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Further improvement of finite-time boundedness based nonfragile state feedback control for generalized neural networks with mixed interval time-varying delays via a new integral inequality

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Abstract

This article investigates new delay-dependent finite-time boundedness for generalized neural networks (GNNs) with mixed-interval time-varying delays based on nonfragile feedback control to achieve the improved stability criterion. We also propose a new integral inequality with an exponential function to estimate the derivative of the Lyapunov–Krasovskii functional (LKF). Furthermore, the well-known Wirtinger's inequality is a particular case of the new integral inequality. Using a toolbox optimization in MATLAB, we derive and solve new delay-dependent conditions in terms of linear matrix inequalities (LMIs). Additionally, we give three numerical examples to show the advantages of our obtained methods. The examples can apply the continuous time-varying delays that do not need to be differentiable. One of them presents the benchmark problem's real-world application, which is a four-tank system.

Keywords: Generalized neural networks; Finite-time stability; New integral inequality; Time-varying delays; Nonfragile control

1 Introduction

Neural networks (NNs) have a large capacity for information processing. NNs have been utilized in various applications such as combinatorial optimization, pattern recognition, associate memory, image processing, fixed-point computations, and signal processing [1–4]. There are two major classes of NNs [5–13]. The first of those are static neural networks (SNNs), which utilize the external states of neurons (neural states of neurons). The second are local-field neural networks (LFNNs), which are the internal states of neurons (local-field states). In recent years, Zhang and Han [14] first combined SNNs and LFNNs into a new unified system of NNs called generalized neural networks (GNNs). Throughout the implementation of NNs, a time delay can occur due to the communication time of neurons or the finite switching speed of the neuron amplifiers. Time delays may cause poor performance, instability, divergence, or even oscillation. Hence, the stability analysis of GNNs with time delay has attracted much attention [14–21]. The delayed GNNs can be classified as constant delay, distributed delay, time-varying delay, interval time-varying delay, and mixed delays.

Finite-time stability can apply to real-life problems such as an industrial weight scale. The system of an industrial weight scale needs to attain its state value within a certain threshold for a finite time, so the system uses a magnetic force to reach the system's equilibrium point faster. Stability analysis of the preceding situation is called finite-time stability. Finite-time stability was introduced in 1961 by Dorato [22]. In 2001, Amato [23] presented finite-time boundedness by extending finite-time stability with the external disturbance. The finite-time stability problem for delayed NNs has received much attention [10, 13, 19, 20, 24–26]. For example, Vadivel et al. [13] investigated the finite-time stability of the recurrent NNs with time-varying delays and leakage terms under the event-triggered controller. Later, Phanlert et al. [19] studied the problem of finite-time mixed H_∞/passivity of the GNNs with mixed interval time-varying delays. After that, the problem of finite-time-based reliable dissipative control for neutral-type artificial NNs with time-varying delays has been presented by Saravanakumar et al. [25]. Past studies on finite-time stability for NNs with time-varying delays mostly assume that delays are in the form of differentiable functions. Consequently, this article focuses on the GNNs with continuous nondifferentiable time-varying delays.

In the real world, the inaccuracy appearing in controller operation is inevitable since perturbations in the controller gain are frequent and may result from actuator deterioration. Thus, precision controllers that are insensitive to the controller's acceptable operating faults are called nonfragile controllers and have been studied by many researchers [5, 9, 10, 18, 27–31]. For instance, Ali et al. [5] studied the problem of NNs for finite-time H_∞ with mixed time-varying delays based on a nonfragile feedback controller. Later, Rajavel et al. presented the problem of finite-time stability and passivity performance for NNs with time-varying delay based on a nonfragile state feedback control [9]. After that, the problem of extended dissipative for GNNs under a nonfragile feedback controller with time-varying delay has been investigated by Manivannan et al. [18]. Recently, Kumar et al. [27] investigated the finite-time stability for a T–S fuzzy flexible spacecraft system with uncertainties and stochastic actuator faults under a sampled-data nonfragile controller.

To reduce the conservatism of the Lyapunov theory's stability criterion, the estimation of the derivative of the presented LKF applied several techniques. For instance, the various inequalities used in the control field are Park's inequality [32], Moon's inequality [33], the free-weighting matrix method [34], and other inequalities [35, 36]. Some well-known inequalities are Jensen's inequality [37] and Wirtinger's inequality [38]. Recently, an integral inequality with an exponential function has been presented by Zamart et al. [39]. Moreover, they presented the novel delay-dependent criteria of finite-time stabilization for linear systems with fewer conservatism stability criteria. However, finite-time stability is an important and pertinent problem for developing integral inequalities to reduce conservatism.

Inspired by the previous discussion, we aim to develop a new integral inequality combined with the LKF technique to improve results. Furthermore, we investigate the finite-time stability for GNNs with mixed-interval time-varying delays via the state feedback controller with a nonfragile issue. The main features of this article are listed as:

- We propose a new inequality with an exponential function to estimate the single integral terms of the derivative of LKFs. The stability criteria in terms of LMIs are less conservatism. Moreover, the new inequality covers the well-known Wirtinger's inequality.
- We can solve new delay-dependent conditions for guaranteeing finite-time stable and finite-time boundedness of the GNNs with mixed-interval time-varying delays that do not need to be differentiable.
- We compare minimum allowable lower bounds (MALBs) of c_2 from the new sufficient conditions between the new inequality, Wirtinger's inequality [38] and the inequality in [39]. Those inequalities apply to improve stability criteria using the same LKFs.
- Our results show that the new inequality can reduce conservatism more than Wirtinger's inequality [38] and the inequality in [39].
- We design the nonfragile state feedback controller for the GNNs with mixed-interval time-varying delays and present an example of a practical application that applies our results on a four-tank system.

The paper is organized as follows. We present the GNNs, some preliminaries, and the new integral inequality with an exponential function, in Sect. 2. Section 3 investigates the new sufficient conditions of finite-time stability, finite-time boundedness, and finite-time boundedness based on the state feedback controller with a nonfragile issue for the delayed GNNs. In Sect. 4, three numerical examples illustrate the effectiveness of our methods. Finally, we conclude and discuss our article in Sect. 5.

Notations This article uses the notations as follows: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ denotes the Euclidean vector norm of a matrix; I indicates the identity matrix; $\mathrm{diag}\{\cdots\}$ refers to a block-diagonal matrix; Q^T and Q^{-1} represent the matrix transport Q and matrix inverse Q, respectively; the notation $Q < 0$ (or $Q \leq 0$) denotes the real symmetric matrix Q is negative definite (or Q is negative-semidefinite); $\lambda_{\min}(Q)$ (or $\lambda_{\max}(Q)$) represents the minimum (or maximum) eigenvalue for real symmetric matrix Q; $\mathcal{L}_2[0,\infty)$ refers to a quadratically integrable function space over $[0,\infty)$; $\mathrm{Sym}\{Q\}$ denotes $Q + Q^T$; $*$ refers to the elements below the main diagonal in a symmetric matrix.

2 Problem statement and preliminaries

This article presents the GNNs with distributed and interval time-varying delays as the following:

$$\begin{aligned}
\dot{x}(t) &= -Ax(t) + L_0 f\big(Wx(t)\big) + L_1 f\big(Wx\big(t-h(t)\big)\big) \\
&\quad + L_2 \int_{t-\eta_2(t)}^{t-\eta_1(t)} f\big(Wx(u)\big)\,du + L_3\omega(t) + Bu(t), \qquad (1)\\
y(t) &= x(t),\\
x(t) &= \phi(t), \quad \forall t \in [-h_M, 0],
\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ represents the state vector at time t; n is the number of neurons; $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ with $a_i > 0$ denotes a positive diagonal matrix; $f(Wx(t)) = [f_1(Wx_1(t)), f_2(Wx_2(t)), \ldots, f_n(Wx_n(t))]^T$ indicates the activation functions; W, L_0, L_1, and L_2 refer to connection weight matrices; L_3 and B refer to real constant matrices that are known; $\omega(t)$

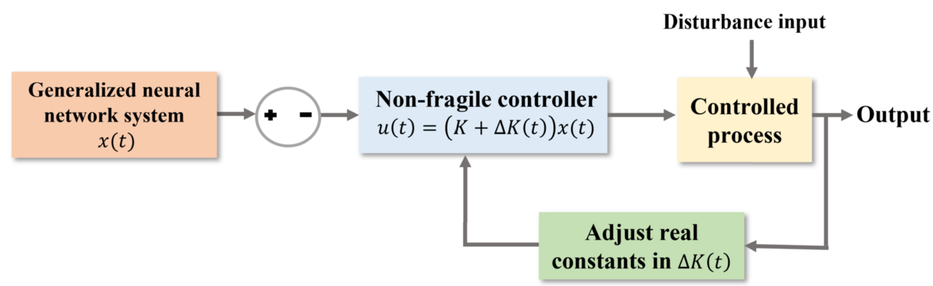

Figure 1 Systematic diagram of the nonfragile controller

refers to the external disturbance input; $u(t) \in \mathbb{R}^m$ denotes the control input; $y(t) \in \mathbb{R}^n$ is the output of the system; $h(t)$ and $\eta_i(t)$ $(i = 1, 2)$ represent the interval time-varying and interval distributed time-varying delays. Both functions are called mixed-interval time-varying delays. The continuous functions $h(t)$ and $\eta_i(t)$ $(i = 1, 2)$ satisfy conditions as follows:

$$0 \le h_m \le h(t) \le h_M \quad \text{and} \quad 0 \le \eta_1 \le \eta_1(t) \le \eta_2(t) \le \eta_2, \quad t \in [0, T],$$

where h_m, h_M, η_1, $\eta_2 \in \mathbb{R}$ refers to known real constants.

This article studies the state feedback controller with a nonfragile issue as the following:

$$u(t) = \big(\mathcal{K} + \Delta\mathcal{K}(t)\big)x(t), \tag{2}$$

where $\mathcal{K}$ denotes the controller matrix that is gained and $\Delta\mathcal{K}(t)$ refers to a perturbed matrix, where it is assumed that the function satisfies $\Delta\mathcal{K}(t) = D_1F(t)D_2$ where D_1 and D_2 represent known real matrices with appropriate dimensions and the unknown time-varying matrix $F(t)$ satisfies $F^T(t)F(t) \le I$.

Remark 1 Figure 1 shows the nonfragile controller mechanism which is a type of control system used in the neural-network model. The controller regulates the flow of information between neurons. It is designed to provide robustness and stability to the model, allowing it to handle unexpected inputs better. The implementation of the controller may obtain some perturbations due to the system faults or the controller gain readjustment circumstances. The nonfragile state feedback controller is the fault-insensitive controller.

Assumption A1 For each $f_i(t)$, $i = 1, 2, \dots, n$ are continuous activation functions that are bounded and satisfy

$$F_i^- \le \frac{f_i(W\kappa_1) - f_i(W\kappa_2)}{W\kappa_1 - W\kappa_2} \le F_i^+, \quad \forall \kappa_1, \kappa_2 \in \mathbb{R}, \kappa_1 \neq \kappa_2,$$

where F_i^- and F_i^+ are known real constants.

Assumption A2 $\omega(t)$ is the external disturbance and satisfies

$$\int_0^T \omega^T(t)\omega(t)\,dt \le d, \quad d \ge 0, T \text{ is a time constant.}$$

Definition 2.1 ([23]) Given positive constants c_1, c_2, and T with $0 < c_1 < c_2$ and H is a symmetric positive-definite matrix. The GNNs (1) are finite-time bounded with respect to (c_1, c_2, H, T), if $\forall t \in [0, T]$

$$\sup_{-h_M \le s \le 0} \left\{ x^T(s)Hx(s), \dot{x}^T(s)H\dot{x}(s) \right\} \le c_1 \quad \Longrightarrow \quad x^T(t)Hx(t) < c_2. \tag{3}$$

Remark 2 When including the external disturbance term, the finite-time stable can be extended to the finite-time boundedness. Thus, the finite-time stability is a particular case of finite-time boundedness. The finite-time stability and finite-time boundedness problems for NNs with time-varying delay have attracted considerable attention [10, 13, 19, 20, 24–26]. Consequently, this article investigates both the finite-time stability and finite-time boundedness for the GNNs with mixed-interval time-varying delays and an external disturbance. Hence, our results are genuinely general.

Lemma 2.2 (Jensen's inequality [37]) *For any scalars d_1 and d_2, any symmetric matrix $M \in \mathbb{R}^{m\times m}$, $M = M^T > 0$, the inequality holds as follows*:

$$(d_2 - d_1) \int_{d_1}^{d_2} x^T(u)Mx(u)\,du \ge \left(\int_{d_1}^{d_2} x(u)\,du \right)^T M \left(\int_{d_1}^{d_2} x(u)\,du \right).$$

Lemma 2.3 (Wirtinger's inequality [38])) *For any symmetric matrix $M \in \mathbb{R}^{m\times m}$, $M = M^T > 0$, any scalars d_1, d_2 and continuously differentiable function $x : [d_1, d_2] \to \mathbb{R}^n$, the inequality holds as follows*:

$$\int_{d_1}^{d_2} \dot{x}^T(u)M\dot{x}(u)\,du \ge \frac{1}{d_2 - d_1} \Omega_0^T M \Omega_0 + \frac{3}{d_2 - d_1} \Omega_1^T M \Omega_1,$$

where $\Omega_0 = x(d_2) - x(d_1)$, $\Omega_1 = x(d_2) + x(d_1) - \frac{2}{d_2 - d_1} \int_{d_1}^{d_2} x(u)\,du$.

Lemma 2.4 ([39]) *For any symmetric matrix $M = M^T > 0$, $M \in \mathbb{R}^{n\times n}$ and scalars $\varrho > 0$, $d_1, d_2 \ge 0$ with $d = d_2 - d_1 > 0$, the inequality holds as follows*:

$$-\int_{t-d_2}^{t-d_1} e^{\varrho(t-u)} \dot{x}^T(u)M\dot{x}(u)\,du \le \begin{bmatrix} x^T(t-d_1) \\ x^T(t-d_2) \end{bmatrix}^T \begin{bmatrix} -k_1 M & k_2 M \\ k_2 M & -k_3 M \end{bmatrix} \begin{bmatrix} x(t-d_1) \\ x(t-d_2) \end{bmatrix},$$

where

$$\begin{aligned}
k_1 &= 2\left(\frac{\varrho}{2} + \frac{1}{d} \right) e^{\varrho d_1} - \varepsilon \left(\frac{\varrho}{2} + \frac{1}{d} \right)^2 e^{2\varrho d_1}, \\
k_2 &= \left(\frac{\varrho}{2} + \frac{1}{d} \right) e^{\varrho d_1} - \left(\frac{\varrho}{2} - \frac{1}{d} \right) e^{\varrho d_2} + \varepsilon \left(\frac{\varrho^2}{4} - \frac{1}{d^2} \right) e^{\varrho(d_1 + d_2)}, \\
k_3 &= -2\left(\frac{\varrho}{2} - \frac{1}{d} \right) e^{\varrho d_2} - \varepsilon \left(\frac{\varrho}{2} - \frac{1}{d} \right)^2 e^{2\varrho d_2}, \\
\varepsilon &= \int_{t-d_2}^{t-d_1} e^{-\varrho(t-u)}\,du = \frac{e^{-\varrho d_1} - e^{-\varrho d_2}}{\varrho}.
\end{aligned}$$

