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$, *then* $f \equiv f^{(k)}$.

To prove our main results, we introduce some notations about the Ahlfors–Shimizu character of a meromorphic function in C:

$$T_0(r,f) = \int_0^r \frac{A(t)}{t}\,dt, \quad A(t) = \frac{1}{\pi}\int_0^{2\pi}\int_0^t \left(\frac{|f'(\rho e^{i\theta})|}{1+|f(\rho e^{i\theta})|^2}\right)^2 d\rho\, d\theta.$$

Nevanlinna theory in an angular domain plays an important role in this paper,so we recall its fundamental notations. Let f be a meromorphic function in $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 \le \beta - \alpha \le 2\pi$. Nevanlinna [4] defined the following symbols:

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+\left|f\left(re^{i\theta}\right)\right|\sin\omega(\theta-\alpha)\,d\theta,$$

$$C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_m|<r}\left(\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_m - \alpha),$$

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f),$$

where $\omega = \frac{\pi}{(\beta-\alpha)}$, and $b_m = |b_m|e^{i\theta_m}$ are the poles of f in D counting multiplicities.

2 Preliminaries

In this section, we prove some lemmas, which will be used in the proof of the main result.

Lemma 2.1 ([2, 12]) *Let $\mathcal{F}$ be a family of meromorphic functions such that for every function $f \in \mathcal{F}$, its zeros of multiplicity are at least k. If $\mathcal{F}$ is not a normal family at the origin 0, then for $0 \le \alpha \le k$, there exist*

(a) *a real number r $(0 < r < 1)$,*
(b) *a sequence of complex numbers $z_n \to 0$, $|z_n| < r$,*
(c) *a sequence of functions $f_n \in \mathcal{F}$, and*
(d) *a sequence of positive numbers $\rho_n \to 0$*

such that

$$g_n(z) = {\rho_n}^{-\alpha} f_n(z_n + \rho_n z)$$

converges locally uniformly with respect to spherical metric to a nonconstant meromorphic function $g(z)$ on $\mathbf{C}$. *Moreover, g is of order at most two.*

For convenience, we use the following notation:

$$LD(r,f:c_1,c_2,c_3) = c_1\left[\sum_{i=1}^4 m\left(r,\frac{f'}{f-a_i}\right)\right] + c_2\left[\sum_{i=1}^4 m\left(r,\frac{f''}{f'-b_i}\right)\right] + c_3\left[\sum_{i=1}^4 m\left(r,\frac{f^{(k+1)}}{f^{(k)}-d_i}\right)\right],$$

where a_i, b_i, c_i, d_i $(i = 1,2,3,4)$ are finite complex numbers, and k is an integer such that $k \ge 2$.

Lemma 2.2 ([11]) *Let f be a meromorphic function in a domain $D = \{z : |z| < R\}$, let a_j $(j = 1,2,3)$ be three distinct finite complex numbers, let t be a positive real number, and let $a \in C$. If*

$$\bar{E}_D(a_j,f) = \bar{E}_D(ta_j,f') \quad \text{for } j = 1,2,3,$$

$a \neq a_j, f(0) \neq a_j, \infty$ $(j = 1,2,3)$, $f'(0) \neq 0, at, f''(0) \neq 0$, and $f'(0) \neq tf(0)$, then for $0 < r < R$, we have

$$\begin{aligned} T(r,f) \le{} & LD(r,f:2,3,0) + \log \frac{\prod_{i=1}^{3} |f(0)-a_i|^2 |f'(0)-ta_i|^3}{|tf(0)-f'(0)|^5 |f'(0)|^2} \\ & + 3\log \frac{1}{|f''(0)|} + \left(\log^+ t + m\left(r, \frac{f''}{f'-ta}\right) + 1\right) O(1), \end{aligned}$$

where $\bar{E}_D(a,f) = \{z : z \in D, f(z) = a\}$ (as a set in $\mathbf{C}$), and $O(1)$ is a complex number depending only on a and a_i $(i = 1,2,3)$.

Lemma 2.3 ([13]) *Let f, g be nonconstant meromorphic functions in the unit disc thath IM share distinct finite complex numbers a_1, a_2, a_3, and $a_4 = \infty$. If $a \neq a_j, f(0) \neq a, a_j$ $(j = 1,2,3,4)$, $f'(0) \neq 0, \infty$, and $f(0) \neq g(0)$, then*

$$\begin{aligned} T(r,f) \le{} & T(r,g) + \log \frac{\prod_{i=1}^{3} |f(0)-a_i|}{|f'(0)||f(0)-g(0)|} \\ & + O(1)\left[m\left(r, \frac{f'}{f-a}\right) + \sum_{i=1}^{3} m\left(r, \frac{f'}{f-a_i}\right) + 1\right], \end{aligned}$$

where $O(1)$ is a complex number depending only on a and a_i $(i = 1,2,3)$.

