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Controlled g-frames and dual g-frames in Hilbert spaces

Hui-Min Liu[1], Yan-Ling Fu[2] and Yu Tian[3*]

*Correspondence:
yutian@zzuli.edu.cn
[3]Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, P.R. China
Full list of author information is available at the end of the article

Abstract

As generalizations of g-frames and controlled frames, the theory of controlled g-frames has been deeply studied. This paper addresses the controlled g-frames and dual g-frames in Hilbert spaces. We first present some equivalent characterizations of controlled g-frames. Then, we introduce the concepts of controlled dual g-frames and controlled dual g-frames operator, get some properties of them. Finally, we obtain some characterizations of the controlled dual g-frames for a given controlled g-frame by the method of operator theory.

MSC: 42C15; 42A38

Keywords: G-frames; Controlled g-frames; Controlled dual g-frames

1 Introduction

A sequence $\{f_j\}_{j\in J}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{j\in J} \left|\langle f, f_j\rangle\right|^2 \le B\|f\|^2$$

for all $f \in \mathcal{H}$. The concept of frames was introduced by Gabor in 1946 and Duffin and Schaeffer in 1952. Gabor in [12] proposed the idea of decomposing a general signal in terms of elementary signals, and Duffin and Schaeffer in [10] abstracted "these elementary signals" as the notion of frame. The frame theory has been developing rapidly since Daubechies, Grossmann, and Meyer [9] had put forward the definition of frames for Hilbert spaces formally in 1986. So far, the theory of frame has achieved fruitful success in pure mathematics, science, and engineering [4, 5, 8, 13, 14, 21, 24]. In the last decades, various generalizations of frame have been put forward for special purposes such as frame of subspaces [6], fusion frame [7], bounded quasi-projector [11], and g-frame [22]. In particular, among these generalizations, a g-frame covers all others, and the research of g-frames has obtained many results [16, 23, 25]. Controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [2]. A sequence $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a C-controlled frame if there exist positive constants

$0 < A_2 \le B_2 < \infty$ such that

$$A_2\|f\|^2 \le \sum_{j\in J}\langle f,f_j\rangle\langle Cf_j,f\rangle \le B_2\|f\|^2$$

for all $f \in \mathcal{H}$, where $C \in \mathcal{GL}(\mathcal{H})$. However, they are only used as a tool to study spherical wavelets [3]. Later, some scholars noticed that these frames can give a generalized way to check the frame conditions while offering numerical advantages in the sense of preconditioning. Since then, controlled frames have been widely studied [15, 17–20]. Rahimi et al. in [18] first introduced the notion of controlled g-frames (see Definition 2.3), which is an extension of g-frames and controlled frames.

Inspired by the above research, in this paper we address the characterization of controlled g-frames and controlled dual g-frames, and it is organized as follows: In Sect. 2, we recall some basic notions, properties, and related results. Section 3 is devoted to the characterization of controlled g-frames, we obtain some equivalent conditions of controlled g-frames. In Sect. 4, we introduce the notion of controlled dual frames in Hilbert spaces and obtain some characterizations of the controlled dual g-frames for a given controlled g-frame by the method of operator theory.

2 Preliminaries

We begin this section with some basic notions and results of g-frames (see [8, 18, 20, 22, 25] for details).

Given separable Hilbert spaces $\mathcal{H}$ and $\mathcal{V}$, let $\{\mathcal{V}_j : j \in J\}$ be a sequence of closed subspaces of $\mathcal{V}$ with J being a subset of integers $\mathbb{Z}$. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$. The set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{V}_j$ is denoted by $L(\mathcal{H},\mathcal{V}_j)$. As a special case, $L(\mathcal{H})$ is a collection of all bounded linear operators on $\mathcal{H}$. The set of all bounded linear operators on $\mathcal{H}$ with a bounded inverse is denoted by $\mathcal{GL}(\mathcal{H})$. If $P, Q \in \mathcal{GL}(\mathcal{H})$, then P^*, P^{-1}, and PQ are also in $\mathcal{GL}(\mathcal{H})$. Let $\mathcal{GL}^+(\mathcal{H})$ be the set of all positive operators in $\mathcal{GL}(\mathcal{H})$. A bounded operator $P : \mathcal{H} \to \mathcal{H}$ is positive if $\langle Pf,f\rangle > 0$ for all $f \neq 0$. In a complex Hilbert space, every bounded positive operator is self-adjoint. In addition, as a technical condition, we also assume that any two positive operators involved in this paper commutate with each other. Define

$$\bigoplus_{j\in J}\mathcal{V}_j = \left\{\{a_j\}_{j\in J} : a_j \in \mathcal{V}_j, \left\|\{a_j\}_{j\in J}\right\|^2 = \sum_{j\in J}\|a_j\|^2 < \infty\right\}.$$

Then $\bigoplus_{j\in J}\mathcal{V}_j$ is a Hilbert space under the following inner product:

$$\left\langle\{a_j\}_{j\in J},\{b_j\}_{j\in J}\right\rangle = \sum_{j\in J}\langle a_j,b_j\rangle \quad \text{for } \{a_j\}_{j\in J},\{b_j\}_{j\in J} \in \bigoplus_{j\in J}\mathcal{V}_j.$$

Suppose that $\{e_{j,k}\}_{k\in K_j}$ is an orthonormal basis (simply o. n. b.) for $\mathcal{V}_j$, where $K_j \subset \mathbb{Z}$, $j \in J$. Define $\tilde{e}_{j,k} = e_{j,k}\delta_j$, where δ is the Kronecker symbol. Then $\{\tilde{e}_{j,k}\}_{j\in J,k\in K_j}$ is an o. n. b. for $\bigoplus_{j\in J}\mathcal{V}_j$ (see [25]).