Lemma 2.5 *For any symmetric matrix $M = M^T > 0$, $M \in \mathbb{R}^{n\times n}$ and positive scalars a, $b > a$, and ϱ, the inequality holds as follows*:

$$\int_a^b e^{\varrho(t-u)}\dot{x}^T(u)M\dot{x}(u)\,du \geq \frac{1}{\Phi_0}\Sigma_0^T M\Sigma_0 + \frac{1}{\Phi_1}\Sigma_1^T M\Sigma_1,$$

where

$$\Sigma_0 = x(b) - x(a), \qquad \Sigma_1 = \varepsilon_1 x(a) + \varepsilon_2 x(b) - \int_a^b x(u)\,du,$$

$$\varepsilon_1 = \frac{(b-a)e^{-\varrho(t-b)}}{e^{-\varrho(t-b)} - e^{-\varrho(t-a)}} - \frac{1}{\varrho}, \qquad \varepsilon_2 = \frac{1}{\varrho} - \frac{(b-a)e^{-\varrho(t-a)}}{e^{-\varrho(t-b)} - e^{-\varrho(t-a)}},$$

$$\Phi_0 = \int_a^b e^{-\varrho(t-u)}\,du = \frac{1}{\varrho}\big(e^{-\varrho(t-b)} - e^{-\varrho(t-a)}\big),$$

$$\begin{aligned}\Phi_1 &= \int_a^b e^{-\varrho(t-u)} l_1^2(u)\,du \\ &= \frac{e^{-2\varrho(t-a)} - (2+\varrho^2(b-a)^2)e^{-\varrho(2t-a-b)} + e^{-2\varrho(t-b)}}{\varrho^3(e^{-\varrho(t-b)} - e^{-\varrho(t-a)})},\end{aligned}$$

$$l_1(u) = u - \left(\int_a^b e^{-\varrho(t-u)}\,du\right)^{-1}\left(\int_a^b e^{-\varrho(t-u)}u\,du\right).$$

Proof Define the function z as

$$\begin{aligned}z(u) = {}& e^{\varrho(t-u)}\dot{x}(u) - \left(\int_a^b e^{-\varrho(t-s)}\,ds\right)^{-1}\big(x(b) - x(a)\big) \\ & - l_1(u)\left(\int_a^b e^{-\varrho(t-s)} l_1^2(s)\,ds\right)^{-1}\left(\int_a^b l_1(s)\dot{x}(s)\,ds\right).\end{aligned}$$

Since $M > 0$, we have $0 \leq \int_a^b e^{-\varrho(t-u)} z^T(u)Mz(u)\,du$ and reinjecting $z(u)$ into the integral, we obtain

$$\begin{aligned}0 \leq {}& \int_a^b e^{\varrho(t-u)}\dot{x}^T(u)M\dot{x}(u)\,du \\ & + \left(\int_a^b e^{-\varrho(t-s)}\,ds\right)^{-1}\big(x(b) - x(a)\big)^T M\big(x(b) - x(a)\big) \\ & + \left(\int_a^b e^{-\varrho(t-s)} l_1^2(s)\,ds\right)^{-1}\left(\int_a^b l_1(s)\dot{x}(s)\,ds\right)^T M\left(\int_a^b l_1(s)\dot{x}(s)\,ds\right) \\ & - 2\left(\int_a^b e^{-\varrho(t-s)}\,ds\right)^{-1}\big(x(b) - x(a)\big)^T M\big(x(b) - x(a)\big) \\ & - 2\left(\int_a^b e^{-\varrho(t-s)} l_1^2(s)\,ds\right)^{-1}\left(\int_a^b l_1(u)\dot{x}(u)\,du\right)^T M\left(\int_a^b l_1(s)\dot{x}(s)\,ds\right) \\ & + 2\left(\int_a^b e^{-\varrho(t-s)}\,ds\right)^{-1}\left(\int_a^b e^{-\varrho(t-s)} l_1^2(s)\,ds\right)^{-1}\big(x(b) - x(a)\big)^T\end{aligned}$$

$$\times M\left(\int_a^b l_1(s)\dot{x}(s)\,ds\right)\left(\int_a^b e^{-\varrho(t-u)}l_1(u)\,du\right). \tag{4}$$

By simple integral calculus and integration by parts, we find that

$$\begin{aligned}
\int_a^b e^{-\varrho(t-u)}l_1(u)\,du &= \int_a^b e^{-\varrho(t-u)}\left(u-\frac{\int_a^b e^{-\varrho(t-u)}u\,du}{\int_a^b e^{-\varrho(t-u)}\,du}\right)du \\
&= \int_a^b e^{-\varrho(t-u)}u\,du-\int_a^b e^{-\varrho(t-u)}\,du\left(\frac{\int_a^b e^{-\varrho(t-u)}u\,du}{\int_a^b e^{-\varrho(t-u)}\,du}\right) \\
&= 0, \qquad (5)
\end{aligned}$$

$$\int_a^b e^{-\varrho(t-s)}\,ds = \frac{1}{\varrho}\left(e^{-\varrho(t-b)}-e^{-\varrho(t-a)}\right) = \Phi_0, \tag{6}$$

$$\begin{aligned}
\int_a^b e^{-\varrho(t-s)}l_1^2(s)\,ds &= \int_a^b e^{-\varrho(t-s)}\left(s-\frac{\int_a^b e^{-\varrho(t-s)}s\,ds}{\int_a^b e^{-\varrho(t-s)}\,ds}\right)^2 ds \\
&= \int_a^b e^{-\varrho(t-s)}s^2\,ds \\
&\quad -2\left[\frac{(be^{-\varrho(t-b)}-ae^{-\varrho(t-a)})}{(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})}-\frac{1}{\varrho}\right]\int_a^b e^{-\varrho(t-s)}s\,ds \\
&\quad +\left[\frac{(be^{-\varrho(t-b)}-ae^{-\varrho(t-a)})}{(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})}-\frac{1}{\varrho}\right]^2\int_a^b e^{-\varrho(t-s)}\,ds \\
&= \left[\frac{1}{\varrho}\left(b^2e^{\varrho(t-b)}-a^2e^{\varrho(t-a)}\right)-\frac{2}{\varrho^2}\left(be^{-\varrho(t-b)}-ae^{\varrho(t-a)}\right)\right. \\
&\quad \left.+\frac{2}{\varrho^3}\left(e^{-\varrho(t-b)}-e^{-\varrho(t-a)}\right)\right]-\frac{2}{\varrho}\left[\frac{(be^{-\varrho(t-b)}-ae^{-\varrho(t-a)})^2}{(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})}\right. \\
&\quad \left.-\frac{2(be^{-\varrho(t-b)}-ae^{-\varrho(t-a)})}{\varrho}+\frac{(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})}{\varrho^2}\right] \\
&\quad +\frac{1}{\varrho}\left[\frac{(be^{-\varrho(t-b)}-ae^{-\varrho(t-a)})^2}{(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})}-\frac{2(be^{-\varrho(t-b)}-ae^{-\varrho(t-a)})}{\varrho}\right. \\
&\quad \left.+\frac{(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})}{\varrho^2}\right] \\
&= \frac{(b^2e^{-\varrho(t-b)}-a^2e^{-\varrho(t-a)})}{\varrho}-\frac{(be^{-\varrho(t-b)}-ae^{-\varrho(t-a)})^2}{\varrho(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})^2} \\
&\quad +\frac{(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})}{\varrho^3} \\
&= \frac{-\varrho^2(b^2-2ab+a^2)e^{-\varrho(2t-a-b)}+(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})^2}{\varrho^3(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})} \\
&= \frac{e^{-2\varrho(t-a)}-(2+\varrho^2(b-a)^2)e^{-\varrho(2t-a-b)}+e^{-2\varrho(t-b)}}{\varrho^3(e^{-\varrho(t-b)}-e^{-\varrho(t-a)})} = \Phi_1, \qquad (7)
\end{aligned}$$

$$\int_a^b l_1(s)\dot{x}(s)\,ds = \int_a^b\left(s-\frac{\int_a^b e^{-\varrho(t-s)}s\,ds}{\int_a^b e^{-\varrho(t-s)}\,ds}\right)\dot{x}(s)\,ds$$

$$
\begin{aligned}
&= \int_a^b s\dot{x}(s)\,ds - \int_a^b \dot{x}(s)\,ds \left(\frac{\int_a^b e^{-\varrho(t-s)} s\,ds}{\int_a^b e^{-\varrho(t-s)}\,ds} \right) \\
&= bx(b) - ax(a) - \int_a^b x(s)\,ds \\
&\quad - \left[\frac{(be^{-\varrho(t-b)} - ae^{-\varrho(t-a)})}{\varrho} - \frac{(e^{-\varrho(t-b)} - e^{-\varrho(t-a)})}{\varrho^2} \right] \\
&\quad \times \frac{\varrho(x(b) - x(a))}{(e^{-\varrho(t-b)} - e^{-\varrho(t-a)})} \\
&= bx(b) - ax(a) - \int_a^b x(s)\,ds \\
&\quad - \frac{(be^{-\varrho(t-b)} - ae^{-\varrho(t-a)})(x(b) - x(a))}{(e^{-\varrho(t-b)} - e^{-\varrho(t-a)})} + \frac{(x(b) - x(a))}{\varrho} \\
&= \varepsilon_1 x(a) + \varepsilon_2 x(b) - \int_a^b x(s)\,ds = \Sigma_1, \qquad (8)
\end{aligned}
$$

From inequality (4) and the above integral, we obtain

$$
\int_a^b e^{\varrho(t-u)} \dot{x}^T(u) M \dot{x}(u)\,du \geq \frac{1}{\Phi_0} \Sigma_0^T M \Sigma_0 + \frac{1}{\Phi_1} \Sigma_1^T M \Sigma_1.
$$

Therefore, the proof is complete. □

Remark 3 If $\varrho = 0$, then $\Phi_0 = b - a$, $\Phi_1 = \frac{(b-a)^3}{12}$, $\Sigma_0 = x(b) - x(a)$ and $\Sigma_1 = \frac{b-a}{2}[x(a) + x(b) - \frac{2}{b-a}\int_a^b x(u)\,du]$. That is, Lemma 2.3 or the well-known Wirtinger's inequality is a particular case of Lemma 2.5.

Lemma 2.6 ([40]) *For any real matrices of appropriate dimensions D_1, D_2, and $F(t)$ satisfying $F^T(t)F(t) \leq I$, then, for any scalar $\varepsilon > 0$,*

$$
D_1 F(t) D_2 + D_2^T F^T(t) D_1^T \leq \varepsilon^{-1} D_1 D_1^T + \varepsilon D_2^T D_2.
$$

Lemma 2.7 (Schur complement [41]) *Given X, Y, and Z are constant matrices with appropriate dimensions and satisfying $X = X^T$, $Y = Y^T > 0$, then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$
\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -Y & Z \\ Z^T & X \end{bmatrix} < 0.
$$

3 Main results

This section proposes new delay-dependent conditions for delayed GNNs of the main theorems. Our first and foremost condition is defining parameters as follows: $h_{Mm} = h_M - h_m$, $h_{Mm} \neq 0$,

$$
\begin{aligned}
\gamma_{1a} &= \frac{1}{\varrho}\left(1 - e^{-\varrho h_m}\right), \qquad \gamma_{1b} = \frac{e^{-2\varrho h_m} - (2 + \varrho^2 h_m^2) e^{-\varrho h_m} + 1}{\varrho^3 (1 - e^{-\varrho h_m})}, \\
\gamma_{2a} &= \frac{1}{\varrho}\left(e^{-\varrho h_m} - e^{-\varrho h(t)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{2b} &= \frac{e^{-2\varrho h(t)} - (2+\varrho^2(h(t)-h_m)^2)e^{-\varrho(h(t)+h_m)} + e^{-2\varrho h_m}}{\varrho^3(e^{-\varrho h_m} - e^{-\varrho h(t)})}, \\
\gamma_{3a} &= \frac{1}{\varrho}\big(e^{-\varrho h(t)} - e^{-\varrho h_M}\big), \\
\gamma_{3b} &= \frac{e^{-2\varrho h_M} - (2+\varrho^2(h_M-h(t))^2)e^{-\varrho(h(t)+h_M)} + e^{-2\varrho h(t)}}{\varrho^3(e^{-\varrho h(t)} - e^{-\varrho h_M})}, \\
\gamma_{4a} &= \frac{1}{\varrho}\big(1 - e^{-\varrho h_M}\big), \qquad \gamma_{4b} = \frac{e^{-2\varrho h_M} - (2+\varrho^2 h_M^2)e^{-\varrho h_M} + 1}{\varrho^3(1-e^{-\varrho h_M})}, \\
\varepsilon_{11} &= \frac{h_m}{1-e^{-\varrho h_m}} - \frac{1}{\varrho}, \qquad \varepsilon_{12} = \frac{1}{\varrho} - \frac{h_m e^{-\varrho h_m}}{1-e^{-\varrho h_m}}, \\
\varepsilon_{21} &= \frac{(h(t)-h_m)e^{-\varrho h_m}}{e^{-\varrho h_m} - e^{-\varrho h(t)}} - \frac{1}{\varrho}, \qquad \varepsilon_{22} = \frac{1}{\varrho} - \frac{(h(t)-h_m)e^{-\varrho h(t)}}{e^{-\varrho h_m} - e^{-\varrho h(t)}}, \\
\varepsilon_{31} &= \frac{(h_M-h(t))e^{-\varrho h(t)}}{e^{-\varrho h(t)} - e^{-\varrho h_M}} - \frac{1}{\varrho}, \qquad \varepsilon_{32} = \frac{1}{\varrho} - \frac{(h_M-h(t))e^{-\varrho h_M}}{e^{-\varrho h(t)} - e^{-\varrho h_M}}, \\
\varepsilon_{41} &= \frac{h_M}{1-e^{-\varrho h_M}} - \frac{1}{\varrho}, \qquad \varepsilon_{42} = \frac{1}{\varrho} - \frac{h_M e^{-\varrho h_M}}{1-e^{-\varrho h_M}}, \\
\Gamma_1 &= \begin{bmatrix} e_1^T & h_m e_{10}^T & h_M e_{11}^T \end{bmatrix}^T, \qquad \Gamma_2 = \begin{bmatrix} e_7^T & e_1^T - e_2^T & e_1^T - e_4^T \end{bmatrix}^T, \\
\Gamma_3 &= \begin{bmatrix} e_1^T & e_7^T \end{bmatrix}^T, \qquad \Gamma_4 = \begin{bmatrix} e_2^T & e_8^T \end{bmatrix}^T, \qquad \Gamma_5 = \begin{bmatrix} e_4^T & e_9^T \end{bmatrix}^T, \qquad \Gamma_6 = \begin{bmatrix} e_1^T - e_2^T \end{bmatrix}^T, \\
\Gamma_7 &= \big[\varepsilon_{11}e_2^T + \varepsilon_{12}e_1^T - h_m e_{10}^T\big]^T, \qquad \Gamma_8 = \big[e_2^T - e_3^T\big]^T, \\
\Gamma_9 &= \big[\varepsilon_{21}e_3^T + \varepsilon_{22}e_2^T - h_{Mm} e_{12}^T\big]^T, \qquad \Gamma_{10} = \big[e_3^T - e_4^T\big]^T, \\
\Gamma_{11} &= \big[\varepsilon_{31}e_4^T + \varepsilon_{32}e_3^T - h_{Mm} e_{13}^T\big]^T, \qquad \Gamma_{12} = \big[e_1^T - e_4^T\big]^T, \\
\Gamma_{13} &= \big[\varepsilon_{41}e_4^T + \varepsilon_{42}e_1^T - h_M e_{11}^T\big]^T, \qquad \Gamma_{14} = \big[e_5^T - e_1^T W^T F_M^T\big]^T, \\
\Gamma_{15} &= [F_P W e_1 - e_5], \qquad \Gamma_{16} = \big[e_6^T - e_3^T W^T F_M^T\big]^T, \qquad \Gamma_{17} = [F_P W e_3 - e_6], \\
\Gamma_{18} &= \big[e_5^T - e_6^T - e_1^T W^T F_M^T + e_3^T W^T F_M^T\big]^T, \\
\Gamma_{19} &= [F_P W e_1 - F_P W e_3 - e_5 + e_6], \\
F_M &= \operatorname{diag}\{F_1^-, \dots, F_n^-\}, \qquad F_P = \operatorname{diag}\{F_1^+, \dots, F_n^+\}, \\
\delta_1 &= \frac{e^{\varrho h_m} - 1}{\varrho}, \qquad \delta_2 = \frac{e^{\varrho h_M} - 1}{\varrho}, \qquad \delta_3 = \frac{e^{\varrho h_m} - \varrho h_m - 1}{\varrho^2}, \\
\delta_4 &= \frac{e^{\varrho h_M} - e^{\varrho h_m} - \varrho h_{Mm}}{\varrho^2}, \qquad \delta_5 = \frac{e^{\varrho h_M} - \varrho h_M - 1}{\varrho^2}, \\
\delta_6 &= \frac{e^{\varrho \eta_2} - e^{\varrho \eta_1} - \varrho \eta_{21}}{\varrho^2}, \qquad \eta_{21} = \eta_2 - \eta_1
\end{aligned}
$$

and we define vectors as follows:

$$
\begin{aligned}
\zeta(t) = \Big[& x^T(t), x^T(t-h_m), x^T\big(t-h(t)\big), x^T(t-h_M), f^T\big(Wx(t)\big), \\
& f^T\big(Wx\big(t-h(t)\big)\big), \dot{x}^T(t), \dot{x}^T(t-h_m), \dot{x}^T(t-h_M),
\end{aligned}
$$

$$\frac{1}{h_m}\int_{t-h_m}^{t} x^T(u)\,du, \frac{1}{h_M}\int_{t-h_M}^{t} x^T(u)\,du, \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m} x^T(u)\,du,$$
$$\left.\frac{1}{h_M-h(t)}\int_{t-h_M}^{t-h(t)} x^T(u)\,du, \int_{t-\eta_2(t)}^{t-\eta_1(t)} f^T\big(Wx(u)\big)\,du, \omega^T(t)\right]^T,$$
$$e_j = \begin{bmatrix} 0_{n\times(j-1)n} & I_n & 0_{n\times(15-j)n}\end{bmatrix}, \quad j = 1,2,\dots,15.$$