Lemma 2.4 *Let f be a meromorphic function in a domain $D = \{z : |z| < R\}$, let a_1, a_2, a_3 be three distinct finite complex numbers, and let t be a positive real number. If*

$$E_D(a_i,f) = E_D(ta_i,f^{(k)}) \quad \text{for } i = 1,2,3,$$

$a \neq a_j, f(0) \neq a_j, \infty$ $(j = 1,2,3)$, $f^{(k)}(0) \neq 0, at, f^{(k+1)}(0) \neq 0$, and $f^{(k)}(0) \neq tf(0)$, then for $0 < r < R$, we have

$$\begin{aligned} T(r,f) \le{} & LD(r,f:1,0,1) + (k+1)\log \frac{\prod_{i=1}^{3} |f(0)-a_i|^2 |f^{(k)}(0)-ta_i|^3}{|tf(0)-f^{(k)}(0)|^5 |f^{(k)}(0)|^2} \\ & + 3(k+1)\log \frac{1}{|f^{(k+1)}(0)|} + \left(\log^+ t + m\left(r, \frac{f^{(k+1)}}{f^{(k)}-ta}\right) + 1\right) O(1), \end{aligned}$$

where $E_D(a,f) = \{z \in D : f(z) = a, \text{counting multiplicity}\}$, and $O(1)$ is a complex number depending only on a and a_i $(i = 1,2,3)$.

Proof Since $E_D(a_i,f) = E_D(ta_i,f^{(k)})$ $(i = 1,2,3)$ with $t \neq 0$, from the assumptions we see that $f^{(k)}(z) \not\equiv tf(z)$. Therefore by the Nevanlinna basic theorem we have

$$\begin{aligned}
&\sum_{j=1}^{3} N\left(r,\frac{1}{f-a_j}\right)\\
&\quad\leq N\left(r,\frac{1}{tf-f^{(k)}}\right)\\
&\quad\leq T\bigl(r,tf-f^{(k)}\bigr)+\log\frac{1}{|tf(0)-f^{(k)}(0)|}=m\bigl(r,tf-f^{(k)}\bigr)+N\bigl(r,tf-f^{(k)}\bigr)\\
&\quad\leq N\bigl(r,f^{(k)}\bigr)+m(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)+\log^+ t+O(1)+\log\frac{1}{|tf(0)-f^{(k)}(0)|}\\
&\quad\leq T(r,f)+k\bar{N}(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)+\log^+ t+O(1)+\log\frac{1}{|tf(0)-f^{(k)}(0)|}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=1}^{3} m\left(r,\frac{1}{f-a_j}\right)&=m\left(r,\frac{1}{f^{(k)}}\sum_{j=1}^{3}\frac{f^{(k)}}{f-a_j}\right)+O(1)\\
&\leq m\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,\sum_{j=1}^{3}\frac{f^{(k)}}{f-a_j}\right)+O(1).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\sum_{j=1}^{3} T\left(r,\frac{1}{f-a_j}\right)&=\sum_{j=1}^{3} m\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{3} N\left(r,\frac{1}{f-a_j}\right)\\
&\leq T(r,f)+k\bar{N}(r,f)+m\left(r,\frac{1}{f^{(k)}}\right)+LD(r,f:1,0,0)\\
&\quad+\log^+ t+\log\frac{1}{|(tf-f^{(k)})(0)|}+O(1).
\end{aligned}$$

Noting that $m(r,\frac{1}{f^{(k)}}) \leq T(r,\frac{1}{f^{(k)}}) = T(r,f^{(k)}) + \log\frac{1}{|f^{(k)}(0)|}$, by Nevanlinna's first fundamental theorem we obtain

$$\begin{aligned}
2T(r,f)\leq{}&T\bigl(r,f^{(k)}\bigr)+k\bar{N}(r,f)+LD(r,f:1,0,0)\\
&+\log\frac{\prod_{i=1}^{3}|f(0)-a_i|}{|(tf-f^{(k)})(0)||f^{(k)}(0)|}+O(1)+\log^+ t.
\end{aligned}$$

Since $T(r,f^{(k)}) \geq N(r,f^{(k)}) = N(r,f) + k\bar{N}(r,f) \geq (k+1)\bar{N}(r,f)$, implying that $\bar{N}(r,f) \leq T(r,f^{(k)})/(k+1)$, we have

$$\begin{aligned}
2T(r,f)\leq{}&\frac{2k+1}{k+1}T\bigl(r,f^{(k)}\bigr)+LD(r,f:1,0,0)\\
&+\log\frac{\prod_{i=1}^{3}|f(0)-a_i|}{|(tf-f^{(k)})(0)||f^{(k)}(0)|}+O(1)+\log^+ t.
\end{aligned}\tag{2.1}$$