Definition 2.1 ([22]) A sequence $\{\Lambda_j \in L(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is called a g-frame for $\mathcal{H}$ with respect to (simply w. r. t.) $\{\mathcal{V}_j\}_{j\in J}$ if

$$A\|f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2 \tag{2.1}$$

for all $f \in \mathcal{H}$ and some positive constants $A \le B$. The numbers A, B are called the frame bounds. If only the right-hand inequality of (2.1) is satisfied, $\{\Lambda_j\}_{j\in J}$ is called a g-Bessel sequence for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with bound B. If $A = B = \lambda$, $\{\Lambda_j\}_{j\in J}$ is called a λ-tight g-frame. In addition, if $\lambda = 1$, $\{\Lambda_j\}_{j\in J}$ is called a Parseval g-frame.

Definition 2.2 ([25]) Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$. A g-frame $\{\Gamma_j\}_{j\in J}$ is called an alternate dual g-frame for $\{\Lambda_j\}_{j\in J}$ if

$$f = \sum_{j\in J} \Gamma_j^* \Lambda_j f \quad \text{for } f \in \mathcal{H}.$$

Moreover, $\{\Lambda_j\}_{j\in J}$ is also an alternate dual g-frame for $\{\Gamma_j\}_{j\in J}$, that is,

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j f \quad \text{for } f \in \mathcal{H}.$$

Definition 2.3 ([8]) Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$. A sequence $\{\Lambda_j\}_{j\in J}$ is called a (P, Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$. If there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{j\in J} \langle \Lambda_j P f, \Lambda_j Q f\rangle \le B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.2}$$

We call A and B the lower and upper frame bounds for (P, Q)-controlled g-frame, respectively.

If the right-hand side of (2.2) holds, then $\{\Lambda_j\}_{j\in J}$ is called a (P, Q)-controlled g-Bessel sequence for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.

If $Q = I_{\mathcal{H}}$, then we call $\{\Lambda_j\}_{j\in J}$ a P-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.

If $P = Q$, then we call $\{\Lambda_j\}_{j\in J}$ a P^2(or (P, P))-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.

Lemma 2.1 ([8]) *Every bounded and positive operator $P : \mathcal{H} \to \mathcal{H}$ has a unique bounded and positive square root W. If P is self-adjoint, then W is self-adjoint. If P is invertible, then W is also invertible.*

For a (P, Q)-controlled g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ with bound B, the operator $T_{P\Lambda Q}$

$$T_{P\Lambda Q} : \bigoplus_{j\in J} \mathcal{V}_j \to \mathcal{H}, \qquad T_{P\Lambda Q} F = \sum_{j\in J} (PQ)^{\frac{1}{2}} \Lambda_j^* f_j, \quad \forall F = \{f_j\}_{j\in J} \in \bigoplus_{j\in J} \mathcal{V}_j$$

is well defined, and its adjoint is given by

$$T_{P\Lambda Q}^* : \mathcal{H} \to \bigoplus_{j\in J} \mathcal{V}_j, \qquad T_{P\Lambda Q}^* f = \bigl\{\Lambda_j (QP)^{\frac{1}{2}} f\bigr\}_{j\in J}, \quad \forall f \in \mathcal{H}.$$

$T_{P\Lambda Q}$ is called the synthesis operator and $T^*_{P\Lambda Q}$ is called the analysis operator of $\{\Lambda_j\}_{j\in J}$. For a (P,Q)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ with bounds A and B, the operator

$$S_{P\Lambda Q}:\mathcal{H}\to\mathcal{H},\qquad S_{P\Lambda Q}f=\sum_{j\in J}Q\Lambda_j^*\Lambda_jPf,\quad \forall f\in\mathcal{H}$$

is called the frame operator of $\{\Lambda_j\}_{j\in J}$. From the definition, $S_{P\Lambda Q}=PS_\Lambda Q$ is positive and invertible, where S_Λ is a frame operator of g-frame $\{\Lambda_j\}_{j\in J}$, and it is bounded, invertible, self-adjoint, positive, and $AI_H\leq S_\Lambda\leq BI_H$. Let $\tilde{\Lambda}_j=\Lambda_jS_\Lambda^{-1}$, then $\{\tilde{\Lambda}_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with frame operator S_Λ^{-1} and frame bounds $\frac{1}{B}$ and $\frac{1}{A}$, respectively. $\{\tilde{\Lambda}_j\}_{j\in J}$ is called the canonical dual g-frame of $\{\Lambda_j\}_{j\in J}$ (see [22]).

Definition 2.4 ([20]) Let $\mathcal{H}$ be a Hilbert space and $C\in\mathcal{GL}(\mathcal{H})$. Suppose that $\{\psi_j\}_{j\in J}\subseteq\mathcal{H}$ is a C-controlled frame and $\{\phi_j\}_{j\in J}\subseteq\mathcal{H}$ is a Bessel sequence. Then $\{\phi_j\}_{j\in J}\subseteq\mathcal{H}$ is said to be a C-controlled dual of $\{\psi_j\}_{j\in J}\subseteq\mathcal{H}$ if the following condition is satisfied:

$$f=\sum_{j\in J}\langle f,\phi_j\rangle C\psi_j$$

for all $f\in\mathcal{H}$.