3.1 Analysis of finite-time boundedness

We first obtain new delay-dependent conditions for the problem of finite-time boundedness of the GNNs (9) with mixed-interval time-varying delays as the following:

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + L_0 f\big(Wx(t)\big) + L_1 f\big(Wx\big(t-h(t)\big)\big)\\ &\quad + L_2\int_{t-\eta_2(t)}^{t-\eta_1(t)} f\big(Wx(u)\big)\,du + L_3\omega(t),\\ x(t) &= \phi(t), \quad \forall t\in[-h_M,0].\end{aligned} \tag{9}$$

Theorem 3.1 *Given positive scalars h_M and ϱ then the delayed GNNs* (9) *are finite-time bounded regarding (c_1,c_2,T,H,d), if there exist symmetric positive-definite matrices $P\in\mathbb{R}^{3n\times 3n}$, $Q_i\in\mathbb{R}^{2n\times 2n}$, $R_j\in\mathbb{R}^{n\times n}$ $(i=1,2, j=1,2,3)$, $Z,X\in\mathbb{R}^{n\times n}$, any matrices N_1, N_2, and positive diagonal matrices S_1, S_2, S_3, such that the conditions hold as follows*:

$$\Xi < 0, \tag{10}$$

$$\begin{aligned}&\lambda_0 I\le \overline{P}_1\le\lambda_1 I, \quad 0\le\overline{P}_2\le\lambda_2 I, \quad 0\le\overline{P}_3\le\lambda_3 I, \quad 0\le\overline{Q}_{11}\le\lambda_4 I,\\ &0\le\overline{Q}_{12}\le\lambda_5 I, \quad 0\le\overline{Q}_{13}\le\lambda_6 I, \quad 0\le\overline{Q}_{21}\le\lambda_7 I, \quad 0\le\overline{Q}_{22}\le\lambda_8 I,\\ &0\le\overline{Q}_{23}\le\lambda_9 I, \quad 0\le\overline{R}_1\le\lambda_{10} I, \quad 0\le\overline{R}_2\le\lambda_{11} I, \quad 0\le\overline{R}_3\le\lambda_{12} I,\\ &0\le\overline{Z}\le\lambda_{13} I, \quad 0\le\overline{X}\le\lambda_{14} I,\end{aligned} \tag{11}$$

$$e^{\varrho T}\big[\Xi_\lambda c_1 + d\lambda_{14}\big(1-e^{-\varrho T}\big)\big] < \lambda_0 c_2, \tag{12}$$

where

$$P = \begin{bmatrix} P_1 & 0 & 0\\ 0 & P_2 & 0\\ 0 & 0 & P_3\end{bmatrix}, \qquad Q_1 = \begin{bmatrix} Q_{11} & Q_{12}\\ * & Q_{13}\end{bmatrix}, \qquad Q_2 = \begin{bmatrix} Q_{21} & Q_{22}\\ * & Q_{23}\end{bmatrix},$$

$$\Xi = \sum_{i=1}^{7}\Xi_i,$$

$$\Xi_1 = \mathrm{Sym}\big\{\Gamma_1^T P\Gamma_2\big\} - \varrho\Gamma_1^T P\Gamma_1,$$

$$\Xi_2 = \Gamma_3^T(Q_1+Q_2)\Gamma_3 - \Gamma_4^T\big(e^{\varrho h_m}Q_1\big)\Gamma_4 - \Gamma_5^T\big(e^{\varrho h_M}Q_2\big)\Gamma_5,$$

$$\begin{aligned}\Xi_3 &= h_m^2 e_7^T R_1 e_7 + h_{Mm}^2 e_7^T R_2 e_7 + h_M^2 e_7^T R_3 e_7 - \frac{h_m}{\gamma_{1a}}\Gamma_6^T R_1\Gamma_6 - \frac{h_m}{\gamma_{1b}}\Gamma_7^T R_1\Gamma_7\\ &\quad - \frac{h_{Mm}}{\gamma_{2a}}\Gamma_8^T R_2\Gamma_8 - \frac{h_{Mm}}{\gamma_{2b}}\Gamma_9^T R_2\Gamma_9 - \frac{h_{Mm}}{\gamma_{3a}}\Gamma_{10}^T R_2\Gamma_{10} - \frac{h_{Mm}}{\gamma_{3b}}\Gamma_{11}^T R_2\Gamma_{11}\\ &\quad - \frac{h_M}{\gamma_{4a}}\Gamma_{12}^T R_3\Gamma_{12} - \frac{h_M}{\gamma_{4b}}\Gamma_{13}^T R_3\Gamma_{13},\end{aligned}$$

$$\Xi_4 = \eta_{21}^2 e_5^T Z e_5 - e^{\varrho\eta_2} e_{14}^T Z e_{14},$$
$$\Xi_5 = 2\Gamma_{14}^T S_1 \Gamma_{15}^T + 2\Gamma_{16}^T S_2 \Gamma_{17}^T + 2\Gamma_{18}^T S_3 \Gamma_{19}^T,$$
$$\Xi_6 = -\varrho e_{15}^T X e_{15},$$
$$\Xi_7 = \mathrm{Sym}\{[e_1^T N_1 + e_7^T N_2][-e_7 - Ae_1 + L_0 e_5 + L_1 e_6 + L_2 e_{14} + L_3 e_{15}]\},$$
$$\begin{aligned}\Xi_\lambda &= \lambda_1 + h_m\lambda_2 + h_M\lambda_3 + \delta_1(\lambda_4 + 2\lambda_5 + \lambda_6) + \delta_2(\lambda_7 + 2\lambda_8 + \lambda_9)\\ &\quad + h_m\delta_3\lambda_{10} + h_{Mm}\delta_4\lambda_{11} + h_M\delta_5\lambda_{12} + \eta_{21}\delta_6\lambda_{13},\end{aligned}$$
$$\lambda_0 = \lambda_{\min}(\overline{P}_1), \qquad \lambda_1 = \lambda_{\max}(\overline{P}_1), \qquad \lambda_2 = \lambda_{\max}(\overline{P}_2), \qquad \lambda_3 = \lambda_{\max}(\overline{P}_3),$$
$$\lambda_4 = \lambda_{\max}(\overline{Q}_{11}), \qquad \lambda_5 = \lambda_{\max}(\overline{Q}_{12}), \qquad \lambda_6 = \lambda_{\max}(\overline{Q}_{13}), \qquad \lambda_7 = \lambda_{\max}(\overline{Q}_{21}),$$
$$\lambda_8 = \lambda_{\max}(\overline{Q}_{22}), \qquad \lambda_9 = \lambda_{\max}(\overline{Q}_{23}), \qquad \lambda_{10} = \lambda_{\max}(\overline{R}_1), \qquad \lambda_{11} = \lambda_{\max}(\overline{R}_2),$$
$$\lambda_{12} = \lambda_{\max}(\overline{R}_3), \qquad \lambda_{13} = \lambda_{\max}(\overline{Z}), \qquad \lambda_{14} = \lambda_{\max}(\overline{X}).$$

Proof We construct the LKFs as the following:

$$V(t,x(t)) = \sum_{i=1}^{4} V_i(t,x(t)), \tag{13}$$

where

$$\begin{aligned}
V_1(t,x(t)) &= \rho_1^T(t)P\rho_1(t),\\
V_2(t,x(t)) &= \int_{t-h_m}^{t} e^{\varrho(t-u)}\rho_2^T(u)Q_1\rho_2(u)\,du\\
&\quad + \int_{t-h_M}^{t} e^{\varrho(t-u)}\rho_2^T(u)Q_2\rho_2(u)\,du,\\
V_3(t,x(t)) &= h_m\int_{-h_m}^{0}\int_{t+u}^{t} e^{\varrho(t-s)}\dot{x}^T(s)R_1\dot{x}(s)\,ds\,du\\
&\quad + h_{Mm}\int_{-h_M}^{-h_m}\int_{t+s}^{t} e^{\varrho(t-s)}\dot{x}^T(s)R_2\dot{x}(s)\,ds\,du\\
&\quad + h_M\int_{-h_M}^{0}\int_{t+s}^{t} e^{\varrho(t-s)}\dot{x}^T(s)R_3\dot{x}(s)\,ds\,du,\\
V_4(t,x(t)) &= \eta_{21}\int_{-\eta_2}^{-\eta_1}\int_{t+u}^{t} e^{\varrho(t-s)}f^T(Wx(s))Zf(Wx(s))\,ds\,du,\\
\rho_1(t) &= \begin{bmatrix} x^T(t) & \int_{t-h_m}^{t} x^T(u)\,du & \int_{t-h_M}^{t} x^T(u)\,du \end{bmatrix}^T,\\
\rho_2(t) &= \begin{bmatrix} x^T(t) & \dot{x}^T(t) \end{bmatrix}^T.
\end{aligned}$$

Taking the derivative of (13) along the trajectory of the GNNs (9), we obtain

$$\dot{V}_1(t,x(t)) = 2\rho_1^T(t)P\begin{bmatrix}\dot{x}(t)\\ x(t)-x(t-h_m)\\ x(t)-x(t-h_M)\end{bmatrix} - \varrho\rho_1^T(t)P\rho_1(t) + \varrho V_1(t,x(t))$$

$$
\begin{aligned}
&= \zeta^T(t)\big\{\mathrm{Sym}\big\{\Gamma_1^T P\Gamma_2\big\} - \varrho\Gamma_1^T P\Gamma_1\big\}\zeta(t) + \varrho V_1\big(t,x(t)\big) \\
&= \zeta^T(t)\Xi_1\zeta(t) + \varrho V_1\big(t,x(t)\big), \qquad (14)\\
\dot V_2\big(t,x(t)\big) &= \rho_2^T(t)Q_1\rho_2(t) - e^{\varrho h_m}\rho_2^T(t-h_m)Q_1\rho_2(t-h_m) + \rho_2^T(t)Q_2\rho_2(t) \\
&\quad - e^{\varrho h_M}\rho_2^T(t-h_M)Q_2\rho_2(t-h_M) + \varrho V_2\big(t,x(t)\big) \\
&= \zeta^T(t)\big\{\Gamma_3^T(Q_1+Q_2)\Gamma_3 - \Gamma_4^T\big(e^{\varrho h_m}Q_1\big)\Gamma_4 - \Gamma_5^T\big(e^{\varrho h_M}Q_2\big)\Gamma_5\big\}\zeta(t) \\
&\quad + \varrho V_2\big(t,x(t)\big) \\
&= \zeta^T(t)\Xi_2\zeta(t) + \varrho V_2\big(t,x(t)\big), \qquad (15)\\
\dot V_3\big(t,x(t)\big) &= h_m^2\dot x^T(t)R_1\dot x(t) - h_m\int_{t-h_m}^{t} e^{\varrho(t-u)}\dot x^T(u)R_1\dot x(u)\,du \\
&\quad + h_{Mm}^2\dot x^T(t)R_2\dot x(t) - h_{Mm}\int_{t-h_M}^{t-h_m} e^{\varrho(t-u)}\dot x^T(u)R_2\dot x(u)\,du \\
&\quad + h_M^2\dot x^T(t)R_3\dot x(t) - h_M\int_{t-h_M}^{t} e^{\varrho(t-u)}\dot x^T(u)R_3\dot x(u)\,du \\
&\quad + \varrho V_3\big(t,x(t)\big), \qquad (16)\\
\dot V_4\big(t,x(t)\big) &= \eta_{21}^2 f^T\big(Wx(t)\big)Zf\big(Wx(t)\big) \\
&\quad - (\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta_1} e^{\varrho(t-u)}f^T\big(Wx(u)\big)Zf\big(Wx(u)\big)\,du + \varrho V_4\big(t,x(t)\big) \\
&\le \eta_{21}^2 f^T\big(Wx(t)\big)Zf\big(Wx(t)\big) \\
&\quad - \big(\eta_2(t)-\eta_1(t)\big)e^{\varrho\eta_2}\int_{t-\eta_2(t)}^{t-\eta_1(t)} f^T\big(Wx(u)\big)Zf\big(Wx(u)\big)\,du \\
&\quad + \varrho V_4\big(t,x(t)\big). \qquad (17)
\end{aligned}
$$

Applying Lemma 2.5 to the integral, we obtain

$$
\begin{aligned}
&-h_m\int_{t-h_m}^{t} e^{\varrho(t-u)}\dot x^T(u)R_1\dot x(u)\,du \\
&\quad\le \zeta^T(t)\bigg\{-\frac{h_m}{\gamma_{1a}}(e_1-e_2)^T R_1(e_1-e_2) \\
&\qquad - \frac{h_m}{\gamma_{1b}}(\varepsilon_{11}e_2+\varepsilon_{12}e_1-h_m e_{10})^T R_1(\varepsilon_{11}e_2+\varepsilon_{12}e_1-h_m e_{10})\bigg\}\zeta(t) \\
&\quad= \zeta^T(t)\bigg\{-\frac{h_m}{\gamma_{1a}}\Gamma_6^T R_1\Gamma_6 - \frac{h_m}{\gamma_{1b}}\Gamma_7^T R_1\Gamma_7)\bigg\}\zeta(t), \qquad (18)\\
&-h_{Mm}\int_{t-h_M}^{t-h_m} e^{\varrho(t-u)}\dot x^T(u)R_2\dot x(u)\,du \\
&\quad= -h_{Mm}\int_{t-h(t)}^{t-h_m} e^{\varrho(t-u)}\dot x^T(u)R_2\dot x(u)\,du \\
&\qquad - h_{Mm}\int_{t-h_M}^{t-h(t)} e^{\varrho(t-u)}\dot x^T(u)R_2\dot x(u)\,du
\end{aligned}
$$

$$\begin{aligned}
&\leq \zeta^T(t)\bigg\{-\frac{h_{Mm}}{\gamma_{2a}}(e_2-e_3)^T R_2(e_2-e_3) \\
&\quad -\frac{h_{Mm}}{\gamma_{2b}}(\varepsilon_{21}e_3+\varepsilon_{22}e_2-h_{Mm}e_{12})^T R_2(\varepsilon_{21}e_3+\varepsilon_{22}e_2-h_{Mm}e_{12}) \\
&\quad -\frac{h_{Mm}}{\gamma_{3a}}(e_3-e_4)^T R_2(e_3-e_4) \\
&\quad -\frac{h_{Mm}}{\gamma_{3b}}(\varepsilon_{31}e_4+\varepsilon_{32}e_3-h_{Mm}e_{13})^T R_2(\varepsilon_{31}e_4+\varepsilon_{32}e_3-h_{Mm}e_{13})\bigg\}\zeta(t) \\
&= \zeta^T(t)\bigg\{-\frac{h_{Mm}}{\gamma_{2a}}\Gamma_8^T R_2\Gamma_8-\frac{h_{Mm}}{\gamma_{2b}}\Gamma_9^T R_2\Gamma_9-\frac{h_{Mm}}{\gamma_{3a}}\Gamma_{10}^T R_2\Gamma_{10} \\
&\quad -\frac{h_{Mm}}{\gamma_{3b}}\Gamma_{11}^T R_2\Gamma_{11}\bigg\}\zeta(t), \qquad (19)
\end{aligned}$$

$$\begin{aligned}
&-h_M\int_{t-h_M}^{t} e^{\varrho(t-u)}\dot{x}^T(u)R_3\dot{x}(u)\,du \\
&\leq \zeta^T(t)\bigg\{-\frac{h_M}{\gamma_{4a}}(e_1-e_4)^T R_3(e_1-e_4) \\
&\quad -\frac{h_M}{\gamma_{4b}}(\varepsilon_{41}e_4+\varepsilon_{42}e_1-h_Me_{11})^T R_3(\varepsilon_{41}e_4+\varepsilon_{42}e_1-h_Me_{11})\bigg\}\zeta(t) \\
&= \zeta^T(t)\bigg\{-\frac{h_M}{\gamma_{4a}}\Gamma_{12}^T R_3\Gamma_{12}-\frac{h_M}{\gamma_{4b}}\Gamma_{13}^T R_3\Gamma_{13}\bigg\}\zeta(t). \qquad (20)
\end{aligned}$$