On the other hand, note that from $\bar{E}_D(a_i,f) = \bar{E}_D(ta_i,f^{(k)})$, $i = 1,2,3$, $\bar{E}_D(\infty,f) = \bar{E}_D(\infty,f^{(k)})$, and $f(0) \neq a_j,\infty$ $(j = 1,2,3)$ it follows that $f^{(k)}(0) \neq ta_j,\infty$ $(j = 1,2,3)$. By application of Lemma 2.3 to $f^{(k)}$ and tf we have

$$\begin{aligned} T\bigl(r,f^{(k)}\bigr) &\leq T(r,f) + LD(r,f:0,0,1)) + \log\frac{\prod_{i=1}^{3}|f^{(k)}(0)-ta_i|}{|tf(0)-f^{(k)}(0)||f^{(k+1)}(0)|} \\ &\quad + \left(\log^+ t + m\left(r,\frac{f^{(k+1)}}{f^{(k)}-ta}\right)+1\right)O(1). \end{aligned} \tag{2.2}$$

Now substituting (2.2) into (2.1) we have

$$\begin{aligned} \frac{T(r,f)}{k+1} &\leq LD(r,f:1,0,1) + \log\frac{\prod_{i=1}^{3}|f(0)-a_i||f^{(k)}(0)-ta_i|}{|f^{(k+1)}(0)||tf(0)-f^{(k)}(0)|^2|f^{(k)}(0)|} \\ &\quad + \left(\log^+ t + m\left(r,\frac{f^{(k+1)}}{f^{(k)}-ta}\right)+1\right)O(1). \end{aligned}$$

Hence we have

$$\begin{aligned} T(r,f) &\leq LD(r,f:1,0,1) + (k+1)\log\frac{\prod_{i=1}^{3}|f(0)-a_i|^2|f^{(k)}(0)-ta_i|^3}{|tf(0)-f^{(k)}(0)|^5|f^{(k)}(0)|^2} \\ &\quad + 3(k+1)\log\frac{1}{|f^{(k+1)}(0)|} + \left(\log^+ t + m\left(r,\frac{f^{(k+1)}}{f^{(k)}-ta}\right)+1\right)O(1). \end{aligned}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5 ([7]) *Let $f(z)$ be a meromorphic function in* $\mathbf{C}$. *Let*

$$\beta_p(r) = \sup_{2\leq t\leq r}\left\{\frac{T_0(t,f)}{(\log t)^p}\right\}, \qquad \varepsilon(r) = \left\{\frac{1}{\beta_p(r)}\right\}^{\frac{1}{q}}$$

with $p \geq 2$ and $q \geq 3$. If $\lim_{r\to\infty}\beta_p(r) = \infty$, then there exist a sequence of positive numbers $\{r_n\}_1^\infty$ and a sequence of points $\{z_n\}_1^\infty$ in $\mathbf{C}$ *such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty}|z_n| = +\infty$ and*

$$A\bigl(\varepsilon\bigl(|z_n|\bigr)|z_n|,z_n,f\bigr) \geq \frac{1}{64\pi^2}\bigl\{\beta_p(r_n)\bigr\}^{1-\frac{2}{q}}(\log r_n)^{p-2} \quad (n = 1,2,\ldots), \tag{2.3}$$

where

$$A(r,a,f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^r\left(\frac{|f'(a+\rho e^{i\theta})|}{1+|f(a+\rho e^{i\theta})|^2}\right)^2 d\rho\, d\theta, \quad |z_n| \leq r_n,$$

and

$$T_0(r,f) = \int_0^r \frac{A(t)}{t}\,dt, \quad A(t) = \frac{1}{\pi}\int_0^{2\pi}\int_0^t\left(\frac{|f'(\rho e^{i\theta})|}{1+|f(\rho e^{i\theta})|^2}\right)^2 d\rho\, d\theta.$$

Lemma 2.6 *Let $f(z)$ be a meromorphic function satisfying the conditions of Lemma 2.5. Then there exist a direction $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$, a sequence of points $\{z_n\}$ $(|z_n| \to \infty)$ with $\lim_{n\to\infty}\arg z_n = \theta_0$, and a sequence of real numbers r_n with $\lim_{n\to\infty} r_n = +\infty$ such that (2.3) holds.*

Proof Set $z_n = |z_n|e^{i\theta_n}$ $(0 \le \theta_n < 2\pi)$ in Lemma 2.5. Since $\{\theta_n\}$ is a bounded sequence, there exists convergent subsequence, still denoted $\{\theta_n\}$. Set $\theta_n \to \theta_0$ $(n \to \infty)$. Thus the lemma follows. □

We say that the direction $\arg z = \theta_0$ is an H direction of $f(z)$.