3 Controlled g-frames in Hilbert spaces

In this section, we present the characterization of controlled dual g-frames, and some equivalent conditions of (P,Q)-controlled g-frames are obtained. For this purpose, we first give some equivalent conditions of bounded and positive operators.

Lemma 3.1 ([8]) *Let $T:\mathcal{H}\to\mathcal{H}$ be a linear operator. Then the following are equivalent*:

(i) *There exist two constants $0<c\leq C<\infty$ such that $cI_{\mathcal{H}}\leq T\leq CI_{\mathcal{H}}$.*

(ii) *T is positive and there exist two constants $0<c\leq C<\infty$ such that*

$$c\|f\|^2\leq\bigl\|T^{\frac{1}{2}}f\bigr\|^2\leq C\|f\|^2.$$

(iii) $T\in\mathcal{GL}^+(\mathcal{H})$.

The following lemma gives a characterization of (P,Q)-controlled g-frames in Hilbert space. By Proposition 2.1 in [1], if $P,Q\in\mathcal{GL}^+(\mathcal{H})$ and $PQ=QP$, then we have $PQ\in\mathcal{GL}^+(\mathcal{H})$.

Lemma 3.2 *Let $P,Q\in\mathcal{GL}^+(\mathcal{H})$. Then $\{\Lambda_j\}_{j\in J}$ is a (P,Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ if and only if $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.*

Proof Suppose that $\{\Lambda_j\}_{j\in J}$ is a (P,Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with bounds A, B. For any $f\in\mathcal{H}$, we have

$$\begin{aligned}A\|f\|^2&=A\bigl\|(PQ)^{\frac{1}{2}}(PQ)^{-\frac{1}{2}}f\bigr\|^2\\&\leq A\bigl\|(PQ)^{\frac{1}{2}}\bigr\|^2\bigl\|(PQ)^{-\frac{1}{2}}f\bigr\|^2\\&\leq\bigl\|(PQ)^{\frac{1}{2}}\bigr\|^2\sum_{j\in J}\bigl\langle\Lambda_jP(PQ)^{-\frac{1}{2}}f,\Lambda_jQ(PQ)^{-\frac{1}{2}}f\bigr\rangle\end{aligned}$$

$$\begin{aligned}
&= \left\|(PQ)^{\frac{1}{2}}\right\|^2\left\langle QS_\Lambda P(PQ)^{-\frac{1}{2}}f,(PQ)^{-\frac{1}{2}}f\right\rangle \\
&= \left\|(PQ)^{\frac{1}{2}}\right\|^2\left\langle S_\Lambda P(PQ)^{-\frac{1}{2}}f,Q(PQ)^{-\frac{1}{2}}f\right\rangle \\
&= \left\|(PQ)^{\frac{1}{2}}\right\|^2\left\langle S_\Lambda P^{\frac{1}{2}}(Q)^{-\frac{1}{2}}f,Q^{\frac{1}{2}}(P)^{-\frac{1}{2}}f\right\rangle \\
&= \left\|(PQ)^{\frac{1}{2}}\right\|^2\left\langle (P)^{-\frac{1}{2}}Q^{\frac{1}{2}}S_\Lambda P^{\frac{1}{2}}(Q)^{-\frac{1}{2}}f,f\right\rangle = \left\|(PQ)^{\frac{1}{2}}\right\|^2\langle S_\Lambda f,f\rangle.
\end{aligned}$$

Thus

$$\frac{A}{\|(PQ)^{\frac{1}{2}}\|^2}\|f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2, \quad \forall f\in\mathcal{H}.$$

For any $f\in\mathcal{H}$, it follows that

$$\begin{aligned}
\sum_{j\in J}\|\Lambda_j f\|^2 = \langle S_\Lambda f,f\rangle &= \left\langle (PQ)^{-\frac{1}{2}}(PQ)^{\frac{1}{2}}S_\Lambda f,f\right\rangle \\
&= \left\langle (PQ)^{\frac{1}{2}}S_\Lambda f,(PQ)^{-\frac{1}{2}}f\right\rangle \\
&= \left\langle S_\Lambda (PQ)(PQ)^{-\frac{1}{2}}f,(PQ)^{-\frac{1}{2}}f\right\rangle \\
&= \left\langle PS_\Lambda Q(PQ)^{-\frac{1}{2}}f,(PQ)^{-\frac{1}{2}}f\right\rangle \\
&\le B\left\|(PQ)^{-\frac{1}{2}}f\right\|^2 \le B\left\|(PQ)^{-\frac{1}{2}}\right\|^2\|f\|^2.
\end{aligned}$$

Hence $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with bounds $\frac{A}{\|(PQ)^{\frac{1}{2}}\|^2}$ and $B\|(PQ)^{-\frac{1}{2}}\|^2$.