Applying Lemma 2.2, we obtain

$$\begin{aligned}
&-\big(\eta_2(t)-\eta_1(t)\big)e^{\varrho\eta_2}\int_{t-\eta_2(t)}^{t-\eta_1(t)} f^T\big(Wx(u)\big)Zf\big(Wx(u)\big)\,du \\
&\leq -e^{\varrho\eta_2}\bigg(\int_{t-\eta_2(t)}^{t-\eta_1(t)} f^T\big(Wx(u)\big)\,du\bigg)^T Z\bigg(\int_{t-\eta_2(t)}^{t-\eta_1(t)} f^T\big(Wx(u)\big)\,du\bigg) \\
&= \zeta^T(t)\big\{-e_{14}^T\big(e^{\varrho\eta_2}Z\big)e_{14}\big\}\zeta(t). \qquad (21)
\end{aligned}$$

From Assumption A1, it can be inferred that for any $\beta_{1i},\beta_{2i},\beta_{3i}>0$, $i=1,2,\dots,n$, we have

$$2\big[f_i\big(W_ix(t)\big)-F_i^-W_ix(t)\big]\beta_{1i}\big[F_i^+W_ix(t)-f_i\big(W_ix(t)\big)\big]\geq 0, \qquad (22)$$

$$\begin{aligned}
&2\big[f_i\big(W_ix\big(t-h(t)\big)\big)-F_i^-W_ix\big(t-h(t)\big)\big]\beta_{2i} \\
&\quad\times\big[F_i^+W_ix\big(t-h(t)\big)-f_i\big(W_ix\big(t-h(t)\big)\big)\big]\geq 0, \qquad (23)
\end{aligned}$$

$$\begin{aligned}
&2\big[f_i\big(W_ix(t)\big)-f_i\big(W_ix\big(t-h(t)\big)\big)-F_i^-\big(W_ix(t)-W_ix\big(t-h(t)\big)\big)\big]\beta_{3i} \\
&\quad\times\big[F_i^+\big(W_ix(t)-W_ix\big(t-h(t)\big)\big)-f_i\big(W_ix(t)\big)+f_i\big(W_ix\big(t-h(t)\big)\big)\big]\geq 0, \qquad (24)
\end{aligned}$$

which imply

$$2\zeta^T(t)\Gamma_{14}^T S_1\Gamma_{15}\zeta(t)\geq 0, \qquad (25)$$

$$2\zeta^T(t)\Gamma_{16}^T S_2\Gamma_{17}\zeta(t)\geq 0, \qquad (26)$$

$$2\zeta^T(t)\Gamma_{18}^T S_3\Gamma_{19}\zeta(t)\geq 0. \qquad (27)$$

Thus, we have

$$0 \leq \zeta^T(t)\{2\Gamma_{14}^T S_1\Gamma_{15} + 2\Gamma_{16}^T S_2\Gamma_{17} + 2\Gamma_{18}^T S_3\Gamma_{19}\}\zeta(t)$$
$$= \zeta^T(t)\Xi_5\zeta(t),$$

where $S_1 = \mathrm{diag}\{\beta_{11},\beta_{12},\ldots,\beta_{1n}\}$, $S_2 = \mathrm{diag}\{\beta_{21},\beta_{22},\ldots,\beta_{2n}\}$, and $S_3 = \mathrm{diag}\{\beta_{31},\beta_{32},\ldots,\beta_{3n}\}$.

Moreover, for any appropriate dimensions matrices N_1, N_2, we obtain

$$\begin{aligned}0 &= 2\big[x^T(t)N_1 + \dot{x}(t)N_2\big]\bigg[-\dot{x}(t) - Ax(t) + L_0 f\big(Wx(t)\big)\\ &\quad + L_1 f\big(Wx\big(t-h(t)\big)\big) + L_2\int_{t-\eta_2(t)}^{t-\eta_1(t)} f\big(Wx(u)\big)\,du + L_3\omega(t)\bigg]\\ &= \zeta^T(t)\big\{\mathrm{Sym}\big\{\big[e_1^T N_1 + e_7^T N_2\big]\\ &\quad \times [-e_7 - Ae_1 + L_0e_5 + L_1e_6 + L_2e_{14} + L_3e_{15}]\big\}\big\}\zeta(t)\\ &= \zeta^T(t)\Xi_7\zeta(t).\end{aligned} \tag{28}$$

Combining (14)–(28), we obtain

$$\dot{V}\big(t,x(t)\big) - \varrho V\big(t,x(t)\big) - \varrho\omega^T(t)X\omega(t) \leq \zeta^T(t)\Xi\zeta(t).$$

From the conditions (10) and (12), we have

$$\dot{V}\big(t,x(t)\big) - \varrho V\big(t,x(t)\big) - \varrho\omega^T(t)X\omega(t) < 0. \tag{29}$$

Multiplying (29) by $e^{-\varrho t}$, we can derive that

$$\frac{d}{dt}\big(e^{-\varrho t}V\big(t,x(t)\big)\big) < \varrho e^{-\varrho t}\omega^T(t)X\omega(t). \tag{30}$$

Using assumption A2 and integrating (30) from 0 to t with $t \in [0,T]$, we have

$$\begin{aligned}V\big(t,x(t)\big) &< e^{\varrho T}\bigg[V\big(0,x(0)\big) + \varrho\int_0^T e^{-\varrho u}\omega^T(u)X\omega(u)\,du\bigg]\\ &< e^{\varrho T}\big[V\big(0,x(0)\big) + d\lambda_{14}\big(1 - e^{-\varrho T}\big)\big].\end{aligned} \tag{31}$$

Considering $V(0,x(0))$, we can derive that

$$\begin{aligned}V\big(0,x(0)\big) &= \rho_1^T(0)P\rho_1(0) + \int_{-hm}^0 e^{-\varrho u}\rho_2(u)Q_1\rho_2(u)\,du\\ &\quad + \int_{h_M}^0 e^{-\varrho u}\rho_2^T(u)Q_2\rho_2(u)\,du\\ &\quad + h_m\int_{-h_m}^0\int_u^0 e^{-\varrho s}\dot{x}(s)R_1\dot{x}(s)\,ds\,du\end{aligned}$$

$$
\begin{aligned}
&+ h_{Mm}\int_{-h_M}^{-h_m}\int_u^0 e^{-\varrho s}\dot{x}(s)R_2\dot{x}(s)\,ds\,du \\
&+ h_M\int_{-h_M}^{0}\int_u^0 e^{-\varrho s}\dot{x}(s)R_3\dot{x}(s)\,ds\,du \\
&+ \eta_{21}\int_{-\eta_2}^{-\eta_1}\int_u^0 e^{-\varrho s}f^T\bigl(Wx(s)\bigr)Zf\bigl(Wx(s)\bigr)\,ds\,du \\
\le{}& \rho_1^T(0)P\rho_1(0) + \int_{-hm}^{0} e^{-\varrho u}\rho_2(u)Q_1\rho_2(u)\,du \\
&+ \int_{h_M}^{0} e^{-\varrho u}\rho_2^T(u)Q_2\rho_2(u)\,du \\
&+ h_m\int_{-h_m}^{0}\int_u^0 e^{-\varrho s}\dot{x}(s)R_1\dot{x}(s)\,ds\,du \\
&+ h_{Mm}\int_{-h_M}^{-h_m}\int_u^0 e^{-\varrho s}\dot{x}(s)R_2\dot{x}(s)\,ds\,du \\
&+ h_M\int_{-h_M}^{0}\int_u^0 e^{-\varrho s}\dot{x}(s)R_3\dot{x}(s)\,ds\,du \\
&+ \eta_{21}\int_{-\eta_2}^{-\eta_1}\int_u^0 e^{-\varrho s}x^T(s)F_wZF_wx(s)\,ds\,du,
\end{aligned}
$$

where $F_w = \operatorname{diag}\{F_1^+, F_2^+, \dots, F_n^+\}W$.

Letting $\overline{P}_i = H^{-\frac{1}{2}}P_iH^{-\frac{1}{2}}$, $\overline{Q}_{1i} = H^{-\frac{1}{2}}Q_{1i}H^{-\frac{1}{2}}$, $\overline{Q}_{2i} = H^{-\frac{1}{2}}Q_{2i}H^{-\frac{1}{2}}$, $\overline{R}_i = H^{-\frac{1}{2}}R_iH^{-\frac{1}{2}}$, $\overline{Z} = H^{-\frac{1}{2}}F_wZF_wH^{-\frac{1}{2}}$, $i = 1, 2, 3$, we have

$$
\begin{aligned}
V\bigl(0, x(0)\bigr) \le{}& \rho_1^T(0)H^{\frac{1}{2}}\overline{P}H^{\frac{1}{2}}\rho_1(0) + \int_{-hm}^{0} e^{\varrho u}\rho_2(u)H^{\frac{1}{2}}\overline{Q}_1H^{\frac{1}{2}}\rho_2(u)\,du \\
&+ \int_{h_M}^{0} e^{-\varrho u}\rho_2^T(u)H^{\frac{1}{2}}\overline{Q}_2H^{\frac{1}{2}}\rho_2(u)\,du \\
&+ h_m\int_{-h_m}^{0}\int_u^0 e^{-\varrho s}\dot{x}^T(s)H^{\frac{1}{2}}\overline{R}_1H^{\frac{1}{2}}\dot{x}(s)\,ds\,du \\
&+ h_{Mm}\int_{-h_M}^{-h_m}\int_u^0 e^{-\varrho s}\dot{x}^T(s)H^{\frac{1}{2}}\overline{R}_2H^{\frac{1}{2}}\dot{x}(s)\,ds\,du \\
&+ h_M\int_{-h_M}^{0}\int_u^0 e^{-\varrho s}\dot{x}^T(s)H^{\frac{1}{2}}\overline{R}_3H^{\frac{1}{2}}\dot{x}^T(s)\,ds\,du \\
&+ \eta_{21}\int_{-\eta_2}^{-\eta_1}\int_u^0 e^{-\varrho s}x^T(s)H^{\frac{1}{2}}\overline{Z}H^{\frac{1}{2}}x(s)\,ds\,du \\
\le{}& \bigl\{\lambda_{\max}(\overline{P}_1) + h_m\lambda_{\max}(\overline{P}_2) + h_M\lambda_{\max}(\overline{P}_3) \\
&+ \delta_1\bigl[\lambda_{\max}(\overline{Q}_{11}) + 2\lambda_{\max}(\overline{Q}_{12}) + \lambda_{\max}(\overline{Q}_{13})\bigr] \\
&+ \delta_2\bigl[\lambda_{\max}(\overline{Q}_{21}) + 2\lambda_{\max}(\overline{Q}_{22}) + \lambda_{\max}(\overline{Q}_{23})\bigr] + h_m\delta_3\lambda_{\max}(\overline{R}_1) \\
&+ h_{Mm}\delta_4\lambda_{\max}(\overline{R}_2) + h_M\delta_5\lambda_{\max}(\overline{R}_3) + \eta_{21}\delta_6\lambda_{\max}(\overline{Z})\bigr\} \\
&\times \sup_{-h_M\le u\le 0}\bigl\{x^T(u)Hx(u), \dot{x}^T(u)H\dot{x}(u)\bigr\}
\end{aligned}
$$

$$\begin{aligned}&\le \big\{\lambda_1 + h_m\lambda_2 + h_M\lambda_3 + \delta_1(\lambda_4 + 2\lambda_5 + \lambda_6)\\&\quad + \delta_2(\lambda_7 + 2\lambda_8 + \lambda_9) + h_m\delta_3\lambda_{10} + h_{Mm}\delta_4\lambda_{11}\\&\quad + h_M\delta_5\lambda_{12} + \eta_{21}\delta_6\lambda_{13}\big\}c_1\\&= \Xi_\lambda c_1.\end{aligned}$$

Moreover, from (13), we obtain

$$V\big(t,x(t)\big) \ge x^T(t)P_1x(t) \ge \lambda_{\min}(\overline{P}_1)x^T(t)Hx(t) = \lambda_0 x^T(t)Hx(t). \tag{32}$$

Then, from (31), (32), and LMI (12), we obtain

$$x^T(t)Hx(t) \le \frac{e^{\varrho T}}{\lambda_0}\big[\Xi_\lambda c_1 + d\lambda_{14}\big(1 - e^{-\varrho T}\big)\big] < c_2.$$

Therefore, the delayed GNNs (9) are finite-time bounded respecting (c_1, c_2, T, H, d). The proof is complete. □

Remark 4 The activation function in Assumption A1 does not need to be nonmonotonic and differentiable since the constants F_i^- and F_i^+, $i = 1, 2, \dots, n$ can be either positive, zero, or negative. Since Assumption A1 has been considered in (22)–(24) of this article, not only $F_i^- \le \frac{f_i(Wx(t))}{Wx(t)} \le F_i^+$ and $F_i^- \le \frac{f_i(Wx(t-h(t)))}{Wx(t-h(t))} \le F_i^+$ but also $F_i^- \le \frac{f_i(Wx_1)-f_i(Wx_2)}{Wx_1-Wx_2} \le F_i^+$. Thus, the assumption is weaker and more general than the usual Lipschitz condition ($|f(Wx_1) - f(Wx_2)| \le F|Wx_1 - Wx_2|$).