Lemma 2.7 ([6, 17]) *Let $f(z)$ be a meromorphic function in a domain $D = \{z : |z| < R\}$. If $f(0) \neq \infty$, then for $0 < r < R$, we have*

$$\left|T(t,f) - T_0(t,f) - \log^+\left|f(0)\right|\right| \le \frac{1}{2}\log 2,$$

where $\log^+|f(0)|$ is replaced by $\log|c(0)|$ when $f(0) = \infty$, $c(0)$ is the coefficient of the Laurent series of $f(z)$ at 0, and $T_0(t,f)$ is defined in (1.2).

Lemma 2.8 ([9]) *Let $f(z)$ be a nonconstant meromorphic function in the complex plane, and let a_1, a_2, a_3 be three distinct finite complex numbers. Assume that f and f' IM share a_i $(i = 1,2,3)$ in $\Omega(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta < 2\pi$. Then one of the following two cases holds:* (i) *$f \equiv f'$, or* (ii) *$S_{\alpha,\beta}(r,f) = Q(r,f)$, where $Q(r,f)$ is a quantity such that if $f(z)$ is of finite order, then $Q(r,f) = O(1)$ as $r \to \infty$, and if $f(z)$ is of infinite order, then $Q(r,f) = O(\log(rT(r,f)))$ as $r \notin E$ and $r \to \infty$, where E is a set of positive real numbers with finite linear measure.*

Lemma 2.9 ([4, 8]) *Let f be a meromorphic function on $\overline{\Omega}(\alpha,\beta)$. If $S_{\alpha,\beta}(r,f) = O(1)$, then*

$$\log\left|f\left(re^{i\phi}\right)\right| = r^{\omega}c\sin\left(\omega(\phi - \alpha)\right) + o\left(r^{\omega}\right)$$

uniformly for $\alpha \le \phi \le \beta$ as $r \notin F$ and $r \to \infty$, where c is a positive constant, $\omega = \frac{\pi}{\beta-\alpha}$, F is a set of finite logarithmic measure, and $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \le \arg z \le \beta\}$.

Lemma 2.10 ([1]) *Let f be a meromorphic function in C, let a_j $(j = 1,2,3)$ be three distinct finite complex numbers, and let $L[f]$ be given by (1.2). Suppose that f and $L[f]$ CM share a_j $(j = 1,2,3)$ in $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. If $f \not\equiv L[f]$, then $S_{\alpha,\beta}(r,f) = R(r,f)$, where $R(r,f)$ is a quantity such that if $f(z)$ is of finite order, then $R(r,f) = O(1)$ as $r \to \infty$, and if $f(z)$ is of infinite order, then $R(r,f) = O(\log(rT(r,f)))$ as $r \notin E$ and $r \to \infty$, where E is a set of positive real numbers with finite linear measure.*

Lemma 2.11 ([14]) *Let $f(z)$ be a meromorphic function in disc $D(0,R)$ centered at 0 with radius R. If $f(0) \neq 0,\infty$, then for $0 < r < \rho < R$, we have*

$$m\left(r,\frac{f^{(k)}}{f}\right) < c_k\left\{1 + \log^+\log^+\left|\frac{1}{f(0)}\right| + \log^+\frac{1}{r} + \log^+\frac{1}{\rho - r} + \log^+\rho + \log^+ T(\rho,f)\right\},$$

where k is a positive integer, and c_k is a constant depending only on k.

Lemma 2.12 ([14]) *Let $T(r)$ be a continuous nondecreasing nonnegative function, and let $a(r)$ be a nonincreasing nonnegative function on $[r_0,R]$ $(0 < r_0 < R < \infty)$. If there exist constants b, c such that*

$$T(r) < a(r) + b\log^+\frac{1}{\rho - r} + c\log^+ T(\rho)$$

for $r_0 < r < \rho < R$, then

$$T(r) < 2a(r) + B\log^+ \frac{2}{R-r} + C,$$

where B, C are two constants depending only on b, c.

The following inequalities in Lemmas 2.13 and 2.14 play an important role in the proof of the theorem.

Lemma 2.13 *Let $f(z)$ be a meromorphic function with finite order $\lambda > 0$, let $\arg z = \theta_0$ be a direction, let $\Gamma_n = \{z|z - z_n| < \varepsilon_n\}$ $(n = 1,2,\ldots)$ be a series of circles, where $z_n = |z_n|e^{i\theta_n}$, $\theta_n \to \theta_0$, $\lim_{n\to\infty} |z_n| = +\infty$, $\varepsilon_n = \epsilon_n|z_n|$, and $\lim_{n\to\infty} \epsilon_n = 0$. Suppose that f and f' IM share three distinct finite complex numbers a_j $(j = 1,2,3)$ in $A(\theta_0,\varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$. If $f \not\equiv f'$, then for every sufficiently large n $(n \geq n_0)$,*

$$A(\varepsilon_n, z_n, f) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr). \tag{2.4}$$

Proof Set $f_n(z) = f(z_n + \varepsilon_n z)$. We distinguish two cases.