On the other hand, suppose that $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with bounds A_1, B_1. Then

$$\langle A_1 f,f\rangle \le \langle S_\Lambda f,f\rangle \le \langle B_1 f,f\rangle \quad \text{for any } f\in\mathcal{H}.$$

Since $P,Q\in\mathcal{GL}^+(\mathcal{H})$, by Lemma 3.1, there exist constants c,c_1,C,C_1 $(0<c,c_1,C,C_1<\infty)$ such that

$$cI_{\mathcal{H}} \le P \le CI_{\mathcal{H}}, \qquad c_1I_{\mathcal{H}} \le Q \le C_1I_{\mathcal{H}}.$$

Using $\langle PS_\Lambda f,f\rangle = \langle f,S_\Lambda Pf\rangle = \langle f,PS_\Lambda f\rangle$, we get

$$cA \le S_\Lambda P = PS_\Lambda \le CB.$$

Similarly, we have

$$cc_1A \le QS_\Lambda P \le CC_1B.$$

It follows that

$$cc_1A\|f\|^2 \le \sum_{j\in J}\langle \Lambda_j Pf,\Lambda_j Qf\rangle \le CC_1B\|f\|^2, \quad \forall f\in\mathcal{H}.$$

Therefore, $\{\Lambda_j\}_{j\in J}$ is a (P,Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$. The proof is completed. □

Lemma 3.3 *Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$. Then $\{\Lambda_j\}_{j\in J}$ is a (P, Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ if and only if $\{\Lambda_j\}_{j\in J}$ is a $((QP)^{\frac{1}{2}}, (QP)^{\frac{1}{2}})$-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.*

Proof For any $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in J}\langle \Lambda_j Pf, \Lambda_j Qf\rangle &= \Big\langle \sum_{j\in J} Q\Lambda_j^*\Lambda_j Pf, f\Big\rangle = \langle QS_\Lambda Pf, f\rangle \\ &= \langle QPS_\Lambda f, f\rangle = \big\langle (QP)^{\frac{1}{2}} S_\Lambda (QP)^{\frac{1}{2}} f, f\big\rangle \\ &= \Big\langle \sum_{j\in J} (QP)^{\frac{1}{2}} \Lambda_j^*\Lambda_j (QP)^{\frac{1}{2}} f, f\Big\rangle \\ &= \sum_{j\in J}\big\langle \Lambda_j (QP)^{\frac{1}{2}} f, \Lambda_j (QP)^{\frac{1}{2}} f\big\rangle.\end{aligned}$$

Hence, $\{\Lambda_j\}_{j\in J}$ is a (P, Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ is equivalent to

$$A\|f\|^2 \le \sum_{j\in J}\big\langle \Lambda_j (QP)^{\frac{1}{2}} f, \Lambda_j (QP)^{\frac{1}{2}} f\big\rangle \le B\|f\|^2, \quad \forall f \in \mathcal{H},$$

where A and B are frame bounds of $\{\Lambda_j\}_{j\in J}$. Thus $\{\Lambda_j\}_{j\in J}$ is a $((QP)^{\frac{1}{2}}, (QP)^{\frac{1}{2}})$-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$. The proof is completed. □

Lemma 3.4 *Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$. Then $\{\Lambda_j\}_{j\in J}$ is a (P, Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ if and only if $\{\Lambda_j\}_{j\in J}$ is a QP-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.*

Proof The proof is similar to that of Lemma 3.3. □

Lemma 3.5 *Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$. Then $\{\Lambda_j\}_{j\in J}$ is a (P, Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ if and only if $\{u_{j,k}\}_{j\in J,k\in K_j}$ is a (P, Q)-controlled g-frame for $\mathcal{H}$, where $\{u_{j,k}\}_{j\in J,k\in K_j}$ is the sequence induced by $\{\Lambda_j\}_{j\in J}$ w. r. t. $\{e_{j,k}\}_{j\in J,k\in K_j}$ (i.e., $u_{j,k} = \Lambda_j^* e_{j,k}$).*

Proof Noting that $\{e_{j,k}\}_{k\in K_j}$ is an o.n.b. for $\mathcal{V}_j$ for each $j \in J$, for any $f \in \mathcal{H}$, we have $\Lambda_j f \in \mathcal{V}_j$. It follows that

$$\Lambda_j Pf = \sum_{k\in K_j}\langle \Lambda_j Pf, e_{j,k}\rangle e_{j,k} = \sum_{k\in K_j}\big\langle f, P\Lambda_j^* e_{j,k}\big\rangle e_{j,k}$$

and

$$\Lambda_j Qf = \sum_{k\in K_j}\langle \Lambda_j Qf, e_{j,k}\rangle e_{j,k} = \sum_{k\in K_j}\big\langle f, Q\Lambda_j^* e_{j,k}\big\rangle e_{j,k}.$$

It is easy to check that

$$\langle \Lambda_j Pf, \Lambda_j Qf\rangle = \sum_{k\in K_j}\big\langle f, P\Lambda_j^* e_{j,k}\big\rangle\big\langle Q\Lambda_j^* e_{j,k}, f\big\rangle = \sum_{k\in K_j}\langle f, Pu_{j,k}\rangle\langle Qu_{j,k}, f\rangle.$$

Hence

$$\sum_{j\in J}\langle \Lambda_j Pf, \Lambda_j Qf\rangle = \sum_{j\in J}\sum_{k\in K_j}\langle f, Pu_{j,k}\rangle\langle Qu_{j,k}, f\rangle.$$

Thus

$$A\|f\|^2 \le \sum_{j\in J}\langle \Lambda_j Pf, \Lambda_j Qf\rangle \le B\|f\|^2 \quad \text{for any } f\in\mathcal{H}$$

is equivalent to

$$A\|f\|^2 \le \sum_{j\in J}\sum_{k\in K_j}\langle f, Pu_{j,k}\rangle\langle Qu_{j,k}, f\rangle \le B\|f\|^2 \quad \text{for any } f\in\mathcal{H}.$$