Moreover, we derive the new sufficient conditions of finite-time boundedness of the GNNs (9) with mixed-interval time-varying delays by applying the Wirtinger-based integral inequality as the following:

Corollary 3.2 *Given positive scalars h_M and ϱ then the delayed GNNs* (9) *are finite-time bounded regarding (c_1, c_2, T, H, d), if there exist symmetric positive-definite matrices $P \in \mathbb{R}^{3n\times 3n}$, $Q_i \in \mathbb{R}^{2n\times 2n}$, $R_j \in \mathbb{R}^{n\times n}$ $(i = 1, 2, j = 1, 2, 3)$, Z, X, any matrices N_1, N_2, and positive diagonal matrices S_1, S_2, S_3 satisfying LMIs* (11) *and* (12) *and*

$$\widetilde{\Xi} < 0, \tag{33}$$

where $\widetilde{\Xi} = \Xi_1 + \Xi_2 + \widetilde{\Xi}_3 + \Xi_4 + \Xi_5 + \Xi_6 + \Xi_7$,

$$\begin{aligned}\widetilde{\Xi}_3 &= h_m^2 e_7^T R_1 e_7 + h_{Mm}^2 e_7^T R_2 e_7 + h_M^2 e_7^T R_3 e_7 - \Gamma_6^T R_1 \Gamma_6 - 3\widetilde{\Gamma}_7^T R_1 \widetilde{\Gamma}_7\\&\quad - \Gamma_8^T R_2 \Gamma_8 - 3\widetilde{\Gamma}_9^T R_2 \widetilde{\Gamma}_9 - \Gamma_{10}^T R_2 \Gamma_{10} - 3\widetilde{\Gamma}_{11}^T R_2 \widetilde{\Gamma}_{11} - \Gamma_{12}^T R_3 \Gamma_{12}\\&\quad - 3\widetilde{\Gamma}_{13}^T R_3 \widetilde{\Gamma}_{13},\\\widetilde{\Gamma}_7 &= \big[e_2^T + e_1^T - 2e_{10}^T\big], \qquad \widetilde{\Gamma}_9 = \big[e_3^T + e_2^T - 2e_{12}^T\big], \qquad \widetilde{\Gamma}_{11} = \big[e_4^T + e_3^T - 2e_{13}^T\big],\\\widetilde{\Gamma}_{13} &= \big[e_4^T + e_1^T - 2e_{11}^T\big],\end{aligned}$$

and the others as given in Theorem 3.1.

Proof We apply the similarity of proof as in Theorem 3.1, except that we apply Lemma 2.3 (Wirtinger's inequality) to the single integral in Equation (16). Therefore, it is omitted here. □

Furthermore, we derive the finite-time boundedness of the delayed GNNs (9) by applying the inequality in Lemma 2.4 [39], and we define vectors as follows:

$$\begin{aligned}
\zeta(t) = \Big[&x^T(t), x^T(t-h_m), x^T\big(t-h(t)\big), x^T(t-h_M), f^T\big(Wx(t)\big),\\
&f^T\big(Wx\big(t-h(t)\big)\big), \dot{x}^T(t), \dot{x}^T(t-h_m), \dot{x}^T(t-h_M),\\
&\frac{1}{h_m}\int_{t-h_m}^{t} x^T(u)\,du, \frac{1}{h_M}\int_{t-h_M}^{t} x^T(u)\,du,\\
&\int_{t-\eta_2(t)}^{t-\eta_1(t)} f^T\big(Wx(u)\big)\,du, \omega^T(t)\Big]^T,\\
e_j = \Big[&0_{n\times(j-1)n} \quad I_n \quad 0_{n\times(13-j)n}\Big], \quad j = 1,2,\ldots,13.
\end{aligned}$$

Corollary 3.3 *Given positive scalars h_M and ϱ then the delayed GNNs (9) are finite-time bounded regarding (c_1, c_2, T, H, d), if there exist symmetric positive-definite matrices $P \in \mathbb{R}^{3n\times 3n}$, $Q_i \in \mathbb{R}^{2n\times 2n}$, $R_j \in \mathbb{R}^{n\times n}$ $(i = 1,2, j = 1,2,3)$, Z, X, any matrices N_1, N_2, and positive diagonal matrices S_1, S_2, S_3 satisfying LMIs (11) and (12) and*

$$\overline{\Xi} < 0, \tag{34}$$

where $\overline{\Xi} = \Xi_1 + \Xi_2 + \overline{\Xi}_3 + \Xi_4 + \Xi_5 + \Xi_6 + \Xi_7$,

$$\begin{aligned}
\overline{\Xi}_3 = {}& h_m^2 e_7^T R_1 e_7 + h_{Mm}^2 e_7^T R_2 e_7 + h_M^2 e_7^T R_3 e_7\\
&+ h_m \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix}^T \begin{bmatrix} -u_1 R_1 & u_2 R_1 \\ u_2 R_1 & -u_3 R_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}\\
&+ h_{Mm} \begin{bmatrix} e_2^T \\ e_3^T \end{bmatrix}^T \begin{bmatrix} -m_1 R_2 & m_2 R_2 \\ m_2 R_2 & -m_3 R_2 \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix}\\
&+ h_{Mm} \begin{bmatrix} e_3^T \\ e_4^T \end{bmatrix}^T \begin{bmatrix} -m_1 R_2 & m_2 R_2 \\ m_2 R_2 & -m_3 R_2 \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix}\\
&+ h_M \begin{bmatrix} e_1^T \\ e_4^T \end{bmatrix}^T \begin{bmatrix} -v_1 R_3 & v_2 R_3 \\ v_2 R_3 & -v_3 R_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_4 \end{bmatrix},\\
u_1 = {}& 2\left(\frac{\varrho}{2} + \frac{1}{h_m}\right) - \rho_1\left(\frac{\varrho}{2} + \frac{1}{h_m}\right)^2,\\
u_2 = {}& \left(\frac{\varrho}{2} + \frac{1}{h_m}\right) - \left(\frac{\varrho}{2} - \frac{1}{h_m}\right)e^{\varrho h_m} + \rho_1\left(\frac{\varrho^2}{4} - \frac{1}{h_m^2}\right)e^{\varrho h_m},\\
u_3 = {}& -2\left(\frac{\varrho}{2} - \frac{1}{h_m}\right)e^{\varrho h_m} - \rho_1\left(\frac{\varrho}{2} - \frac{1}{h_m}\right)^2 e^{2\varrho h_m},\\
v_1 = {}& 2\left(\frac{\varrho}{2} + \frac{1}{h_M}\right) - \rho_2\left(\frac{\varrho}{2} + \frac{1}{h_M}\right)^2,
\end{aligned}$$

$$\begin{aligned}
\nu_2 &= \left(\frac{\varrho}{2}+\frac{1}{h_M}\right)-\left(\frac{\varrho}{2}-\frac{1}{h_M}\right)e^{\varrho h_M}+\rho_2\left(\frac{\varrho^2}{4}-\frac{1}{h_M^2}\right)e^{\varrho h_M},\\
\nu_3 &= -2\left(\frac{\varrho}{2}-\frac{1}{h_M}\right)e^{\varrho h_M}-\rho_2\left(\frac{\varrho}{2}-\frac{1}{h_M}\right)^2 e^{2\varrho h_M},\\
m_1 &= 2\left(\frac{\varrho}{2}+\frac{1}{h_{Mm}}\right)e^{\varrho h_m}-\rho_3\left(\frac{\varrho}{2}+\frac{1}{h_{Mm}}\right)^2 e^{2\varrho h_m},\\
m_2 &= \left(\frac{\varrho}{2}+\frac{1}{h_{Mm}}\right)e^{\varrho h_m}-\left(\frac{\varrho}{2}-\frac{1}{h_{Mm}}\right)e^{\varrho h_M}+\rho_3\left(\frac{\varrho^2}{4}-\frac{1}{h_{Mm}^2}\right)e^{\varrho(h_m+h_M)},\\
m_3 &= -2\left(\frac{\varrho}{2}-\frac{1}{h_{Mm}}\right)e^{\varrho h_M}-\rho_3\left(\frac{\varrho}{2}-\frac{1}{h_{Mm}}\right)^2 e^{2\varrho h_M},\\
\rho_1 &= \frac{1-e^{-\varrho h_m}}{\varrho},\qquad \rho_2=\frac{1-e^{-\varrho h_M}}{\varrho},\qquad \rho_3=\frac{e^{-\varrho h_m}-e^{-\varrho h_M}}{\varrho},
\end{aligned}$$

and the others as given in Theorem 3.1.

Proof We follow the similarity of proof as in Theorem 3.1, except that we apply Lemma 2.4 [39] to the single integral in Equation (16). Therefore, it is omitted here. □

3.2 Analysis of finite-time stability

This part presents the new delay-dependent criteria for guaranteeing the finite-time stability of the GNNs (35) with interval time-varying delay. If we let $L_2 = 0$ and $L_3 = 0$ in the GNNs (9), the GNNs (9) can be written as

$$\begin{aligned}
\dot{x}(t) &= -Ax(t)+L_0 f\big(Wx(t)\big)+L_1 f\big(Wx\big(t-h(t)\big)\big),\\
x(t) &= \phi(t),\quad \forall t\in[-h_M,0],
\end{aligned}\tag{35}$$

which in (35) is a particular case for the delayed GNNs (9) and they can be encountered as in [12, 15, 20].

Furthermore, we define vectors as follows:

$$\begin{aligned}
\zeta(t) = \bigg[&x^T(t), x^T(t-h_m), x^T\big(t-h(t)\big), x^T(t-h_M), f^T\big(Wx(t)\big),\\
&f^T\big(Wx\big(t-h(t)\big)\big), \dot{x}^T(t), \dot{x}^T(t-h_m), \dot{x}^T(t-h_M),\\
&\frac{1}{h_m}\int_{t-h_m}^{t} x^T(u)\,du, \frac{1}{h_M}\int_{t-h_M}^{t} x^T(u)\,du, \frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m} x^T(u)\,du,\\
&\frac{1}{h_M-h(t)}\int_{t-h_M}^{t-h(t)} x^T(u)\,du\bigg]^T,\\
e_j = \Big[&0_{n\times(j-1)n}\quad I_n\quad 0_{n\times(13-j)n}\Big],\quad j=1,2,\ldots,13.
\end{aligned}$$

We derive the new delay-dependent conditions of the finite-time stability for the GNNs (9) with interval time-varying delay by using the new integral inequality in Lemma 2.5 as the following.

Corollary 3.4 *Given positive scalars h_M and ϱ then the delayed GNNs (35) are finite-time stable regarding (c_1, c_2, T, H), if there exist symmetric positive-definite matrices $P\in\mathbb{R}^{3n\times 3n}$,*

$Q_i \in \mathbb{R}^{2n\times 2n}$, $R_j \in \mathbb{R}^{n\times n}$ $(i = 1,2, j = 1,2,3)$, *any matrices* N_1, N_2, *and positive diagonal matrices* S_1, S_2, S_3, *such that the following conditions hold*:

$$\Xi_s < 0, \tag{36}$$

$$\begin{aligned} &\lambda_0 I \le \overline{P}_1 \le \lambda_1 I, \quad 0 \le \overline{P}_2 \le \lambda_2 I, \quad 0 \le \overline{P}_3 \le \lambda_3 I, \quad 0 \le \overline{Q}_{11} \le \lambda_4 I, \\ &0 \le \overline{Q}_{12} \le \lambda_5 I, \quad 0 \le \overline{Q}_{13} \le \lambda_6 I, \quad 0 \le \overline{Q}_{21} \le \lambda_7 I, \quad 0 \le \overline{Q}_{22} \le \lambda_8 I, \\ &0 \le \overline{Q}_{23} \le \lambda_9 I, \quad 0 \le \overline{R}_1 \le \lambda_{10} I, \quad 0 \le \overline{R}_2 \le \lambda_{11} I, \quad 0 \le \overline{R}_3 \le \lambda_{12} I, \end{aligned} \tag{37}$$

$$e^{\varrho T}\Xi_{s\lambda} c_1 < \lambda_0 c_2, \tag{38}$$

where

$$\begin{aligned} \Xi_s &= \Xi_1 + \Xi_2 + \Xi_3 + \Xi_5 + \Xi_{7s} \\ \Xi_{7s} &= \mathrm{Sym}\big\{\big[e_1^T N_1 + e_7^T N_2\big][-e_7 - Ae_1 + L_0 e_5 + L_1 e_6]\big\}, \\ \Xi_{s\lambda} &= \lambda_1 + h_m\lambda_2 + h_M\lambda_3 + \delta_1(\lambda_4 + 2\lambda_5 + \lambda_6) + \delta_2(\lambda_7 + 2\lambda_8 + \lambda_9) \\ &\quad + h_m\delta_3\lambda_{10} + h_{Mm}\delta_4\lambda_{11} + h_M\delta_5\lambda_{12}, \end{aligned}$$

and the others as given in Theorem 3.1.

Proof We follow the similarity of proof as in Theorem 3.1, except $Z = 0$ or $V_4 = 0$. This corollary uses the new inequality to improve stability criteria. Therefore, it is omitted here. □

In addition, we derive the new sufficient conditions of the finite-time stability for the delayed GNNs (35) that apply the Wirtinger-based integral inequality as the following:

Corollary 3.5 *Given positive scalars* h_M *and* ϱ *then the delayed GNNs* (35) *are finite-time stable regarding* (c_1, c_2, T, H), *if there exist symmetric positive-definite matrices* $P \in \mathbb{R}^{3n\times 3n}$, $Q_i \in \mathbb{R}^{2n\times 2n}$, $R_j \in \mathbb{R}^{n\times n}$ $(i = 1,2, j = 1,2,3)$, *any matrices* N_1, N_2, *and positive diagonal matrices* S_1, S_2, S_3 *satisfying LMIs* (37) *and* (38) *and*

$$\widetilde{\Xi}_s < 0, \tag{39}$$

where $\widetilde{\Xi}_s = \Xi_1 + \Xi_2 + \widetilde{\Xi}_3 + \Xi_5 + \Xi_{7s}$,*and the others as given in Theorem* 3.1, *and Corollaries* 3.2 *and* 3.4.

Proof We follow the same proof as in Corollary 3.4, except that we apply Lemma 2.3 (Wirtinger's inequality) to the single integral in Equation (16). Therefore, it is omitted here. □

Furthermore, we derive the finite-time stability for the GNNs (35) with interval time-varying delay by applying the inequality in Lemma 2.4 [39], and we define the following vectors:

$$\zeta(t) = \Big[x^T(t), x^T(t-h_m), x^T\big(t-h(t)\big), x^T(t-h_M), f^T\big(Wx(t)\big),$$

$$f^T\big(Wx\big(t-h(t)\big)\big), \dot{x}^T(t), \dot{x}^T(t-h_m), \dot{x}^T(t-h_M),$$
$$\left.\frac{1}{h_m}\int_{t-h_m}^{t} x^T(u)\,du, \frac{1}{h_M}\int_{t-h_M}^{t} x^T(u)\,du, \right]^T,$$
$$e_j = \begin{bmatrix} 0_{n\times(j-1)n} & I_n & 0_{n\times(11-j)n}\end{bmatrix}, \quad j = 1,2,\ldots,11.$$

Corollary 3.6 *Given positive scalars h_M and ϱ then the delayed GNNs (35) are finite-time stable regarding (c_1, c_2, T, H), if there exist symmetric positive-definite matrices $P \in \mathbb{R}^{3n\times 3n}$, $Q_i \in \mathbb{R}^{2n\times 2n}$, $R_j \in \mathbb{R}^{n\times n}$ $(i = 1,2, j = 1,2,3)$, any matrices N_1, N_2, and positive diagonal matrices S_1, S_2, S_3 satisfying LMIs (37) and (38) and*

$$\overline{\Xi}_s < 0, \tag{40}$$

where $\overline{\Xi}_s = \Xi_1 + \Xi_2 + \overline{\Xi}_3 + \Xi_5 + \Xi_{7s}$, and the others as given in Theorem 3.1, and Corollaries 3.3 and 3.4.