Case 1. Assume that $f_n(z)$ is normal in $|z| \leq 1$, implying that

$$\frac{|f_n'(z)|}{1 + |f_n(z)|^2} = \frac{\varepsilon_n|f'(z_n + \varepsilon_n z)|}{1 + |f(z_n + \varepsilon_n z)|^2} \leq M \quad (n = 1,2,\ldots)$$

in $|z| \leq 1$, where M is a positive number. Then we have

$$A(\varepsilon_n, z_n, f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon_n}\left(\frac{|f'(z_n + \rho e^{i\theta})|}{1 + |f(z_n + \rho e^{i\theta})|^2}\right)^2 \rho\, d\rho\, d\theta \leq 2M^2.$$

So (2.4) holds.

Case 2. Assume that $f_n(z)$ is not normal in $|z| \leq 1$. By Lemma 2.1 there exist

(1) a sequence of points $\{z_n'\} \subset \{|z| < 1\}$;

(2) a subsequence of $\{f_n(z)\}_1^\infty$ (without loss of generality, we still denote it by $\{f_n(z)\}$); and

(3) positive numbers ρ_n with $\rho_n \to 0$ $(n \to \infty)$ such that

$$h_n(z) = f_n\bigl(z_n' + \rho_n z\bigr) \to g(z) \tag{2.5}$$

in spherical metric uniformly on a compact subset of $\mathbf{C}$ as $n \to \infty$, where $g(z)$ is a nonconstant meromorphic function. Thus for any positive integer k, we have

$$h_n^{(k)}(\xi) = {\rho_n}^k f_n^{(k)}\bigl(z_n' + \rho_n\xi\bigr) \to g^{(k)}(\xi).$$

We claim that $g''(\xi) \not\equiv 0$. Otherwise, $g(z) = cz + d$ ($c, d \in \mathbf{C}$ and $c \neq 0$). We can choose ξ_0 with $g(\xi_0) = a_1$. By Hurwitz's theorem there exists a sequence $\xi_n \to \xi_0$ such that

$$h_n(\xi_n) = f_n\bigl(z_n' + \rho_n\xi_n\bigr) = g(\xi_0) = a_1.$$

Notice that f and f' IM share a_1 in $\{z : |\arg z - \theta_0| < \varepsilon\}$ and $s \neq \infty$, so we have

$$\begin{aligned} c = g'(\xi_0) &= \lim_{n\to\infty} h'_n(\xi_n) = \lim_{n\to\infty} \rho_n\varepsilon_n f'\bigl(z_n + \varepsilon_n\bigl(z'_n + \rho_n\xi_n\bigr)\bigr) \\ &= \lim_{n\to\infty} \rho_n\varepsilon_n f\bigl(z_n + \varepsilon_n\bigl(z'_n + \rho_n\xi_n\bigr)\bigr) = \lim_{n\to\infty} \rho_n\varepsilon_n a_1, \end{aligned}$$

and thus

$$\lim_{n\to\infty} \rho_n\varepsilon_n = \frac{c}{a_1}.$$

Likewise, we get

$$\lim_{n\to\infty} \rho_n\varepsilon_n = \frac{c}{a_2},$$

which gives a contradiction.

For a sequence of positive numbers $\rho_n\varepsilon_n$, it is easy to snow that there exists a subsequence, still denoted by $\rho_n\varepsilon_n$, such that $\lim_{n\to\infty} \rho_n\varepsilon_n = a_0$, where $a_0 \in [0, +\infty) \cup \{+\infty\}$. Now we consider two cases: $a_0 = 0$ or $+\infty$, and $0 < a_0 < +\infty$.

Case 1. Assume that $\lim_{n\to\infty} \rho_n\varepsilon_n = 0$ or ∞.

We choose $\xi_0 \in C$ such that

$$g(\xi_0) \neq 0, a_1, a_2, a_3, \infty, \qquad g'(\xi_0) \neq 0, \infty, \qquad g''(\xi_0) \neq 0, \infty.$$

Let

$$p_n(z) = f_n\bigl(z'_n + \rho_n\xi_0 + z\bigr)$$

for arbitrary small $\varepsilon > 0$. In view of

$$\overline{E}_{A(\theta_0,\varepsilon)}(a_j, f) = \overline{E}_{A(\theta_0,\varepsilon)}\bigl(a_j, f'\bigr), \quad j = 1, 2, 3,$$

and $\lim_{n\to\infty} \epsilon_n = 0$, for sufficiently large n, we have

$$\Gamma_n = \bigl\{z|z - z_n| < \epsilon_n|z_n|, z_n = |z_n|e^{i\theta_0}\bigr\} \subseteq A(\theta_0, \varepsilon/2).$$