The proof is completed. □

Lemma 3.6 *Let $P, Q\in\mathcal{GL}^+(\mathcal{H})$. Then $\{\Lambda_j\}_{j\in J}$ is a (P,Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ if and only if $\{Pu_{j,k}\}_{j\in J,k\in K_j}$ is a QP^{-1}-controlled frame for $\mathcal{H}$, where $\{u_{j,k}\}_{j\in J,k\in K_j}$ is the sequence induced by $\{\Lambda_j\}_{j\in J}$ w. r. t. $\{e_{j,k}\}_{j\in J,k\in K_j}$ (i.e., $u_{j,k}=\Lambda_j^*e_{j,k}$).*

Proof From the proof of Theorem 3.5, we have

$$\sum_{j\in J}\langle \Lambda_j Pf, \Lambda_j Qf\rangle = \sum_{j\in J}\sum_{k\in K_j}\left\langle f, P\Lambda_j^*e_{j,k}\right\rangle\left\langle Q\Lambda_j^*e_{j,k}, f\right\rangle.$$

If we take $u_{j,k}=\Lambda_j^*e_{j,k}$, $f_{j,k}=Pu_{j,k}$, then

$$A\|f\|^2 \le \sum_{j\in J}\langle \Lambda_j Pf, \Lambda_j Qf\rangle \le B\|f\|^2 \quad \text{for any } f\in\mathcal{H}$$

is equivalent to

$$A\|f\|^2 \le \sum_{j\in J}\sum_{k\in K_j}\langle f, Pu_{j,k}\rangle\left\langle QP^{-1}Pu_{j,k}, f\right\rangle \le B\|f\|^2 \quad \text{for any } f\in\mathcal{H}.$$

The proof is completed. □

Combining Lemmas 3.2–3.6, we get Theorem 3.1.

Theorem 3.1 *Let $P, Q\in\mathcal{GL}^+(\mathcal{H})$. Then the following are equivalent:*
(i) *$\{\Lambda_j\}_{j\in J}$ is a (P,Q)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.*
(ii) *$\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.*
(iii) *$\{\Lambda_j\}_{j\in J}$ is a $((QP)^{\frac{1}{2}},(QP)^{\frac{1}{2}})$-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.*
(iv) *$\{\Lambda_j\}_{j\in J}$ is a QP-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$.*
(v) *$\{u_{j,k}\}_{j\in J,k\in K_j}$ is a (P,Q)-controlled frame for $\mathcal{H}$, where $\{u_{j,k}\}_{j\in J,k\in K_j}$ is the sequence induced by $\{\Lambda_j\}_{j\in J}$ w. r. t. $\{e_{j,k}\}_{j\in J,k\in K_j}$.*
(vi) *$\{Pu_{j,k}\}_{j\in J,k\in K_j}$ is a QP^{-1}-controlled frame for $\mathcal{H}$, where $\{u_{j,k}\}_{j\in J,k\in K_j}$ is the sequence induced by $\{\Lambda_j\}_{j\in J}$ w. r. t. $\{e_{j,k}\}_{j\in J,k\in K_j}$.*

4 Controlled dual g-frames in Hilbert spaces

In this section, we introduce the notion of controlled dual frames and obtain some characterizations of the controlled dual g-frames for a given controlled g-frame by the method of operator theory.

Definition 4.1 Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be (P,P)-controlled and (Q,Q)-controlled g-Bessel sequences for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$, respectively. If for any $f \in \mathcal{H}$

$$f = \sum_{j\in J} P\Lambda_j^*\Gamma_j Qf,$$

then $\{\Gamma_j\}_{j\in J}$ is called a (P,Q)-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$. In particular, if $Q = I_{\mathcal{H}}$, then $\{\Gamma_j\}_{j\in J}$ is called a P-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$.

Definition 4.2 Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be (P,P)-controlled and (Q,Q)-controlled g-Bessel sequence for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$, respectively. We define a (P,Q)-controlled dual g-frame operator for this pair of controlled g-Bessel sequence as follows:

$$S_{P\Lambda\Gamma Q}f = \sum_{j\in J} P\Lambda_j^*\Gamma_j Qf, \quad \forall f \in \mathcal{H}.$$

As mentioned before, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are also two g-Bessel sequences. It is easy to check that $S_{P\Lambda\Gamma Q}$ is a well-defined and bounded operator, and

$$S_{P\Lambda\Gamma Q} = T_{P\Lambda P}T_{Q\Gamma Q}^* = PT_\Lambda T_\Gamma^* Q = PS_{\Lambda\Gamma}Q,$$

where $S_{\Lambda\Gamma} = \sum_{j\in J}\Lambda_j^*\Gamma_j$. From Definition 4.1, $\{\Gamma_j\}_{j\in J}$ is a (P,Q)-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$ if and only if $S_{P\Lambda\Gamma Q} = I_{\mathcal{H}}$.

Proposition 4.1 *Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be (P,P)-controlled and (Q,Q)-controlled g-Bessel sequences with bounds B_Λ and B_Γ, respectively. If $S_{P\Lambda\Gamma Q}$ is bounded below, then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are (P,P)-controlled and (Q,Q)-controlled g-frames, respectively.*

Proof Suppose that there exists a constant $\lambda > 0$ such that

$$\|S_{P\Lambda\Gamma Q}f\| \geq \lambda\|f\| \quad \text{for all } f \in \mathcal{H}.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\lambda\|f\| \leq \|S_{P\Lambda\Gamma Q}f\| &= \sup_{\|g\|=1}\left|\left\langle\sum_{j\in J} P\Lambda_j^*\Gamma_j Qf, g\right\rangle\right| \\
&= \sup_{\|g\|=1}\left|\sum_{j\in J}\langle\Gamma_j Qf, \Lambda_j Pg\rangle\right| \\
&\leq \sup_{\|g\|=1}\left(\sum_{j\in J}\|\Gamma_j Qf\|^2\right)^{\frac{1}{2}}\left(\sum_{j\in J}\|\Lambda_j Pg\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \sqrt{B_\Lambda}\left(\sum_{j\in J}\|\Gamma_j Qf\|^2\right)^{\frac{1}{2}}.$$