Proof We follow the same proof as in Corollary 3.4, except that we apply Lemma 2.4 [39] to the single integral in Equation (16). Therefore, it is omitted here. □

3.3 Analysis of nonfragile finite-time boundedness

This part presents the new delay-dependent criteria of the finite-time boundedness for the GNNs under a nonfragile feedback controller with delays like the following:

$$\begin{aligned}\dot{x}(t) &= \big(-A + B\big(\mathcal{K} + \Delta\mathcal{K}(t)\big)\big)x(t) + L_0 f\big(Wx(t)\big) + L_1 f\big(Wx\big(t-h(t)\big)\big) \\ &\quad + L_2\int_{t-\eta_2(t)}^{t-\eta_1(t)} f\big(Wx(u)\big)\,du + L_3\omega(t) + Bu(t), \qquad (41)\\ y(t) &= x(t), \\ x(t) &= \phi(t), \quad \forall t \in [-h_M, 0].\end{aligned}$$

Theorem 3.7 *Given positive scalars h_M and ϱ then the delayed GNNs (41) are finite-time bounded respecting (c_1, c_2, T, H, d), if there exist positive symmetric definite matrices $\widetilde{P}, \widetilde{Q}_i, \widetilde{R}_j$ $(i = 1,2, j = 1,2,3)$, $\widetilde{Z}$, $\widetilde{X}$, and positive diagonal matrices $\widetilde{S}_1$, $\widetilde{S}_2$, $\widetilde{S}_3$, such that the conditions hold as follows:*

$$\Xi_u < 0, \tag{42}$$
$$e^{\varrho T}\big[\Xi_\lambda c_1 + d\lambda_{14}\big(1 - e^{-\varrho T}\big)\big] < \lambda_0 c_2, \tag{43}$$

where

$$\begin{aligned}\Xi_u &= \sum_{i=1}^{8} \Xi_{ui}, \\ \Xi_{u1} &= \mathrm{Sym}\big\{\Gamma_1^T\widetilde{P}\Gamma_2^T\big\} - \varrho\Gamma_1^T\widetilde{P}\Gamma_1, \\ \Xi_{u2} &= \Gamma_3^T(\widetilde{Q}_1 + \widetilde{Q}_2)\Gamma_3 - \Gamma_4^T\big(e^{\varrho h_m}\widetilde{Q}_1\big)\Gamma_4 - \Gamma_5^T\big(e^{\varrho h_M}\widetilde{Q}_2\big)\Gamma_5,\end{aligned}$$

$$\begin{aligned}
\Xi_{u3} &= h_m^2 e_7^T \widetilde{R}_1 e_1 + h_{Mm}^2 e_7^T \widetilde{R}_2 e_7 + h_M^2 e_7^T R_3 e_7 - \frac{h_m}{\gamma_{1a}} \Gamma_6^T \widetilde{R}_1 \Gamma_6 - \frac{h_m}{\gamma_{1b}} \Gamma_7^T \widetilde{R}_2 \Gamma_7 \\
&\quad - \frac{h_{Mm}}{\gamma_{2a}} \Gamma_8^T \widetilde{R}_2 \Gamma_8 - \frac{h_{Mm}}{\gamma_{2b}} \Gamma_9^T \widetilde{R}_2 \Gamma_9 - \frac{h_{Mm}}{\gamma_{3a}} \Gamma_{10}^T \widetilde{R}_2 \Gamma_{10} - \frac{h_{Mm}}{\gamma_{3b}} \Gamma_{11}^T \widetilde{R}_2 \Gamma_{11} \\
&\quad - \frac{h_M}{\gamma_{4a}} \Gamma_{12}^T \widetilde{R}_2 \Gamma_{12} - \frac{h_M}{\gamma_{4b}} \Gamma_{13}^T \widetilde{R}_3 \Gamma_{13}, \\
\Xi_{u4} &= \eta_{21}^2 e_5^T \widetilde{Z} e_5 - e^{\varrho \eta_2} e_{14}^T \widetilde{Z} e_{14}, \\
\Xi_{u5} &= 2\Gamma_{14}^T \widetilde{S}_1 \Gamma_{15} + 2\Gamma_{16}^T \widetilde{S}_2 \Gamma_{17} + 2\Gamma_{18}^T \widetilde{S}_3 \Gamma_{19}, \qquad \Xi_{u6} = -\varrho e_{15}^T \widetilde{X} e_{15}, \\
\Xi_{u7} &= \mathrm{Sym}\big\{-e_1^T U^T e_7 - e_1^T A U^T e_1 + e_1^T B Y e_1 + e_1^T L_o U^T e_5 + e_1^T L_1 U^T e_6 \\
&\quad + e_1^T L_2 U^T e_{14} + e_1^T L_3 U^T e_{15} - e_7^T U e_7 - e_7^T A U^T e_1 + e_7^T B Y e_1 \\
&\quad + e_7^T L_0 U^T e_5 + e_7^T L_1 U^T e_6 + e_7^T L_2 U^T e_{14} + e_7^T L_3 U^T e_{15}\big\}, \\
\Xi_{u8} &= \mathrm{Sym}\big\{e_1^T B D_1 e_{16} + e_7^T B D_1 e_{16} + \alpha e_1^T U D_2^T e_{17}\big\} - e_{16}^T (\alpha I) e_{16} - e_{17}^T (\alpha I) e_{17}.
\end{aligned}$$

Additionally, the gain matrix $\mathcal{K}$ of the feedback controller with a nonfragile issue can be created as $\mathcal{K} = YU^{-1}$.

Proof We follow the method of proof as in Theorem 3.1 and replace A by $A - B(\mathcal{K} + \Delta\mathcal{K}(t))$ in Ξ, and we obtain

$$\widetilde{\Xi} + \Pi_1 F(t) \Pi_2 + \Pi_2^T F^T(t) \Pi_1^T < 0,$$

where

$$\begin{aligned}
\widetilde{\Xi} &= \sum_{i=1}^{6} \Xi_i + \widetilde{\Xi}_7, \\
\widetilde{\Xi}_7 &= \mathrm{Sym}\big\{-e_1^T N_1 e_7 - e_1^T N_1 A e_1 + e_1^T N_1 B \mathcal{K} e_1 + e_1^T N_1 L_0 e_5 \\
&\quad + e_1^T N_1 L_1 e_6 + e_1^T N_2 L_2 e_{14} + e_1^T N_1 L_3 e_{15} - e_7^T N_2 e_7 - e_7^T N_2 A e_1 \\
&\quad + e_7^T N_2 B \mathcal{K} e_1 + e_7^T N_2 L_0 e_5 + e_7^T N_2 L_1 e_6 + e_7^T N_2 L_2 e_{14} + e_7^T N_2 L_3 e_{15}\big\}, \\
\Pi_1 &= \big[(N_1 B D_1)^T, \underbrace{0 \cdots 0}_{5\text{ times}}, (N_2 B D_1)^T, \underbrace{0 \cdots 0}_{8\text{ times}}\big]^T, \\
\Pi_2 &= [D_2, \underbrace{0 \cdots 0}_{14\text{ times}}]^T.
\end{aligned}$$

Applying Lemma 2.6, there exists $\alpha > 0$, so that

$$\widetilde{\Xi} + \alpha^{-1} \Pi_1 \Pi_1^T + \alpha \Pi_2^T \Pi_2 < 0.$$

Applying Lemma 2.7 (the Schur complement), we obtain

$$\begin{bmatrix} \widetilde{\Xi} & \Pi_1 & \alpha \Pi_2^T \\ \Pi_1^T & -\alpha I & 0 \\ \alpha \Pi_2 & 0 & -\alpha I \end{bmatrix} < 0. \tag{44}$$

It can be seen that the LMI conditions in (44) cannot be directly applied to the controllers. Hence, we need to convert the conditions to be the LMI terms. We let $N_1 = N_2 = \mathcal{U}^{-1}$. Then, the criteria (44) are pre- and postmultiplied by $\operatorname{diag}\{\underbrace{\mathcal{U}\cdots\mathcal{U}}_{15\text{ times}}, I, I\}$ and its transpose, describe variables as the following:

$$\begin{aligned}
&\widetilde{P} = \mathcal{U}P\mathcal{U}^T, \quad \widetilde{Q}_1 = \mathcal{U}Q_1\mathcal{U}^T, \quad \widetilde{Q}_2 = \mathcal{U}Q_2\mathcal{U}^T, \quad \widetilde{R}_1 = \mathcal{U}R_1\mathcal{U}^T,\\
&\widetilde{R}_2 = \mathcal{U}R_2\mathcal{U}^T, \quad \widetilde{R}_3 = \mathcal{U}R_3\mathcal{U}^T, \quad \widetilde{Z} = \mathcal{U}Z\mathcal{U}^T, \quad \widetilde{X} = \mathcal{U}X\mathcal{U}^T,\\
&\widetilde{S}_1 = \mathcal{U}S_1\mathcal{U}^T, \quad \widetilde{S}_2 = \mathcal{U}S_2\mathcal{U}^T, \quad \widetilde{S}_3 = \mathcal{U}S_3\mathcal{U}^T, \quad Y = \mathcal{K}\mathcal{U}.
\end{aligned}$$

Thus, we can obtain $\Xi_u < 0$. The proof is complete. □

4 Numerical examples

This section demonstrates the effectiveness of our approaches with three numerical examples.

Example 4.1 Consider the following parameters for GNNs (9):

$$\begin{aligned}
&A = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad L_0 = \begin{bmatrix} -1 & 1 \\ 0.5 & -1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -0.5 & 0.6 \\ 0.7 & 0.8 \end{bmatrix},\\
&L_2 = \begin{bmatrix} 0.15 & 0.1 \\ 0 & -0.3 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0.05 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad W = \begin{bmatrix} 1.28 & 0.35 \\ 0.28 & 0.35 \end{bmatrix},\\
&F_M = \operatorname{diag}\{0, 0\} \quad \text{and} \quad F_P = \operatorname{diag}\{0.6, 0.8\}.
\end{aligned}$$

Let $f(x) = \tanh(x(t))$, $\eta_1(t) = 0.4 + 0.3\sin(t)$, $\eta_2(t) = 0.5 + 0.2\sin(t)$, and $\omega(t) = \sqrt{0.5}\cos(t)$. Given scalars $c_1 = 0.72$, $d = 0.5$, $\eta_1 = 0.1$, $\eta_2 = 0.7$ and matrix $H = I$. From the parameters as mentioned above, we solve the LMIs in Theorem 3.1, and Corollaries 3.2 and 3.3 to obtain the feasible solution guaranteeing finite-time boundedness regarding (c_1, c_2, T, H, d) for comparing the minimum allowable lower bounds (MALBs) of c_2. In this example, we investigate four cases to demonstrate the effectiveness of our results as follows:

Case I: Let $h(t) = 1.25|\sin(t)| + 1$, which means $[h_m, h_M] = [1.0, 2.25]$ for $t \in [0, T]$, $T = 10, 20, 30, 40, 50$. We solve the LMIs in Theorem 3.1, and Corollaries 3.2 and 3.3 to obtain the MALBs of c_2 for different values of final time $T = 10, 20, 30, 40, 50$, as displayed in Table 1. Theorem 3.1 provides the smallest MALBs of c_2 for various values of T, which are smaller than those from Corollaries 3.2 and 3.3. By applying the new integral inequality, Theorem 3.1 is less conservative than Corollaries 3.2 and 3.3.

Table 1 MALBs of c_2 for different values of T in Example 4.1 (Case I). Numbers in parentheses are ϱ

T	10	20	30	40	50
Corollary 3.2 (Wirtinger's inequality)	1.5570 (0.3)	1.7186 (0.1)	1.9238 (0.05)	2.1604 (0.03)	2.4135 (0.02)
Corollary 3.3 (Lemma 2.4 [39])	1.4050 (0.3)	1.5701 (0.09)	1.7321 (0.04)	1.8801 (0.02)	2.0181 (0.01)
Theorem 3.1 (New inequality)	1.3055 (0.3)	1.4306 (0.09)	1.5686 (0.04)	1.6982 (0.02)	1.8203 (0.01)

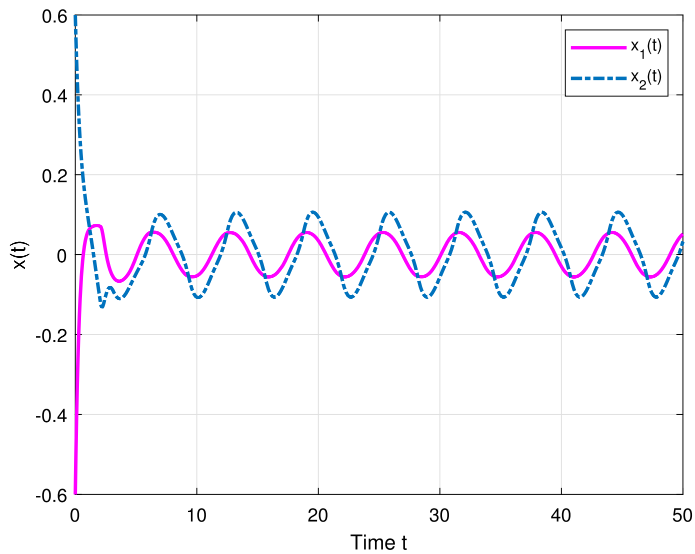

Figure 2 State responses of $x(t)$ for the GNNs (9) in Example 4.1

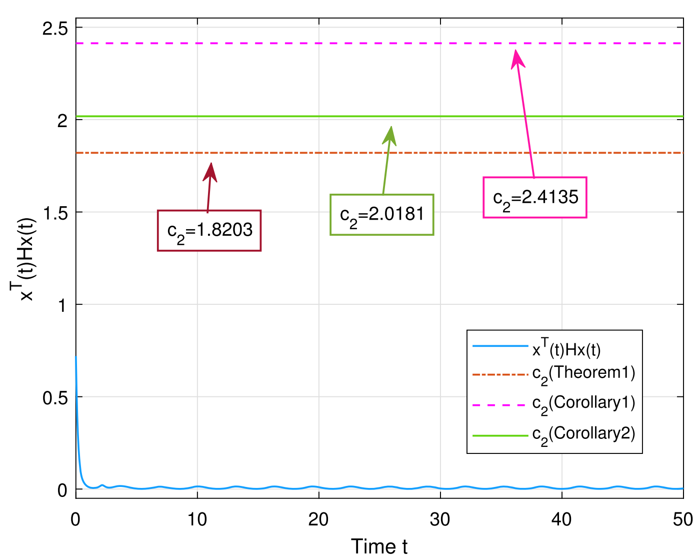

Figure 3 Time history of $x^T(t)Hx(t)$ for the GNNs (9) in Case I Example 4.1, $T = 50$

Figures 2 and 3 display the state responses of $x(t)$ and the time history of $x^T(t)Hx(t)$ for the GNNs (9) in Example 4.1 with an initial condition $\phi(t) = \begin{bmatrix} -0.6\cos(t) & 0.6\cos(t) \end{bmatrix}^T$. From Fig. 3, the time history of $x^T(t)Hx(t)$ does not exceed the MALBs of c_2 in Table 1. Thus, the accuracy of the proposed results is confirmed.

Case II: Let $h(t) = 0.25l|\sin(t)| + 1$, $l = 2, 3, 4, 5, 6$, for $t \in [0, 10]$. In this case, we investigate the effect of ranges of time delay $[h_m, h_M]$ for $h_{Mm} = h_M - h_m = 0.5, 0.75, 1.0, 1.25, 1.5$. For fixed lower bounds $h_m = 1.0$ and different upper bounds $h_M = 1.5, 1.75, 2.0, 2.25, 2.50$, we solve the LMIs in Theorem 3.1, and Corollaries 3.2 and 3.3 to obtain the MALBs of c_2, as shown in Table 2. From the table, we observe that the MALBs of c_2 from Theorem 3.1 are

Table 2 MALBs of c_2 for $T = 10$, $\varrho = 0.3$ and different values of $[h_m, h_M]$ in Example 4.1 (Case II)

$[h_m, h_M]$ h_{Mm}	[1.0, 1.5] 0.5	[1.0, 1.75] 0.75	[1.0, 2.0] 1.0	[1.0, 2.25] 1.25	[1.0, 2.5] 1.5
Corollary 3.2 (Wirtinger's inequality)	1.0198	1.1518	1.3260	1.5570	1.8742
Corollary 3.3 (Lemma 2.4 [39])	0.9297	1.0286	1.1783	1.4050	1.7571
Theorem 3.1 (New inequality)	0.9805	1.0708	1.1783	1.3055	1.4553

Table 3 MALBs of c_2 for $T = 10$, $\varrho = 0.3$ and different values of $[h_m, h_M]$ ($h_{Mm} = 0.5$) in Example 4.1 (Case III)

$[h_m, h_M]$	[1.1, 1.6]	[1.2, 1.7]	[1.3, 1.8]	[1.4, 1.9]	[1.5, 2.0]
Corollary 3.2 (Wirtinger's inequality)	1.0156	1.0114	1.0073	1.0033	0.9994
Corollary 3.3 (Lemma 2.4 [39])	0.9241	0.9187	0.9134	0.9083	0.9033
Theorem 3.1 (New inequality)	0.9737	0.9670	0.9606	0.9543	0.9482

Table 4 MALBs of c_2 for $T = 10$, $\varrho = 0.3$ and different values of $[h_m, h_M]$ ($h_{Mm} = 1.5$) in Example 4.1 (Case IV)

$[h_m, h_M]$	[1.1, 2.6]	[1.2, 2.7]	[1.3, 2.8]	[1.4, 2.9]	[1.5, 3.0]
Corollary 3.2 (Wirtinger's inequality)	1.8670	1.8600	1.8531	1.8464	1.8398
Corollary 3.3 (Lemma 2.4 [39])	1.7220	1.6884	1.6564	1.6257	1.5963
Theorem 3.1 (New inequality)	1.4331	1.4117	1.3911	1.3712	1.3521

smaller than those from Corollaries 3.2 and 3.3 in the ranges of delay $h_{Mm} = 1.25, 1.5$. In the range of delay $h_{Mm} = 1.0$, the MALBs of c_2 from Theorem 3.1 are equal to those from Corollary 3.3 but smaller than those from Corollary 3.2. On the other hand, the MALBs of c_2 from Theorem 3.1 are greater than those from Corollary 3.3 but smaller than those from Corollary 3.2 in the ranges of delay $h_{Mm} = 0.5, 0.75$. Moreover, the MALBs of c_2 from our results increase as h_{Mm} increases.