Therefore for every sufficiently large n ($n \geq n_0$), on $|z| \leq 4$, we have

$$\bar{E}_D\bigl(a_i, p_n(z)\bigr) = \bar{E}_D\bigl(\varepsilon_n a_i, p'_n(z)\bigr) \quad (i = 1, 2, 3).$$

Note that

$$\begin{aligned} p_n(0) &= f_n\bigl(z'_n + \rho_n\xi_0\bigr) = h_n(\xi_0) \to g(\xi_0) \neq a_1, a_2, a_3, \infty, \\ p'_n(0) &= f'_n\bigl(z'_n + \rho_n\xi_0\bigr) = \frac{h'_n(\xi_0)}{\rho_n}, \qquad h'_n(\xi_0) \to g'(\xi_0), \\ p''_n(0) &= f''_n\bigl(z'_n + \rho_n\xi_0\bigr) = \frac{h''_n(\xi_0)}{\rho_n^2}, \qquad h''_n(\xi_0) \to g''(\xi_0), \end{aligned}$$

$$\varepsilon_n p_n(0) - p_n'(0) = \frac{\varepsilon_n \rho_n h_n(\xi_0) - h_n'(\xi_0)}{\rho_n}.$$

Thus we have

$$\begin{aligned}
&\log \frac{\prod_{i=1}^{3} |p_n(0) - a_i|^2 |p_n'(0) - \varepsilon_n a_i|^3}{|\varepsilon_n p_n(0) - p_n'(0)|^5 |p_n'(0)|^2} + 3\log \frac{1}{|p_n''(0)|} \\
&\quad = \log \frac{\prod_{i=1}^{3} |p_n(0) - a_i|^2 |p_n'(0) - \varepsilon_n a_i|^3}{|\varepsilon_n p_n(0) - p_n'(0)|^5 |p_n'(0)|^2 |p_n''(0)|^3} \\
&\quad = 4\log \rho_n + \log \frac{\prod_{i=1}^{3} |h_n(\xi_0) - a_i|^2 |h_n'(\xi_0) - \rho_n \varepsilon_n a_i|^3}{|\rho_n \varepsilon_n h_n(\xi_0) - h_n'(\xi_0)|^5 |h_n'(\xi_0)|^2 |h_n''(\xi_0)|^3}.
\end{aligned} \tag{2.6}$$

Since $\lim_{n\to\infty} \rho_n \varepsilon_n = 0$ or ∞, we deduce

$$\begin{aligned}
&\lim_{n\to\infty} \log \frac{\prod_{i=1}^{3} |h_n(\xi_0) - a_i|^2 |h_n'(\xi_0) - \rho_n \varepsilon_n a_i|^3}{|\rho_n \varepsilon_n h_n(\xi_0) - h_n'(\xi_0)|^5 |h_n'(\xi_0)|^2 |h_n''(\xi_0)|^3)} \\
&\quad \leq \log \frac{\prod_{i=1}^{3} |g(\xi_0) - a_i|^2}{|g'(\xi_0)|^{-2} |g''(\xi_0)|^3} \quad \text{as } n \to \infty.
\end{aligned} \tag{2.7}$$

Applying Lemma 2.2 to $p_n(z)$ with (2.6) and (2.7), we have

$$T(r,p_n) \leq LD(r,p_n;2,3,0) + O(1)\left(\log^+ |z_n| + m\left(r, \frac{p_n''}{p_n' - \varepsilon_n a}\right) + 1\right)$$

for $0 < r \leq 3$ and sufficiently large n, where $a \neq a_j$ ($j = 1,2,3$) and $a \in C$.
By Lemmas 2.11 and 2.12 we have

$$T(r,p_n) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr).$$

In view of Lemma 2.8, we obtain

$$T_0(r,p_n) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr).$$

Thus we get

$$T_0\bigl(3\varepsilon_n, z_n + \varepsilon_n\bigl(z_n' + \rho_n \xi_0\bigr), f\bigr) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr).$$

It follows that

$$A\bigl(2\varepsilon_n, z_n + \varepsilon_n\bigl(z_n' + \rho_n \xi_0\bigr), f\bigr) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr).$$

Noting that $z_n' + \rho_n \xi_0 \to 0$, we get

$$\bigl\{z : |z - z_n| < \varepsilon_n\bigr\} \subseteq \bigl\{z : \bigl|z - z_n - \varepsilon_n\bigl(z_n' - \rho_n \xi_0\bigr)\bigr| < 2\varepsilon_n\bigr\}.$$

Therefore we have

$$A(\varepsilon_n, z_n, f) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr).$$

Case 2. Assume that $\lim_{n\to\infty} \rho_n\varepsilon_n = a_0 \neq 0, \infty$.