Thus

$$\frac{\lambda^2}{B_\Lambda}\|f\|^2 \leq \sum_{j\in J}\|\Gamma_j Qf\|^2 \quad \text{for } f \in \mathcal{H}.$$

On the other hand, since

$$S^*_{P\Lambda\Gamma Q} = (PS_{\Lambda\Gamma}Q)^* = QS^*_{\Lambda\Gamma}P = QS_{\Gamma\Lambda}P = S_{Q\Gamma\Lambda P},$$

then $S_{Q\Gamma\Lambda P}$ is also bounded below. Similarly, we can prove that $\{\Lambda_j\}_{j\in J}$ is a (P,P)-controlled g-frame. The proof is completed. □

Theorem 4.1 *Let $P,Q \in \mathcal{GL}^+(\mathcal{H})$, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be (P,P)-controlled and (Q,Q)-controlled g-Bessel sequences for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$, respectively. Then the following conditions are equivalent*:

(i) $f = \sum_{j\in J} P\Lambda_j^*\Gamma_j Qf$, $\forall f \in \mathcal{H}$;

(ii) $f = \sum_{j\in J} Q\Gamma_j^*\Lambda_j Pf$, $\forall f \in \mathcal{H}$;

(iii) $\langle f,g\rangle = \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qg\rangle = \sum_{j\in J}\langle \Gamma_j Qf, \Lambda_j Pg\rangle$, $\forall f,g \in \mathcal{H}$;

(iv) $\|f\|^2 = \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qf\rangle = \sum_{j\in J}\langle \Gamma_j Qf, \Lambda_j Pf\rangle$, $\forall f \in \mathcal{H}$.

In case the equivalent conditions are satisfied, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are (P,P)-controlled and (Q,Q)-controlled g-frames, respectively.

Proof (i)⇔(ii). Let $T_{P\Lambda P}$ be the synthesis operator of the (P,P)-controlled g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ and $T_{Q\Gamma Q}$ be the synthesis operator of the (Q,Q)-controlled g-Bessel sequence $\{\Gamma_j\}_{j\in J}$. In these conditions (i) means that $T_{P\Lambda P}T^*_{Q\Gamma Q} = I_{\mathcal{H}}$, this is equivalent to $T_{Q\Gamma Q}T^*_{P\Lambda P} = I_{\mathcal{H}}$, which is identical to statement (ii). Conversely, (ii) implies (i) similarly.

(ii)⇔(iii). It is clear that (ii)⇒(iii). Next we prove (iii) implies (ii) for any $f,g \in \mathcal{H}$, $\langle f,g\rangle = \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qg\rangle$ shows that

$$\left\langle f - \sum_{j\in J} Q\Gamma_j^*\Lambda_j Pf, g\right\rangle = 0, \quad \forall g \in \mathcal{H}.$$

Hence (ii) is followed.

(iii)⇔(iv). (iii)⇒(iv) is obvious. To prove that (iv)⇒(iii), applying condition (iv), we have

$$\begin{aligned}
\|f+g\|^2 &= \sum_{j\in J}\bigl\langle \Lambda_j P(f+g), \Gamma_j Q(f+g)\bigr\rangle \\
&= \sum_{j\in J}\langle \Lambda_j Pf + \Lambda_j Pg, \Gamma_j Qf + \Gamma_j Qg\rangle \\
&= \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qf\rangle + \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qg\rangle \\
&\quad + \sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qf\rangle + \sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qg\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}\|f-g\|^2 &= \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qf\rangle - \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qg\rangle \\ &\quad - \sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qf\rangle + \sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qg\rangle, \\ \|f+\mathbf{i}g\|^2 &= \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qf\rangle - \mathbf{i}\sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qg\rangle \\ &\quad + \mathbf{i}\sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qf\rangle + \sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qg\rangle, \\ \|f-\mathbf{i}g\|^2 &= \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qf\rangle + \mathbf{i}\sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qg\rangle \\ &\quad - \mathbf{i}\sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qf\rangle + \sum_{j\in J}\langle \Lambda_j Pg, \Gamma_j Qg\rangle.\end{aligned}$$

By polarization identity,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{4}\big(\|f+g\|^2 - \|f-g\|^2 + \mathbf{i}\|f+\mathbf{i}g\|^2 - \mathbf{i}\|f-\mathbf{i}g\|^2\big) \\ &= \sum_{j\in J}\langle \Lambda_j Pf, \Gamma_j Qg\rangle.\end{aligned}$$

In case the equivalent conditions are satisfied, $S_{Q\Gamma\Lambda P} = I_{\mathcal{H}}$ implies $\|S_{Q\Gamma\Lambda P}\| = 1$, by Proposition 4.1, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are (P,P)-controlled and (Q,Q)-controlled g-frames, respectively. The proof is completed. □

Lemma 4.1 *Let $P, Q \in \mathcal{GL}^+(\mathcal{H})$. A sequence $\{\Lambda_j\}_{j\in J}$ is a (P,Q)-controlled g-Bessel sequence for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with bound B if and only if the operator*

$$T_{P\Lambda Q}: \bigoplus_{j\in J}\mathcal{V}_j \to \mathcal{H}, \qquad T_{P\Lambda Q}\big(\{f_j\}_{j\in J}\big) = \sum_{j\in J}(PQ)^{\frac{1}{2}}\Lambda_j^* f_j$$

is well defined and bounded with $\|T_{P\Lambda Q}\| \le \sqrt{B}$.