Case III: Let $h(t) = 0.5|\sin(t)| + 1 + 0.1l$, $l = 1, 2, 3, 4, 5$, for $t \in [0, 10]$. In this case, the effect of changing the interval time-delay range $[h_m, h_M]$ for $h_{Mm} = 0.5$ is investigated. We solve the LMIs in Theorem 3.1, Corollaries 3.2 and 3.3 to obtain the MALBs of c_2 with a fixed range of interval time delay $h_{Mm} = 0.5$ and various lower bounds $h_m = 1.1, 1.2, 1.3, 1.4, 1.5$. From Table 3, the MALBs of c_2 from Theorem 3.1 are greater than those from Corollary 3.3 but smaller than those from Corollary 3.2 for the delay range of $h_{Mm} = 0.5$. Furthermore, the MALBs of c_2 from our results decrease as the lower bound h_m increases.

Case IV: Let $h(t) = 1.5|\sin(t)| + 1 + 0.1l$, $l = 1, 2, 3, 4, 5$, for $t \in [0, 10]$. In this case, the effect of changing the interval time-delay range $[h_m, h_M]$ for $h_{Mm} = 1.5$ is analyzed. We solve the LMIs in Theorem 3.1, and Corollaries 3.2 and 3.3 to obtain the MALBs of c_2 with a defined interval time delay $h_{Mm} = 1.5$ and various lower bounds $h_m = 1.1, 1.2, 1.3, 1.4, 1.5$, as shown in Table 4. In the delay range $h_{Mm} = 1.5$, the MALBs of c_2 from Theorem 3.1 are

Table 5 MALBs of c_2 for different values of T in Example 4.2 (Case I). Numbers in parentheses are ϱ

T	10	20	30	40	50
Corollary 3.5 (Wirtinger's inequality)	2.0068 (1.2)	2.4544 (0.65)	3.2058 (0.45)	4.7698 (0.35)	7.4311 (0.29)
Corollary 3.6 (Lemma 2.4 [39])	1.9687 (1.2)	2.3854 (0.65)	3.1061 (0.45)	4.6231 (0.35)	7.2162 (0.29)
Corollary 3.4 (New inequality)	1.9676 (1.2)	2.3807 (0.65)	3.0918 (0.45)	4.5885 (0.35)	7.1423 (0.29)

smaller than those from Corollaries 3.2 and 3.3. Additionally, the MALBs of c_2 decrease, when the lower bound h_m increases.

Example 4.2 Consider the following parameters for GNNs (35):

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.7 \end{bmatrix}, \qquad L_0 = \begin{bmatrix} 0.2 & -0.1 \\ -0.5 & 0.1 \end{bmatrix}, \qquad L_1 = \begin{bmatrix} -0.5 & 0 \\ -0.3 & -0.2 \end{bmatrix},$$

$$W = \begin{bmatrix} 1.5 & 0.25 \\ 0.68 & 0.66 \end{bmatrix}, \qquad F_M = \operatorname{diag}\{0, 0\} \quad \text{and} \quad F_P = \operatorname{diag}\{0.6, 1.6\}.$$

The activation function is given by $f(x) = \tanh(x(t))$. Given scalars $c_1 = 1.28$ and matrix $H = I$. From the parameters as mentioned above, we solve the LMIs in Corollaries 3.4, 3.5, and 3.6 to obtain the feasible solution guaranteeing finite-time stability regarding (c_1, c_2, T, H, d) for comparing the MALBs of c_2. In this example, we investigate four cases to demonstrate the effectiveness of our results as follows:

Case I: Let $h(t) = 1.25|\sin(t)| + 1.5$, which mean $[h_m, h_M] = [1.5, 2.75]$ for $t \in [0, T]$, $T = 10, 20, 30, 40, 50$. We solve the LMIs in Corollaries 3.4, 3.5, and 3.6 to obtain the MALBs of c_2 for various values of final time $T = 10, 20, 30, 40, 50$, as displayed in Table 5. Corollary 3.4 provides the smallest MALBs of c_2 for different values of T, which are smaller than those from Corollaries 3.5 and 3.6. By applying the new integral inequality, Corollary 3.4 is less conservative than Corollaries 3.5 and 3.6.

Figures 4 and 5 illustrate the state responses of $x(t)$ and time history of $x^T(t)Hx(t)$ for the GNNs (35) in Example 4.2 with an initial condition $\phi(t) = [-0.8\cos(t) \quad 0.8\cos(t)]^T$. From Fig. 5, the time history of $x^T(t)Hx(t)$ does not exceed the MALBs of c_2 in Table 5. Thus, the correctness of the proposed results is confirmed.

Case II: Let $h(t) = 0.25l|\sin(t)| + 1.5$, $l = 2, 3, 4, 5, 6$, for $t \in [0, 30]$. In this case, we analyze the effect of ranges of time delays $[h_m, h_M]$ for $h_{Mm} = h_M - h_m = 0.5, 0.75, 1.0, 1.25, 1.5$. For fixed lower bound $h_m = 1.5$ and various upper bounds $h_M = 2.0, 2.25, 2.5, 2.75, 3.0$, we solve the LMIs in Corollaries 3.4, 3.5, and 3.6 to obtain the MALBs of c_2, as displayed in Table 6. From the table, we can see that the MALBs of c_2 from Corollary 3.4 are smaller than those from Corollaries 3.5 and 3.6 in the ranges of delay $h_{Mm} = 1.0, 1.25, 1.5$. However, the MALBs of c_2 from Corollary 3.4 are greater than those from Corollary 3.6 but smaller than those from Corollary 3.5 in the ranges of delay $h_{Mm} = 0.5, 0.75$. Additionally, the MALBs of c_2 from our results increase as h_{Mm} increases.

Case III: Let $h(t) = 0.25|\sin(t)| + 1.5 + 0.1l$, $l = 1, 2, 3, 4, 5$, for $t \in [0, 30]$. In this case, the effect of changing the interval time-delay range $[h_m, h_M]$ for $h_{Mm} = 0.25$ is examined. We solve the LMIs in Corollaries 3.4, 3.5, and 3.6 to obtain the MALBs of c_2 with a fixed range

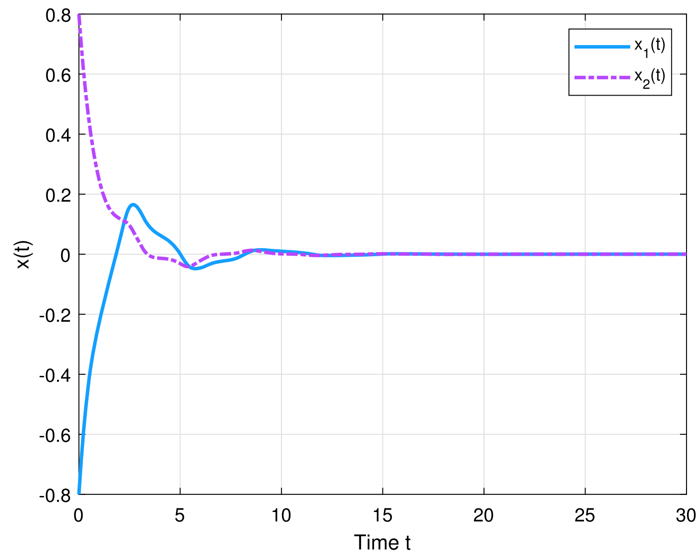

Figure 4 State responses of $x(t)$ for the GNNs (35) in Example 4.2

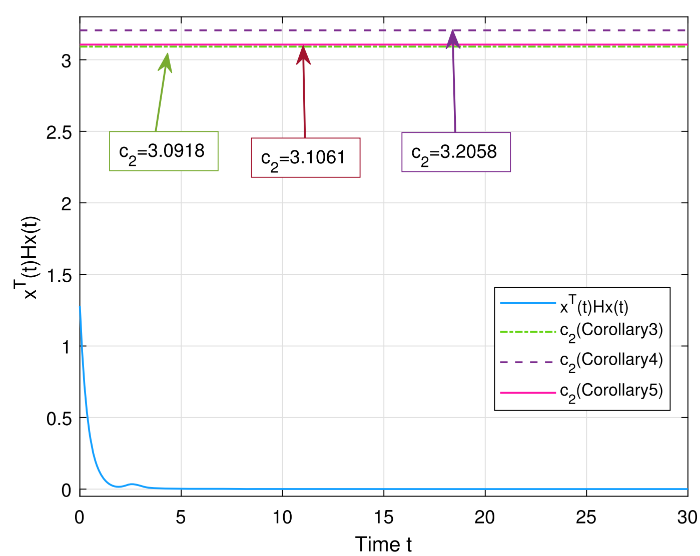

Figure 5 Time history of $x^T(t)Hx(t)$ for the GNNs (35) in Case I Example 4.2, $T = 30$

of interval time delay $h_{Mm} = 0.25$ and different lower bounds $h_m = 1.6, 1.7, 1.8, 1.9, 2.0$. From Table 7, the MALBs of c_2 from Corollary 3.4 are greater than those from Corollary 3.6 but smaller than those from Corollary 3.5 for the delay range of $h_{Mm} = 0.25$. Moreover, the MALBs of c_2 from our results increase as the lower bound h_m increases.

Case IV: Let $h(t) = 1.25|\sin(t)| + 1.5 + 0.1l$, $l = 1, 2, 3, 4, 5$, for $t \in [0, 30]$. In this case, the effect of changing the interval time-delay range $[h_m, h_M]$ for $h_{Mm} = 1.25$ is investigated. We solve the LMIs in Corollaries 3.4, 3.5, and 3.6 to obtain the MALBs of c_2 with a fixed interval time delay $h_{Mm} = 1.25$ and different lower bounds $h_m = 1.6, 1.7, 1.8, 1.9, 2.0$, as shown in Table 8. In the delay range $h_{Mm} = 1.25$, the MALBs of c_2 from Corollary 3.4 are smaller

Table 6 MALBs of c_2 for $T = 30$, $\varrho = 0.45$ and different values of $[h_m, h_M]$ in Example 4.2 (Case II)

$[h_m, h_M]$ h_{Mm}	[1.5, 2.0] 0.5	[1.5, 2.25] 0.75	[1.5, 2.5] 1.0	[1.5, 2.75] 1.25	[1.5, 3.0] 1.5
Corollary 3.5 (Wirtinger's inequality)	1.7118	2.0795	2.5630	3.2058	4.1721
Corollary 3.6 (Lemma 2.4 [39])	1.6814	2.0405	2.5062	3.1061	3.8808
Corollary 3.4 (New inequality)	1.6884	2.0447	2.5048	3.0918	3.8328

Table 7 MALBs of c_2 for $T = 30$, $\varrho = 0.45$ and different values of $[h_m, h_M]$ ($h_{Mm} = 0.25$) in Example 4.2 (Case III)

$[h_m, h_M]$	[1.6, 1.85]	[1.7, 1.95]	[1.8, 2.05]	[1.9, 2.15]	[2.0, 2.25]
Corollary 3.5 (Wirtinger's inequality)	1.5933	1.7665	1.9547	2.1591	2.3809
Corollary 3.6 (Lemma 2.4 [39])	1.5642	1.7348	1.9202	2.1217	2.3404
Corollary 3.4 (New inequality)	1.5742	1.7457	1.9318	2.1338	2.3530

Table 8 MALBs of c_2 for $T = 30$, $\varrho = 0.45$ and different values of $[h_m, h_M]$ ($h_{Mm} = 1.25$) in Example 4.2 (Case IV)

$[h_m, h_M]$	[1.6, 2.85]	[1.7, 2.95]	[1.8, 3.05]	[1.9, 3.15]	[2.0, 3.25]
Corollary 3.5 (Wirtinger's inequality)	3.5054	3.8282	4.1761	4.5503	4.9530
Corollary 3.6 (Lemma 2.4 [39])	3.3966	3.7098	4.0474	4.4111	4.8025
Corollary 3.4 (New inequality)	3.3830	3.6966	4.0343	4.3981	4.7894

than those from Corollaries 3.5 and 3.6. Furthermore, the MALBs of c_2 increase, when the lower bound h_m increases.

Remark 5 In Lemma 2.5, we derive the new integral inequality with the exponential function to estimate the single integral terms of the derivative of LKFs in Theorem 3.1 and Corollary 3.4. In contrast, we use the approximation $-e^{\varrho(t-u)} \leq -e^{\varrho d_2}$, $t - d_1 \leq u \leq t - d_2$ and Wirtinger's integral inequality without the exponential term such as $-\int_{t-d_1}^{t-d_2} e^{\varrho(t-u)}\dot{x}^T(t)M\dot{x}(t)\,du \leq -e^{\varrho d_2}\int_{t-d_1}^{t-d_2}\dot{x}^T(t)M\dot{x}(t)\,du$ in Corollaries 3.2 and 3.5. In Examples 4.1 and 4.2, the MALBs of c_2 from the new integral inequality are smaller than those from Wirtinger's integral inequality in all cases. Thus, the results obtained by the new inequality are less conservative than those obtained by Wirtinger's inequality.

Remark 6 Note that the similarities between Lemma 2.4 [39] and the new inequality are in the form of exponential functions. In contrast, the differences are dimension-free matrices from an estimate of the single integral terms of the derivative of LKFs. While Lemma 2.4 [39] desires only a 2-dimensional-free matrix $[x^T(t-h_m) \quad x^T(t-h_M)]^T$, the new inequality requires a 3-dimensional-free matrix $[\varepsilon_1 x^T(t-h_m) \quad \varepsilon_2 x^T(t-h_M) \quad \int_{t-h_M}^{t-h_m} x^T(u)\,du]^T$. In Case III of Examples 4.1 and 4.2, the MALBs of c_2 obtained by Lemma 2.4 [39] are smaller than those from Lemma 2.5 (New inequality), where $h_{Mm} = 0.25, 0.50$. However, the MALBs of c_2 obtained by Lemma 2.4 [39] are greater than those from the new inequality, where $h_{Mm} = 1.25, 1.50$ in Case IV. Similar to Case II of Examples 4.1 and 4.2, we observe that the MALBs of c_2 by applying the new inequality are smaller than those from Lemma 2.4 [39], where $h_{Mm} > 1.0$. Thus, the results obtained by the new integral inequality can open up the possibility of overcoming results obtained by Lemma 2.4 [39].