Now we distinguish two subcases, $a_0 g(z) \not\equiv g'(z)$ and $a_0 g(z) \equiv g'(z)$.

Case 2.1. $a_0 g(z) \not\equiv g'(z)$.

We can choose $\xi_0 \in C$ such that

$$g(\xi_0) \neq 0, a_1, a_2, a_3, \infty, \qquad g'(\xi_0) \neq 0, \infty,$$
$$g''(\xi_0) \neq 0, \infty, \qquad a_0 g(\xi_0) - g'(\xi_0) \neq 0, \infty.$$

Let

$$p_n(z) = f_n\bigl(z'_n + \rho_n \xi_0 + z\bigr).$$

By the same arguments as in case 1, we can get

$$A(\varepsilon_n, z_n, f) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr).$$

Case 2.2. $a_0 g(z) \equiv g'(z)$.

We can derive that $g(z) = e^{a_0 z + b_0}$, where $b_0 \in C$. From (2.5) we obtain

$$h_n(z) = f_n\bigl(z'_n + \rho_n z\bigr) = f\bigl(z_n + \varepsilon_n\bigl(z'_n + \rho_n z\bigr)\bigr) = f\bigl(z_n + \varepsilon_n z'_n + \varepsilon_n \rho_n z\bigr) \to g(z) \tag{2.8}$$

in spherical metric uniformly on compact subsets of $\mathbf{C}$ as $n \to \infty$,

On the other hand, noting that f and f' share a_i $(i = 1, 2, 3)$ in $A(\theta_0, \varepsilon)$ and $f \not\equiv f'$, by Lemma 2.8 we have $S_{\theta-\varepsilon,\theta+\varepsilon}(r, f) = O(1)$. Therefore, applying Lemma 2.9 to f in $A(\theta_0, \varepsilon)$ we obtain

$$\log\bigl|f\bigl(re^{i\phi}\bigr)\bigr| = r^{\omega} c \sin\bigl(\omega(\phi - \alpha)\bigr) + o\bigl(r^{\omega}\bigr)$$

uniformly for $\theta_0 - \varepsilon = \alpha \leq \phi \leq \beta = \theta_0 + \varepsilon$ as $r \notin F$ and $r \to \infty$, where c is a positive constant, $\omega = \frac{\pi}{\beta - \alpha} = \frac{\pi}{2\varepsilon}$, and F is a set of finite logarithmic measure.

Since F is a set of finite logarithmic measure, there exist a real number R $(0 < R < \infty)$ and a sequence of complex numbers u_n, $0 < |u_n| < R$ for every sufficiently large n, such that

$$\log\bigl|f\bigl(z_n + \varepsilon_n z'_n + \varepsilon_n \rho_n u_n\bigr)\bigr| = r_n^{\omega} c \sin\bigl(\omega(\phi - \alpha)\bigr) + o\bigl(r_n^{\omega}\bigr), \tag{2.9}$$

where $r_n = |z_n + \varepsilon_n z'_n + \varepsilon_n \rho_n u_n| \notin F$, $\phi_n = \arg(z_n + \varepsilon_n z'_n + \varepsilon_n \rho_n u_n)$, $\theta_0 - \varepsilon/2 \leq \phi \leq \theta_0 + \varepsilon/2$, and $\alpha = \theta_0 - \varepsilon$.

By (2.8), $h_n(z) = f_n(z'_n + \rho_n z) \to g(z)$ uniformly on $|z| \leq R$ as $n \to \infty$, and therefore $\lim_{n\to\infty}(f(z_n + \varepsilon_n z'_n + \varepsilon_n \rho_n u_n) - g(u_n)) = 0$. Noting that u_n is a bounded sequence, there exists convergent subsequence, still denoted by u_n. Setting $u_n \to u_0$ $(n \to \infty)$, we have that $\lim_{n\to\infty} g(u_n) = \lim_{n\to\infty} e^{a_0 u_n + b_0} = e^{a_0 u_0 + b_0}$, so it follows that

$$\lim_{n\to\infty} \frac{\log|f(z_n + \varepsilon_n z'_n + \varepsilon_n \rho_n u_n)|}{r_n^{\omega}} = 0.$$

On the other hand, by (2.8) we obtain that

$$\lim_{n\to\infty} \frac{\log|f(z_n + \varepsilon_n z'_n + \varepsilon_n \rho_n u_n)|}{r_n^{\omega}} = \lim_{n\to\infty} c \sin \omega(\phi - \alpha) \geq c \sin \frac{\pi}{4} > 0.$$

We obtain a contradiction, and so case 2.2 is impossible.