Proof The necessary condition follows from the definition of (P,Q)-controlled g-Bessel sequence. We only need to prove that the sufficient condition holds. Suppose that $T_{P\Lambda Q}$ is well defined and bounded operator with $\|T_{P\Lambda Q}\| \le \sqrt{B}$. For any $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in J}\langle \Lambda_j Pf, \Lambda_j Qf\rangle &= \sum_{j\in J}\big\langle Q\Lambda_j^*\Lambda_j Pf, f\big\rangle = \langle QS_\Lambda Pf, f\rangle \\ &= \big\langle (QP)^{\frac{1}{2}} S_\Lambda (QP)^{\frac{1}{2}} f, f\big\rangle \\ &= \left\langle \sum_{j\in J}(QP)^{\frac{1}{2}}\Lambda_j^*\Lambda_j (QP)^{\frac{1}{2}} f, f\right\rangle \\ &\le \|T_{P\Lambda Q}\|\left(\sum_{j\in J}\big\|\Lambda_i (QP)^{\frac{1}{2}} f\big\|^2\right)^{\frac{1}{2}}\|f\|\end{aligned}$$

$$= \|T_{P\Lambda Q}\| \left(\sum_{j\in J} \langle \Lambda_j Pf, \Lambda_j Qf\rangle \right)^{\frac{1}{2}} \|f\|.$$

Hence we get

$$\sum_{j\in J} \langle \Lambda_j Pf, \Lambda_j Qf\rangle \leq \|T_{P\Lambda Q}\|^2 \|f\|^2 \leq B\|f\|^2.$$

This shows that $\{\Lambda_j\}_{j\in J}$ is a (P,Q)-controlled g-Bessel sequence for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with bound B. The proof is completed. □

Theorem 4.2 *Let $P,Q \in \mathcal{GL}^+(\mathcal{H})$, $\{\Lambda_j\}_{j\in J}$ be a (P,P)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with the synthesis operator $T_{P\Lambda P}$. Then a (Q,Q)-controlled g-frame $\{\Gamma_j\}_{j\in J}$ is a (P,Q)-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$ if and only if*

$$Q\Gamma_j^* e_{j,k} = U(e_{j,k}\delta_j), \quad j \in J, k \in K_j,$$

where $U : \bigoplus_{j\in J} \mathcal{V}_j \to \mathcal{H}$ is a bounded left-inverse of $T^*_{P\Lambda P}$.

Proof If $\{g_j\}_{j\in J} \in \bigoplus_{j\in J} \mathcal{V}_j$, then

$$\{g_j\}_{j\in J} = \sum_{j\in J} g_j\delta_j = \sum_{j\in J}\sum_{k\in K_j} \langle g_j, e_{j,k}\rangle e_{j,k}\delta_j.$$

Roughly speaking, $\{e_{j,k}\delta_j\}_{j\in J,k\in K_j}$ is an o. n. b. of $\bigoplus_{j\in J} \mathcal{V}_j$. If there exist $U : \bigoplus_{j\in J} \mathcal{V}_j \to \mathcal{H}$ is a bounded left-inverse of $T^*_{P\Lambda P}$ such that

$$Q\Gamma_j^* e_{j,k} = U(e_{j,k}\delta_j), \quad j \in J, k \in K_j.$$

By Lemma 4.1, $\{\Gamma_j\}_{j\in J}$ is a (Q,Q)-controlled g-Bessel sequence for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$. For any $f \in \mathcal{H}$, we have

$$\begin{aligned}
f = UT^*_{P\Lambda P} f &= U\left(\sum_{j\in J}\sum_{k\in K_j} \langle \Lambda_j Pf, e_{j,k}\rangle e_{j,k}\delta_j\right) \\
&= \sum_{j\in J}\sum_{k\in K_j} \langle f, P\Lambda_j^* e_{j,k}\rangle U(e_{j,k}\delta_j) \\
&= \sum_{j\in J}\sum_{k\in K_j} \langle f, Pu_{j,k}\rangle Q\Gamma_j^* e_{j,k} \\
&= \sum_{j\in J} Q\Gamma_j^* \left(\sum_{k\in K_j} \langle Pf, u_{j,k}\rangle e_{j,k}\right) = \sum_{j\in J} Q\Gamma_j^*\Lambda_j Pf,
\end{aligned}$$

where $u_{j,k} = \Lambda_j^* e_{j,k}$. By the definition of controlled dual g-frame, $\{\Gamma_j\}_{j\in J}$ is a (P,Q)-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$.