Remark 7 We use a specific form of the Lyapunov–Krasovskii functional (13) with exponential functions to obtain finite-time stability results, simplifying the application of our results to analyze the finite-time stability and finite-time boundedness in practice. The proofs of Theorem 3.1 and Corollary 3.4 show that the new inequality and the given Lyapunov–Krasovskii functional with both exponential functions can be used to quickly derive the derivative condition of finite-time stability and finite-time boundedness, which is an advantage of the approach used in this work. However, our method is theoretically difficult to determine the upper bound of the delay of settling time T except for certain particular cases. Furthermore, our new integral inequality is complex in practice, making it difficult to solve and requiring much time and effort to find a solution. These are the downsides of our method.

Remark 8 If we choose $L_0 = 0$, $L_1 = 1$, $L_2 = 0$ and the external disturbance input is equal to zero in the GNNs (9), we obtain

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + f\big(x\big(t-h(t)\big)\big), \\ x(t) &= \phi(t), \quad \forall t \in [-h_M, 0], \end{aligned} \tag{45}$$

then (45) is a particular case of the GNNs (9) and can be found as in [11, 42–44].

Example 4.3 Consider the following parameters for the system (45):

$$\begin{aligned} &A = \operatorname{diag}\{0.8, 5.3\}, \qquad W_0 = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.1 \end{bmatrix}, \\ &h_m = 1, \qquad F_m = \operatorname{diag}\{0, 0\} \quad \text{and} \quad F_M = \operatorname{diag}\{1, 1\}. \end{aligned}$$

This example examines the stability criterion of the system (45). Using the parameters mentioned above, we solve the method of this paper with the new integral inequality to obtain the maximum allowable bounds of h_M for different values of μ, as shown in Table 9. Our method provides the greatest maximum allowable bounds of h_M for different values of μ, which are greater than those in [11, 42–44]. Therefore, our approach in this paper is less conservative than those in [11, 42–44].

Example 4.4 Neural Networks have been widely applied in several applications. In particular, the four-tank system is a fascinating neural-network application. In 2000, Johansson [45] first proposed the four-tank system. Johansson's four-tank system consists of 4 correspondingly connected water tanks with valves and two batches of pumps, as illustrated in Fig. 6. The purpose of the four-tank system is to manage the water level with two pumps. The voltages to the water pumps are the process inputs. Pump 1 is responsible for adding water to Tanks 1 and 4. Pump 2 is responsible for filling water to Tanks 2 and 3. Water

Table 9 Maximum allowable bounds of h_M for different values of μ in Example 4.3

Method	$\mu = 0.95$	$\mu = 0.99$	Unknown μ
[42]	–	–	3.0465
[43] ($m = 2$)	8.4119	5.4834	4.9471
[44] Theorem 1	15.5432	12.7286	12.7274
[44] Theorem 2	15.6611	12.7978	12.7970
[11] ($m = r = 2$)	17.8264	14.2548	14.2541
This paper	–	–	17.0400

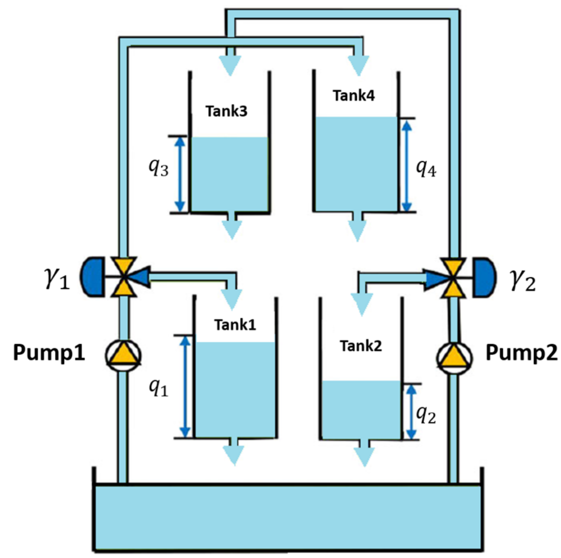

Figure 6 Schematic representation of the four-tank system. Source: Johansson [45]

flows from Tank 3 to Tank 1 and Tank 4 to Tank 2 by gravity. The water levels for Tank 1 (q_1) and Tank 2 (q_2) are evaluated together as outputs. The four-tank system has received a great deal of attention, as in [13, 17, 18, 26, 46]. The four-tank system and the present controller can be written as:

$$\dot{\tilde{x}}(t) = \widetilde{A}_0\tilde{x}(t) + \widetilde{A}_1\tilde{x}(t-\tilde{d}_1) + \widetilde{B}_0\tilde{u}(t-\tilde{d}_2) + \widetilde{B}_1\tilde{u}(t-\tilde{d}_3), \tag{46}$$

where

$$\widetilde{A}_0 = \begin{bmatrix} -0.0021 & 0 & 0 & 0 \\ 0 & -0.0021 & 0 & 0 \\ 0 & 0 & -0.0424 & 0 \\ 0 & 0 & 0 & -0.0424 \end{bmatrix},$$

$$\widetilde{A}_1 = \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{B}_0 = \begin{bmatrix} 0.1113\gamma_1 & 0 & 0 & 0 \\ 0 & 0.1042\gamma_2 & 0 & 0 \end{bmatrix}^T,$$

$$\widetilde{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0.1113(1-\gamma_1) \\ 0 & 0 & 0.1042(1-\gamma_2) & 0 \end{bmatrix}^T,$$

$$\gamma_1 = 0.333, \qquad \gamma_2 = 0.307, \qquad \widetilde{u}(t) = \widetilde{\mathcal{K}}\widetilde{x}(t) \qquad \text{and}$$

$$\widetilde{\mathcal{K}} = \begin{bmatrix} -0.1609 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix}.$$

In addition, this example illustrates how transportation delays occur between the tanks and the valves, which are interval time-delay signals. Suppose the controller $\widetilde{u}(t)$ is the water quantity from the pumps and $\widetilde{d}_1 = 0$, $\widetilde{d}_2 = 0$, $\widetilde{d}_3 = h(t)$ ($h_m < h(t) < h_M$). Hence, $\widetilde{u}(t)$ is a nonlinear function as the following: $\widetilde{u}(t) = \widetilde{\mathcal{K}}\widetilde{f}(\widetilde{x}(t))$, $\widetilde{u}(t-h(t)) = \widetilde{\mathcal{K}}\widetilde{f}(\widetilde{x}(t-h(t)))$, $\widetilde{f}(\widetilde{x}(t)) = [\widetilde{f}(\widetilde{x}_1(t)),\dots,\widetilde{f}(\widetilde{x}_4(t))]^T$, $\widetilde{f}_i(\widetilde{x}_i(t)) = 0.01(|\widetilde{x}_i(t)+1| - |\widetilde{x}_i(t)-1|)$, $\forall i = 1,2,3,4$.

We modify the four-tank system (46) as the following delayed NNs (47) as follows:

$$\dot{x}(t) = \bigl(-A + B\bigl(\mathcal{K} + \Delta\mathcal{K}(t)\bigr)\bigr)x(t) + L_0 f\bigl(x(t)\bigr) + L_1 f\bigl(x\bigl(t-h(t)\bigr)\bigr) + L_3\omega(t), \tag{47}$$

where $A = -\widetilde{A}_0 - \widetilde{A}_1$, $L_0 = \widetilde{B}_0\widetilde{\mathcal{K}}$, $L_1 = \widetilde{B}_1\widetilde{\mathcal{K}}$, $f(\cdot) = \widetilde{f}(\cdot)$, $W = I$, $D_1 = D_2 = -0.5I$, $B = -I$, $F_M = 0$, $F_P = 0.5I$. Given $h(t) = 1.0 + 0.8\cos(t)$, which means $h_m = 0.2$, $h_M = 1.8$. Given scalars $d = 0.1$, $\varrho = 0.01$, $c_1 = 0.5$, $c_2 = 5$, $T = 30$ and matrix $H = I$. From the parameters mentioned above, we can compute the gain matrix $\mathcal{K}$ of the state feedback controller with a nonfragile issue by Theorem 3.7 as the following:

$$\mathcal{K} = \begin{bmatrix} 2.0236 & -0.0028 & 0.0408 & -0.0034 \\ -0.0028 & 2.0240 & -0.0032 & 0.0410 \\ -0.0008 & -0.0010 & 1.9876 & 0.0019 \\ -0.0010 & -0.0007 & 0.0019 & 1.9875 \end{bmatrix}. \tag{48}$$

We show the effectiveness of our results in Example 4.4. Figure 7 illustrates the state responses of $x(t)$ for the four-tank system (47) without $u(t)$. Figure 8 shows the state responses for $x(t)$ of the four-tank system (47) with $u(t)$. Moreover, we present the control inputs in Fig. 9. The proposed controller internally stabilizes the four-tank system (47) with external disturbance. Thus, our results are of a consistently high effectiveness non-fragile feedback control scheme while maintaining state-response stability.

Remark 9 We select the time-delay functions $h(t)$ that are continuous functions and satisfy the condition $h_m \le h(t) \le h_M$. In Examples 4.1, 4.2, and 4.4, we use $h(t) = 1.25|\sin(t)| + 1.0$, $h(t) = 1.25|\sin(t)| + 1.5$, and $h(t) = 1.0 + 0.8\cos(t)$ for $t \in [0, 50]$, respectively (see Fig. 10). From Fig. 10, our time-delay functions $h(t)$ are continuous functions and satisfy $h_m \le h(t) \le h_M$. However, our delay functions do not need to be differentiable. In contrast to other previous studies [9, 10, 19], the time-delay function is always differentiable.

5 Conclusions

The conservatism of the finite-time stability criterion in Lyapunov theory is an important topic. Developing integral inequalities leads to reduced conservatism. Hence, this article proposes the new integral inequality with an exponential function to estimate the derivative of the LKFs. The well-known Wirtinger's inequality is a particular case of the new

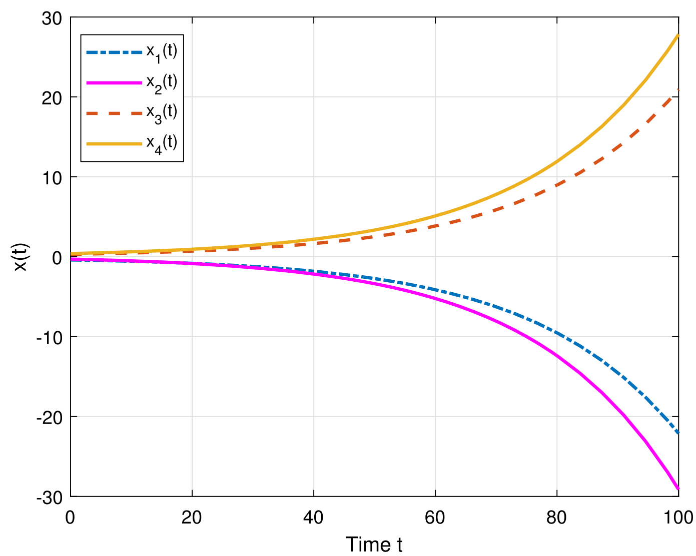

Figure 7 State responses of $x(t)$ against $u(t) = 0$ of the GNNs (47) in Example 4.4

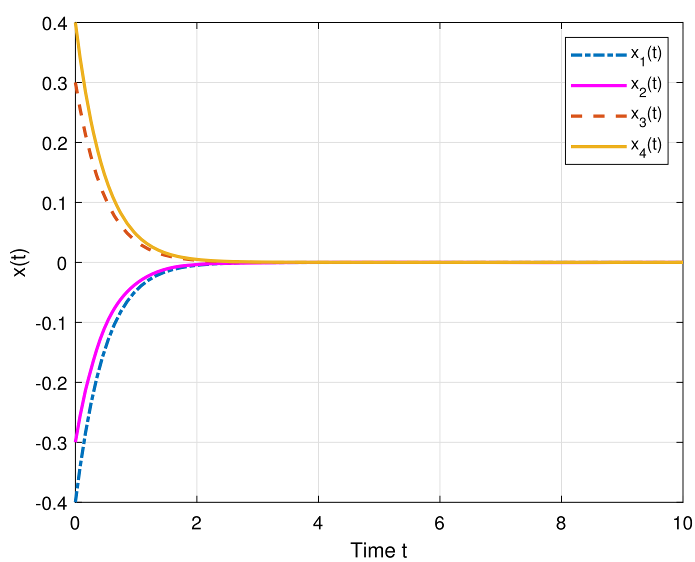

Figure 8 State responses of $x(t)$ for the GNNs with $u(t)$ (47) in Example 4.4

integral inequality. Furthermore, we investigate the new delay dependence for guaranteeing finite-time stability of the GNNs with mixed-interval time-varying delays that do not need to be differentiable and design the state feedback controller with a nonfragile issue. Numerical examples show the MALBs of c_2 obtained by several inequalities, including our new inequality, Wirtinger's inequality [38], and the inequality with an exponential function in [39]. Our new inequality efficiently reduces conservatism more than using Wirtinger's inequality [38] and the inequality in [39]. Moreover, one of the examples presents a practical implementation that applies our results on the four-tank system. For future work, this article can be applied to various dynamical systems such as T–S fuzzy NNs [26], neutral-

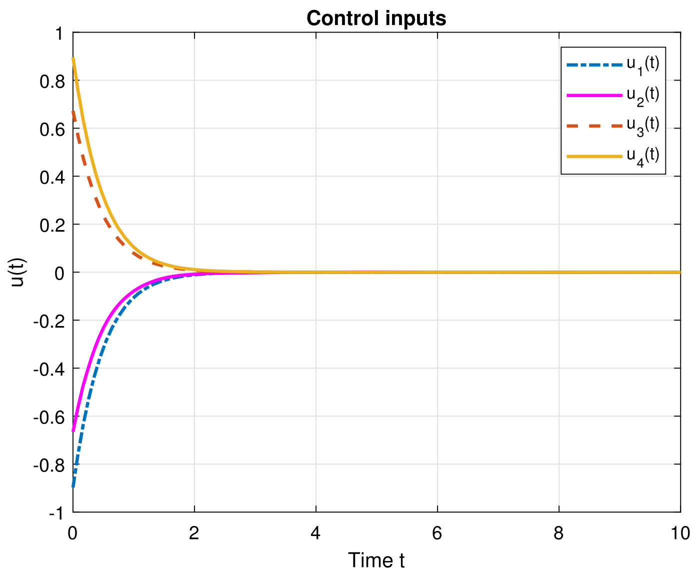

Figure 9 Control inputs for the GNNs (47) in Example 4.4

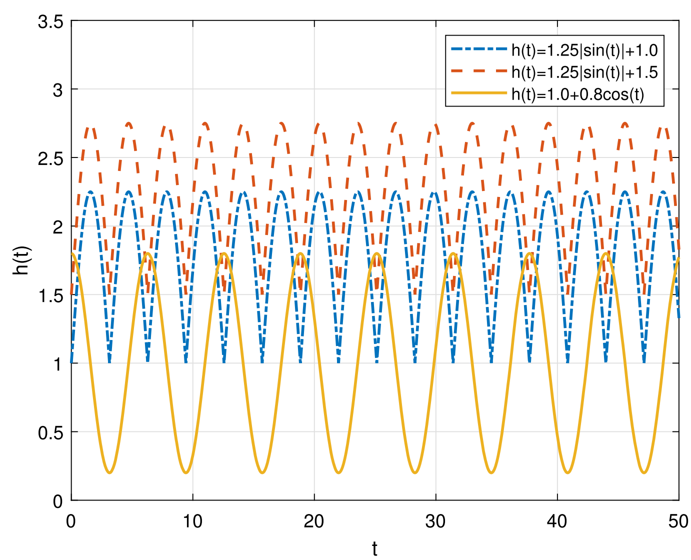

Figure 10 Continuous time-varying delay functions for $t \in [0, 50]$

type NNs [46], uncertain NNs [47], and neutral high-order Hopfield NNs [48] or several time delays, such as additive time delay [49–51] and leakage time-delay [13, 46, 48]. Additionally, future work could potentially design a sample-data nonfragile controller [30] for the delayed dynamical systems.

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Declarations

Competing interests
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Author contributions
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