This completes the proof of Lemma 2.13. □

Lemma 2.14 *Let $f(z)$ be a meromorphic function with finite order $\lambda > 0$, $\arg z = \theta_0$ be a direction, and let $\Gamma_n = \{z|z - z_n| < \varepsilon_n\}$ $(n = 1, 2, \ldots)$ be a series of circles, where $z_n = |z_n|e^{i\theta_n}$, $\theta_n \to \theta_0$, $\lim_{n\to\infty} |z_n| = +\infty$, and $\varepsilon_n = \epsilon_n|z_n|$, $\lim_{n\to\infty} \epsilon_n = 0$. Suppose that f and $f^{(k)}$ CM share three distinct finite complex numbers a_j $(j = 1, 2, 3)$ in $A(\theta_0, \varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$. If $f \not\equiv f^{(k)}$, then for every sufficiently large n $(n \geq n_0)$,*

$$A(\varepsilon_n, z_n, f) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr), \tag{2.10}$$

where $\varepsilon_n = |z_n|\epsilon_n$.

Proof Suppose that f and $f^{(k)}$ *CM* share three distinct finite complex numbers a_j $(j = 1, 2, 3)$ in $A(\theta_0, \varepsilon)$. Then, as in the proof of Lemma 2.13, by replacing f' in Lemma 2.13 with $f^{(k)}$ and using Lemmas 2.4, 2.10, and 2.9 in $A(\theta_0, \varepsilon)$, we can deduce (2.10). □

3 Proof of Theorem 1.1

Suppose that $f(z) \not\equiv f'(z)$. By Lemma 2.6, there exist a direction $\arg z = \theta_0$ and sequences z_n and r_n such that

$$A\bigl(\varepsilon\bigl(|z_n|\bigr)|z_n|, z_n, f\bigr) \geq \frac{1}{64\pi^2}\bigl\{\beta_p(r_n)\bigr\}^{1-\frac{2}{q}}(\log r_n)^{p-2} \quad (n = 1, 2, \ldots).$$

Set $\varepsilon_n = |z_n|\varepsilon(r_n)$, where $\varepsilon(r_n)$ is defined in (1.2).

For arbitrary small $\varepsilon > 0$, if there are three distinct complex numbers a_1, a_2, a_3 such that

$$\overline{E}_{A(\theta_0,\varepsilon)}(a_j, f) = \overline{E}_{A(\theta_0,\varepsilon)}\bigl(a_j, f'\bigr), \quad j = 1, 2, 3,$$

where $A(\theta_0, \varepsilon) = \{z| \arg z - \theta_0| < \varepsilon\}$, then by Lemma 2.13 the following inequality holds:

$$A(\varepsilon_n, z_n, f) \leq O(1)\bigl(1 + \log^+ |z_n|\bigr), \tag{3.1}$$

where $|z| \leq 1$ and $\varepsilon_n = |z_n|\varepsilon(|z_n|)$. Combining this with (2.3), we have

$$\beta_p(r_n)^{1-\frac{2}{q}}(\log r_n)^{p-2} \leq O(1)\bigl(1 + \log^+ |z_n|\bigr),$$

where $p \geq 3$ and $q \geq 2$.

Taking $p = 3$ and noting that $|z_n| \leq r_n$ and $\lim_{n\to\infty} \beta_p(r_n) = \infty$, we arrive at a contradiction. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Suppose that $f(z) \not\equiv f^{(k)}(z)$. By Lemma 2.6 there exist a direction $\arg z = \theta_0$ and sequences z_n and r_n such that

$$A\bigl(\varepsilon\bigl(|z_n|\bigr)|z_n|, z_n, f\bigr) \geq \frac{1}{64\pi^2}\bigl\{\beta_p(r_n)\bigr\}^{1-\frac{2}{q}}(\log r_n)^{p-2} \quad (n = 1, 2, \ldots).$$

Set $\varepsilon_n = |z_n|\varepsilon(r_n)$, where $\varepsilon(r_n)$ is defined in (1.2).

Next, since f and $f^{(k)}$ CM share three distinct finite complex numbers a_j $(j = 1, 2, 3)$ in $A(\theta_0, \varepsilon) = \{z| \arg z - \theta_0| < \varepsilon\}$, by Lemma 2.14 the following inequality holds:

$$A(\varepsilon_n, z_n, f) \le O(1)\bigl(1 + \log^+ |z_n|\bigr), \tag{4.1}$$

where $|z| \le 1$ and $\varepsilon_n = |z_n|\varepsilon(|z_n|)$. Combining this with (2.3), we have

$$\beta_p(r_n)^{1-\frac{2}{q}}(\log r_n)^{p-2} \le O(1)\bigl(1 + \log^+ |z_n|\bigr),$$

where $p \ge 3$ and $q \ge 2$.

Taking $p = 3$ and noting that $|z_n| \le r_n$ and $\lim_{n\to\infty} \beta_p(r_n) = \infty$, we arrive at a contradiction. This completes the proof of Theorem 1.2.

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