On the other hand, suppose that a (Q,Q)-controlled g-frame $\{\Gamma_j\}_{j\in J}$ is a (P,Q)-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$. For any $f \in \mathcal{H}$, we have

$$f = \sum_{j\in J} P\Lambda_j^*\Gamma_j Qf = \sum_{j\in J} Q\Gamma_j^*\Lambda_j Pf,$$

that is, $T_{Q\Gamma Q}T^*_{P\Lambda P} = I_{\mathcal{H}}$. Let $U = T_{Q\Gamma Q}$, then $U : \bigoplus_{j\in J}\mathcal{V}_j \to \mathcal{H}$ is a bounded left-inverse of $T^*_{P\Lambda P}$. A calculation as above shows that

$$\sum_{j\in J}\sum_{k\in K_j}\langle f, Pu_{j,k}\rangle Q\Gamma_j^* e_{j,k} = f = \sum_{j\in J}\sum_{k\in K_j}\langle f, Pu_{j,k}\rangle U(e_{j,k}\delta_j), \quad \forall f \in \mathcal{H}.$$

Combining this with the fact $\{e_{j,k}\}_{k\in K_j}$ is an o. n. b. of $\mathcal{V}_j$, we have

$$Q\Gamma_j^* e_{j,k} = U(e_{j,k}\delta_j), \quad j \in J, k \in K_j.$$

The proof is completed. □

Theorem 4.3 *Let $P \in \mathcal{GL}^+(\mathcal{H})$, $\{\Lambda_j\}_{j\in J}$ be a (P,P)-controlled g-frame for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$ with the synthesis operator and frame operator $T_{P\Lambda P}$ and $S_{P\Lambda P}$, respectively. Then $\{\Gamma_j \in L(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is a P-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$ if and only if*

$$\Gamma_j f = (Tf)_j + \Lambda_j S^{-1}_{P\Lambda P}Pf, \quad j \in J, f \in \mathcal{H},$$

where $T : \mathcal{H} \to \bigoplus_{j\in J}\mathcal{V}_j$ is a bounded linear operator satisfying $T_{P\Lambda P}T = 0$.

Proof If $T : \mathcal{H} \to \bigoplus_{j\in J}\mathcal{V}_j$ is a bounded linear operator satisfying $T_{P\Lambda P}T = 0$, then $\{\Gamma_j \in L(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}$ w. r. t. $\{\mathcal{V}_j\}_{j\in J}$. In fact, for any $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in J}\|\Gamma_j f\|^2 &= \sum_{j\in J}\left\|(Tf)_j + \Lambda_j S^{-1}_{P\Lambda P}Pf\right\|^2 \\ &\le 2\left(\sum_{j\in J}\left\|\Lambda_j S^{-1}_{P\Lambda P}Pf\right\|^2 + \|Tf\|^2\right) \\ &\le 2\left(B\left\|S^{-1}_{P\Lambda P}P\right\|^2 + \|T\|^2\right)\|f\|^2,\end{aligned}$$

where B is the upper bound of $\{\Lambda_j\}_{j\in J}$. Furthermore,

$$\begin{aligned}\sum_{j\in J}P\Lambda_j^*\Gamma_j f &= \sum_{j\in J}P\Lambda_j^*\left((Tf)_j + \Lambda_j S^{-1}_{P\Lambda P}Pf\right) \\ &= T_{P\Lambda T}Tf + \sum_{j\in J}P\Lambda_j^*\Lambda_j S^{-1}_{P\Lambda P}Pf = f.\end{aligned}$$

Thus $\{\Gamma_j \in L(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is a P-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$.

Now we prove the converse. Assume that $\{\Gamma_j \in L(\mathcal{H}, \mathcal{V}_j)\}_{j\in J}$ is a P-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$. Define the operator T as follows:

$$T : \mathcal{H} \to \bigoplus_{j\in J}\mathcal{V}_j, \qquad f \mapsto Sf \quad (\forall f \in \mathcal{H})$$

satisfying

$$\Gamma_j f = (Tf)_j + \Lambda_j S^{-1}_{P\Lambda P}Pf, \quad j \in J.$$

For any $f \in \mathcal{H}$, we have

$$\begin{aligned}\|Tf\|^2 &= \sum_{j\in J}\left\|\Gamma_j f - \Lambda_j S_{P\Lambda P}^{-1} Pf\right\|^2 \\ &\leq \sum_{j\in J}\|\Gamma_j f\|^2 + \sum_{j\in J}\left\|\Lambda_j S_{P\Lambda P}^{-1} Pf\right\|^2 + 2\left(\sum_{j\in J}\|\Gamma_j f\|^2\right)^{\frac{1}{2}}\left(\sum_{j\in J}\left\|\Lambda_j S_{P\Lambda P}^{-1} Pf\right\|^2\right)^{\frac{1}{2}} \\ &\leq \left(B_1 + A^{-1} + 2\sqrt{B_1 A^{-1}}\right)\|f\|^2,\end{aligned}$$

where B_1 is the frame upper bound of $\{\Gamma_j\}_{j\in J}$, A is the frame lower bound of $\{\Lambda_j\}_{j\in J}$. Thus T is a linear bounded operator. Moreover, for any $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\langle T_{P\Lambda P} Tf, g\rangle &= \sum_{j\in J}\left\langle P\Lambda_j^* Tf, g\right\rangle = \sum_{j\in J}\left\langle P\Lambda_j^*\left(\Gamma_j f - \Lambda_j S_{P\Lambda P}^{-1} Pf\right), g\right\rangle \\ &= \sum_{j\in J}\left\langle P\Lambda_j^*\Gamma_j f, g\right\rangle - \sum_{j\in J}\left\langle P\Lambda_j^*\Lambda_j S_{P\Lambda P}^{-1} Pf, g\right\rangle \\ &= \langle f, g\rangle - \langle f, g\rangle = 0.\end{aligned}$$

That is, $T_{P\Lambda P} T = 0$. The proof is completed. □

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Author details
[1]Institute of Applied Mathematics, Zhengzhou Shengda University, Zhengzhou, Henan 451191, P.R. China. [2]School of Statistics and Mathematics, Henan Finance University, Zhengzhou, Henan 450000, P.R. China. [3]Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, P.R. China.

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